Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Symbolic Computation 39 (2005) 127–153

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

The Gröbner fan and Gröbner walk for modules

Ruth L. Auerbach*

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

Received 25 October 2003; accepted 11 July 2004
Available online 18 January 2005

Abstract

This paper extends the theory of the Gröbner fan and Gröbner walk for ideals in polynomial rings to the case of submodules of free modules over a polynomial ring. The Gröbner fan for a submodule creates a correspondence between a pair consisting of a cone in the fan and a point in the support of the cone and a pair consisting of a leading monomial submodule (or equivalently, a reduced marked Gröbner basis) and a grading of the free module over the ring that is compatible with the choice of leading monomials. The Gröbner walk is an algorithm based on the Gröbner fan that converts a given Gröbner basis to a Gröbner basis with respect to a different monomial order; the point being that the Gröbner walk can be more efficient than the standard algorithms for Gröbner basis computations with difficult monomial orders. Algorithms for generating the Gröbner fan and for the Gröbner walk are given.

Keywords: Gröbner fan for modules; Gröbner walk

1. Introduction

Gröbner basis theory is a fundamental tool of computational commutative algebra. The theory has been advanced by the introduction of techniques from combinatorics and polyhedral geometry. In particular, such techniques were used to create the concept of the Gröbner fan and Gröbner walk for an ideal of a polynomial ring.

* Tel.: +1 301 405 5101; fax: +1 301 405 5058.
E-mail address: rauerb@math.umd.edu.

0747-7171/$ - see front matter © 2005 Published by Elsevier Ltd
doi:10.1016/j.jsc.2004.07.002

These advances were sparked by the rediscovery of a classification of term orders on a polynomial ring by Robbiano (1985). He showed that term orders are in one-to-one correspondence with a certain subset of real matrices.

On the basis of this classification, Mora and Robbiano (1988) created the Gröbner fan of an ideal of a polynomial ring.

Based on the Gröbner fan, the Gröbner walk is an algorithm, introduced by Collart et al. (1997), that converts one Gröbner basis of an ideal of a polynomial ring to another. This technique is particularly useful for computing Gröbner bases with respect to elimination term orders. Gröbner bases with respect to elimination term orders are necessary in many applications, but are notorious for their inefficiency when used with the standard algorithms.

This paper extends the Gröbner fan and the Gröbner walk to the case of submodules of free modules over a polynomial ring. (Previous work by Assi et al. (2000, 2001) and Smith (2001) has extended the Gröbner fan to the Weyl algebra and $\mathcal{D}$-modules, although without a complete consideration of all possible monomial orders. Also see Saito et al. (2000).)

Section 1.1 describes the classifications of term orders and monomial orders and states the known result that every submodule has finitely many reduced marked Gröbner bases. Section 1.2 discusses the relevant properties of graded modules and their Gröbner bases, and it introduces the concept of compatibility that is used to associate a monomial order with a leading term submodule. Background on polyhedral cones is given in Section 1.3. Sections 2 and 3 expand the Gröbner fan and Gröbner walk to the case of submodules of free modules of finite rank.

1.1. Monomial orders and Gröbner bases

Consider a polynomial ring $R = k[x_1, \dots, x_n]$, where k is a field. A *term* is a power product $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \in R$, where $\alpha = (\alpha_1, \dots, \alpha_n)$. Let $\mathbb{Z}$ be the integers, $\mathbb{Q}$ the rationals, $\mathbb{R}$ the reals. Let $\mathbb{Z}^+$, $\mathbb{Q}^+$, and $\mathbb{R}^+$ be the non-negative integers, rationals, and reals, respectively.

A *term order* is a total order on the terms of R such that 1 is the least element and multiplication by a term does not change the relative order of the terms. Robbiano (1985) rediscovered the classification of term orders on R as $m \times n$ matrices with real entries. This classification was originally done by Riquier (1910), Kolchin (1973), Trevisan (1953), and Zaĭceva (1953). Robbiano classified total orders on $\mathbb{Q}^n$ that are compatible with its properties as a $\mathbb{Z}$-module. Such an ordering can be restricted to a term order if $\alpha > (0, \dots, 0)$ for all $\alpha \in (\mathbb{Z}^+)^n \setminus \{(0, \dots, 0)\}$. The following is Robbiano's theorem.

Theorem 1. *There is a one-to-one correspondence between linear orders $>$ on $\mathbb{Q}^n$ compatible with its $\mathbb{Z}$-module structure and $k \times n$ matrices over $\mathbb{R}$ with rows $(u_1, u_2, \dots, u_k)$ satisfying:*

(1) $k \le n$;
(2) *let d_i be the dimension of the $\mathbb{Q}$-vector space spanned by the entries of u_i; then* $d_1 + d_2 + \cdots + d_k = n$;
(3) $|u_i| = 1$, *for* $i = 1, \dots, k$;

(4) u_i *is in the real completion of the rational subspace orthogonal to the real space generated by* $u_1, \ldots, u_{i-1}$, *for* $i = 2, \ldots, k$.

The correspondence is given by $X^\alpha > X^\beta$ *if and only if*

$$(\alpha \cdot u_1, \alpha \cdot u_2, \ldots, \alpha \cdot u_k) >_{\texttt{lex}} (\beta \cdot u_1, \beta \cdot u_2, \ldots, \beta \cdot u_k).$$

where $>_{\texttt{lex}}$ *is the usual lexicographic order.*

In addition, such an order on $\mathbb{Q}^n$ *restricts to a term order on* R *if and only if the first non-zero entry in each column of the matrix is positive. In particular, a term order has a matrix in which the first row* u_1 *has non-negative entries.*

Next we generalize to monomials in a free module over a polynomial ring. Consider R^t, the free module on R with t components. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_t$ be the usual standard basis vectors on R^t. A *monomial* of R^t is an element $\mathbf{X} = X^\alpha \mathbf{e}_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mathbf{e}_i$, $1 \le i \le t$. After Robbiano's result, several people worked on classifying monomial orders for free modules over polynomial rings. After several partial classifications (e.g. Carrà Ferro and Sit (1994), Caboara and Silvestri (1999)), Rust and Reid (1997), Rust (1998), and independently Horn (1998), classified all monomial orders on free modules.

In this paper, Rust's and Reid's classification of monomial orders will be used. Their classification is as follows:

Theorem 2. *Consider a set of matrices with real entries* $U_1, U_2, \ldots, U_t$, *where the matrix* U_i *is an* $n_i \times n$ *matrix describing a term order on* R *as in Theorem 1, a set of vectors* $\gamma_1, \gamma_2, \ldots, \gamma_t$ *with* $\gamma_i \in \mathbb{R}^{n_i}$ *for* $1 \le i \le t$, *a* $t \times t$ *matrix of non-negative integers* (t_{ij}), *and an element* $\sigma \in S_t$, *the symmetric group on the set* $\{1, 2, \ldots, t\}$. *Define* m_{ij} *to be the largest non-negative integer such that matrices* U_i *and* U_j *have the first* m_{ij} *rows in common. Suppose the entries of the matrix* (t_{ij}) *and* $\sigma \in S_t$ *satisfy:*

(1) $0 \le t_{ij} \le m_{ij}$ *for* $1 \le i, j \le t$;
(2) $t_{ii} = m_{ii} = n_i$ *for* $1 \le i \le t$;
(3) $t_{ij} = t_{ji}$ *for* $1 \le i, j \le t$;
(4) $t_{ik} \ge \min(t_{ij}, t_{jk})$ *for* $1 \le i, j, k \le t$.
(5) *Whenever* $t_{ik} > \max(t_{ij}, t_{jk})$ *and* $\sigma(i) < \sigma(j)$ *for some* $1 \le i, j, k \le t$, *then* $\sigma(k) < \sigma(j)$.

Let $\texttt{Pr}_d(\alpha)$ *denote projection onto the first* d *coordinates of the vector* α. *Then the following defines a monomial order* $>$ *on* R^t:

$$X^\alpha \mathbf{e}_i > X^\beta \mathbf{e}_j \Leftrightarrow \begin{cases} \texttt{Pr}_{t_{ij}}(U_i\alpha + \gamma_i) >_{\texttt{lex}} \texttt{Pr}_{t_{ij}}(U_j\beta + \gamma_j), \text{ or} \\ \texttt{Pr}_{t_{ij}}(U_i\alpha + \gamma_i) = \texttt{Pr}_{t_{ij}}(U_j\beta + \gamma_j) \text{ and } \sigma(i) > \sigma(j). \end{cases}$$

Conversely, any monomial order on R^t, *can be represented as above by a set of matrices* $U_1, \ldots, U_t$, *vectors* $\gamma_1, \ldots, \gamma_t$, *a* $t \times t$ *matrix of non-negative integers* (t_{ij}), *and* $\sigma \in S_t$ *satisfying the conditions above.*

The *leading monomial* with respect to monomial order $>$ of $\mathbf{f} \in R^t$, denoted as $\texttt{lm}_>(\mathbf{f})$, is the monomial in $\mathbf{f}$ which is the largest with respect to $>$. The leading monomial notation will also be applied to sets, i.e. if $S \subseteq R^t$, $\mathtt{lm}_>(S) = \{\mathtt{lm}_>(\mathbf{f}) | \mathbf{f} \in S\}$. Let $M \subseteq R^t$ be a submodule. A set $G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_a\} \subseteq M$ is a *Gröbner basis* for M with respect to monomial order $>$ if $\langle \mathtt{lm}_>(G) \rangle = \langle \mathtt{lm}_>(M) \rangle$. A *reduced Gröbner basis G* for a submodule $M \subseteq R^t$, is a Gröbner basis with respect to $>$ such that for each $\mathbf{g} \in G$, there are no monomials in $\mathbf{g}$ that are divisible by any monomials in $\mathtt{lm}_>(H)$, for $H = G \setminus \{\mathbf{g}\}$, and for each $\mathbf{g} \in G$, the coefficient for the monomial $\mathtt{lm}_>(\mathbf{g})$ is 1. A *marked Gröbner basis G* for a submodule $M \subseteq R^t$ is a Gröbner basis with respect to some monomial order, such that each $\mathbf{g} \in G$ has its leading monomial identified. For more background on Gröbner bases for submodules of free modules over polynomial rings, see Adams and Loustaunau (1994), Kreuzer and Robbiano (2000), Cox et al. (1998), or Eisenbud (1995).

For the existence of either a Gröbner fan or a Gröbner walk, the following theorem is necessary.

Theorem 3. *For any submodule $M \subseteq R^t$, there are only finitely many reduced marked Gröbner bases.*

The proof of the result is essentially the same as the proof for the case of reduced marked Gröbner bases of an ideal $I \subseteq R$. This proof for the ideal case can be found in the first chapter of Sturmfels (1996) and its accompanying note.

*1.2. Graded **R**-modules and monomial orders*

We let $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. $\hat{\mathbb{R}}$ has the usual abelian binary operation of addition with the extra property $-\infty + c = -\infty$ for all $c \in \hat{\mathbb{R}}$, and the order on $\hat{\mathbb{R}}$ is the usual order with $-\infty < c$ for all $c \in \mathbb{R}$. Note that $\hat{\mathbb{R}}$ is an ordered abelian monoid. When $R = k[x_1, x_2, \ldots, x_n]$ is graded over the abelian monoid $\mathbb{R}^a$, R_α denotes the subspace of homogeneous components of degree α. For an R-module M that is an $\hat{\mathbb{R}}^a$-graded R-module, the k-subspace of homogeneous components of degree α will be denoted as M_α. All the gradings of R that are considered in this paper have the property that the variables $x_1, \ldots, x_n$ are homogeneous. Also, all the gradings of R^t that are considered have the property that the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_t$ are homogeneous.

Suppose we have an $\mathbb{R}^a$-grading on R and an $\hat{\mathbb{R}}^a$-grading on the R-module R^t. Let $\mathtt{Terms}(R)$ be the multiplicative abelian monoid of the terms of R. The grading on R can be represented by the homomorphism $\tau : \mathtt{Terms}(R) \to \mathbb{R}^a$ given by $X^\alpha \mapsto \beta$ where $X^\alpha \in R_\beta$. Let $\mathtt{Mon}(R^t)$ be the monomials of R^t. The grading on R^t can be represented by the homomorphism $\phi : \mathtt{Mon}(R^t) \to \hat{\mathbb{R}}^a$ such that $\mathbf{X} \mapsto \alpha$ if and only if $\mathbf{X} \in (R^t)_\alpha$. The map ϕ is *compatible* with the map τ if $\phi(Y\mathbf{X}) = \tau(Y) + \phi(\mathbf{X})$ for $Y \in \mathtt{Terms}(R)$ and $\mathbf{X} \in \mathtt{Mon}(R^t)$. To eliminate ambiguity, when there is more than one grading considered on a module, the word **ϕ-degree** will be used to refer to the image of a monomial by ϕ. Also, an element of R^t will be called **ϕ-homogeneous** if every monomial in the element has the same image under ϕ. Furthermore, the notation $\mathtt{deg}_\phi(\mathbf{X}) = \phi(\mathbf{X})$ for $\mathbf{X} \in \mathtt{Mon}(R^t)$ will be used in the paper.

Definition 4. Extend the lexicographic order to $\hat{\mathbb{R}}^a$ in the obvious way. Let $\mathbf{f} = \sum_{i=1}^t f_i \mathbf{e}_i \in R^t$, with $f_i = \sum_j a_{ij} X^{\alpha_{ij}} \in R$, $i = 1, \ldots, t$, $a_{ij} \neq 0$. Then define the **leading monomials with respect to grading ϕ** as

$$\mathtt{lm}_\phi(\mathbf{f}) = \sum_{i,j \text{ s.t. } \phi(X^{a_{ij}}\mathbf{e}_i) \geq \phi(X^{a_{kl}}\mathbf{e}_k) \forall k,l} a_{ij} X^{a_{ij}} \mathbf{e}_i.$$

(The leading monomial notation will also be applied to sets, for which we mean the set of leading monomials, one for each element of the set.)

Definition 5. A monomial order $>$ on R^t is **compatible** with an $\hat{\mathbb{R}}^a$-grading given by the map ϕ if, given monomials $\mathbf{X}, \mathbf{Y} \in \mathtt{Mon}(R^t)$,

$$\mathbf{X} > \mathbf{Y} \Rightarrow \phi(\mathbf{X}) \geq_{\mathtt{lex}} \phi(\mathbf{Y}).$$

More generally, an $\hat{\mathbb{R}}^a$-grading of R^t given by the map ϕ is **compatible on a marked set** $S \subseteq R^t$ if, given an $\mathbf{f} \in S$ with marked leading monomial $\mathbf{X}$, then $\phi(\mathbf{X}) \geq_{\mathtt{lex}} \phi(\mathbf{Y})$ for all monomials $\mathbf{Y}$ in $\mathbf{f}$, or equivalently $\mathtt{lm}_>\left(\mathtt{lm}_\phi(\mathbf{f})\right) = \mathtt{lm}_>(\mathbf{f})$.

In this paper, we will frequently be considering the case of a monomial order $>$ on R^t with reduced marked Gröbner basis G for a submodule $M \subseteq R^t$ that is compatible with an $\hat{\mathbb{R}}^a$-grading on G.

The standard results regarding Gröbner bases and the property of homogeneity apply. Specifically, a ϕ-homogeneous submodule has a ϕ-homogeneous Gröbner basis. Also, if G is a reduced marked Gröbner basis with respect to monomial order $>$ that is compatible with a ϕ-grading on G, then $\mathtt{lm}_\phi(G)$ is a reduced Gröbner basis for $\left\langle \mathtt{lm}_\phi(G) \right\rangle$ with respect to $>$.

The following theorem is the main tool for the Gröbner walk for modules. The proof is essentially the same as in the ideal case. See Cox et al. (2001).

Theorem 6. *Let $M \subseteq R^t$ be a submodule. Let there be an $\hat{\mathbb{R}}^a$-grading on R^t defined by ϕ. Let $>_1$ and $>_2$ be monomial orders which are compatible with ϕ, and let G be a Gröbner basis for M with respect to $>_2$. Let H be a Gröbner basis for $\left\langle \mathtt{lm}_\phi(M) \right\rangle$ with respect to $>_1$. Using the division algorithm with respect to $>_2$, write each $\mathbf{h} \in H$ as*

$$\mathbf{h} = \sum_{\mathbf{g} \in G} p_{\mathbf{g},\mathbf{h}} \mathtt{lm}_\phi(\mathbf{g}),$$

with $p_{\mathbf{g},\mathbf{h}} \in R$. For each $\mathbf{h} \in H$ define $\mathbf{f_h}$ by

$$\mathbf{f_h} = \sum_{\mathbf{g} \in G} p_{\mathbf{g},\mathbf{h}} \mathbf{g}.$$

Then the set $F = \{\mathbf{f_h} | \mathbf{h} \in H\}$ forms a Gröbner basis for M with respect to $>_1$.

1.3. *Polyhedral geometry*

This section is background for the polyhedral geometry that is used in the paper. See Sturmfels (1996) for more details. Let $\mathbb{R}$ be the real numbers. Let $\mathbb{R}^+$ be the non-negative real numbers.

A *polyhedron* in $\mathbb{R}^t$ is a finite intersection of closed half-spaces in $\mathbb{R}^t$. Thus a polyhedron can be written as $P = \{\omega \in \mathbb{R}^t | A \cdot \omega \leq \gamma\}$, where A is a matrix with t columns and $\gamma \in \mathbb{R}^n$. If each of the *supporting* hyperplanes of the polyhedron intersects the origin, or, equivalently, $P = \{\omega \in \mathbb{R}^t | A \cdot \omega \leq (0, 0, \ldots, 0)\}$, then the polyhedron is a *(polyhedral) cone*. For any polyhedral cone P there exist vectors $\omega_1, \omega_2, \ldots, \omega_m \in \mathbb{R}^t$ such that

$$P = \{a_1\omega_1 + \cdots + a_m\omega_m | a_1, \ldots, a_m \in \mathbb{R}^+\}.$$

A *face* of a polyhedron $P \subseteq \mathbb{R}^t$ is a subset of P which maximizes some linear functional, i.e. for every $\omega \in \mathbb{R}^t$,

$$\texttt{face}_\omega(P) = \{u \in P | \omega \cdot u \geq \omega \cdot v \text{ for all } v \in P\}$$

is a face of P. The dimension zero faces are called *vertices*, and the codimension one faces are called *facets*. Note that the property of being a face is transitive, i.e. if F is a face of P and P is a face of Q, then F is a face of Q. The proof of this fact is straightforward. A *(polyhedral) complex* Δ is a finite collection of polyhedra in R^t such that if $P \in \Delta$ and F is a face of P, then $F \in \Delta$, and if $P_1, P_2 \in \Delta$ and $P_1 \cap P_2 \neq \emptyset$, then $P_1 \cap P_2$ is a face of P_1 and P_2. The *support* of a complex Δ is $|\Delta| = \cup_{P \in \Delta} P$. A complex which consists of cones is called a **fan**.

Example 7. The Gröbner fan for an ideal is an example of a fan. See Cox et al. (2001), Mora and Robbiano (1988), or Sturmfels (1996) for more details. A main result of this paper is a generalization of the fan for submodules of R^t.

The following construction will be used to create fans.

Definition 8. Let $P_i \subseteq \mathbb{R}^{t_i}$ be a polyhedron for $1 \leq i \leq m$. Then the **product polyhedron** of the set $\{P_1, P_2, \ldots, P_m\}$ is

$$\prod_{i=1}^{m} P_i := \{(\alpha_1, \alpha_2, \ldots, \alpha_m) | \alpha_i \in P_i \text{ for } 1 \leq i \leq m\} \subseteq \mathbb{R}^{t_1 + \cdots + t_m}.$$

Moreover, let Δ_i be a complex in $\mathbb{R}^{t_i}$, for $1 \leq i \leq m$. Then the **product complex** $\prod_{i=1}^m \Delta_i$ is a complex in $\prod_{i=1}^m \mathbb{R}^{t_i}$ where $Q \in \prod_{i=1}^m \Delta_i$ if and only if $Q = \prod_{i=1}^m P_i$ for some choice of $P_i \in \Delta_i$ with $1 \leq i \leq m$.

2. The Gröbner fan for submodules of R^t

The aim of this section is to generalize the concept of a Gröbner fan to include submodules of free modules of finite rank. We generalize the notion of the position over term type monomial order to be any monomial order for which there exists $1 \leq i \neq j \leq t$ such that $X\mathbf{e}_i > Y\mathbf{e}_j$ for all $X, Y \in \texttt{Terms}(R)$. This notion corresponds to the case of $t_{ij} = 0$ in the classification of monomial orders in Theorem 2.

Proposition 9. *Given a representation of the monomial order as in Theorem 2, let $\sim$ be a relation on the set $\{1, \ldots, t\}$ defined by*

$$i \sim j \textit{ if and only if } t_{ij} \neq 0, 1 \leq i, j \leq t.$$

The relation $\sim$ is an equivalence relation. Furthermore, $\sim$ is a well-defined property of the monomial order, i.e. the equivalence relation $\sim$ is independent of its representation *in Theorem 2. Moreover, the monomial order gives a natural, well-defined ordering of the equivalence classes: for $i \not\sim j$, the equivalence class of i is greater than the equivalence class of j, if $\mathbf{e}_i > \mathbf{e}_j$.*

Proof. It is straightforward to check that $\sim$ is an equivalence relation.

Suppose a monomial order $>$ has the following two sets of descriptors:

(1) Let one set of descriptors for $>$ be the matrices U_i, vectors γ_i, integers t_{ij} for $1 \le i, j \le t$, and $\sigma \in S_t$.
(2) Let the second set of descriptors for $>$ be the matrices V_i, vectors δ_i, integers v_{ij} for $1 \le i, j \le t$, and $\tau \in S_t$.

Suppose there exist $1 \le i, j \le t$ such that $t_{ij} = 0$ and $v_{ij} \neq 0$. Without loss of generality, assume $\sigma(i) > \sigma(j)$. Hence, the first set of descriptors says that $X^\alpha \mathbf{e}_i > X^\beta \mathbf{e}_j$ for all $\alpha, \beta \in (\mathbb{Z}^+)^n$. However, using the second set of descriptors, because non-trivial linear functions are unbounded, there exists $\alpha_1, \alpha_2 \in (\mathbb{Z}^+)^n$ such that $X^{\alpha_1} \mathbf{e}_i < X^{\alpha_2} \mathbf{e}_j$, contradicting the first set of descriptors. So there cannot be $t_{ij} = 0$ in one set of descriptors and $v_{ij} \neq 0$ in the other set. Therefore, $\sim$ is a well-defined property of a monomial order.

It suffices to show that for $i, j \in \{1, \dots, t\}$ representatives of distinct equivalence classes for a monomial order $>$ such that $\mathbf{e}_i > \mathbf{e}_j$, if $i', j' \in \{1, \dots, t\}$ such that $i' \sim i$ and $j' \sim j$, then $\mathbf{e}_{i'} > \mathbf{e}_{j'}$. Suppose not. Since $i \sim i'$, there exists $\alpha, \alpha' \in (\mathbb{Z}^+)^n$ such that $X^{\alpha'} \mathbf{e}_{i'} > X^\alpha \mathbf{e}_i$, and similarly, there exists $\beta, \beta' \in (\mathbb{Z}^+)^n$ such that $X^\beta \mathbf{e}_j > X^{\beta'} \mathbf{e}_{j'}$. Therefore,

$$X^{\alpha'+\beta} \mathbf{e}_j > X^{\alpha'+\beta'} \mathbf{e}_{j'} > X^{\alpha'+\beta'} \mathbf{e}_{i'} > X^{\alpha+\beta'} \mathbf{e}_i.$$

However, since i and j are in distinct equivalence classes with respect to $>$ and $\mathbf{e}_i > \mathbf{e}_j$, any set of matrices U_a, vectors γ_a, integers t_{ab} for $1 \le a, b \le t$, and $\sigma \in S_t$ representing $>$, as in Theorem 2, has the property that $t_{i,j} = 0$ and $\sigma(i) > \sigma(j)$. However, such descriptors require that $X^{\alpha+\beta'} \mathbf{e}_i > X^{\alpha'+\beta} \mathbf{e}_j$, contradicting the inequality above. Thus the $\sim$-equivalence classes with respect to $>$ have a natural, well-defined order with respect to $>$. $\square$

The following definition is used to classify the monomial orders based on the equivalence relation $\sim$.

Definition 10. Suppose a monomial order $>$ has q equivalence classes with respect to $\sim$, and $h_1, h_2, \dots, h_q \in \{1, \dots, t\}$ are representatives for each $\sim$ equivalence class. Let $[h_j]$ denote the equivalence class that h_j represents. Then we say the monomial order $>$ is of **type** $\big([h_1], [h_2], \dots, [h_q]\big)$, if

$$[h_1] > [h_2] > \cdots > [h_q].$$

Note that for monomial orders for which $i \sim j$ for all $i, j \in \{1, \dots, t\}$, instead of referring to them as type $([i])$, the more compact notation $[i]$ will be used.

There will be separate Gröbner fans for each configuration of equivalence classes.

2.1. The Gröbner fans

The fans are based on the following set of gradings of R and R^t.

Definition 11. Let $(W, r) \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$, where $\mathtt{Mat}_{q\times n}(\mathbb{R}^+)$ denotes the set of $q \times n$ matrices with non-negative real entries and $r = (r_1, \ldots, r_t)$. The matrix W with rows $(\omega_1, \ldots, \omega_q)$ defines an $\mathbb{R}^q$-grading on R given by the map

$$X^\alpha \mapsto (\alpha \cdot \omega_1, \alpha \cdot \omega_2, \ldots, \alpha \cdot \omega_q).$$

Call this grading of R the W**-grading**.

Note that $\hat{\mathbb{R}}$ is an ordered abelian monoid. Let $P = ([h_1], [h_2], \ldots, [h_q])$ be a partition of $\{1, 2, \ldots, t\}$, $1 \le q \le t$. Furthermore, (W, r, P) defines an $\hat{\mathbb{R}}^q$-grading on R^t given by the map $X^\alpha \mathbf{e}_b \mapsto (i_1, \ldots, i_q)$, where

$$i_j = \begin{cases} \alpha \cdot \omega_j + r_b & \text{if } b \in [h_j], \\ -\infty & \text{otherwise.} \end{cases}$$

Call this grading of R^t the (W, r, P)**-grading**.

The definition above also establishes the identification of a (W, r, P)-grading with a point $(W, r) \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$.

Similarly, the proposition below establishes the identification of a monomial order of type $P = ([h_1], [h_2], \ldots, [h_q])$ and a grading by a point in $\mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$.

Proposition 12. *Let $>$ be a type $P = ([h_1], [h_2], \ldots, [h_q])$ monomial order on R^t. As in Theorem 2, let matrices U_i, vectors γ_i, integers t_{ij} ($1 \le i, j \le t$), and $\sigma \in S_t$ define the monomial order $>$. Then $>$ is compatible with the (W, r, P)-grading, where the vector $r = (r_1, r_2, \ldots, r_t)$ with r_j the first component of the vector γ_j and the matrix $W = (\omega_1, \ldots, \omega_q)$ with ω_a the common first row of U_j for $j \in [h_a]$. In particular, $\mathtt{lm}_>\left(\mathtt{lm}_{(W,r,P)}(\mathbf{f})\right) = \mathtt{lm}_>(\mathbf{f})$ for each $\mathbf{f} \in R^t$.*

Proof. Suppose $X^\alpha \mathbf{e}_c > X^\beta \mathbf{e}_d$ with $c \in \left[h_{c_*}\right]$, $d \in \left[h_{d_*}\right]$, $X^\alpha \mathbf{e}_c \in R^t_a$, and $X^\beta \mathbf{e}_d \in R^t_b$.

If $c_* \ne d_*$, then the result follows by looking at which components of a and b are not $-\infty$.

If $c_* = d_*$, the jth component of a and b is $-\infty$ for $j \ne c_*$. The values $(\omega_{c_*} \cdot \alpha) + r_c$ and $(\omega_{c_*} \cdot \beta) + r_d$ are the c_*th components of a and b, respectively. Since $c \sim d$, we have $t_{cd} \ge 1$. Since

$$\mathtt{Pr}_{t_{cd}}(U_c\alpha + \gamma_c) >_{\mathtt{lex}} \mathtt{Pr}_{t_{cd}}(U_d\beta + \gamma_d),$$

by looking at the first coordinates of each, we get $a \ge_{\mathtt{lex}} b$. $\square$

Next, we define the cones for the fan, a construction that is essentially induced by the identifications above.

Definition 13. Each reduced marked Gröbner basis G for a submodule $M \subseteq R^t$ with respect to a monomial order $>$ of type $P = \big([h_1], [h_2], \ldots, [h_q]\big)$ is associated with a subset $C_G \subseteq \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$, called a G**-cone**, defined by

$$C_G = \left\{ (W, r) \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t \;\middle|\; \begin{array}{l} > \text{ is compatible on } G \\ \text{with a } (W, r, P)\text{-grading} \end{array} \right\}.$$

The definition of the cones also shows how to identify leading monomial submodules with cones in the fan. Specifically, for a leading monomial submodule, take its associated reduced marked Gröbner basis G, and identify it with the cone C_G.

Next, it is shown that the sets C_G truly are cones.

Proposition 14. *For any reduced marked Gröbner basis G of a submodule $M \subseteq R^t$ with respect to a monomial order of type $P = \big([h_1], [h_2], \ldots, [h_q]\big)$, $C_G \subseteq \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$ is a polyhedral cone.*

Proof. Suppose $X^\alpha \mathbf{e}_i$ is the identified leading monomial of some $\mathbf{g} \in G$. Then by the equivalent definition of compatibility, $(W, r) \in C_G$ if and only if $X^\alpha \mathbf{e}_i$ is a monomial in $\mathtt{lm}_{(W,r,P)}(\mathbf{g})$. Let $\mathbf{g} = \sum_{y=1}^t f_i \mathbf{e}_i$.

Let matrix $W = (\omega_1, \omega_2, \ldots, \omega_q)$, and vector $r = (r_1, r_2, \ldots, r_t)$. The monomial $X^\alpha \mathbf{e}_i$ is in $\mathtt{lm}_{(W,r,P)}(\mathbf{g})$ if and only if every term X^β in f_j with $i \in [h_a]$ and $j \in [h_b]$ satisfies either

(1) $a < b$ or
(2) $a = b$ and $\alpha \cdot \omega_a + r_i \geq \beta \cdot \omega_a + r_j$.

The collection of these linear inequalities in (2) forms a polyhedral cone.

For a $(W, r) \in \big((\mathbb{R}^+)^n\big)^q \times \mathbb{R}^t$ to be in C_G, it has to be in the polyhedral cone for each $\mathbf{g} \in G$. Hence (W, r) is in the intersection of a collection of polyhedral cones, which itself is a polyhedral cone. □

Below, the necessary intersection property of the fan is shown.

Proposition 15. *Let G and H be distinct reduced marked Gröbner bases of a submodule $M \subseteq R^t$ with respect to monomial orders $>_G$ and $>_H$, respectively, of type $P = \big([h_1], [h_2], \ldots, [h_q]\big)$. Then $C_G \cap C_H$ is both a face of C_G and a face of C_H.*

Proof. It suffices to show that $C_G \cap C_H$ is a face of C_G.

Let F be the smallest face of C_G which contains $C_G \cap C_H$. It suffices to show that $F \subseteq C_H$. Suppose not.

Let

$$F' = F \Bigg\backslash \left(\bigcup_{E \text{ s.t. } E \text{ is a proper face of } F} E \right),$$

i.e. F' is the relative interior of F. Using topological considerations and the convexity of polyhedral cones, it can be shown that $F' \cap C_H \neq \emptyset$ and $F' \not\subseteq C_H$. Also, just as in the ideal case, for any two points $(U, p), (V, s) \in F'$,

$$\mathtt{lm}_{(U,p,P)}(G) = \mathtt{lm}_{(V,s,P)}(G).$$

Now we are ready to obtain the contradiction to our assumption that $F \not\subseteq C_H$. By the above, we can choose $(W, r) \in C_H \cap F'$ and $(V, s) \in F' \setminus C_H$. Since $\mathtt{lm}_{(W,r,P)}(G) = \mathtt{lm}_{(V,s,P)}(G)$, we have

$$\langle \mathtt{lm}_{(W,r,P)}(M) \rangle = \langle \mathtt{lm}_{(W,r,P)}(G) \rangle = \langle \mathtt{lm}_{(V,s,P)}(G) \rangle = \langle \mathtt{lm}_{(V,s,P)}(M) \rangle .$$

Also we know that $\mathtt{lm}_{(W,r,P)}(H)$ is a Gröbner basis with respect to $>_H$ for the submodule $\langle \mathtt{lm}_{(W,r,P)}(M) \rangle$. Therefore, it must be that for each $\mathbf{g} \in H$, the monomial $\mathtt{lm}_{>_H}\left(\mathtt{lm}_{(V,s,P)}(\mathbf{g})\right)$ is divisible by an element of

$$\mathtt{lm}_{>_H}\left(\mathtt{lm}_{(W,r,P)}(H)\right) = \mathtt{lm}_{>_H}(H).$$

However, since H is a reduced Gröbner basis, the only possible divisor is $\mathtt{lm}_{>_H}(\mathbf{g})$. Therefore, it must be that $\mathtt{lm}_{>_H}(\mathbf{g})$ is a monomial in $\mathtt{lm}_{(V,s,P)}(\mathbf{g})$. Hence, we have that $(V, s) \in C_H$, contradicting our assumption.

Therefore, it must be that $F \subseteq (C_G \cap C_H)$. So we have that $C_G \cap C_H$ is exactly F. $\square$

The next proposition shows how to construct a monomial order on R^t of type $P = \left([h_1], [h_2], \ldots, [h_q]\right)$ from the set of monomial orders on R^{ζ_i} of type $[h_i]$, where $\zeta_i = |[h_i]|$, for $1 \le i \le q$.

Proposition 16. *Let $P = \left([h_1], [h_2], \ldots, [h_q]\right)$ be a partition of $\{1, 2, \ldots, t\}$, where the h_i's are representatives of each subset in the partition. Let matrices U_c, vectors γ_c, integers t_{cd} for $c, d \in [h_j]$, and $\sigma^{(j)} \in S_{[h_j]}$, where $S_{[h_j]}$ is the symmetric group on the set $[h_j]$, define the monomial order $>^{(j)}$ with one $\sim$ equivalence class on R^{ζ_j} as in Theorem 2, where $\zeta_j = \left|[h_j]\right|$. Define $t_{ij} = 0$, for $i \in [h_a]$, $j \in [h_b]$, $a \neq b$. Define $\sigma \in S_t$ as $\sigma(j) = \zeta_{a+1} + \cdots + \zeta_q + s_j^{(a)}$, where $s_j^{(a)} = |\{i \in [h_a] : \sigma^{(a)}(i) \le \sigma^{(a)}(j)\}|$ for $j \in [h_a]$.*

Then the matrices $U_1, \ldots, U_t$, vectors $\gamma_1, \ldots, \gamma_t$, integers t_{ij} for $1 \le i, j \le t$, and $\sigma \in S_t$ define a monomial order $>$ of type P, as in Theorem 2.

Proof. It is straightforward to check that t_{ij}, $1 \le i, j \le t$, satisfy conditions (1)–(4) of Theorem 2.

It remains to show that σ satisfies condition (5) of Theorem 2:

$$t_{ik} > \max(t_{ij}, t_{jk}) \text{ and } \sigma(i) < \sigma(j) \Rightarrow \sigma(k) < \sigma(j).$$

If $t_{ik} = 0$, the condition is trivially satisfied. So we may assume $t_{ik} \neq 0$. Hence $i \sim k$. Let $i, k \in [h_a]$ and $j \in [h_b]$. Assume $\sigma(i) < \sigma(j)$. Therefore

$$\zeta_{a+1} + \cdots + \zeta_q + s_i^{(a)} < \zeta_{b+1} + \cdots + \zeta_q + s_j^{(b)}.$$

Since $0 < s_j^{(b)} \le \zeta_b$, we have $b \le a$. The case $b < a$ follows from $s_k^{(a)} \le \zeta_a$. The case $a = b$ follows because $\sigma^{(a)}(k) < \sigma^{(a)}(j)$ implies that $s_k^{(a)} < s_j^{(a)}$. $\square$

The result below shows the support of the fan for the case of monomial orders with one $\sim$ equivalence class.

Proposition 17. *A grading by $(\omega, r, \{1, 2, \ldots, t\})$, with $\omega \in \mathtt{Mat}_{1\times n}(\mathbb{R}^+)$ and $r \in \mathbb{R}^t$, is compatible with some monomial order $>$ that has only one $\sim$ equivalence class.*

Proof. By Proposition 12 and Theorem 2, any set of matrices $U_1, \ldots, U_t$, vectors $\gamma_1, \ldots, \gamma_t \in \mathbb{R}^n$, non-negative integers $\{t_{ij}\}_{1\le i,j\le t}$, and $\sigma \in S_t$ where the first row of U_i is $\frac{\omega}{|\omega|}$ and the first coordinate of γ_i is $\frac{r_i}{|\omega|}$ and which satisfy Theorem 2 shows the existence of the monomial order. Set $U_i = \frac{\omega}{|\omega|}$, $1 \le i \le t$, the $1 \times n$ matrices, and set $\gamma_i = \left(\frac{r_i}{|\omega|}\right)$, $1 \le i \le t$, the vector of length one. Set $t_{ij} = 1$ for $1 \le i, j \le t$. Choose any $\sigma \in S_t$. Then this collection of matrices, vectors, integers, and σ form a monomial order by Theorem 2. □

The next proposition shows the support of the fans for the case of general monomial orders.

Proposition 18. *Let $M \subseteq R^t$ be a submodule. Let $P = ([h_1], [h_2], \ldots, [h_q])$ be an ordered partition of $\{1, 2, \ldots, t\}$. Every $(W, r) \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$ defines a (W, r, P)-grading of R^t that is compatible with some monomial order $>$ of type P.*

Proof. Such a monomial order can be constructed in the following way. Let vector $r = (r_1, r_2, \ldots, r_t)$ and matrix $W = (\omega_1, \omega_2, \ldots, \omega_q)$. For each $1 \le j \le q$, consider the $\rho_j = (\omega_j, r^{(j)}, \{z_1, z_2, \ldots, z_{\zeta_j}\})$-grading on R^{ζ_j}, where $r^{(j)} = (r_{z_1}, r_{z_2}, \ldots, r_{z_{\zeta_j}})$ with $[h_j] = \{z_1 < z_2 < \cdots < z_{\zeta_j}\}$. By Proposition 17, there exists a monomial order $>^{(j)}$ on R^{ζ_j} that has one $\sim$ equivalence class and is compatible with the ρ_j-grading.

Then combine the monomial orders $>^{(1)}, \ldots, >^{(q)}$ as in Proposition 16 to define a monomial order $>$ of type P. By construction, this $>$ is compatible with a (W, r, P)-grading. □

Finally this leads to the main result:

Theorem 19. *For any submodule $M \subseteq R^t$, the set*

$$\left\{ C_G \;\middle|\; \begin{array}{c} G \text{ is a reduced marked Gröbner basis for } M \text{ with respect} \\ \text{to a monomial order } > \text{ of type } ([h_1], [h_2], \ldots, [h_q]) \end{array} \right\}$$

is a fan in the space $\left((\mathbb{R}^+)^n\right)^q \times \mathbb{R}^t = \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$. Call the fan the $([h_1], [h_2], \ldots, [h_q])$ **Gröbner fan for** *M.*

Furthermore, the support of the fan is $\mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$.

One unexpected difference between the submodule case and the ideal case is that the G-cones may not have interior points. The example below illustrates this property.

Example 20. Let $R = \mathbf{Q}[x, y, z]$. Let $M \subseteq R^2$ be the submodule generated by

$$G = \left\{ \begin{array}{c} \mathbf{g}_1 = (x^2 + z^2)\mathbf{e}_1 + y\mathbf{e}_2, \mathbf{g}_2 = y\mathbf{e}_1 + (x^2 + z^2)\mathbf{e}_2, \\ \mathbf{g}_3 = (y^2 z - yz^2)\mathbf{e}_2, \mathbf{g}_4 = (y^2 z^2 - y^3 z)\mathbf{e}_1 \end{array} \right\}.$$

Consider the monomial order $>$ given by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix},$$

the integer matrix $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, the vectors $\gamma_1 = (0, 0, 0)$ and $\gamma_2 = (-2, 1, 0)$, and $\sigma = (1\ \ 2) \in S_2$. We compute $\mathtt{lm}_>(\mathbf{g}_1) = x^2\mathbf{e}_1$, $\mathtt{lm}_>(\mathbf{g}_2) = x^2\mathbf{e}_2$, $\mathtt{lm}_>(\mathbf{g}_3) = y^2z\mathbf{e}_2$, $\mathtt{lm}_>(\mathbf{g}_4) = y^2z^2\mathbf{e}_1$. Furthermore, it can be checked that G is a reduced Gröbner basis with respect to $>$.

The bounds for C_G from the vector $\mathbf{g}_1$ are $(2, 0, -2)\cdot\omega \geq 0$ and $(2, -1, 0)\cdot\omega - r \geq 0$. The bounds from the vector $\mathbf{g}_2$ are $(2, 0, -2)\cdot\omega \geq 0$ and $(-2, 1, 0)\cdot\omega - r \leq 0$. The bound from the vector $\mathbf{g}_3$ is $(0, 1, -1)\cdot\omega \geq 0$. The bound from the vector $\mathbf{g}_4$ is $(0, -1, 1)\cdot\omega \geq 0$.

From the bounds for C_G from $\mathbf{g}_1$ and $\mathbf{g}_2$ we get for $\omega = (\omega_1, \omega_2, \omega_3)$ the inequalities $\omega_1 \geq \omega_3$ and $-2\omega_1 + \omega_2 \leq r \leq 2\omega_1 - \omega_2$. From the bounds from $\mathbf{g}_3$ and $\mathbf{g}_4$ we get $\omega_2 = \omega_3$. This last restriction $\omega_2 = \omega_3$ shows that C_G does not have interior points.

Without loss of generality, in the remainder of this section, we define the integers $1 \leq v_1 < v_2 < \cdots < v_q = t$ such that

$$
\begin{aligned}
[h_1] &= \{1, 2, \ldots, v_1\} \\
[h_2] &= \{v_1 + 1, \ldots, v_2\} \\
&\ \ \vdots \\
\left[h_q\right] &= \{v_{q-1} + 1, \ldots, v_q\},
\end{aligned}
$$

which is justified by relabelling the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_t$. Thus, $|[h_1]| = v_1$, and $|[h_i]| = v_i - v_{i-1}$, for each $2 \leq i \leq q$.

Also, in the remainder of this section, the definition of a product of polyhedra (or cones) (see Definition 8) is slightly altered. For C_i, a cone in the $[h_i]$ Gröbner fan, $1 \leq i \leq q$, we define the cone

$$
\prod_{i=1}^{q} C_i := \left\{ (W, r) \;\middle|\; \begin{array}{c} W \text{ is a matrix with rows } \omega_1, \omega_2, \ldots, \omega_q, \\ \text{and } r \text{ is the concatenation of } r_1, r_2, \ldots, r_q, \\ \text{such that } (\omega_i, r_i) \in C_i,\ 1 \leq i \leq q \end{array} \right\}.
$$

Another way to view these Gröbner fans is as product fans (see Definition 8 of the product complex) of one $\sim$ equivalence class Gröbner fans.

Theorem 21. *Let F be the $P = \big([h_1], [h_2], \ldots, [h_q]\big)$ Gröbner fan for a submodule $M \subseteq R^t$. Let F_i be the $[h_i]$ Gröbner fan for the submodule*

$$
N_i = \left\{ \mathbf{g} = \sum_{j \in [h_i]} f_j \mathbf{e}_j \;\middle|\; \begin{array}{l} \textit{there exists } \mathbf{h} = \left(\displaystyle\sum_{v_i + 1 \leq c \leq v_q} f_c \mathbf{e}_c \right) \\ \textit{where each } f_c \in R, \textit{ and } \mathbf{g} + \mathbf{h} \in M \end{array} \right\} \subseteq R^{|[h_i]|},
$$

for $1 \leq i \leq q$. Then $F = \prod_{i=1}^{q} F_i$.

Proof. First, it will be shown that each cone in the P Gröbner fan for a submodule $M \subseteq R^t$ is a product of cones in the $[h_i]$ Gröbner fans, F_i, $1 \leq i \leq q$. Let G be a reduced marked Gröbner basis for M with respect to any monomial order $>$ of type P. Define a set of maps $\phi_a : M \to N_a$ for $1 \leq a \leq q$ by

$$
\sum_{i=1}^{t} f_i \mathbf{e}_i \mapsto \begin{cases} \displaystyle\sum_{i \in [h_a]} f_i \mathbf{e}_i & \text{if } \ f_j = 0 \ \text{ for } \ 1 \leq j \leq v_{a-1}, \\ 0 & \text{otherwise.} \end{cases}
$$

Note that the maps ϕ_a are not homomorphisms for $a > 1$. Furthermore, $\phi_a(G)$ is a Gröbner basis for N_a with respect to $>$. (See this fact by checking the equivalence of the leading term submodules.)

The cone C_G is the set of points $(W, r) \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t = \left((\mathbb{R}^+)^n\right)^q \times \mathbb{R}^t$, with matrix $W = (\omega_1, \omega_2, \ldots, \omega_q)$ and vector $r = (r_1, \ldots, r_t)$, that satisfy all the inequalities

$$(\alpha - \beta) \cdot \omega_a + r_i - r_j \geq 0,$$

where $X^\alpha \mathbf{e}_i = \mathtt{lm}_>(\mathbf{g})$ and $X^\beta \mathbf{e}_j$ is a monomial in $\mathbf{g}$ for some $\mathbf{g} \in G$, with $i, j \in [h_a]$ for some $1 \leq a \leq q$. However, the inequalities where $a = a_0$ define the cone $C_{\phi_{a_0}(G)}$ in the $[h_{a_0}]$ Gröbner fan of N_{a_0}. So $(W, r) \in C_G$ implies that $(\omega_a, r^{(a)}) \in C_{\phi_a(G)}$ for each $1 \leq a \leq q$, where $r^{(a)} = (r_{v_{a-1}+1}, \ldots, r_{v_a})$. Hence C_G is the product of the cones $C_{\phi_a(G)}$ for each $1 \leq a \leq q$.

Since the support of the P Gröbner fan for M and the support of F is the same, we can conclude that all codimension zero G-cones in F are G-cones in the P Gröbner fan for M. $\square$

2.2. *Algorithms for computing fans*

The algorithm for computing a general Gröbner fan in this article computes a product of one $\sim$ equivalence class Gröbner fans. Once the computation is broken up into computations of one $\sim$ equivalence class Gröbner fans, there are two more steps. The next step is finding all the codimension zero cones. The final step is finding the cones of higher codimension. Therefore, the algorithm will be given in three parts.

Below is an algorithm for finding the codimension zero cones of a one $\sim$ equivalence class Gröbner fan and their corresponding reduced marked Gröbner bases for a given submodule $M \subseteq R^t$. This algorithm finds all the cones in the fan, but not necessarily all the reduced marked Gröbner bases with respect to one $\sim$ equivalence class monomial orders for the submodule. Specifically, if there is a G-cone of codimension greater than zero, then most probably the cone and its corresponding Gröbner basis will not be found. However, by Proposition 15 such a G-cone is a face of a codimension zero G-cone. So the fan itself has been found, but not all the G-cones will necessarily be identified.

Algorithm 1. ONE_CLASS_FAN_SHAPE

INPUT: Generators $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ of a submodule $M \subseteq R^t$.

OUTPUT: The maximal (codimension zero) G-cones of the one $\sim$ equivalence class Gröbner fan of M and the associated reduced marked Gröbner basis for each cone, along with any G-cones of codimension greater than zero that are fortuitously found in the process.

INITIALIZATION: $GF := \emptyset$, $\mathtt{SPAN} := \emptyset$.

WHILE $\mathtt{SPAN} \cap \left((\mathbb{R}^+)^n \times \mathbb{R}^t\right) \neq (\mathbb{R}^+)^n \times \mathbb{R}^t$ **DO**

Choose $(\omega, r) \in \left((\mathbb{R}^+)^n \times \mathbb{R}^t\right) \setminus \mathtt{SPAN}$.

Let U be a matrix with first row ω and refined by the degree reverse lexicographic order and let $\gamma_i := (r_i)$ for $1 \leq i \leq t$.

Let $>$ be the monomial order given by t copies of the matrix U, vectors γ_i for $1 \leq i \leq t$, integers $t_{ij} = 1$ for $1 \leq i, j \leq t$, and $\sigma = \mathtt{identity} \in S_t$.

$G :=$ reduced marked Gröbner basis for M with respect to $>$.
$D := \left\{ (\mathbf{X}, \mathbf{Y}) \,\middle|\, \begin{array}{l} \exists \mathbf{g} \in G \text{ such that } \mathbf{X} = \mathtt{lm}_{>}(\mathbf{g}) \text{ and} \\ \mathbf{Y} \text{ is a non-leading monomial in } \mathbf{g} \end{array} \right\}.$
$C_G := \{(\omega, r) | \deg_{(\omega,r,\{1,2,\dots,t\})}(\mathbf{X}) \geq \deg_{(\omega,r,\{1,2,\dots,t\})}(\mathbf{Y}) \forall (\mathbf{X}, \mathbf{Y}) \in D\}.$
$GF := GF \cup \{(G, C_G)\}.$
$\mathtt{SPAN} := \mathtt{SPAN} \cup C_G$

RETURN: *GF*

This algorithm will stop by Theorem 3 and Proposition 18. Also note that any algorithm for the computation of reduced marked Gröbner bases can be used in this algorithm.

Next an algorithm for finding the cones in the fan of codimension greater than zero is given. Such an algorithm finds all the reduced marked Gröbner bases with respect to one $\sim$ equivalence class monomial orders and each basis's corresponding G-cone is presented.

The algorithm uses the following construction of a new monomial order from another monomial order and a ϕ-grading. This construction will be used frequently throughout the paper.

Definition 22. The monomial order $>_{[\phi,>]}$ on R^t, where ϕ is a grading of $\mathtt{Mon}(R^t)$ and $>$ is a monomial order on R^t, is defined by

$$\mathbf{X} >_{[\phi,>]} \mathbf{Y} \Leftrightarrow \begin{cases} \deg_\phi(\mathbf{X}) > \deg_\phi(\mathbf{Y}), \text{ or} \\ \deg_\phi(\mathbf{X}) = \deg_\phi(\mathbf{Y}) \text{ and } \mathbf{X} > \mathbf{Y}. \end{cases}$$

Note that the monomial order $>_{[\phi,>]}$ is a monomial order that is compatible with an ϕ-grading on R^t.

Since there is a one-to-one correspondence between reduced Gröbner bases for a given module and the leading monomial submodules, one approach to the computation is to search for leading monomial submodules. Furthermore, if we want to look for a reduced Gröbner basis that is compatible with a certain τ-grading, it suffices to look at the leading monomial submodules of $\langle \mathtt{lm}_\tau(M) \rangle$. In particular, if one is looking for a Gröbner basis that corresponds to a certain face in the Gröbner fan, it suffices to pick a τ on the relative interior of the face and compute the leading monomial submodules for $\langle \mathtt{lm}_\tau(M) \rangle$. If the monomial order $>'$ gives a new leading monomial submodule of $\langle \mathtt{lm}_\tau(M) \rangle$, then the monomial order $>_{[\tau,>']}$ will give the same leading monomial submodule for M. The leading monomial submodules can be found by computing the Gröbner fans for $\langle \mathtt{lm}_\tau(M) \rangle$. However, it is only necessary to compute the Gröbner fans for types with more than one $\sim$ equivalence class, because the leading monomial submodules found from a computation of the one $\sim$ equivalence class Gröbner fan for $\langle \mathtt{lm}_\tau(M) \rangle$ are the same ones as are found by a computation of the codimension zero cones in the Gröbner fan for M.

This idea can be used together with Theorem 21 to find the G-cones of codimension greater than zero. Specifically, Theorem 21 states that the Gröbner fans for monomial orders with more than one $\sim$ equivalence class are products of one $\sim$ equivalence class Gröbner fans for submodules of lesser rank. So to find all the possible leading monomial submodules, the idea above will be used recursively on these submodules of progressively lesser rank until the rank one case is reached. The rank one case is the ideal case, for which all the reduced marked Gröbner bases correspond to codimension zero cones (see Cox et al. (2001), Mora and Robbiano (1988), Sturmfels (1996)), and hence the algorithm will stop.

Note that the faces of highest dimension must be checked first. Otherwise if a proper face of a G-cone is checked before the true G-cone, the algorithm will mistake the smaller subset for the G-cone.

The following is the algorithm (note that Algorithm 3 `GENERAL_FAN` is used):

Algorithm 2. ONE_CLASS_FAN

INPUT: Generators $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ of a submodule $M \subseteq R^t$.

OUTPUT: The one $\sim$ equivalence class Gröbner fan of M, along with the reduced marked Gröbner basis corresponding to each G-cone in the fan.

INITIALIZATION:

GF :=ONE_CLASS_FAN_SHAPE($\mathbf{f}_1, \ldots, \mathbf{f}_s$).
$D :=$ set of codimension one faces of cones of GF.
$L :=$ list of $F \in \text{PowerSet}(D)$ that satisfy $\left(\bigcap_{X \in F} X\right) \Big\backslash \left(\bigcup_{X \in D \setminus F} X\right) \neq \emptyset$, with the list ordered by reverse inclusion, i.e. if $\left(\bigcap_{X \in A} X\right) \supset \left(\bigcap_{X \in B} X\right)$, then A is before B.

FOR EACH CONSECUTIVE $F \in L$ **DO**

Let $(\omega, r) \in \left(\bigcap_{X \in F} X\right) \Big\backslash \left(\bigcup_{X \in D \setminus F} X\right)$.
Let $N := \left\langle \mathtt{lm}_{(\omega, r, \{1,2,\ldots,t\})}(M) \right\rangle$.
For each ordered partition $([h_1], [h_2], \ldots, [h_q])$ of $\{1, 2, \ldots, t\}$ with $q > 1$ do

Let A := the $([h_1], [h_2], \ldots, [h_q])$ Gröbner fan of N, computed using `GENERAL_FAN`.

If for some reduced marked Gröbner basis H of N with respect to the $([h_1], [h_2], \ldots, [h_q])$ monomial order $>$, the set of marked leading monomials of H is not a set of marked leading monomials of some reduced marked Gröbner basis in GF, then

$G :=$ the reduced marked Gröbner basis for M with respect to $>_{[(\omega, r, \{1,2,\ldots,t\}), >]}$.

$GF := GF \cup \left\{\left(G, \left(\bigcap_{X \in F} X\right)\right)\right\}$.

RETURN: GF.

This algorithm ends because L is a finite list, Theorem 3, and because the set of ordered partition of $\{1, \ldots, t\}$ is finite. Also, Algorithm 3 GENERAL_FAN, which computes A, is given below. As in the previous algorithms, any method can be used for the computation of the reduced marked Gröbner basis.

The following is an algorithm for computing the general $([h_1], [h_2], \ldots, [h_q])$ Gröbner fan. It is based on Theorem 21.

Algorithm 3. GENERAL_FAN

INPUT: Generators $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ of a submodule $M \subseteq R^t$. The ordered partition $([h_1], [h_2], \ldots, [h_q])$ indicating the type of monomial orders in the fan.
OUTPUT: The $([h_1], [h_2], \ldots, [h_q])$ Gröbner fan of M.
INITIALIZATION:

$>:=$ the monomial order of type $([h_1], [h_2], \ldots, [h_q])$ with the order on the components is degree reverse lexicographic order

$G :=$ the reduced marked Gröbner basis for $\langle \mathbf{f}_1, \ldots, \mathbf{f}_s \rangle$ with respect to $>$

FOR $i := 1$ **TO** q **DO**

$$J_i := \left\{ \mathbf{g} = \sum_{r\in[h_i]} g_r \mathbf{e}_r \in R^t \,\middle|\, \mathbf{f}_j - \mathbf{g} = \sum_{r\in[h_{i+1}]\cup\cdots\cup[h_q]} g_r \mathbf{e}_r \text{ for some } f_j \in G \right\}$$

$F_i := \texttt{ONE\_CLASS\_FAN}(J_i)$

RETURN: $\prod_{i=1}^{q} F_i$

As in the previous algorithms, any method can be used for the computation of the reduced marked Gröbner basis.

2.3. An example of a Gröbner fan computation

Let $R = \mathbb{R}[x, y, z]$. Consider the following submodule $M \subseteq R^4$ generated by the columns of the matrix

$$\begin{pmatrix} x & y & z & 0 \\ -y & x & 0 & z \\ -z & 0 & x & -y \\ 0 & -z & y & x \end{pmatrix}.$$

We will compute the $([h_1], [h_2], [h_3])$ Gröbner fan for M, with the three equivalence classes $[h_1] = \{1\}$, $[h_2] = \{3, 4\}$, and $[h_3] = \{2\}$.

Let $>$ be a monomial order of type $(\{1\}, \{3, 4\}, \{2\})$ with the order on each component a degree reverse lexicographic order with $x > y > z$. We follow Algorithm 3 in the computation. Then the reduced marked Gröbner basis for M with respect to $>$ is

$$G = \left\{ \begin{array}{c} \boxed{x\mathbf{e}_1} - y\mathbf{e}_2 - z\mathbf{e}_3, \boxed{y\mathbf{e}_1} + x\mathbf{e}_2 - z\mathbf{e}_4, \boxed{z\mathbf{e}_1} + x\mathbf{e}_3 + y\mathbf{e}_4, \\ z\mathbf{e}_2 - y\mathbf{e}_3 + \boxed{x\mathbf{e}_4}, \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_2, \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_3, \\ -xz\mathbf{e}_2 + \boxed{xy\mathbf{e}_3} + (y^2 + z^2)\mathbf{e}_4 \end{array} \right\},$$

with the marked leading monomials in boxes. So we can observe that

$$\begin{gathered} J_1 = \langle x, y, z \rangle, \\ J_2 = \left\langle \left(x^2 + y^2 + z^2\right)\mathbf{e}_1, xy\mathbf{e}_1 + \left(y^2 + z^2\right)\mathbf{e}_2, -y\mathbf{e}_1 + x\mathbf{e}_2 \right\rangle, \\ J_3 = \left\langle x^2 + y^2 + z^2 \right\rangle. \end{gathered}$$

The ideal J_1 has reduced marked Gröbner basis $\left\{\boxed{x}, \boxed{y}, \boxed{z}\right\}$ for any term order. So F_1, the Gröbner fan for J_1, has only one cone.

In the case of the ideal J_3, the set $\{x^2 + y^2 + z^2\}$ is the only reduced Gröbner basis. However, there are three choices for the leading term. So there will be three cones in F_3, the Gröbner fan for J_3.

For the submodule J_2 we do a one $\sim$ equivalence class Gröbner fan computation for the submodule $J_2 \subseteq R^2$. The first step is the `ONE_CLASS_FAN_SHAPE` computation. That computation finds the following cones of codimension zero in a space parametrized by $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^+$ and $r_1, r_2 \in \mathbb{R}$:

Point	Reduced marked Gröbner basis	Gröbner region
$((1, 0, 0), (0, 0))$	$\left(\boxed{x^2} + y^2 + z^2\right) \mathbf{e}_1$ $\boxed{xy\mathbf{e}_1} + \left(y^2 + z^2\right) \mathbf{e}_2$ $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$	$\omega_1 - \omega_2 - r_1 + r_2 \geq 0$ $\omega_1 - \omega_2 + r_1 - r_2 \geq 0$ $\omega_1 + \omega_2 - 2\omega_3 + r_1 - r_2 \geq 0$
$((1, 0, 0), (0, -2))$	$\left(\boxed{x^2} + z^2\right) \mathbf{e}_1 + xy\mathbf{e}_2$ $\left(\boxed{x^2} + y^2 + z^2\right) \mathbf{e}_2$ $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$	$\omega_1 - \omega_2 - r_1 + r_2 \leq 0$ $\omega_1 - \omega_3 \geq 0$ $\omega_1 - \omega_2 \geq 0$
$((2, 1, 0), (0, 2))$	$\left(\boxed{x^2} + y^2 + z^2\right) \mathbf{e}_1$ $xy\mathbf{e}_1 + \left(\boxed{y^2} + z^2\right) \mathbf{e}_2$ $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$	$\omega_1 - \omega_2 + r_1 - r_2 \leq 0$ $\omega_1 - \omega_2 \geq 0$ $\omega_2 - \omega_3 \geq 0$
$((2, 0, 1), (0, 2))$	$\left(\boxed{x^2} + y^2 + z^2\right) \mathbf{e}_1$ $xy\mathbf{e}_1 + \left(\boxed{z^2} + y^2\right) \mathbf{e}_2$ $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$	$\omega_1 - \omega_3 \geq 0$ $\omega_2 - \omega_3 \leq 0$ $\omega_1 + \omega_2 - 2\omega_3 + r_1 - r_2 \leq 0$
$((0, 1, 0), (0, 0))$	$\left(x^2 + z^2\right) \mathbf{e}_1 + \boxed{xy\mathbf{e}_2}$ $\left(\boxed{y^2} + z^2 + x^2\right) \mathbf{e}_2$ $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$	$\omega_1 - \omega_2 - r_1 + r_2 \leq 0$ $\omega_1 - \omega_2 + r_1 - r_2 \leq 0$ $\omega_1 + \omega_2 - 2\omega_3 - r_1 + r_2 \geq 0$
$((0, 1, 0), (0, 2))$	$\left(\boxed{y^2} + x^2 + z^2\right) \mathbf{e}_1$ $xy\mathbf{e}_1 + \left(\boxed{y^2} + z^2\right) \mathbf{e}_2$ $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$	$\omega_1 - \omega_2 - r_1 + r_2 \geq 0$ $\omega_1 - \omega_2 \leq 0$ $\omega_2 - \omega_3 \geq 0$
$((0.1, 1, 0), (0, -2))$	$\left(\boxed{x^2} + z^2\right) \mathbf{e}_1 + xy\mathbf{e}_2$ $\left(\boxed{y^2} + x^2 + z^2\right) \mathbf{e}_2$ $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$	$\omega_1 - \omega_2 + r_1 - r_2 \geq 0$ $\omega_1 - \omega_2 \leq 0$ $\omega_1 - \omega_3 \geq 0$

Point	Reduced marked Gröbner basis	Gröbner region
((0, 1, 0.1), (0, –2))	$\left(\boxed{z^2} + x^2\right)\mathbf{e}_1 + xy\mathbf{e}_2$ $\left(\boxed{y^2} + x^2 + z^2\right)\mathbf{e}_2$ $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$	$\omega_2 - \omega_3 \geq 0$ $\omega_1 - \omega_3 \leq 0$ $\omega_1 + \omega_2 - 2\omega_3 - r_1 + r_2 \leq 0$
((0.1, 0, 1), (0, 0))	$\left(\boxed{z^2} + x^2 + y^2\right)\mathbf{e}_1$ $xy\mathbf{e}_1 + \left(\boxed{z^2} + y^2\right)\mathbf{e}_2$ $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$	$\omega_1 - \omega_2 - r_1 + r_2 \geq 0$ $\omega_2 - \omega_3 \leq 0$ $\omega_1 - \omega_3 \leq 0$
((0, 0.1, 1), (0, 0))	$\left(\boxed{z^2} + x^2\right)\mathbf{e}_1 + xy\mathbf{e}_2$ $\left(\boxed{z^2} + x^2 + y^2\right)\mathbf{e}_2$ $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$	$\omega_1 - \omega_2 - r_1 + r_2 \leq 0$ $\omega_2 - \omega_3 \leq 0$ $\omega_1 - \omega_3 \leq 0$

Now it remains to check for G-cones of higher codimension by checking the proper faces of the codimension zero cones. There are 23 proper faces to check. Of the fifteen proper faces of codimension one, two are G-cones. The proper face $\omega_3 - \omega_2 = 0$ satisfying $\omega_1 - \omega_3 \leq 0$ and $\omega_1 - \omega_2 - r_1 + r_2 \geq 0$ is a G-cone. It is a face of the sixth and ninth G-cones listed above. The following are the two reduced marked Gröbner bases that have this proper face as a G-cone:

- $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$, $\left(\boxed{y^2} + x^2 + z^2\right)\mathbf{e}_1$, $xy\mathbf{e}_1 + \left(\boxed{z^2} + y^2\right)\mathbf{e}_2$.
- $-y\mathbf{e}_1 + \boxed{x\mathbf{e}_2}$, $\left(\boxed{z^2} + x^2 + y^2\right)\mathbf{e}_1$, $xy\mathbf{e}_1 + \left(\boxed{y^2} + z^2\right)\mathbf{e}_2$.

The proper face $\omega_1 - \omega_3 = 0$ satisfying $\omega_2 - \omega_3 \leq 0$ and $\omega_1 - \omega_2 - r_1 + r_2 \leq 0$ is also a G-cone for two distinct reduced marked Gröbner bases. It is a face of the second and tenth G-cones listed above. The following are those reduced marked Gröbner bases:

- $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$, $\left(\boxed{z^2} + x^2\right)\mathbf{e}_1 + xy\mathbf{e}_2$, $\left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_2$.
- $\boxed{y\mathbf{e}_1} - x\mathbf{e}_2$, $\left(\boxed{x^2} + z^2\right)\mathbf{e}_1 - xy\mathbf{e}_2$, $\left(\boxed{z^2} + y^2 + x^2\right)\mathbf{e}_2$.

Next, look at the proper faces of codimension two, followed by the proper faces of codimension three. By looking at the leading monomial submodules with respect to monomial orders of types ({1}, {2}) and ({2}, {1}), you can check that there are no further reduced marked Gröbner bases found.

The final step is putting the three fans F_1, F_2, and F_3 together as a product. Since there are twelve G-cones in F_2, three G-cones in F_3, and one G-cone in F_1, there will be a total of 36 G-cones in the ({1}, {3, 4}, {2}) Gröbner fan for M. However, since two of the G-cones in F_2 each correspond to two distinct reduced marked Gröbner bases, there are a total of 42 distinct reduced marked Gröbner bases for M.

3. Gröbner walk on submodules of free modules

This section is about the Gröbner walk on the Gröbner fan of a submodule M of a free module of finite rank. The discussion of the walk will be broken into subsections. First, walking between monomial orders of the same type is discussed in Section 3.1. Section 3.2 covers walking between monomial orders of different type. A detailed example of the algorithm is given in Section 3.3.

Again, consider a polynomial ring $R = k[x_1, \ldots, x_n]$, where k is a field. Let $\mathbb{Z}$ be the integers and $\mathbb{R}$ the reals. Let $\mathbb{Z}^+$ and $\mathbb{R}^+$ be the non-negative integers and non-negative reals, respectively.

3.1. *Walking between monomial orders of the same type*

Let $M \subseteq R^t$ be a submodule. Suppose that it is easier to compute a Gröbner basis for M with respect to the monomial order $>_b$, but one would like to have the Gröbner basis for M with respect to the monomial order $>_e$, where $>_b$ and $>_e$ are the same type $P = \big([h_1], [h_2], \ldots, [h_q]\big)$. Let (W, r) and (V, s) be the points corresponding to $>_b$ and $>_e$ in the P Gröbner fan, respectively, as in Proposition 12.

The walk uses a monomial order $>_{[(W',r',P'),>]}$ on M, as in Definition 22, defined with respect to $(W', r') \in \big((\mathbb{R}^+)^n\big)^{q'} \times \mathbb{R}^t$, an ordered partition P' of $\{1, 2, \ldots, t\}$ into q' non-empty sets, and a monomial order $>$ on M. Let G_b be the reduced marked Gröbner basis for M with respect to $>_b$. At the end of the walk, a Gröbner basis for M with respect to $>_e$ will have been computed.

The idea, as in the ideal case, is to follow the linear path from (W, r) to (V, s),

$$v(t) = (1 - t)(W, r) + t(V, s), t \in [0, 1],$$

in the P Gröbner fan. Similar to the ideal case, each time the path crosses a boundary of a cone, Theorem 6 is used to find the reduced marked Gröbner basis for the adjacent cone. Eventually, this will lead to computing the reduced marked Gröbner basis for the cone containing the monomial order $>_e$.

Theorem 6 is the basis of the following algorithm for converting from a Gröbner basis G with a cone in the P Gröbner fan to a Gröbner basis for the adjacent cone along the path towards $>_e$, where (W, a) is a point on the path common to the two cones:

Algorithm 4. CROSSING

INPUT: The initial Gröbner basis G for a module $M \subseteq R^t$ with respect to a monomial order $>_b$. The monomial order $>_e$ which is the ultimate final monomial order. The type P. A point (W, r) in the P Gröbner fan for M on the boundary of the cone for G.
OUTPUT: The reduced marked Gröbner basis for M with respect to the monomial order $>_{[(W,r,P),>_e]}$.
STEPS:

$LM := \mathtt{lm}_{(W,r,P)}(G)$
$H :=$ reduced marked Gröbner basis for $\langle LM\rangle$ with respect to $>_{[(W,r,P),>_e]}$
$LF := \mathtt{DivisionAlgorithm}(LM, H, >_b)$

$G := \mathtt{Expand}(LF, G)$
$G := \mathtt{Reduce}(G, >_{[(W,r,P),>_e]})$
RETURN G

In the above algorithm, the procedure `DivisionAlgorithm` takes each $\mathbf{h} \in H$, and applies the division algorithm with respect to $>_b$, and returns a list of pairs $(p_\mathbf{g}, \mathbf{g}) \in R \times LM$ such that

$$\mathbf{h} = \sum_{\mathbf{g} \in LM} p_\mathbf{g}\mathbf{g}.$$

The procedure `Expand`, takes each pair $(p_\mathbf{g}, \mathbf{g}) \in LF$, and replaces $\mathbf{g}$ with the $\mathbf{g}' \in G$ satisfying $\mathtt{lm}_{(W,r,P)}(\mathbf{g}') = \mathbf{g}$. Then it returns the sum of each list:

$$\mathbf{f_h} = \sum_{\mathbf{g} \in LM} p_\mathbf{g}\mathbf{g}'.$$

The procedure `Reduce` interreduces the vectors with respect to the given term order, to get a reduced marked list. As in the previous algorithms, Buchberger's or any other Gröbner basis algorithm can be used for the Gröbner basis computations.

The following is the algorithm for the walk:

Algorithm 5. SAME_TYPE_G-WALK

INPUT: An initial type P monomial order $>_b$ given by matrices $U_1, \ldots, U_t$, vectors $\gamma_1, \ldots, \gamma_t$, a $t \times t$ integer matrix T_b, and an element $\sigma_b \in S_t$ as in Theorem 2. A final type P monomial order $>_e$ given by matrices $V_1, V_2, \ldots, V_t$, vectors $\delta_1, \delta_2, \ldots, \delta_t$, a $t \times t$ integer matrix T_e, and an element $\sigma_e \in S_t$ as in Theorem 2. A reduced marked Gröbner basis G_b of M with respect to $>_b$. The ordered partition $P = ([h_1], [h_2], \ldots, [h_q])$ of $\{1, 2, \ldots, t\}$ which is the type of both $>_b$ and $>_e$.
OUTPUT: The reduced marked Gröbner basis for M with respect to $>_e$.
INITIALIZATION:
Set $>$ to be the monomial order $>_b$
For $i := 1$ to q do
  Let $j \in [h_i]$
  $\omega_i :=$ first row of U_j
  $\rho_i :=$ first row of V_j
For $i := 1$ to t do
  $r_i :=$ first coordinate of γ_i
  $s_i :=$ first coordinate of δ_i
$W :=$ the matrix with rows $\omega_1, \omega_2, \ldots, \omega_q$
$r := (r_1, r_2, \ldots, r_t)$
`finished` := `false`
$G := G_b$
If $0 \in \left\{ \omega_a \cdot (\alpha - \beta) + r_i - r_j \;\middle|\; \begin{array}{c} \exists \mathbf{g} \in G \text{ such that } X^\alpha \mathbf{e}_i \text{ is the} \\ \text{identified leading monomial of } \mathbf{g} \\ \text{and } X^\beta \mathbf{e}_j \text{ is a non-leading} \\ \text{monomial of } \mathbf{g}, i, j \in [h_a] \end{array} \right\}$ then

$G := \mathtt{CROSSING}(G, >, >_e, W, r, P)$
Set $>$ to be the monomial order $>_{[(W,r,P),>_e]}$
WHILE $\mathtt{finished} = \mathtt{false}$ **DO**

$$D := \left\{ \frac{(\beta-\alpha)\cdot\omega_a + r_j - r_i}{\begin{pmatrix}(\alpha-\beta)\cdot(\rho_a-\omega_a)+\\ s_i - r_i - s_j + r_j\end{pmatrix}} \;\middle|\; \begin{array}{c} \exists \mathbf{g} \in G \text{ such that } X^\alpha \mathbf{e}_i \text{ is the} \\ \text{identified leading monomial of } \mathbf{g} \text{ and} \\ X^\beta \mathbf{e}_j \text{ is a non-leading monomial of} \\ \mathbf{g}, i, j \in [h_a], \text{ and } (\alpha-\beta)\cdot(\rho_a-\omega_a) \\ + s_i - r_i - s_j + r_j \neq 0 \end{array} \right\}$$

If $\{t \in D | 0 < t \leq 1\} = \emptyset$ then $\mathtt{finished} := \mathtt{true}$
Else
$d := \min\{t \in D | 0 < t \leq 1\}$
For $i := 1$ to q do $\omega_i := (1-d)\omega_i + d\rho_i$
For $i := 1$ to t do $r_i := (1-d)r_i + ds_i$
$W :=$ the matrix with rows $\omega_1, \omega_2, \ldots, \omega_q$
$r := (r_1, r_2, \ldots, r_t)$
$G := \mathtt{CROSSING}(G, >, >_e, W, r, P)$
Set $>$ to be the monomial order $>_{[(W,r,P),>_e]}$
If $d = 1$ then $\mathtt{finished} := \mathtt{true}$
RETURN G

Note that the function CROSSING is Algorithm 4. The algorithm ends by Theorem 3.

3.2. Walking between monomial orders of different type

It is also possible to do a Gröbner walk between a monomial order $>_b$ of type $P_b = ([h_1], [h_2], \ldots, [h_{q_b}])$ and a monomial order $>_e$ of a different type $P_e = ([h'_1], [h'_2], \ldots, [h'_{q_e}])$. The idea is to create the list of types

$$(P_b, \{1, 2, \ldots, t\}, ([h'_1], [h'_2] \cup \cdots \cup [h'_{q_e}]), ([h'_1], [h'_2], [h'_3] \cup \cdots \cup [h'_{q_e}]), \\ \ldots, ([h'_1], [h'_2], \ldots, [h'_{q_e-2}], [h'_{q_e-1}] \cup [h'_{q_e}]), P_e).$$

For each successive type P in the list, walk within the P Gröbner fan from a known reduced marked Gröbner basis to a reduced marked Gröbner basis with respect to both a monomial order of type P and a monomial order of the next type in the list. The starting point of the Gröbner walk in the next Gröbner fan is this reduced marked Gröbner basis.

Suppose $>_b$ is given by matrices $U_1, U_2, \ldots, U_t$, vectors $\gamma_1, \gamma_2, \ldots, \gamma_t$, a $t \times t$ integer matrix T_b, and an element $\sigma_b \in S_t$ as in Theorem 2. Denote by $\sim_b$ the equivalence relation on $\{1, 2, \ldots, t\}$ given by $>_b$. Suppose we would like to compute the reduced marked Gröbner basis of a submodule $M \subseteq R^t$ with respect to $>_e$.

The first step is to do a Gröbner walk in the P_b Gröbner fan from $>_b$ to a reduced marked Gröbner basis that is also a reduced marked Gröbner basis with respect to a monomial order with one $\sim$ equivalence class. Monomial orders with one $\sim$ equivalence class have the property that the matrices describing the term orders as in Theorem 2 all have the same first row. So for convenience, we pick a $j_0 \in \{1, 2, \ldots, t\}$ and walk to the reduced marked Gröbner basis with respect to a monomial order $>_2$ where the term order on each component is given by the matrix for the j_0 component, U_{j_0}. Let l be the number of rows of U_{j_0}. Specifically, the monomial order $>_2$ is described as in Theorem 2 by t copies of the matrix U_{j_0}, t zero-vectors of length l, a $t \times t$ integer matrix with the ijth entry equal to zero if the ijth entry of T_b is zero, otherwise the entry is one, and $\sigma_b \in S_t$.

The reduced marked Gröbner basis G_2 for M with respect to $>_2$ is also a reduced marked Gröbner basis with respect to a one $\sim$ equivalence class monomial order $>_3$. The monomial order $>_3$ can be described as in Theorem 2 by t copies of the matrix U_{j_0}, a $t \times t$ matrix T_2 of all ones, $\sigma_b \in S_t$, and a set of vectors $\zeta_1, \ldots, \zeta_t$ of length l which will be computed below. Since T_2 is a matrix of all ones, only the first coordinates of the vectors ζ_i are important, so we may assign the remaining coordinates to zero. Let ω_1 be the first row of the matrix U_{j_0}. The first coordinates of the vectors $\zeta_1, \ldots, \zeta_t$ must satisfy:

(1) If $i \sim_b j$ then the first coordinates of ζ_i and ζ_j are the same.
(2) For each $\mathbf{g} \in G_2$ with marked leading term $X^\alpha \mathbf{e}_i$, given any other term $X^\beta \mathbf{e}_j$ in $\mathbf{g}$ with $i \not\sim_b j$, the first coordinate of ζ_i must be greater than the first coordinate of ζ_j plus $\omega_1 \cdot (\beta - \alpha)$.

To find a set of values to satisfy these conditions, first, for each $i \in [h_1]$, set the first coordinate of ζ_i equal to zero. For each $j \in \{2, 3, 4, \ldots, q_b\}$, compute the maximum m_j of the set

$$\left\{ \omega_1 \cdot (\beta - \alpha) \,\middle|\, \begin{array}{l} X^\alpha \mathbf{e}_c \text{ is the leading monomial for some } \mathbf{g} \in G \text{ with} \\ c \in [h_{j-1}] \text{ and } X^\beta \mathbf{e}_d \text{ a term in } \mathbf{g} \text{ with } d \notin [h_{j-1}] \end{array} \right\}.$$

For each $i \in [h_j]$, $2 \le j \le q$, set the first coordinate of ζ_i equal to $-j - \sum_{v=2}^{j} m_v$.

For each subsequent $i := 1, 2, 3, \ldots, q_e - 1$, we perform the following steps to get from a type $([h'_1], \ldots, [h'_{i-1}], [h'_i] \cup \cdots \cup [h'_{q_e}])$ Gröbner fan to a type $([h'_1], \ldots, [h'_i], [h'_{i+1}] \cup \cdots \cup [h'_{q_e}])$ Gröbner fan. Let $\rho(m)$ be a vector of length t, where the jth component is m if $j \in [h'_i]$, and zero otherwise. We let W be an $i \times n$ matrix where each row is ω_1. We do a Gröbner walk in the $([h'_1], \ldots, [h'_{i-1}], [h'_i] \cup \cdots \cup [h'_{q_e}])$ Gröbner fan along the ray $(W, \zeta + \rho(m))$, $m \ge 0$, until we reach an m_0 such that the ray $(W, \zeta + \rho(m))$, $m \ge m_0$, is contained in a single cone. The reduced marked Gröbner basis for this cone is a Gröbner basis with respect to some monomial order of type $([h'_1], \ldots, [h'_i], [h'_{i+1}] \cup \cdots \cup [h'_{q_e}])$.

Finally, the walk ends in the P_e Gröbner fan, where we follow Algorithm 5 `SAME_TYPE_G-WALK` to get to the reduced marked Gröbner basis for $>_e$.

Here is the algorithm for this walk:

Algorithm 6. TYPE_CHANGE_G-WALK

INPUT: An initial monomial order $>_b$ given by matrices $U_1, \ldots, U_t$, vectors $\gamma_1, \ldots, \gamma_t$, a $t \times t$ integer matrix T_b, and an element $\sigma_b \in S_t$ as in Theorem 2, with $P_b = ([h_1], [h_2], \ldots, [h_{q_b}])$ as its type. A final monomial order $>_e$ given by matrices $V_1, \ldots, V_t$, vectors $\delta_1, \ldots, \delta_t$, a $t \times t$ integer matrix T_e, and an element $\sigma_e \in S_t$ as in Theorem 2, with $P_e = ([h'_1], [h'_2], \ldots, [h'_{q_e}])$ as its type. The reduced marked Gröbner basis G_b of M with respect to $>_b$.
OUTPUT: The reduced marked Gröbner basis G_e of M with respect to $>_e$.
INITIALIZATION:
$G := G_b$
Let $j_0 \in [h_1]$

$\omega_1 :=$ the first row of U_{j_0}
$l :=$ the number of rows of U_{j_0}
$\zeta :=$ zero-vector of length l
$A := t \times t$ matrix with the ijth entry equal to zero if the ijth entry of T_b is zero, otherwise the ijth entry is 1
$G := \texttt{SAME\_TYPE\_G-WALK}\begin{pmatrix}(U_1, \ldots, U_t, \gamma_1, \ldots, \gamma_t, T_b, \sigma_b), G, \\ (U_{j_0}, \ldots, U_{j_0}, \zeta, \ldots, \zeta, A, \sigma_b), P_b\end{pmatrix}$
$>:=$ the monomial order defined by t copies of the matrix U_{j_0}, t copies of the vector ζ, the matrix A, and $\sigma_b \in S_t$ as in Theorem 2
$a_1 := 0$
For each $j \in [h_1]$ do
  $\zeta_j := \zeta$
  $b_j := 0$
For $i := 2$ to q_b do

$$a_i := -1 + a_{i-1} - \max\left\{ \omega_1 \cdot (\beta - \alpha) \;\middle|\; \begin{array}{c} \exists \mathbf{g} \in G \text{ with } X^\alpha \mathbf{e}_c, \text{ the} \\ \text{identified leading term of } \mathbf{g} \\ \text{with } c \in [h_{i-1}], \text{ and } X^\beta \mathbf{e}_d, \\ \text{a term in } \mathbf{g} \text{ with } d \notin [h_{i-1}] \end{array} \right\}$$

  For each $j \in [h_i]$ do
    $\zeta_j := (a_i, 0, \ldots, 0)$ such that the vector has length l
    $b_j := a_i$
FOR $i := 2$ **TO** q_e **DO**

$$D := \left\{ \omega_1 \cdot (\alpha - \beta) + b_c - b_d \;\middle|\; \begin{array}{c} \exists \mathbf{g} \in G \text{ with } X^\alpha \mathbf{e}_c \text{ the identified leading} \\ \text{term of } \mathbf{g} \text{ with } c \in [h'_p] \text{ for some } p > i \\ \text{and } X^\beta \mathbf{e}_d \text{ a term in } \mathbf{g} \text{ with } d \in [h'_i] \end{array} \right\}$$

  $P := ([h'_1], [h'_2], \ldots, [h'_{i-1}], [h'_i] \cup [h'_{i+1}] \cup \cdots \cup [h'_{q_e}])$
  $W :=$ an $i \times n$ matrix where each row is ω_1
  If $0 \in D$ then
    $a := (b_1, \ldots, b_t)$
    $G := \texttt{CROSSING}(G, >, >_e, W, a, P)$
    Set $>$ to be the monomial order $>_{[(W,a,P),>_e]}$

$$D := \left\{ \omega_1 \cdot (\alpha - \beta) + b_c - b_d \;\middle|\; \begin{array}{c} \exists \mathbf{g} \in G \text{ with } X^\alpha \mathbf{e}_c \text{ the identified} \\ \text{leading term of } \mathbf{g} \text{ with } c \in [h'_p] \text{ for} \\ \text{some } p > i \text{ and } X^\beta \mathbf{e}_d \\ \text{a term in } \mathbf{g} \text{ with } d \in [h'_i] \end{array} \right\}$$

  While $\{x | x \in D, x > 0\} \neq \emptyset$ **do**
    $d := \min\{x | x \in D, x \geq 0\}$
    For $j \in [h'_i]$ do $b_j := b_j + d$
    $a := (b_1, \ldots, b_t)$
    $G := \texttt{CROSSING}(G, >, >_e, W, a, P)$
    Set $>$ to be the monomial order $>_{[(W,a,P),>_e]}$

$$D := \left\{ \omega_1 \cdot (\alpha - \beta) + b_c - b_d \;\middle|\; \begin{array}{c} \exists \mathbf{g} \in G \text{ with } X^\alpha \mathbf{e}_c \text{ the identified} \\ \text{leading term of } \mathbf{g} \text{ with } c \in [h'_p] \text{ for} \\ \text{some } p > i \text{ and } X^\beta \mathbf{e}_d \\ \text{a term in } \mathbf{g} \text{ with } d \in [h'_i] \end{array} \right\}$$

For each $j \in [h'_1] \cup \cdots \cup [h'_i]$ do $\zeta_j := (b_j, 0, 0, \ldots, 0)$ where ζ_j has length l
$A := t \times t$ matrix with the ijth entry equal to zero if the ijth entry of T_e is zero, otherwise the ijth entry is 1

RETURN `SAME_TYPE_G-WALK` $\begin{pmatrix} (U_{j_0}, \ldots, U_{j_0}, \zeta_1, \ldots, \zeta_t, A, \sigma_e), G, \\ (V_1, \ldots, V_t, \delta_1, \ldots, \delta_t, T_e, \sigma_e), P_e \end{pmatrix}$

Note that the function `CROSSING` is Algorithm 4. Also, the algorithm ends by Theorem 3.

3.3. An example of a Gröbner walk

Let $R = \mathbb{R}[x, y, z]$. Consider the submodule $M \subseteq R^3$, from the example in Section 2.3, generated by the columns of the following matrix:

$$\begin{pmatrix} x & y & z & 0 \\ -y & x & 0 & z \\ -z & 0 & x & -y \\ 0 & -z & y & x \end{pmatrix}.$$

The set of marked vectors

$$G = \left\{ \begin{array}{c} \boxed{x\mathbf{e}_1} - y\mathbf{e}_2 - z\mathbf{e}_3, \boxed{y\mathbf{e}_1} + x\mathbf{e}_2 - z\mathbf{e}_4, \boxed{z\mathbf{e}_1} + x\mathbf{e}_3 + y\mathbf{e}_4, \\ z\mathbf{e}_2 - y\mathbf{e}_3 + \boxed{x\mathbf{e}_4}, \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_2, \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_3, \\ -xz\mathbf{e}_2 + \boxed{xy\mathbf{e}_3} + (y^2 + z^2)\mathbf{e}_4 \end{array} \right\}$$

is a reduced marked Gröbner basis for M with respect to a monomial order $>_b$ of type $(\{1\}, \{3, 4\}, \{2\})$ that is defined by the matrices

$$U_1 = U_2 = U_3 = U_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the vectors $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = (0, 0, 0)$, the matrix

$$T_b = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix},$$

and the element $(1\ 4\ 3\ 2) \in S_4$, as in Theorem 2.

Suppose we want to compute the reduced marked Gröbner basis for M with respect to the monomial order $>_e$ of type $(\{1, 3\}, \{2, 4\})$ defined by the matrices

$$V_1 = V_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V_2 = V_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

the vectors $\delta_1 = \delta_2 = \delta_3 = \delta_4 = (0, 0, 0)$, the matrix

$$T_e = \begin{pmatrix} 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix},$$

and the element $(1 \quad 4 \quad 2) \in S_4$, as in Theorem 2. Algorithm 6 will be used to do the computation.

First, notice that since $U_1 = U_2 = U_3 = U_4$, it is not necessary to use the algorithm `SAME_TYPE_G-WALK` because the initial monomial order $>_b$ corresponds to a point in the fan for which there is a monomial order of type $\{1, 2, \ldots, t\}$ with the same reduced marked Gröbner basis for M. We want to find a monomial order of type $\{1, 2, \ldots, t\}$ that has the reduced marked Gröbner basis G for M. We know that the matrices will be four copies of U_1, the integer matrix will be a matrix of all ones, the element of S_t will be $(1 \quad 4 \quad 3 \quad 2)$. We determine the vectors $\zeta_1 = (0, 0, 0)$, $\zeta_3 = \zeta_4 = (-2, 0, 0)$, and $\zeta_2 = (-3, 0, 0)$. So now we have the monomial order of type $\{1, 2, \ldots, t\}$ that was needed.

Next we walk within the $\{1, 2, \ldots, t\}$ Gröbner fan from the point

$$((1, 0, 0), (0, -3, -2, -2)),$$

which corresponds to the monomial order above, to a point in a fan for a reduced marked Gröbner basis that is also a reduced marked Gröbner basis with respect to a monomial order of type $(\{1, 3\}, \{2, 4\})$. The walk is in the direction of the vector $((0, 0, 0), (1, 0, 1, 0))$. The first cone boundary that is crossed by the path is at the point

$$(W, a) := ((1, 0, 0), (1, -3, -1, -2)).$$

So we compute

$$LM = \mathtt{lm}_{(W,a,P)}(G) = \left\{x\mathbf{e}_1, y\mathbf{e}_1, z\mathbf{e}_1, -y\mathbf{e}_3 + x\mathbf{e}_4, x^2\mathbf{e}_2, x^2\mathbf{e}_3, xy\mathbf{e}_3\right\}.$$

The reduced marked Gröbner basis for $\langle LM\rangle$ with respect to $>_e$ is

$$H = \left\{\boxed{x\mathbf{e}_1}, \boxed{y\mathbf{e}_1}, \boxed{z\mathbf{e}_1}, \boxed{y\mathbf{e}_3} - x\mathbf{e}_4, \boxed{x^2\mathbf{e}_2}, \boxed{x^2\mathbf{e}_3}, \boxed{x^2\mathbf{e}_4}\right\}.$$

Next, the division algorithm is used to determine how the vectors in H can be written as a combination of the vectors in LM. Then `Expand` is used to replace the vectors in LM with the original vectors in G in the combinations. This new set of vectors replaces the set G. Then `Reduce` interreduces the vectors in G to get a reduced marked Gröbner basis. The result is the following reduced marked Gröbner basis for M:

$$G = \left\{\begin{array}{c} \boxed{x\mathbf{e}_1} - y\mathbf{e}_2 - z\mathbf{e}_3, \boxed{y\mathbf{e}_1} + x\mathbf{e}_2 - z\mathbf{e}_4, \boxed{z\mathbf{e}_1} + x\mathbf{e}_3 + y\mathbf{e}_4, \\ -z\mathbf{e}_2 + \boxed{y\mathbf{e}_3} - x\mathbf{e}_4, \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_2, \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_3, \\ \left(\boxed{x^2} + y^2 + z^2\right)\mathbf{e}_4 \end{array}\right\}.$$

Continuing along the path we determine that the remainder of the path is contained in a single cone in the Gröbner fan. Thus, G is a reduced marked Gröbner basis for some monomial order of type $(\{1,3\},\{2,4\})$.

The monomial order of type $(\{1,3\},\{2,4\})$ that has G as a reduced marked Gröbner basis is defined by four copies of U_1, the vectors $\zeta_1 := (1,0,0)$, ζ_2, $\zeta_3 := (-1,0,0)$, ζ_4, the element $(1\ \ 4\ \ 3\ \ 2) \in S_t$, and the integer matrix $\frac{1}{3}T_e$, as in Theorem 2.

The final stage of the algorithm is the `SAME_TYPE_G-WALK` to the point corresponding to $>_e$. The walk is from the point $\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, (1,-3,-1,-2)\right)$ to the point $\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, (0,0,0,0)\right)$. The first cone boundary along the straight line path is at the point $(W,r) := \left(\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix}, (0.5,-1.5,-0.5,-1)\right)$. Using the same procedures as above, we compute the reduced marked Gröbner basis for M with respect to a $>_{[(W,r,(\{1,3\},\{2,4\})),>_e]}$ monomial order

$$G = \left\{ \begin{array}{c} \boxed{x\mathbf{e}_1} - y\mathbf{e}_2 - z\mathbf{e}_3, \boxed{y\mathbf{e}_1} + x\mathbf{e}_2 - z\mathbf{e}_4, z\mathbf{e}_1 + \boxed{x\mathbf{e}_3} + y\mathbf{e}_4, \\ -z\mathbf{e}_2 + \boxed{y\mathbf{e}_3} - x\mathbf{e}_4, \left(\boxed{y^2} + x^2 + z^2\right)\mathbf{e}_2, \left(\boxed{y^2} + x^2 + z^2\right)\mathbf{e}_4 \end{array} \right\}.$$

So this step of the `SAME_TYPE_G-WALK` is complete.

Finally we determine that the points $\left(\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix}, (0.5,-1.5,-0.5,-1)\right)$ and $\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, (0,0,0,0)\right)$ are in the same cone. So the algorithm is complete and G is the reduced marked Gröbner basis for M with respect to $>_e$.

Acknowledgement

The results of this paper form a part of my Ph.D. thesis at the University of Maryland, College Park under the direction of Dr. William W. Adams.

References

Adams, W.W., Loustaunau, P., 1994. An Introduction to Gröbner Bases. Graduate Studies in Mathematics, vol. 3. American Mathematical Society, Providence, RI.

Assi, A., Castro-Jiménez, F.J., Granger, M., 2000. The Gröbner fan of an A_n-module. J. Pure Appl. Algebra 150 (1), 27–39.

Assi, A., Castro-Jiménez, F.J., Granger, M., 2001. The analytic standard fan of a $\mathcal{D}$-module. J. Pure Appl. Algebra 164 (1–2), 3–21. Effective methods in algebraic geometry (Bath, 2000).

Caboara, M., Silvestri, M., 1999. Classification of compatible module orderings. J. Pure Appl. Algebra 142 (1), 13–24.

Carrà Ferro, G., Sit, W.Y., 1994. On term-orderings and rankings. In: Computational Algebra. Fairfax, VA, 1993. Lecture Notes in Pure and Appl. Math., vol. 151. Dekker, New York, pp. 31–77.

Collart, S., Kalkbrener, M., Mall, D., 1997. Converting bases with the Gröbner walk. J. Symbolic Comput. 24 (3–4), 465–469. Computational algebra and number theory (London, 1993).

Cox, D., Little, J., O'Shea, D., 1998. Using Algebraic Geometry. Graduate Texts in Mathematics, vol. 185. Springer-Verlag, New York.

Cox, D., Little, J., O'Shea, D., 2001. Using algebraic geometry. Draft sections on the Gröbner fan and Gröbner walk for the second edition (preprint).

Eisenbud, D., 1995. Commutative Algebra. Graduate Texts in Mathematics, vol. 150. Springer-Verlag, New York, with a view toward algebraic geometry.

Horn, K., 1998. Classification of term orders on a module. Ph.D. Thesis, University of Maryland, College Park.

Kolchin, E.R., 1973. Differential Algebra and Algebraic Groups. Pure and Applied Mathematics, vol. 54. Academic Press, New York.

Kreuzer, M., Robbiano, L., 2000. Computational Commutative Algebra. 1. Springer-Verlag, Berlin.

Mora, T., Robbiano, L., 1988. The Gröbner fan of an ideal. J. Symbolic Comput. 6 (2–3), 183–208. Computational aspects of commutative algebra.

Riquier, C., 1910. Les systèmes d'équations aux dérivées partielles. Gauthier-Villars.

Robbiano, L., Term orderings on the polynomial ring. In: EUROCAL'85, Linz, 1985, vol. 2. Lecture Notes in Comput. Sci., vol. 204, Springer, Berlin, pp. 513–517.

Rust, C.J., 1998. Rankings of derivatives for elimination algorithms and formal solvability of analytic partial differential equations. Ph.D. Thesis, University of Chicago.

Rust, C.J., Reid, G.J., 1997. Rankings of partial derivatives. In: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation. Kihei, HI. ACM, New York, pp. 9–16 (electronic).

Saito, M., Sturmfels, B., Takayama, N., 2000. Gröbner deformations of hypergeometric differential equations. In: Algorithms and Computation in Mathematics, vol. 1. Springer-Verlag, Berlin.

Smith, G.G., 2001. Irreducible components of characteristic varieties. J. Pure Appl. Algebra 165 (3), 291–306.

Sturmfels, B., 1996. Gröbner Bases and Convex Polytopes. University Lecture Series, vol. 8. American Mathematical Society, Providence, RI.

Trevisan, G., 1953. Classificazione dei semplici ordinamenti di un gruppo libero commutativo con *n* generatori. Rend. Sem. Mat. Univ. Padova 22, 143–156.

Zaĭceva, M.I., 1953. On the set of ordered Abelian groups. Uspekhi Mat. Nauk (N.S.) 8 (1(53)), 135–137.