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An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation

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Abstract

We give an algorithm to compute the following cohomology groups on $U = \mathbf{C}^n \setminus V(f)$ for any non-zero polynomial $f \in \mathbf{Q}[x_1,\dots,x_n]$:

1. $H^k(U, \mathbf{C}_U)$, $\mathbf{C}_U$ is the constant sheaf on U with stalk $\mathbf{C}$.
2. $H^k(U, \mathscr{V})$, $\mathscr{V}$ is a locally constant sheaf of rank 1 on U.

We also give partial results on computation of cohomology groups on U for a locally constant sheaf of general rank and on computation of $H^k(\mathbf{C}^n \setminus Z, \mathbf{C})$ where Z is a general algebraic set. Our algorithm is based on computations of Gröbner bases in the ring of differential operators with polynomial coefficients.

MSC: 14F40; 14Q99; 55N30

0. Introduction

In this paper, we give an algorithm to compute the following cohomology groups on $U = \mathbf{C}^n \setminus V(f)$ for any non-zero polynomial $f \in \mathbf{Q}[x_1,\dots,x_n]$:

1. $H^k(U, \mathbf{C}_U)$, where $\mathbf{C}_U$ is the constant sheaf on U with stalk $\mathbf{C}$.
2. $H^k(U, \mathscr{V})$, where $\mathscr{V}$ is the locally constant sheaf on U of rank one defined by a multi-valued function $f_1^{a_1} \cdots f_d^{a_d}$ with polynomials $f_1,\dots,f_d \in \mathbf{Q}[x]$ such that $f = f_1 \cdots f_d$ and $a_1,\dots,a_d \in \mathbf{Q}$.

We also give partial results on the computation of cohomology groups on U for a locally constant sheaf of general rank as well as on the computation of $H^k(\mathbf{C}^n \setminus Z, \mathbf{C})$, where Z is a general algebraic set of $\mathbf{C}^n$.

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Our algorithm is based on computations of Gröbner bases in the ring of differential operators with polynomial coefficients, algorithms for functors in the theory of $\mathscr{D}$-modules [32,33], and Grothendieck–Deligne comparison theorem [10,15], which relates sheaf cohomology groups and algebraic de Rham cohomology groups.

One advantage of the use of the ring of differential operators in algebraic geometry is that, for example, $\mathbf{Q}[x, 1/x]$, which is the localized module of $\mathbf{Q}[x]$ along x, is not finitely generated as a $\mathbf{Q}[x]$-module, but it can be regarded as a finitely generated $\mathbf{Q}\langle x, \partial_x\rangle$-module with $\partial_x = \partial/\partial x$. In fact, we have $1/x^k = (-1)^{k-1}(1/(k-1)!)(\frac{\partial}{\partial x})^{k-1}\frac{1}{x}$. Computation of the localization of a given $\mathscr{D}$-module and computation of the integration functor are the most important part of our algorithm. See [11] for a classical approach.

As an introduction to this paper, it will be the best to mention how we started this project.

A connection between de Rham cohomology groups and $\mathscr{D}$-modules is well-understood theoretically. In fact, the connection is given by the Riemann–Hilbert correspondence by Kashiwara and Mebkhout [20,27] between the derived category of constructible sheaves and the derived category of bounded complexes of $\mathscr{D}$-modules whose cohomology groups are regular holonomic. The correspondence has yielded fruitful results in algebraic geometry and the representation theory. The authors were convinced that it should also give fruitful results in computational algebraic geometry. However, there had been only a few results to this direction because we have to deal with left and right modules simultaneously. After [32], this difficulty was essentially removed and we tried the following computation.

Let us consider the differential equation for the function $f = (x-u)^a(x-v)^b$ where u, v, a, b are rational numbers with $u < v$ and $a + b \notin \mathbf{Z}$. The function f satisfies the differential equation

$$pf = 0, \quad p = (x-u)(x-v)\partial_x - a(x-v) - b(x-u).$$

Let $\hat{p}$ be the formal Fourier transform of this operator:

$$\begin{aligned}\hat{p} &= (-\partial_x - u)(-\partial_x - v)x - a(-\partial_x - v) - b(-\partial_x - u)\\ &= x\partial_x^2 + (ux + vx + 2 + a + b)\partial_x + uvx + u + v + av + bu\end{aligned}$$

and A be the ring of differential operators $\mathbf{Q}\langle x, \partial_x\rangle$. We want to evaluate the dimension of the $\mathbf{Q}$-vector space

$$A/(A\hat{p} + xA) \simeq (A/A\hat{p})/x(A/A\hat{p}) \simeq (A/A\hat{p})/\mathrm{Im}\, x\cdot,$$

which is called *the* (0th) restriction of $A/A\hat{p}$ along $x = 0$. Now, we can apply the algorithm for the $\mathscr{D}$-module theoretic restriction in [32, Section 5] to evaluate the dimension. Here, we need what is called a b-function for the evaluation, which is nothing but the indicial (characteristic) polynomial at $x = 0$ of the ordinary differential operator $\hat{p}$ that appears in the classical method of Frobenius; here the b-function is $s(s - a - b)$. Applying Proposition 5.2 in [32] with this b-function, we conclude that the dimension is equal to one, which coincides with the number of the bounded segments of $\mathbf{R}\setminus\{u, v\}$.

Next, we tried to evaluate the dimension of $A_2/(A_2\hat{p} + A_2\hat{q} + xA_2 + yA_2)$ where A_2 is the ring of differential operators generated by x, y, ∂_x and ∂_y, and p and q

are differential operators which annihilate the function $f = x^a y^b (1-x-y)^c$; we take $p = x(1-x-y)(\partial_x - a/x + c/(1-x-y))$ and $q = y(1-x-y)(\partial_y - b/y + c/(1-x-y))$. We evaluate the dimension, this time with a computer program [41], by iterating to apply the algorithm for computing the 0th restriction in [32] firstly for x and secondly for y. The result is again one, which is equal to the number of the bounded cells of the hyperplane arrangement $\mathbf{R}^2 \setminus \{(x,y) \mid xy(1-x-y) = 0\}$. It is well known in the theory of hypergeometric functions that the number of bounded cells is equal to the dimension of the middle dimensional twisted cohomology group associated with f, which is equal to the rank of the corresponding hypergeometric system. (Strictly speaking, it turns out that p and q do not generate the annihilating ideal $\{l \in A_2 \mid lf = 0\}$ for f (see Example 4.3); however, the ideal generated by p, q happens to be 'close enough' to the annihilating ideal.)

Inspired by the observation above, we started the project to obtain an algorithm for computing the cohomology groups of the complement of an affine variety by elaborating the method sketched above.

1. Computation of cohomology groups of the complement of an affine hypersurface

For any non-zero polynomial $f \in \mathbf{Q}[x_1, \ldots, x_n]$, we prove the following theorem.

Theorem 1.1. *Put* $X = \mathbf{C}^n$, $Y = V(f) := \{x \in X \mid f(x) = 0\}$, *and* $U = X \setminus Y$. *Then the cohomology group* $H^k(U, \mathbf{C}_U)$ *is computable for any integer* k, *where* $\mathbf{C}_U$ *denotes the constant sheaf on* U *with stalk* $\mathbf{C}$.

Note that $H^k(U, \mathbf{C}_U)$ is the kth cohomology group (with coefficients in $\mathbf{C}$) of the $2n$-dimensional real C^∞-manifold U_{cl} underlying U. Also note that $H^k(U, \mathbf{C}_U) = 0$ for $k > n$ since U is affine and that $H^0(U, \mathbf{C}_U) = \mathbf{C}$ since U is connected.

In the theorem above, we may replace $\mathbf{Q}$ by any computable field. Here, we mean by a computable field a subfield K of $\mathbf{C}$ such that each element of K can be expressed by a finite set of data so that we can decide whether two such expressions correspond to the same element, and that the addition, subtraction, multiplication, and division in K are computable by the Turing machine. For example, any algebraic extension field of $\mathbf{Q}$ of finite rank is a computable field by virtue of Gröbner bases and factorization algorithms over algebraic number fields.

In this section, we illustrate an algorithm to compute the cohomology groups. Correctness will be proved as a special case of the corresponding theorem for cohomology groups with coefficients in a locally constant sheaf of rank one. In order to compute the cohomology groups, we translate the problem to that of computations of functors, especially to that of the de Rham functor, of A_n-modules, which are studied in a series of papers [32,33]. Here, A_n is the ring of differential operators with polynomial coefficients and is called *the Weyl algebra*. The computations of functors are based on the Buchberger algorithm to compute Gröbner bases in the Weyl algebra. We shall quickly review the definition of Weyl algebra and the Gröbner basis. See [7,8,14,25,29,39] for details, [38] for an introduction, and [41] for implementations.

The Weyl algebra

$$A_n = \mathbf{Q}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

is the ring of non-commutative polynomials generated by $2n$ elements x_i, ∂_i, $(i = 1, \ldots, n)$ satisfying the relations

$$\begin{aligned} x_i x_j &= x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \\ \partial_i x_j &= x_j \partial_i + \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases} \end{aligned}$$

The theory (and practice) of Gröbner bases works perfectly well for left ideals in the Weyl algebra A_n. We quickly review the relevant basics. Every element p in A_n can be written uniquely as a $\mathbf{Q}$-linear combination of *normally ordered* monomials $x^a \partial^b$. This representation of p is called the *normally ordered representation.* For example, the monomial $\partial_1 x_1 \partial_1$ is not normally ordered. Its normally ordered representation is $x_1 \partial_1^2 + \partial_1$.

Consider the commutative polynomial ring in $2n$ variables

$$\mathrm{gr}(A_n) = \mathbf{Q}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$$

and the $\mathbf{Q}$-linear map $\mathrm{gr} : A_n \to \mathrm{gr}(A_n)$, $x^a \partial^b \mapsto x^a \xi^b$. Let $<$ be any term order on $\mathrm{gr}(A_n)$. This gives a total order among normally ordered monomials in A_n via $x^A \partial^B > x^a \partial^b \Leftrightarrow x^A \xi^B > x^a \xi^b$. For any element $p \in A_n$ let $in_<(p)$ denote the highest monomial $x^A \partial^B$ in the normally ordered representation of p. If I is a left ideal in A_n then its *initial ideal* is the ideal $\mathrm{gr}(in_<(I))$ in $\mathrm{gr}(A_n)$ generated by all monomials $\mathrm{gr}(in_<(p))$ for $p \in I$. Clearly, $\mathrm{gr}(in_<(I))$ is generated by finitely many monomials $x^a \xi^b$. A finite subset G of I is called *a Gröbner basis* of I with respect to the term order $<$ if $\{\mathrm{gr}(in_<(q)) \mid q \in G\}$ generates $\mathrm{gr}(in_<(I))$. Noting that $in_<(p) \leq in_<(q)$ implies $in_<(hp) \leq in_<(hq)$ for all $h \in A_n$, one proves that the reduced Gröbner basis of I is unique and finite, and can be computed using Buchberger's algorithm. Any left (or right) ideal in A_n is finitely generated and we denote by $\langle p_1, \ldots, p_m\rangle$ the left ideal in A_n generated by $p_1, \ldots, p_m \in A_n$.

Most constructions in the commutative algebra can be reduced to computations of Gröbner bases. This is also the case with some constructions for modules over the Weyl algebra. For example, the construction of a free resolution of a left coherent A_n-module is a straightforward generalization of algorithms of constructing free resolutions of modules over the ring of polynomials. As to algorithms to construct a free resolution by *the Schreyer order*, see [1, p. 167 Theorem 3.7.13], [12, Theorem 15.10] and [37]. We note that an algorithm to compute a sheaf cohomology on the n-dimensional projective space is given based on computation of syzygies in the ring of polynomials [13]. Computation of *an elimination ideal* in the Weyl algebra is also a straightforward generalization of computation of an elimination ideal in the ring of polynomials, (see, e.g., [1, p. 69 Theorem 2.3.4], [9, p. 114 Theorem 2]). These two constructions will be used in our algorithm to obtain cohomology groups (see Algorithm 1.2 Step 3, Procedure 1.4 Step 2 and Procedure 2.2 Step 2).

However, the non-commutativity causes some difficulty in constructing various objects in the category of modules over the Weyl algebra. For example, to compute the tensor product of right and left A_n-modules in the derived category, special care must be taken. This problem has been an open problem since [40]. As a special (but important) case of the tensor product computation as above, we give an algorithm for the $\mathscr{D}$-module theoretic restriction of an A_n-module by using the V-filtration and the b-function (or the indicial polynomial). As to details, see [32, Proposition 5.2, Theorem 5.7, Algorithm 5.10] and [33]. Walther [42] solved a related problem of computing algebraic local cohomology groups based on V-filtration, b-function and the Čech complex. As we will see in Section 7, his algorithm gives an algorithm for Theorem 1.1 different from ours explained below.

We have explained a general background on an algorithmic treatment of modules over the Weyl algebra. Now let us explain our algorithm to compute cohomology groups by a top-down expansion.

Algorithm 1.2 (Computation of the cohomology groups $H^k(U, \mathbf{C}_U)$).

Input: a polynomial $f \in \mathbf{Q}[x_1, \ldots, x_n]$.

Output: $H^k(U, \mathbf{C}_U)$ for $0 \le k \le n$ where $U = \mathbf{C}^n \setminus V(f)$.

1. Find a left ideal I such that

$$\mathbf{Q}\left[x, \frac{1}{f}\right] \simeq A_n/I$$

as a left A_n-module.

2. Let J be the formal Fourier transform of I;

$$J = I_{x_i \mapsto -\partial_i, \partial_i \mapsto x_i}.$$

3. Compute a free resolution

$$\cdots \longrightarrow A_n^{p_{-(n+1)}} \xrightarrow{\cdot L^{-(n+1)}} A_n^{p_{-n}} \xrightarrow{\cdot L^{-n}} A_n^{p_{-(n-1)}} \cdots \xrightarrow{\cdot L^{-1}} A_n^{p_0} \longrightarrow A_n/J \longrightarrow 0,$$

($p_0 = 1$) of A_n/J by using Schreyer's theorem [12, Theorem 15.10] with an order which refines the partial order defined by the weight vector

$$\begin{matrix} \partial_1 & \cdots & \partial_n & x_1 & \cdots & x_n \\ (1 & \cdots & 1 & -1 & \cdots & -1). \end{matrix}$$

The length may be more than $n+1$, but we discard higher order syzygies $A_n^{p_{-m}} \xrightarrow{L^{-m}}$, $m > n+1$.

4. Compute the cohomology groups of the complex of $\mathbf{Q}$-vector spaces

$$\left(A_n/(x_1 A_n + \cdots + x_n A_n) \otimes_{A_n} A_n^{p_{-k}}, \xrightarrow{1 \otimes L^{-k}} \right).$$

Then, the $(k-n)$-th cohomology group $\mathrm{Ker}(1 \otimes L^{k-n})/\mathrm{Im}(1 \otimes L^{k-n-1})$ of the complex above tensored with $\mathbf{C}$ gives $H^k(U, \mathbf{C}_U)$.

The step 1 will be explained in Procedure 1.4 in detail and the steps 2, 3 and 4 will be explained in Procedure 1.8 in detail.

The main purpose of this paper is to show the correctness of this algorithm and related generalized algorithms. Here is a good place to overview the contents of each sections.

In step 1, we derive systems of differential equations for f^{-r_0}. We will prove Theorem 1.1 in a more general form Theorem 2.1, which deals with cohomology groups with coefficients in a locally constant sheaf instead of **C**-coefficients. Although an algorithm to derive differential equations for f^{-r_0} is discussed in [31], in order to compute the cohomology groups with coefficients in a locally constant sheaf, step 1 should be replaced by a more general algorithm, which will be discussed in Sections 2–4 with a proof of correctness.

Sections 5 and 6 are for steps 2, 3 and 4. The correctness of steps 3 and 4 are shown by utilizing results of [33]. We note that the steps 2, 3 and 4 are nothing but the computation of

$$H^{k-n}(A_n/(\partial_1 A_n + \cdots + \partial_n A_n) \otimes^L_{A_n} A_n/I),$$

which is denoted by

$$\int^{k-n}_{\mathbf{C}^n} A_n/I = \int^{k-n}_{\mathbf{C}^n} \mathbf{Q}[x, 1/f]$$

in the theory of $\mathscr{D}$-modules. We shall prove that this cohomology group tensored with **C** is equal to $H^k(U, \mathbf{C}_U)$ by the Grothendieck–Deligne comparison theorem in Section 5. Here, for a left A_n-module M and a right A_n-module N, we denote by $N \otimes^L_{A_n} M$ the complex

$$(N \otimes_{A_n} M^i, 1 \otimes d^{i-1}; i = 1, 0, -1, -2, \ldots),$$

where

$$\cdots \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} \cdots \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \to M \to 0, \quad \text{(exact)}$$

M^i is a free A_n-module, d^i is a left A_n-morphism and $M^1 = 0$, $d^0 = 0$. It is known that there exists a finite length free resolution for a given finitely generated left A_n-module M (e.g., apply the method of [12, p. 336 Corollary 15.11] to our case).

Remark 1.3. For a left A_n-module $M = A_n/I$, the left A_{n-1}-module $(A_n/\partial_n A_n) \otimes_{A_n} M = A_n/(I + \partial_n A_n)$ is called the 0-th integral of M with respect to x_n. Why is it called the integral? Let us explain an intuition behind this terminology.

Let f be a function of $x_1, \ldots, x_n$. We suppose that the function f is rapidly decreasing with respect to the variable x_n and put $I = \operatorname{Ann} f = \{l \in A_n \mid lf = 0\}$. Then the A_n-module generated by the function f is isomorphic to the left A_n-module A_n/I. Put $g(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_n$. Then, we have $[(I + \partial_n A_n) \cap A_{n-1}]g = 0$. In fact, since any element l in $(I + \partial_n A_n) \cap A_{n-1}$ can be written as $l = l_1 + \partial_n l_2$, $l_1 \in I, l_2 \in A_n$, we have $lg = \int_{-\infty}^{\infty} (l_1 + \partial_n l_2) f dx_n = \int_{-\infty}^{\infty} \partial_n (l_2 f) dx_n = 0$. Therefore, g can be regarded as a solution of the differential equations corresponding to the left A_{n-1}-submodule

$A_{n-1}/(I+\partial_n A_n)\cap A_{n-1}$ of $A_n/(\partial_n A_n)\otimes_{A_n} M$. Note that $A_n/(\partial_n A_n)\otimes_{A_n} M$ itself describes a system of differential equations for $\int_{-\infty}^{\infty} x_n^j f\,dx_n$ with $j\geq 0$.

Let us explain in detail the step 1 of Algorithm 1.2. This algorithm is given in [31].

Procedure 1.4 (Computing the differential equations for $1/f^{-r_0}$; step 1 of Algorithm 1.2).

Input: f.

Output: a left ideal I of A_n such that $\mathbf{Q}[x,1/f]\simeq A_n/I$.

1. (Computation of the annihilating ideal of f^s)
   Compute
   $$\left\langle t-f(x), \frac{\partial f}{\partial x_1}\partial_t+\partial_1,\ldots,\frac{\partial f}{\partial x_n}\partial_t+\partial_n\right\rangle\cap\mathbf{Q}[t\partial_t]\langle x,\partial_x\rangle.$$
   Replacing $t\partial_t$ by $-s-1$, we obtain the left ideal $\mathrm{Ann}\,f^s$ in $\mathbf{Q}[s]\langle x,\partial_x\rangle$. (Call Procedure 4.1 with $d=1$ to compute the intersection of the left ideal and the subring $\mathbf{Q}[t\partial_t]\langle x,\partial_x\rangle$.)
2. (Computation of the b-function of f)
   Compute the generator $b(s)$ of
   $$\langle\mathrm{Ann}\,f^s, f\rangle\cap\mathbf{Q}[s]$$
   by an elimination order $x,\partial_x>s$.
3. Let r_0 be the minimum integral root of $b(s)=0$. Put $I=(\mathrm{Ann}\,f^s)_{s\to r_0}$. Then, we have $\mathbf{Q}[x,\frac{1}{f}]\simeq A_n/I$.

The polynomial $b(s)$ is called the (global) *Bernstein–Sato polynomial* or the *b-function* of f. This polynomial is the minimal degree polynomial satisfying the relation

$$Lf^{s+1}=b(s)f^s,\quad \exists L\in\mathbf{Q}[s]\langle x,\partial_x\rangle.$$

The left module A_n/I is a *holonomic* A_n-module (or called a module belonging to the Bernstein class). The holonomicity of A_n/I and the existence of the b-function were shown by I.N. Bernstein. See, e.g., [4] and [5, p. 13, 5.5 Theorem]. It is known that when $f\neq$ const, $b(s)$ always has $s+1$ as a factor. M. Kashiwara proved that all the roots of $b(s)=0$ are negative rational numbers for any $f\in\mathbf{C}[x_1,\ldots,x_n]$ in [18].

Remark 1.5. The left A_n-isomorphism $\mathbf{Q}[x,1/f]\xrightarrow{\varphi}A_n/I$ is expressed as $\varphi=\varphi_2\circ\varphi_1$ where the left A_n-isomorphisms φ_1 and φ_2

$$\mathbf{Q}[x,1/f]\xrightarrow{\varphi_1}A_nf^{r_0}\xrightarrow{\varphi_2}A_n/I$$

are defined as

$$\varphi_2(f^{r_0})=1\in A_n/I,\quad \varphi_1(f^{r_0})=f^{r_0}\in A_nf^{r_0},$$

$$\varphi_1(f^{r_0-k})=\frac{L(r_0-k)\cdots L(r_0-1)}{b(r_0-k)\cdots b(r_0-1)}f^{r_0},\quad (k=1,2,\ldots).$$

Hence, for example, we have

$$\varphi^{-1}(\partial_i) = r_0(\partial f/\partial x_i) f^{r_0-1} = \varphi_1^{-1}(r_0(\partial f/\partial x_i) L(r_0-1) f^{r_0}/b(r_0-1)).$$

Example 1.6. For $f = x(1-x)$, we have

$$\mathrm{Ann}\, f^s = \langle x(1-x)\partial_x - s(1-2x)\rangle.$$

The b-function of f is $s+1$ with

$$((1-2x)\partial_x + 4(1+s)) f^{s-1} = (s+1) f^s,$$

and hence we get

$$\mathbf{Q}[x, 1/f] \simeq \mathbf{Q}\langle x, \partial_x\rangle / \langle x(1-x)\partial_x + (1-2x)\rangle.$$

Example 1.7. Put $f = x^3 - y^2$. We compute the left ideal I such that $\mathbf{Q}[x, y, 1/f] \simeq A_2/I$. Here is a log of the output of kan/k0, which may be self-explanatory. The system k0 is a translator that compiles Java like inputs to codes for kan/sm1, which is a Postscript like language for computations in the ring of differential operators [41].

In(9) = a = annfs($x^3 - y^2, [x, y]$);
Computing the Groebner basis of
$[v*t + x^3 - y^2, -v*u + 1, -3*u*x^2*Dt + Dx, 2*u*y*Dt + Dy]$
with the order $u, v >$ other elements.
In(10) = a :
$[3*x^2*Dy + 2*y*Dx, -6*(-1-s) - 2*x*Dx - 3*y*Dy - 6]$
In(11) = b = ReducedBase(Eliminatev(Groebner(Append[$a, y^2 - x^3$]),
$[x, y, Dx, Dy]$));
In(12) = b:
$[-216*s^3 - 648*s^2 - 642*s - 210]$
In(13) = Factor(b[0]):
$[[-6, 1], [6*s+5, 1], [6*s+7, 1], [s+1, 1]]$

Since $s = -1$ is the minimum integral root of the b-function, we have

$$\mathbf{Q}[x, y, 1/f] \simeq A_2/\langle 3x^2\partial_y + 2y\partial_x, -2x\partial_x - 3y\partial_y - 6\rangle.$$

Finally, let us explain our algorithm for computing

$$H^{k-n}(A_n/(\partial_1 A_n + \cdots + \partial_n A_n) \otimes^L_{A_n} A_n/I).$$

This is a detailed explanation of steps 2, 3 and 4 of Algorithm 1.2. We can compute the cohomology groups by applying [33, Theorem 5.3] to the Fourier transformed ideal J of I. Correctness will be discussed in Sections 5 and 6.

Put

$$w = \begin{pmatrix} \partial_1 & \cdots & \partial_n & x_1 & \cdots & x_n \\ (1 & \cdots & 1 & -1 & \cdots & -1) \end{pmatrix},$$

$$F_k = \{ f \in A_n \mid ord_w(f) \le k \},$$

where

$$ord_w(x^a\partial^b) := -|a| + |b|.$$

$\{F_k\}$ is called the V-filtration.

Procedure 1.8 (Oaku and Takayama [33], Computing the D-module theoretic integral of A_n/I; steps 2, 3 and 4 in Algorithm 1.2).

Input: a left ideal I of A_n. (A_n/I is holonomic.)

Output: The $-k$th cohomology groups of $A_n/(\partial_1 A_n + \cdots + \partial_n A_n) \otimes^L_{A_n} A_n/I$ for $0 \le k \le n$.

1. Let J be the formal Fourier transform of I;

$$J = I|_{x_i \mapsto -\partial_i, \partial_i \mapsto x_i (i=1,\ldots,n)}.$$

2. Let G be a Gröbner basis of the left ideal J with the weight vector w. Find the generator $b(\theta_1 + \cdots + \theta_n)$ of

$$\langle in_w(G)\rangle \cap \mathbf{Q}[\theta_1 + \cdots + \theta_n], \quad \theta_i = x_i\partial_i.$$

3. Let k_1 be the maximum integral root of $b(s) = 0$. If there exists no integral root, then quit; the cohomology groups are all zero in that case.
4. Let $<_w$ be a refinement of the partial order by w. Construct a free resolution

$$\cdots \longrightarrow A_n^{p_{-(n+1)}} \xrightarrow{\cdot L^{-(n+1)}} A_n^{p_{-n}} \xrightarrow{\cdot L^{-n}} A_n^{p_{-(n-1)}} \cdots \xrightarrow{\cdot L^{-1}} A_n^{p_0} \to A_n/J \to 0$$

with $p_0 = 1$ by using the Schreyer orders associated with $<_w$. The length may be more than $n+1$, but we do not need higher order syzygies $A_n^{p_{-m}} \xrightarrow{L^{-m}}$, $m > n+1$.
5. (Computation of degree shifts) Put $s_1^0 = 0$ and

$$s_i^{k+1} = \max_{1 \le j \le p_{-k}} \left(ord_w(L_{ij}^{-(k+1)}) + s_j^k \right) \quad (1 \le i \le p_{-(k+1)})$$

successively.
6. Compute the cohomology groups of the induced complex

$$\cdots \xrightarrow{\cdot \bar{L}^{-2}} F_{k_1 - s_1^1}/(F_{-1} + xA_n) \bigoplus \cdots \bigoplus F_{k_1 - s_{p_{-1}}^1}/(F_{-1} + xA_n)$$
$$\xrightarrow{\cdot \bar{L}^{-1}} F_{k_1}/(F_{-1} + xA_n) \xrightarrow{\bar{L}^0} 0$$

as a complex of $\mathbf{Q}$-vector space where $xA_n = x_1A_n + \cdots + x_nA_n$. Then, the $(k-n)$th cohomology group

$$\mathrm{Ker}\, \bar{L}^{k-n}/\mathrm{Im}\, \bar{L}^{k-n-1}$$

of this complex tensored by $\mathbf{C}$ gives $H^k(U, \mathbf{C}_U)$.

In step 2, we denote by $in_w(\sum_{(a,b)\in J} c_{ab}x^a\partial^b)$ the w-leading form

$$\sum_{\langle w,(a,b)\rangle = m} c_{ab}x^a\partial^b, \quad m = \max_{(a,b)\in J} \langle w, (a,b)\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{Z}^{2n}$. Put $in_w(G) = \{in_w(g) | g \in G\}$. One needs to compute the intersection of the left ideal $in_w(J) = \langle in_w(G) \rangle$ and the subring $\mathbf{Q}[\theta_1 + \cdots + \theta_n]$. This can be done in a procedure similar to the one explained in Section 4.

In step 4, we compute Gröbner bases with the Schreyer orders over the order $<_w$ to construct a resolution. Note that $<_w$ is not a well-order, which causes a difficulty in computation of Gröbner basis in our non-commutative situation. There are two ways to overcome this difficulty; one is to use the F-homogenization introduced in [28] (see also [30, Section 3]) and the other is the use of the homogenized Weyl algebra which has a homogenization variable h so that the relation $\partial_i x_i = x_i \partial_i + h^2$ holds. The homogenized Weyl algebra was introduced in `kan/sm1` [41] in version 2 released in 1994. See [3] on a theoretical study on this homogenization technique.

In Step 6, we truncate the complex from above by using k_1; we could also make truncation from below by using the minimum integral root of $b(s) = 0$, which would somewhat reduce the complexity of Step 6. See [33] for details. Moreover, if we need to compute $H^k(U, \mathbf{C}_U)$ for $k \geq \ell$, then we have only to compute L^{-i} for $i \leq n - \ell + 1$ in Step 4.

Example 1.9. We take $f = x(1-x)$ in $\mathbf{Q}[x]$. We denote by A the Weyl algebra $\mathbf{Q}\langle x, \partial_x \rangle$. As we have seen, we have $\mathbf{Q}[x, 1/f] \simeq A/\langle p \rangle$, $p = x(1-x)\partial_x - (2x-1)$. The formal Fourier transform of p is $\hat{p} = -x\partial_x^2 - x\partial_x$. By multiplying $-x$ from the left, we have

$$-x\hat{p} = \theta(\theta - 1) + x\theta, \quad \theta = x\partial_x.$$

Therefore the b-function is equal to $s(s-1)$ and $k_1 = 1$. A resolution of $A/\langle \hat{p} \rangle$ is

$$0 \to A \xrightarrow{\cdot(x\partial_x^2 - x\partial_x)} A \to A/\langle \hat{p} \rangle \to 0.$$

Since $k_1 = 1$ and the degree shift by $x\partial_x^2 - x\partial_x$ is equal to 1, the truncated complex is

$$0 \to F_0/(F_{-1} + xA) \xrightarrow{\cdot(x\partial_x^2 - x\partial_x)} F_1/(F_{-1} + xA) \to 0.$$

Since

$$F_0/(F_{-1} + xA) = \mathbf{Q}, \qquad F_1/(F_{-1} + xA) = \mathbf{Q} + \mathbf{Q}\partial_x$$

and $1 \cdot (x\partial_x^2 - x\partial_x) \equiv 0$ in $F_1/(F_{-1} + xA)$, we conclude that

$$H^{-1} = F_0/(F_{-1} + xA) = \mathbf{Q}, \qquad H^0 = F_1/(F_{-1} + xA) = \mathbf{Q}^2.$$

Hence, the cohomology groups of $U = \mathbf{C} \setminus \{0, 1\}$ are

$$H^0(U, \mathbf{C}_U) = \mathbf{C}, \qquad H^1(U, \mathbf{C}_U) = \mathbf{C}^2.$$

The two generators of H^1 correspond to two loops that encircle the points $x = 0$ and $x = 1$, respectively, in view of the Poincaré duality of homology groups and cohomology groups.

Example 1.10 (*Cohomology groups of* $\mathbf{C}^2 \setminus V(x^3 - y^2)$). This is the output of `kan/k0`.

```
In(43) = bb = bfunctionForIntegral([3*x^2*Dy+2*y*Dx, -2*x*Dx-3*y*
              Dy - 6], [x, y]);
In(44) = bb:
[ - 216 * s^3 + 432 * s^2 - 264 * s + 48]
In(45) = Factor(bb):
[[ - 24, 1], [3 * s - 2, 1], [s - 1, 1], [3 * s - 1, 1]]
In(46) = integralOfModule([3 * x^2 * Dy + 2 * y * Dx, -2 * x * Dx - 3 * y * Dy - 6],
         [x, y], 1, 1, 2) :
```

Here, 1,1,2 specify the minimum and the maximum integral roots, and the length of the resolution respectively.

```
0-th cohomology: [0,[  ]]
-1-th cohomology: [1,[  ]]
-2-th cohomology: [1,[  ]]
```

The output means that

$$H^0(U, \mathbf{C}_U) = \mathbf{C}, \qquad H^1(U, \mathbf{C}_U) = \mathbf{C}, \qquad H^2(U, \mathbf{C}_U) = 0.$$

Let us explain this example a little more precisely. For $f = x^3 - y^2$, we have $\mathbf{Q}[x, y, 1/f] \simeq A_2/I$ with

$$I = \langle 2x\partial_x + 3y\partial_y + 6,\ 3x^2\partial_y + 2y\partial_x \rangle.$$

Its Fourier transform is A_2/J with

$$J = \langle -2x\partial_x - 3y\partial_y + 1,\ 3y\partial_x^2 - 2x\partial_y \rangle.$$

The b-function of A_2/J is $(s-1)(3s-1)(3s-2)$. Hence we put $k_1 = 1$. A Schreyer (adapted) free resolution of A_2/J with the weight vector $(-1,-1,1,1)$ is given by

$$0 \to A_2 \xrightarrow{\cdot L^{-3}} A_2^4 \xrightarrow{\cdot L^{-2}} A_2^4 \xrightarrow{\cdot L^{-1}} A_2 \to A_2/J \to 0$$

with

$$L^{-1} = \begin{pmatrix} -2x\partial_x - 3y\partial_y + 1 \\ 3y\partial_x^2 - 2x\partial_y \\ -9y^2\partial_y\partial_x - 3y\partial_x - 4x^2\partial_y \\ -27y^3\partial_y^2 - 27y^2\partial_y + 3y + 8x^3\partial_y \end{pmatrix},$$

$$L^{-2} = \begin{pmatrix} 3y\partial_x & 2x & -1 & 0 \\ 2x\partial_y & -3y\partial_y + 2 & -\partial_x & 0 \\ -9y^2\partial_y - 3y & 0 & 2x & 1 \\ 4x^2\partial_y & 0 & -3y\partial_y + 4 & \partial_x \end{pmatrix},$$

$$L^{-3} = (-3y\partial_y + 2, -2x, -\partial_x, 1).$$

The shift vectors are given by

$$(s_1^1, s_2^1, s_3^1, s_4^1) = (0, 1, 0, -1),$$

$$(s_1^2, s_2^2, s_3^2, s_4^2) = (0, 1, -1, 0),$$

$$s_1^3 = 0.$$

By computing the truncated complex, which is a complex of finite dimensional vector spaces and linear maps, we obtain the result.

Example 1.11 (*Cohomology groups of* $\mathbf{C}^3 \setminus V(x^3 - y^2z^2)$). Put $U := \{(x, y, z) \in \mathbf{C}^3 \mid f(x, y, z) \neq 0\}$ with $f = x^3 - y^2z^2$. Then $\mathbf{Q}[x, y, z, 1/f] \simeq A_3/I$ with I being the left ideal generated by

$$-2x\partial_x - 3y\partial_y - 6, \quad -3y\partial_y + 3z\partial_z, \quad -z^2y^2\partial_z + x^3\partial_z - 2zy^2,$$

$$-2z^2y\partial_x - 3x^2\partial_y, \quad -2zy^2\partial_x - 3x^2\partial_z, \quad -6z^3\partial_z\partial_x - 9x^2\partial_y^2 - 6z^2\partial_x,$$

$$-3z^3y\partial_z + 3x^3\partial_y - 6z^2y, \quad -3z^4\partial_z^2 + 3x^3\partial_y^2 - 12z^3\partial_z - 6z^2.$$

Its Fourier transform is A_3/J with J generated by

$$2x\partial_x + 3y\partial_y - 1, \quad 3y\partial_y - 3z\partial_z, \quad -z\partial_z^2\partial_y^2 - z\partial_x^3,$$
$$2x\partial_z^2\partial_y - 3y\partial_x^2, \quad 2x\partial_z\partial_y^2 - 3z\partial_x^2, \quad 6zx\partial_z^3 - 9y^2\partial_x^2 + 12x\partial_z^2,$$
$$-3z\partial_z^3\partial_y - 3y\partial_x^3 - 3\partial_z^2\partial_y, \quad -3z^2\partial_z^4 - 3y^2\partial_x^3 - 12z\partial_z^3 - 6\partial_z^2.$$

The b-function of A_3/J is $(s-1)(2s-1)$. Hence we put $k_1 = 1$. A Schreyer resolution of A_3/J with the weight vector $(-1, -1, -1, 1, 1, 1)$ is given by

$$0 \to A_3 \xrightarrow{\cdot L^{-5}} A_3^3 \xrightarrow{\cdot L^{-4}} A_3^{11} \xrightarrow{\cdot L^{-3}} A_3^{15} \xrightarrow{\cdot L^{-2}} A_3^8 \xrightarrow{\cdot L^{-1}} A_3 \to A_3/J \to 0$$

with

$$\begin{aligned} L^{-1} = (&2x\partial_x + 3y\partial_y - 1, 3y\partial_y - 3z\partial_z, 2x\partial_z^2\partial_y - 3y\partial_x^2, \\ &2x\partial_z\partial_y^2 - 3z\partial_x^2, -z\partial_z^2\partial_y^2 - z\partial_x^3, 6zx\partial_z^3 + 12x\partial_z^2 - 9y^2\partial_x^2, \\ &-3z\partial_z^3\partial_y - 3\partial_z^2\partial_y - 3y\partial_x^3, -3z^2\partial_z^4 - 12z\partial_z^3 - 6\partial_z^2 - 3y^2\partial_x^3), \end{aligned}$$

$$L^{-2} = \begin{pmatrix} 0, \partial_x^3, 0, 0, -3\partial_z, 0, \partial_y, 0 \\ -3y\partial_x^2, 0, 3, 0, 0, -\partial_y, -2x, 0 \\ 0, \partial_x^2, \partial_y, -\partial_z, 0, 0, 0, 0 \\ 0, -y\partial_x^3, 0, 0, 0, 0, z\partial_z + 2, -\partial_y \\ -z\partial_x^2, z\partial_x^2, 0, -z\partial_z, -2x, 0, 0, 0 \\ -3y\partial_x^2, 3y\partial_x^2, -3z\partial_z - 3, 0, 0, 0, -2x, 0 \\ -3y^2\partial_x^2, 3y^2\partial_x^2, 0, 0, 0, -z\partial_z - 1, 0, -2x \\ 3y\partial_y - 3z\partial_z, -2x\partial_x - 3y\partial_y + 1, 0, 0, 0, 0, 0, 0 \\ -\partial_z^2\partial_y, \partial_z^2\partial_y, \partial_x, 0, 0, 0, -1, 0 \\ 0, 2x\partial_z^2, -3y, 0, 0, 1, 0, 0 \\ -\partial_z\partial_y^2, \partial_z\partial_y^2, 0, \partial_x, -3, 0, 0, 0 \\ 0, 2x\partial_z\partial_y, 3z, -3y, 0, 0, 0, 0 \\ -3z\partial_z^3 - 6\partial_z^2, 3z\partial_z^3 + 6\partial_z^2, 0, 0, 0, \partial_x, 0, -3 \\ 0, z\partial_z^3 + \partial_z^2, 0, 0, 0, 0, y, -1 \\ 0, z\partial_z^2\partial_y, 0, 0, 3y, 0, -z, 0 \end{pmatrix}.$$

and so on. From this resolution, we get the final result

$$H^i(U, \mathbf{C}_U) = \begin{cases} \mathbf{C} & (i = 0, 1), \\ 0 & (i = 2, 3). \end{cases}$$

Programs written in the user language of `kan/sm1` for algorithms in the present paper are contained in the `lib` directory of `kan/sm1`.

2. Computation of cohomology groups with coefficients in a locally constant sheaf of rank one

A sheaf $\mathscr{V}$ on U is called *a locally constant sheaf* of rank m if for any $x \in U$, there exists an open set $W \ni x$ such that the restriction $\mathscr{V}|_W$ is a constant sheaf $\mathbf{C}_W^m$.

Let $f_1, \ldots, f_d \in \mathbf{Q}[x]$ be (not necessarily irreducible) factors of f satisfying $f = f_1 \cdots f_d$. Let $a_1, \ldots, a_d$ be complex numbers which lie in a computable field.

The left A_n-module

$$L(a) = \mathbf{Q}[x, 1/f] f_1^{a_1} \cdots f_d^{a_d}$$

is defined as follows: we define the action of ∂_k and x_k by

$$\partial_k \cdot ((g(x)/f^p) \cdot m) = \left(\sum_{i=1}^{d} \frac{a_i \frac{\partial f_i}{\partial x_k}}{f_i} \right) (g(x)/f^p) \cdot m + \frac{\partial (g/f^p)}{\partial x_k} \cdot m,$$

$$x_k \cdot ((g(x)/f^p) \cdot m) = x_k g(x)/f^p \cdot m,$$

where $m = f_1^{a_1} \cdots f_d^{a_d}$ and $g(x)$ is an arbitrary polynomial. In fact, we can easily check that

$$\partial_k \cdot (x_k(g/f^p)m) = (\partial_k x_k) \cdot ((g/f^p)m)$$

and hence our definition of the action is well-defined.

The left A_n-module

$$P(a) = A_n f_1^{a_1} \cdots f_d^{a_d}$$

is the left A_n-submodule of $L(a)$ generated by m.

Put

$$\mathscr{V} = \mathscr{H}\mathrm{om}_{A_n}(P(a), \mathscr{O}_U^{an}),$$

where $\mathscr{O}_U^{an}$ is the sheaf of *holomorphic* functions on the complex manifold $U = \mathbf{C}^n \setminus V(f)$. Here we endow U with the topology as $2n$-dimensional real smooth manifold (the classical topology) instead of the Zariski topology and we denote it U_{cl}. When the left A_n-module $P(a)$ is expressed as $A_n/I(a)$, we can regard $\mathscr{V}$ as a sheaf of holomorphic solutions on U of the system of linear partial differential equations $I(a)$; we have, for a simply connected open set $u \subset U_{cl}$,

$$\mathscr{V}(u) \simeq \{f \in \mathscr{O}_U^{an}(u) | lf = 0 \text{ for all } l \in I(a)\},$$

where the isomorphism is given by

$$\mathscr{V}(u) \ni \varphi \mapsto \varphi(1) \in \mathscr{O}_U^{an}(u).$$

The $\mathbf{C}$-vector space $\mathscr{V}(u)$ is one dimensional and spanned by the function $f_1^{a_1} \cdots f_d^{a_d}$, which is a multi-valued analytic function on U, since $I(a)$ contains

$$f\left(\partial_k - \sum_{i=1}^{d} \frac{a_i}{f_i}\frac{\partial f_i}{\partial x_k}\right) \qquad (k = 1, \ldots, n).$$

Thus, $\mathscr{V}$ is a locally constant sheaf of rank one.

Theorem 2.1. *The cohomology group $H^k(U, \mathscr{V})$ is computable for any $k \geq 0$.*

This theorem is a generalization of Theorem 1.1. In fact, when $a_1 = \cdots = a_d = 0$, the locally constant sheaf $\mathscr{V}$ is the constant sheaf $\mathbf{C}_U$. In order to prove this theorem, we need to generalize Procedure 1.4 to compute a left ideal $I(a)$ of A_n such that $L(a) = A_n/I(a)$ where $I(a)$ is, intuitively speaking, the differential equations for $(f^{-\nu})f_1^{a_1} \cdots f_d^{a_d}$ with an appropriate nonnegative integer ν.

We introduce the Weyl algebra

$$A_{d+n} = \mathbf{Q}\langle t_1, \ldots, t_d, x_1, \ldots, x_n, \partial_{t_1}, \ldots, \partial_{t_d}, \partial_1, \ldots, \partial_n \rangle.$$

for our computation of $I(a)$.

Procedure 2.2 (Computing $L(a)$).

Input: $f, f_1, \ldots, f_d, a_1, \ldots, a_d$.

Output: a left ideal $I(a)$ of A_n such that $L(a) = \mathbf{Q}[x, 1/f]f_1^{a_1} \cdots f_d^{a_d} \simeq A_n/I(a)$.

1. (Computation of the annihilating ideal of $f_1^{s_1}\cdots f_d^{s_d}$ with indeterminates $s_1,\ldots,s_d$) Compute

$$\langle t_j - f_j(x)\ (j=1,\ldots,d), \frac{\partial f_j}{\partial x_i}\partial_{t_j} + \partial_i\ (i=1,\ldots,n, j=1,\ldots,d)\rangle$$
$$\cap\ \mathbf{Q}[t_1\partial_{t_1},\ldots,t_d\partial_{t_d}]\langle x,\partial_x\rangle.$$

Replacing each $t_i\partial_{t_i}$ by the indeterminate $-s_i-1$ in generators of the intersection, we obtain the set

$$G_0(-s_1-1,\ldots,-s_d-1)=\{Q_1(x,\partial_x,-s-1),\ldots,Q_k(x,\partial_x,-s-1)\}.$$

(Call Procedure 4.1 to compute the intersection of a left ideal and the subring $\mathbf{Q}[t_1\partial_{t_1},\ldots,t_d\partial_{t_d}]\langle x,\partial_x\rangle$.) The left ideal $I(s)$ of $\mathbf{Q}[t_1\partial_{t_1},\ldots,t_d\partial_{t_d}]\langle x,\partial_x\rangle$ generated by $G_0(-s-1)$ gives the annihilating ideal for $f_1^{s_1}\cdots f_d^{s_d}$.

2. Compute

$$\langle I(s), f_1(x)\ldots f_d(x)\rangle \cap \mathbf{Q}[s_1,\ldots,s_d]$$

by an elimination order $x,\partial_x > s_1,\ldots,s_d$. Let $G_1(s)$ be a set of generators of the elimination ideal above.

3. Choose a positive integer v such that the set

$$(a_1-v,\ldots,a_d-v)-\mathbf{Z}_{>0}(1,\ldots,1)$$

does not meet the zero set

$$V(G_1(s))=\{v\in\mathbf{C}^d \mid g(v)=0 \text{ for all } g(s)\in G_1(s)\}.$$

4. Output

$$I(a):=\langle G_0(-a_1+v-1,\ldots,-a_d+v-1)\rangle$$

In the above procedure, $I(a)$ is the annihilating ideal of $f_1^{a_1-v}\cdots f_d^{a_d-v}$. The annihilating ideal of $f_1^{a_1}\cdots f_d^{a_d}$ can be computed as the ideal quotient $I(a):(A_n f^v)$ through syzygy computation by means of Gröbner bases.

Let us present an algorithm to compute the cohomology groups $H^k(U,\mathscr{V})$.

Algorithm 2.3 (Computing the cohomology groups $H^k(U,\mathscr{V})$).
Input: $f, f_1,\ldots,f_d,a_1,\ldots,a_d$.
Output: the cohomology groups $H^k(U,\mathscr{V})$.

1. Call Procedure 2.2 with the input $f,f_1,\ldots,f_d,-a_1,\ldots,-a_d$. Get the output $I(-a)$.
2. Call Procedure 1.8 with the input $I=I(-a)$.

3. Computation of $P(a)$ and its localization

Put $X=\mathbf{C}^n$ and let Y be an algebraic set of X defined by the polynomial $f\in\mathbf{Q}[x]$ with $x=(x_1,\ldots,x_n)$. Let $\partial=(\partial_1,\ldots,\partial_n)$ be the corresponding differentiations. We denote by $\mathcal{O}_X$ and $\mathscr{D}_X=\mathcal{O}_X\langle\partial_1,\ldots,\partial_n\rangle$ the sheaf of regular functions, and the sheaf of algebraic differential operators on X respectively (see, e.g., [16, p.15 and p.70] and [17, p.15]). We note that the set of the global sections $\Gamma(X, \mathscr{D}_X)$ coincides with $\mathbf{C} \otimes_{\mathbf{Q}} A_n$, which is the Weyl algebra with coefficients in the complex numbers. We will denote it also by A_n if there is no risk of confusion.

In the sequel, we shall work in the category of algebraic $\mathscr{D}_X$-modules and prove isomorphisms for sheaves of $\mathscr{D}_X$-modules. Correctness of algorithms and procedures given in preceding sections follows by taking global section on X in isomorphisms of propositions.

Put

$$\mathscr{M} = \mathscr{P}(a) = \mathscr{D}_X f_1^{a_1} \cdots f_d^{a_d}.$$

The left coherent $\mathscr{D}_X$-module $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module of rank one on $X \setminus Y$, which is called an integrable connection and $\mathscr{M}_{|X \setminus Y}$ has regular singularities along Y (see, e.g., [10,22], [6, pp. 151–172], [17, pp. 94–100] on regular singularities). Our purpose in this section is to give a proof of correctness of Procedure 2.2, which also gives an algorithm to compute the localization $\mathscr{M}[1/f] := \mathscr{O}_X[1/f] \otimes_{\mathscr{O}_X} \mathscr{M}$. $\mathscr{M}[1/f]$ is a holonomic system on X (Theorem 1.3 of Kashiwara [19]) and coincides with $\mathscr{M}$ on $X \setminus Y$.

We outline a method to compute $\mathscr{P}(a)[1/f]$ for given non-constant polynomials $f_1, \ldots, f_d$ and $a = (a_1, \ldots, a_d)$ with $f := f_1 \cdots f_d$. Here, we assume that a_i lies in a computable field.

Let $s = (s_1, \ldots, s_d)$ be commutative indeterminates and put

$$\mathscr{L}(s) := \mathscr{O}_X[s, 1/f] f_1^{s_1} \cdots f_d^{s_d},$$

which we regard as a free $\mathscr{O}_X[s, 1/f]$-module. Put $\mathscr{P}(s) := \mathscr{D}_X[s] f_1^{s_1} \cdots f_d^{s_d}$.

Then the set of global sections $\Gamma(X, \mathscr{L}(s))$ of $\mathscr{L}(s)$ coincides with $\mathbf{C}[x, s, 1/f] f_1^{s_1} \cdots f_d^{s_d}$, and that of $\mathscr{P}(s)$ with $A_n[s] f_1^{s_1} \cdots f_d^{s_d}$.

Definition 3.1. The (global) Bernstein–Sato ideal $B(f_1, \ldots, f_d)$ in $\mathbf{Q}[s]$ is defined by

$$B(f_1, \ldots, f_d) := \{b(s) \in \mathbf{Q}[s] | b(s) f_1^{s_1} \cdots f_d^{s_d} \in A_n[s] f_1^{s_1 - 1} \cdots f_d^{s_d + 1}\}.$$

The step 2 of Procedure 2.2 gives an algorithm to compute the Bernstein–Sato ideal.

Proposition 3.2 (Sabbah [35]). *There exist a finite number of linear forms $L_1(s), \ldots, L_\kappa(s)$ in s with nonnegative integer coefficients, and nonzero univariate polynomials $b_1, \ldots, b_\kappa$, such that*

$$b(s) := b_1(L_1(s)) \cdots b_\kappa(L_\kappa(s)) \in B(f_1, \ldots, f_d).$$

In particular, for any $a = (a_1, \ldots, a_d) \in \mathbf{C}^d$, the intersection of $\{(a_1 - \nu, \ldots, a_d - \nu) | \nu \in \mathbf{N}\}$ with

$$V(B(f_1, \ldots, f_d)) := \{s = (s_1, \ldots, s_d) \in \mathbf{C}^d | b(s) = 0 \text{ for all } b \in B(f_1, \ldots, f_d)\}$$

is a finite set.

The following proposition tells us that if a is generic, then the localization $\mathscr{L}(a)$ of $\mathscr{P}(a)$ agrees with $\mathscr{P}(a)$.

Proposition 3.3. *Assume that $a=(a_1,\dots,a_d)\in\mathbf{C}^d$ satisfy that $(a_1-v,\dots,a_d-v)$ is not contained in $V(B(f_1,\dots,f_d))$ for all $v=1,2,3,\dots$. Then $\mathscr{P}(a)=\mathscr{L}(a)$ holds. In particular, the $\mathscr{O}_X$-homomorphism $f:\mathscr{P}(a)\to\mathscr{P}(a)$ is an isomorphism.*

Proof. In the notation of Proposition 3.2, there exist $b(s)\in B(f_1,\dots,f_d)$ and $p(s)\in A_n[s]$ such that

$$p(s)f_1^{s_1+1}\cdots f_d^{s_d+1}=b(s)f_1^{s_1}\cdots f_d^{s_d}.$$

and $b(a_1-1,\dots,a_d-1)\neq 0$. Then we have

$$f_1^{a_1-1}\cdots f_d^{a_d-1}=b(a_1-1,\dots,a_d-1)^{-1}p(a_1-1,\dots,a_d-1)f_1^{a_1}\cdots f_d^{a_d}.$$

Proceeding in the same way by using the assumption, we know that $f_1^{a_1-v}\cdots f_d^{a_d-v}$ is contained in $\mathscr{P}(a)$ for $v=1,2,3,\dots$. This implies $\mathscr{P}(a)=\mathscr{L}(a)$. □

Next, we shall see that the localization $\mathscr{L}(a)$ agrees with $\mathscr{P}(a_1-v_0,\dots,a_d-v_0)$ for an integer v_0 determined by the zero set of the Berndstein–Sato ideal. In order to prove this fact, we need a lemma.

Lemma 3.4. *$\mathscr{O}_X[1/f]$ is a flat $\mathscr{O}_X$-module.*

Proof. This should be well-known (e.g. this is a special case of Lemma 1.1 of [21]). Here we give a direct proof. Let $\iota:\mathscr{K}\to\mathscr{N}$ be an arbitrary injective $\mathscr{O}_X$-homomorphism. We have only to show that $1\otimes\iota:\mathscr{K}[1/f]\to\mathscr{N}[1/f]$ is also injective. An arbitrary element of $\mathscr{K}[1/f]=\mathscr{O}_X[1/f]\otimes_{\mathscr{O}_X}\mathscr{K}$ is written in a form $f^{-v}\otimes u$ with some $u\in\mathscr{K}$ and $v\in\mathbf{N}$. Then $f^{-v}\otimes\iota(u)=0$ if and only if $f^{\mu}\iota(u)=0$ for some $\mu\in\mathbf{N}$ (cf. Lemma 7.2 of [32]). This implies that $1\otimes\iota$ is injective. This completes the proof. □

Proposition 3.5. *Fix an arbitrary $a=(a_1,\dots,a_d)\in\mathbf{C}^d$. Let v_0 be a positive integer such that $(a_1-v,\dots,a_d-v)$ is not contained in $V(B(f_1,\dots,f_d))$ for any integer $v>v_0$. Then we have*

$$\mathscr{P}(a)[1/f]=\mathscr{L}(a)=\mathscr{P}(a_1-v_0,\dots,a_d-v_0).$$

Proof. Consider the short exact sequence

$$0\to\mathscr{P}(a)\xrightarrow{\iota}\mathscr{L}(a)\to\mathscr{L}(a)/\mathscr{P}(a)\to 0, \tag{3.1}$$

where ι is the inclusion. First note that $(\mathscr{L}(a)/\mathscr{P}(a))[1/f]=0$. In fact, any section v of $\mathscr{L}(a)$ can be written in the form $v=gf_1^{a_1-v}\dots f_d^{a_d-v}$ with $g\in\mathscr{O}_X$ and $v\in\mathbf{N}$. Hence we have $f^v v\in\mathscr{P}(a)$. This implies $(\mathscr{L}(a)/\mathscr{P}(a))[1/f]=0$.

Since $\mathscr{O}_X[1/f]$ is a flat $\mathscr{O}_X$-module, we have from (3.1) an exact sequence

$$0\to\mathscr{P}(a)[1/f]\xrightarrow{1\otimes\iota}\mathscr{L}(a)[1/f]\to 0.$$

Since $\mathscr{L}(a)[1/f]=\mathscr{L}(a)$, we have proved the first equality of the proposition. The second one follows from Proposition 3.3 since $\mathscr{L}(a)=\mathscr{L}(a_1-v_0,\dots,a_d-v_0)$ (f is invertible in $\mathscr{L}(a)$). □

Proposition 3.6. *Under the same assumption as in Proposition* 3.3, *the* $\mathscr{D}_X$-*homomorphism* (*specialization* $s=a$)

$$\rho:\mathscr{P}(s)/((s_1-a_1)\mathscr{P}(s)+\cdots+(s_d-a_d)\mathscr{P}(s))\to\mathscr{P}(a)$$

is an isomorphism.

Proof. Assume that a section $u:=p(s)f_1^{s_1}\cdots f_d^{s_d}$ of $\mathscr{P}(s)$ satisfies $\rho(\overline{u})=0$, where $\overline{u}$ denotes the modulo class of u. Then there exist $g_1(s),\dots,g_d(s)\in\mathscr{O}_X[s]$ and $v\in\mathbf{N}$ such that

$$u=\sum_{j=1}^{d}(s_j-a_j)g_j(s)f_1^{s_1-v}\cdots f_d^{s_d-v}.$$

By the same argument as the proof of Proposition 3.3, we can find $\tilde{b}(s)\in\mathbf{Q}[s]$ and $Q(s)\in\mathscr{D}_X[s]$ such that

$$\tilde{b}(s)f_1^{s_1-v}\cdots f_d^{s_d-v}=Q(s)f_1^{s_1}\cdots f_d^{s_d}$$

and $\tilde{b}(a)\neq 0$.

There exist $c_1(s),\dots,c_d(s)\in\mathbf{C}[s]$ which satisfy

$$\tilde{b}(a)-\tilde{b}(s)=\sum_{j=1}^{d}(s_j-a_j)c_j(s).$$

Hence we get

$$\begin{aligned}\tilde{b}(a)u &= \left(\tilde{b}(s)p(s)+\sum_{j=1}^{d}(s_j-a_j)c_j(s)p(s)\right)f_1^{s_1}\cdots f_d^{s_d}\\ &=\sum_{j=1}^{d}(s_j-a_j)(\tilde{b}(s)g_j(s)f_1^{s_1-v}\cdots f_d^{s_d-v}+c_j(s)p(s)f_1^{s_1}\cdots f_d^{s_d})\\ &=\sum_{j=1}^{d}(s_j-a_j)(g_j(s)Q(s)+c_j(s)p(s))f_1^{s_1}\cdots f_d^{s_d}.\end{aligned}$$

Since $\tilde{b}(a)\neq 0$ by the assumption, we conclude that $u\in(s_1-a_1)\mathscr{P}(s)+\cdots+(s_d-a_d)\mathscr{P}(s)$. Hence ρ is injective. The surjectivity is obvious. □

Let us consider the problem of finding the annihilating ideal of $f_1^{s_1}\cdots f_d^{s_d}$.

Let A_d be the Weyl algebra on the variables $t=(t_1,\dots,t_d)$. We denote by $A_d\mathscr{D}_X:=A_d\otimes_{\mathbf{C}}\mathscr{D}_X$ the sheaf on X of the differential operators in variables (t,x) which are polynomials in t. We follow an argument of Malgrange [26] for the case of $d=1$.

We can endow $\mathscr{L}(s)$ with a structure of left $A_d\mathscr{D}_X$-module by

$$t_j(g(x,s)f_1^{s_1}\cdots f_d^{s_d}) = g(x,s_1,\ldots,s_j+1,\ldots,s_d)f_1^{s_1}\cdots f_j^{s_j-1}\cdots f_d^{s_d}, \tag{3.2}$$

$$\partial_{t_j}(g(x,s)f_1^{s_1}\cdots f_d^{s_d}) = -s_j g(x,s_1,\ldots,s_j-1,\ldots,s_d)f_1^{s_1}\cdots f_j^{s_j-1}\cdots f_d^{s_d} \tag{3.3}$$

for $g(x,s)\in\mathcal{O}_X[s,1/f]$ and $j=1,\ldots,d$.

Lemma 3.7. *Let $\mathcal{N}$ be the sheaf of left ideals of $A_d\mathscr{D}_X$ generated by*

$$t_j - f_j(x) \quad (j=1,\ldots,d), \tag{3.4}$$

$$\partial_{x_i} + \sum_{j=1}^{d}\frac{\partial f_j}{\partial x_i}\partial_{t_j} \quad (i=1,\ldots,n). \tag{3.5}$$

Then each stalk of $\mathcal{N}$ is a maximal left ideal.

Proof. By a coordinate transformation

$$t_j' = t_j - f_j(x) \quad (j=1,\ldots,d), \qquad x'=x,$$

we can reduce to the case where $f_1=\cdots=f_d=0$. In that case, the statement is obvious. □

Proposition 3.8. *We have*

$$\mathcal{N} = \{p\in A_d\mathscr{D}_X \mid pf_1^{s_1}\cdots f_d^{s_d}=0\}.$$

Proof. It is easy to verify the inclusion $\subset$ by using Eqs. (3.2) and (3.3). Since $\mathcal{N}$ is maximal, we obtain the equality. □

We put

$$\mathscr{I}(s) := \{p(s)\in\mathscr{D}_X[s] \mid p(s)f_1^{s_1}\cdots f_d^{s_d}=0\}.$$

Proposition 3.9. *For a Zariski open set u of X, we have*

$$\Gamma(u,\mathscr{I}(s)) = \{p(-s_1-1,\ldots,-s_d-1) \mid p(t_1\partial_{t_1},\ldots,t_d\partial_{t_d})\in\Gamma(u,\mathcal{N}\cap\mathscr{D}_X[t_1\partial_{t_1},\ldots,t_d\partial_{t_d}])\}.$$

Proof. By Eqs. (3.2) and (3.3), we get the relations

$$s_j = -\partial_{t_j}t_j = -t_j\partial_{t_j} - 1 \quad (j=1,\ldots,d).$$

Hence $\mathscr{D}_X[s]$ is isomorphic to the subring $\mathscr{D}_X[t_1\partial_{t_1},\ldots,t_d\partial_{t_d}]$ of $A_d\mathscr{D}_X$. This implies the conclusion. □

Proposition 3.10. *Procedure* 2.2 *is correct.*

Proof. The correctness of step 1 follows from Proposition 3.9.

To verify the correctness of step 2 of Procedure 2.2, one has only to note that for $b(s) \in \mathbf{Q}[s]$, we have $b(s) \in B(f_1, \dots, f_d)$ if and only if $b(s)$ belongs to $\Gamma(X, \mathscr{I}(s)) + A_n[s]f$.

The correctness of steps 3 and 4 can be shown by taking global sections in sheaf isomorphisms given in Propositions 3.5 and 3.6. ☐

As to our experiments, it is more efficient that one eliminates ∂_x first, and then elminates x in step 2 of Procedure 2.2. However, even with this, the complexity of Procedure 2.2 is huge.

4. Computation of the intersection of a left ideal and a subring

In this section, we give a procedure to compute the intersection of the left ideal

$$\left\langle t_j - f_j(x)\ (j = 1, \dots, d), \frac{\partial f_j}{\partial x_i}\partial_{t_j} + \partial_i\ (i = 1, \dots, n, j = 1, \dots, d)\right\rangle$$

in A_{d+n} and the subring $\mathbf{Q}[t_1\partial_{t_1}, \dots, t_d\partial_{t_d}]\langle x, \partial_x\rangle$ of A_{d+n}. The intersection gives the annihilating ideal for $f_1^{s_1} \cdots f_d^{s_d}$ with the replacement $t_i\partial_{t_i} \mapsto -s_i - 1$.

Procedure 4.1. *Input*: polynomials $f_1, \dots, f_d$ in $x = (x_1, \dots, x_n)$.

Output: a set of generators of the annihilating ideal $\mathscr{I}(s)$ of $f_1^{s_1} \cdots f_d^{s_d}$.

1. Introducing indeterminates $t = (t_1, \dots, t_d)$, $u = (u_1, \dots, u_d)$, $v = (v_1, \dots, v_d)$, let I be the left ideal of $A_{n-d}[u, v] = \mathbf{Q}[u, v]\langle x, t, \partial_x, \partial_t\rangle$ generated by

$$t_j - u_j f_j, \quad (j = 1, \dots, d), \tag{4.1}$$

$$\partial_{x_i} + \sum_{j=1}^{d} \frac{\partial f_j}{\partial x_i} u_j \partial_{t_j} \quad (i = 1, \dots, n), \tag{4.2}$$

$$1 - u_j v_j, \quad (j = 1, \dots, d). \tag{4.3}$$

2. Take any term order on $A_{n+d}[u, v]$ for eliminating u, v. Let G be a Gröbner basis of I with respect to this term order. Put $G_0 = \{P_1, \dots, P_k\} := G \cap A_{n-d}$.
3. For each $i = 1, \dots, k$, there exist $Q_i \in \mathscr{D}_X[s]$ and $v_{i1}, \dots, v_{id} \in \mathbf{Z}$ such that

$$S_{1,v_{i1}} \cdots S_{d,v_{id}} P_i = Q_i(x, \partial_x, t_1\partial_{t_1}, \dots, t_d\partial_{t_d})$$

holds, where $S_{j,v} := \partial_{t_j}^{v}$ if $v \geq 0$, and $S_{j,v} := t_j^{-v}$ otherwise. Set

$$G_0(s) := \{Q_1(x, \partial_x, s), \dots, Q_k(x, \partial_x, s)\}.$$

Output: $G_0(-s_1 - 1, \dots, -s_d - 1)$ is a set of generators of $\mathscr{I}(s)$.

Proposition 4.2. *Procedure* 4.1 *is correct.*

Proof. First, we must show that for each i we can find $S_{1,v_{i1}},\ldots,S_{d,v_{id}}$ and Q_i as in the step 3 of Procedure 4.1. Fix any j with $1\le j\le d$. Then the generators of I given in the step 1 are homogeneous with respect to the weight table $\mathscr{W}_j$ below:

$\mathscr{W}_j$:

Variables	x_i, ∂_{x_i} $(1\le i\le n)$	t_j	∂_{t_j}	u_j	v_j	$t_k, \partial_{t_k}, u_k, v_k$ $(k\ne j)$
Weight	0	-1	1	-1	1	0

Moreover, the product of two operators preserves the homogeneity with respect to $\mathscr{W}_j$. Hence each element of G_0 is homogeneous with respect to $\mathscr{W}_j$ and free of u and v. This enables us to find a suitable multiple Q_i of P_i as in the step 3.

Now let us show that each $Q_i(x,\partial_x,-s-1)$ belongs to $\mathscr{I}(s)$ with the notation $-s-1=(-s_1-1,\ldots,-s_d-1)$. By the definition, P_i is contained in the ideal generated by (4.1)–(4.3). Substituting 1 for every u_i and v_i, we know that P_i belongs to $\mathscr{N}$, which is the annihilating ideal sheaf of $f_1^{s_1}\cdots f_d^{s_d}$, since it does not depend on u,v. Hence $Q_i(-s-1)$ belongs to $\mathscr{I}(s)$ in view of Proposition 3.9.

Conversely, let $p(-s-1)$ be an arbitrary section of $\mathscr{I}(s)$. Multiplying by a polynomial, we may assume that $p(t_1\partial_{t_1},\ldots,t_d\partial_{t_d})$ belongs to the left ideal of A_{n-d} generated by (3.4) and (3.5) making use of Proposition 3.9 again. That is, there exist $R_j,S_i\in A_{n+d}$ such that

$$p(t_1\partial_{t_1},\ldots,t_d\partial_{t_d})=\sum_{j=1}^{d}R_j\cdot(t_j-f_j)+\sum_{i=1}^{n}S_i\cdot\left(\partial_{x_i}+\sum_{j=1}^{d}\frac{\partial f_j}{\partial x_i}\right). \tag{4.4}$$

We can homogenize the both sides of Eq. (4.4) by adding u with respect to the weight table $\mathscr{W}_j$. By performing this procedure for every $j=1,\ldots,d$, we obtain a homogenization of Eq. (4.4) with respect to all $\mathscr{W}_1,\ldots,\mathscr{W}_d$. The left hand side of this homogenization is in the form $u_1^{\mu_1}\cdots u_d^{\mu_d}p$ with nonnegative integers $\mu_1,\ldots,\mu_d$ since p itself is homogeneous. Thus $u_1^{\mu_1}\cdots u_d^{\mu_d}p$ is contained in the ideal of $A_n[u]$ generated by (4.1) and (4.2). This implies that

$$p=(1-u_1^{\mu_1}\cdots u_d^{\mu_d}v_1^{\mu_1}\cdots v_d^{\mu_d})p+u_1^{\mu_1}\cdots u_d^{\mu_d}v_1^{\mu_1}\cdots v_d^{\mu_d}p$$

belongs to I. Since G is a Gröbner basis of I with respect to a term order for eliminating u,v, there exist $U_1,\ldots,U_k\in A_{n+d}$ such that

$$p(t_1\partial_{t_1},\ldots,t_d\partial_{t_d})=\sum_{i=1}^{k}U_iP_i.$$

Since p and P_i are homogeneous with respect to each $\mathscr{W}_j$, we may assume that so is U_i. Moreover, since the weight of p is zero with respect to each $\mathscr{W}_j$, all U_i are written in the form

$$U_i=U_i'(t_1\partial_{t_1},\ldots,t_d\partial_{t_d})S_{1,v_{i1}}\cdots S_{d,v_{id}}$$

with some $U_i' \in A_n[t_1\partial_{t_1}, \ldots, t_d\partial_{t_d}]$. Hence $p(s)$ belongs to the left ideal of $A_n[s]$ generated by $G_0(s)$. This completes the proof. □

Example 4.3. Consider $f = x^{s_1}y^{s_2}(1-x-y)^{s_3}$. $\mathscr{I}(s)$ is generated by

$$ys_1 + ys_2 + ys_3 - s_2 - yx\partial_x - y^2\partial_y + y\partial_y,$$
$$xs_1 + xs_2 + xs_3 - s_1 - x^2\partial_x - yx\partial_y + x\partial_x,$$
$$xs_2 + ys_2 + ys_3 - s_2 - yx\partial_y - y^2\partial_y + y\partial_y.$$

Note that this ideal is strictly larger than the ideal generated by trivial annihilators

$$x(1-x-y)\partial_x - x(1-x-y)(\partial f/\partial x)/f,$$
$$y(1-x-y)\partial_y - y(1-x-y)(\partial f/\partial y)/f.$$

5. Twisted de Rham cohomology group

In this section, we shall explain that computation of $\mathscr{D}$-module theoretic integrals of $L(a)$ gives the cohomology groups $H^k(U, \mathscr{V})$, which is nothing but what Grothendieck–Deligne comparison theorem says; the contents of this section should be well-known to specialists. However, they are not explicitly explained in the literature.

First let us recall the integration functor for $\mathscr{D}$-modules. In general, let $\mathscr{M}$ be a left $\mathscr{D}_X$-module (or, more generally, a bounded complex of $\mathscr{D}_X$-modules) defined on X. Then the integration of $\mathscr{M}$ over X is defined by

$$\int_X \mathscr{M} := R\Gamma(X, \Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M})$$

as an object of the derived category of **C**-vector spaces, where R and L denote the right and the left derived functors in the derived categories, Γ is the global section functor, and Ω_X is the sheaf of algebraic n-forms on X, which has a natural structure of the right $\mathscr{D}_X$-module and is isomorphic to $\mathscr{D}_X/(\partial_1\mathscr{D}_X + \cdots + \partial_n\mathscr{D}_X)$ since X is the affine space. For $i \in \mathbf{Z}$, the ith cohomology of $\int_X \mathscr{M}$ is denoted by $\int_X^i \mathscr{M}$, which is a **C**-vector space. $R^i\Gamma(X, \mathscr{N})$ is often denoted by $H^i(X, \mathscr{N})$. See, e.g., [12,16] for an introduction to the mechanism of derived functors.

Now put

$$h_i = \sum_{j=1}^{d} a_j \frac{f}{f_j}\frac{\partial f_j}{\partial x_i} \quad (i = 1, \ldots, n).$$

Let $\mathscr{M}$ be the left $\mathscr{D}_X$-module $\mathscr{M} := \mathscr{D}_X/\mathscr{I}$, where $\mathscr{I}$ is the left ideal generated by $f\partial_i - h_i$ $(i = 1, \ldots, n)$ with the polynomials h_i defined above. Here, we note that h_i satisfy the integrability condition

$$\frac{\partial}{\partial x_j}(h_i/f) = \frac{\partial}{\partial x_i}(h_j/f) \quad (1 \le i, j \le n), \tag{5.1}$$

and the function $f_1^{a_1} \cdots f_d^{a_d}$ is annihilated by the operators $f\partial_i - h_i$. $\mathscr{M}$ has regular singularities along (the non-singular locus of) $Y = V(f)$ and also along the hyperplane at infinity of the projective space $\mathbf{P}^n$ [10,22]. $\mathscr{M}$ and $\mathscr{P}(a)$ are isomorphic as

$\mathscr{D}_X$-modules on $X \setminus Y$. In fact, both are simple holonomic systems and there exists a natural $\mathscr{D}_X$-homomorphism of $\mathscr{M}$ to $\mathscr{P}(a)$ which sends the modulo class of $1 \in \mathscr{D}_X$ to $f_1^{a_1} \cdots f_d^{a_d}$. However, these two modules are not isomorphic on X in general.

Let us denote by Ω_X^i the sheaf of regular (algebraic) i-forms on X. We use the notation $\partial = (\partial_1, \ldots, \partial_n)$ with $\partial_i := \partial/\partial x_i$. Let us denote by $\mathrm{DR}(\mathscr{M})$ the complex

$$0 \to \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{M} \xrightarrow{d} \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{M} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \otimes_{\mathscr{O}_X} \mathscr{M} \to 0,$$

where d is defined by

$$d(dx_{k_1} \wedge \cdots \wedge dx_{k_i} \otimes u) = \sum_{j=1}^{n} dx_j \wedge dx_{k_1} \wedge \cdots \wedge dx_{k_i} \otimes (\partial_j u)$$

for $u \in \mathscr{M}$. Here we regard $\Omega^i \otimes_{\mathscr{O}_X} \mathscr{M}$ as being placed at degree $i-n$. In particular, the cohomology groups of $\mathrm{DR}(\mathscr{D}_X)$ are given by

$$\mathscr{H}^i(\mathrm{DR}(\mathscr{D}_X)) = \begin{cases} \Omega_X & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have an isomorphism

$$\Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M} \simeq \mathrm{DR}(\mathscr{D}_X) \otimes_{\mathscr{D}_X} \mathscr{M} = \mathrm{DR}(\mathscr{M}).$$

Since $\Omega_X^i \otimes_{\mathscr{O}_X} \mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module and X is affine, we have

$$H^k(X, \Omega_X^i \otimes_{\mathscr{O}_X} \mathscr{M}) = 0 \quad (k > 0).$$

Hence by using the standard argument for the sheaf cohomology, the integral which is explicitly represented by a complex $\int_X \mathscr{M} = R\Gamma(X; \mathrm{DR}(\mathscr{M}))$ is equivalent to

$$0 \to (\overset{0}{\wedge} \mathbf{Z}^n) \otimes_{\mathbf{Z}} M \xrightarrow{d} (\overset{1}{\wedge} \mathbf{Z}^n) \otimes_{\mathbf{Z}} M \xrightarrow{d} \cdots \xrightarrow{d} (\overset{n}{\wedge} \mathbf{Z}^n) \otimes_{\mathbf{Z}} M \to 0, \tag{5.2}$$

where $M := \Gamma(X, \mathscr{M})$ and

$$d(e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes u) = \sum_{j=1}^{n} e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes (\partial_j u)$$

with the unit vectors $e_1, \ldots, e_n$ of $\mathbf{Z}^n$.

The de Rham complex $\mathrm{DR}(\mathscr{M}[1/f])$ of the localization $\mathscr{M}[1/f] := \mathscr{O}_X[1/f] \otimes_{\mathscr{O}_X} \mathscr{M}$ is defined by

$$0 \to \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{M}[1/f] \xrightarrow{d} \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{M}[1/f] \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \otimes_{\mathscr{O}_X} \mathscr{M}[1/f] \to 0, \tag{5.3}$$

where d is given by

$$d(dx_{k_1} \wedge \cdots \wedge dx_{k_i} \otimes u) = \sum_{j=1}^{n} dx_j \wedge dx_{k_1} \wedge \cdots \wedge dx_{k_i} \otimes (\partial_j u)$$

for $u \in \mathscr{M}[1/f]$. As $\mathscr{D}_X[1/f]$-module (not as $\mathscr{D}_X$-module!), there is an isomorphism

$$\mathscr{M}[1/f] \simeq \mathscr{D}_X[1/f]/(\mathscr{D}_X[1/f](\partial_1 - h_1 f^{-1}) + \cdots + \mathscr{D}_X[1/f](\partial_n - h_n f^{-1})).$$

Let P be a section of $\mathscr{M}[1/f]$. Then there exist $Q_i \in \mathscr{D}_X[1/f]$ and $r \in \mathscr{O}_X[1/f]$ such that

$$P = \sum_{i=1}^{n} Q_i(\partial_i - h_i f^{-1}) + r.$$

Such r is determined uniquely. Then we define $\varphi(P) = r$. Hence

$$\varphi : \mathscr{M}[1/f] \to \mathscr{O}_X[1/f]$$

defines an isomorphism as $\mathscr{O}_X[1/f]$-module. By transforming the complex (5.3) by means of this φ, we get the following complex that is isomorphic to (5.3):

$$0 \to \Omega_X^0[1/f] \xrightarrow{\nabla} \Omega_X^1[1/f] \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^n[1/f] \to 0, \tag{5.4}$$

where ∇, which is called an integrable connection, is defined by

$$\nabla(u\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} + \frac{h_j}{f} u \right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for $u \in \mathscr{O}_X[1/f]$; in fact, we have

$$\partial_j \cdot u = u\partial_j + \frac{\partial u}{\partial x_j} \equiv u\frac{h_j}{f} + \frac{\partial u}{\partial x_j}$$

modulo $\mathscr{D}_X[1/f](\partial_1 - h_1 f^{-1}) + \cdots + \mathscr{D}_X[1/f](\partial_n - h_n f^{-1})$. Thus the integral $\int_X \mathscr{M}[1/f] = R\Gamma(X, (5.4))$ is isomorphic to the complex

$$0 \to \Gamma(X; \Omega_X^0[1/f]) \xrightarrow{\nabla} \Gamma(X; \Omega_X^1[1/f]) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Gamma(X; \Omega_X^n[1/f]) \to 0 \tag{5.5}$$

since $\Omega_X^k[1/f]$ is a quasi-coherent $\mathscr{O}_X$-module, X is affine and hence $H^k(X, \Omega_X^p[1/f]) = 0$ for $k > 0$ (see, e.g., [16, p. 205, Proposition 1.2A, p. 215, Theorem 3.7] and [36]). The cohomology of this complex is nothing but the algebraic twisted de Rham cohomology with respect to the local system on $X \setminus Y$ defined by the equation $\nabla u = 0$ for $u \in \mathscr{O}_X^{an}$. When $\mathscr{M} = \mathscr{P}(a)$ on $X \setminus Y$, (5.5) gives the algebraic twisted de Rham cohomology groups associated with the local system defined by $\mathscr{P}(-a)$, i.e. the cohomology groups of $X \setminus Y$ with coefficients in the locally constant sheaf

$$\mathscr{V} := \mathscr{H}om_{\mathscr{D}_X}(\mathscr{P}(-a), \mathscr{O}_X^{\mathrm{an}}) = \mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{D}_X^{\mathrm{an}} \otimes_{\mathscr{D}_X} \mathscr{P}(a)),$$

where $\mathscr{D}_X^{\mathrm{an}}$ denotes the sheaf of holomorphic differential operators. In fact, by applying the functor $\mathscr{O}_X^{\mathrm{an}} \otimes_{\mathscr{O}_X}$ to the complex (5.3), we obtain a complex of sheaves on $X \setminus Y$ whose kth cohomology group is

$$\{u \in \mathscr{O}_X^{\mathrm{an}} \mid \nabla u = 0\} = \mathscr{H}om_{\mathscr{D}_X}(\mathscr{P}(-a), \mathscr{O}_X^{\mathrm{an}})$$

if $k = 0$ and zero otherwise.

The algebraic twisted de Rham cohomology coincides with the analytic one by virtue of the comparison theorem of Deligne [10, p. 98 Theorem 6.2, p. 99 Corollary 6.3]. Let us summarize what we have explained.

Theorem 5.1 (Comparison theorem [10]).

$$H^k(U,\mathscr{V}) \simeq H^k((\Gamma(X;\Omega_X^\bullet[1/f]),\nabla)) \simeq H^{k-n}(X,\mathrm{DR}(\mathscr{M}[1/f])).$$

As we will see in Proposition 6.1, we have moreover

$$H^{k-n}(X,\mathrm{DR}(\mathscr{M}[1/f])) \simeq H^{k-n}(A_n/(\partial_1 A_n + \cdots + \partial_n A_n) \otimes^L_{A_n} A_n/I(a)).$$

Example 5.2 (*Beta function*). Putting $X=\mathbf{C}$, we consider $\mathscr{P}(a)=\mathscr{D}_X x^{a_1}(1-x)^{a_2}$ for generic complex numbers a_1 and a_2. We have $\mathscr{P}(a)=\mathscr{L}(a)\simeq \mathscr{D}_X/\langle p\rangle$ with $p=(x^2-x)\partial_x-(a_1+a_2)x+a_1$. The Bernstein–Sato ideal for x and $1-x$ is generated by $(s_1+1)(s_2+1)$. The b-function of the Fourier transform $\mathscr{D}_X/\langle\hat{p}\rangle$ with

$$\hat{p}=x\partial_x^2+(x+a_1+a_2+2)\partial_x+a_1+1$$

is $s(s+a_1+a_2+1)$. Hence by applying Procedure 1.8 with $k_1=0$, we have

$$0\to F_{-1}/(F_{-1}+xA_1)\xrightarrow{\cdot\hat{p}} F_0/(F_{-1}+xA_1)\to 0$$

and we get

$$H^0(U,\mathscr{V})=0,\quad H^1(U,\mathscr{V})=\mathbf{C},$$

where $U=\mathbf{C}\backslash\{0,1\}$ and

$$\mathscr{V}(w)=\{u\in\mathscr{O}^{an}(w)\mid \mathrm{d}u/\mathrm{d}x=(-a_1/x+a_2/(1-x))u\}$$

for a simply connected open set w. Note that

$$H^1(U,\mathscr{V})\simeq\frac{\boldsymbol{C}\left[x,\frac{1}{x(1-x)}\right]\mathrm{d}x}{\nabla\boldsymbol{C}\left[x,\frac{1}{x(1-x)}\right]}\simeq\boldsymbol{C}\cdot\left(\frac{1}{x}-\frac{1}{1-x}\right)\mathrm{d}x,$$

where $\nabla=d+(a_1/x-a_2/(1-x))\,\mathrm{d}x\wedge$. The beta function should be regarded as

$$\int_0^1 x^{a_1}(1-x)^{a_2}\varphi,$$

where $\varphi=\mathrm{d}x/(x(1-x))\in H^1(U,\mathscr{V})$.

Example 5.3. For generic complex numbers $a_1,\ldots,a_m$, we consider $\mathscr{P}(a)=\mathscr{D}_X\prod_{i=1}^m(x-c_i)^{a_i}$ where $c_1,\ldots,c_m$ are distinct points in $\boldsymbol{C}$. By applying our algorithm, we can see that $H^1(U,\mathscr{V})=\boldsymbol{C}^{m-1}$ and $H^0(U,\mathscr{V})=0$ where $U=\boldsymbol{C}\setminus\{c_1,\ldots,c_m\}$ and $\mathscr{V}=\mathscr{H}om_{\mathscr{D}_U}(\mathscr{P}(-a),\mathscr{O}_U^{an})$. See [2] for details on these cohomology groups and hypergeometric functions.

Example 5.4 (*Counting the number of bounded chambers by $\mathscr{D}$-module algorithms*). We consider a collection of hyperplanes

$$L_i(x)=\sum_{j=1}^n c_{ij}x_j+c_{i0}=0,\quad (i=1,\ldots,m)$$

in $\boldsymbol{R}^n$ and put $f=\prod_{i=1}^m L_i(x)$. For complex numbers $a_1,\ldots,a_m$, we consider $\mathscr{P}(a)=\mathscr{D}_X\prod_{i=1}^m L_i(x)^{a_i}$. The number of bounded chambers in $U=\boldsymbol{R}^n\setminus\bigcup_{i=1}^m\{x\mid L_i(x)=0\}$

Table 1
$\dim_{\mathbf{C}} H^i(X \setminus Y, \mathbf{C})$

f	$i=2$	$i=1$	$i=0$	Euler ch.
xy	1	2	1	0
$xy(x+y+1)$	3	3	1	1
$xy \cdot (x+y+1) \cdot (x-y-2)$	6	4	1	3

is equal to the Euler number of $H^k(U, \mathscr{V})$ (see [2, p.47 Theorem 2.13.1] and [34]). Although there are several algorithms in computational geometry to count the number, this number can also be counted by our purely algebraic algorithm. Table 1 is an example of computation of Euler numbers by our algorithm and implementation, where $X = \mathbf{C}^2 = \{(x, y)\}$, $Y = \{f = 0\}$.

Of course, our method is far from efficient. However, it is rather surprising that purely algebraic computations in the ring of differential operators can evaluate the number of bounded chambers in a given hyperplane arrangement.

6. Computation of integration

Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module defined on $X := \mathbf{C}^n$. In this section, we explain a method to translate the computation of integrals $\int_X^i \mathscr{M}$ to that of the restriction $H^i((\mathscr{D}_X/(x_1\mathscr{D}_X + \cdots + x_n\mathscr{D}_X) \otimes^L_{A_n} \hat{\mathscr{M}})$, where $\hat{\mathscr{M}}$ is the Fourier transform of $\mathscr{M}$. Our discussion together with the algorithm of computing the restriction in [33] proves the correctness of Procedure 1.8 and consequently the correctness of steps 2, 3 and 4 of Algorithm 1.2.

The Weyl algebra A_n has a ring automorphism Φ defined by

$$\Phi(x_i) = -\partial_i, \quad \Phi(\partial_i) = x_i \qquad (i = 1, \ldots, n).$$

This Φ naturally defines a new left A_n-module $\hat{M} := \Phi(M)$, which is called the Fourier transform of M. Since $\mathscr{M}$ is holonomic, M belongs to the Bernstein class of A_n-modules (cf. [5, p. 125]). Since the Bernstein class is invariant under the Fourier transform, we know that $\hat{\mathscr{M}} := \mathscr{D}_X \otimes_{A_n} \Phi(M)$ is a holonomic $\mathscr{D}_X$-module on X. By applying Φ to the complex (5.2), we obtain another complex

$$0 \to (\overset{0}{\wedge} \mathbf{Z}^n) \otimes_{\mathbf{Z}} \Phi(M) \overset{\delta}{\to} (\overset{1}{\wedge} \mathbf{Z}^n) \otimes_{\mathbf{Z}} \Phi(M) \overset{\delta}{\to} \cdots \overset{\delta}{\to} (\overset{n}{\wedge} \mathbf{Z}^n) \otimes_{\mathbf{Z}} \Phi(M) \to 0, \tag{6.1}$$

where

$$\delta(e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes u) = \sum_{j=1}^{n} e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes (x_j u).$$

Since the complexes (5.2) and (6.1) are isomorphic, we have only to compute the cohomology groups of (6.1). Here note that (6.1) is a complex defining the restriction of $\hat{\mathscr{M}}$ to the origin of X. Thus, we have the following proposition.

Proposition 6.1. *We have for any i,*

$$H^i(X, \mathrm{DR}(\mathscr{M})) \simeq H^i((A_n/(x_1 A_n + \cdots + x_n A_n)) \otimes^L_{A_n} \hat{M}).$$

Note that $\hat{\mathscr{M}}$ is specializable to the origin (i.e., a nonzero b-function exists) since $\hat{\mathscr{M}}$ is holonomic (cf. [22]). Hence the cohomology groups of (6.1) are computable by steps 2–6 of Procedure 1.8 as shown in [33]. Thus each $\int_X^i \mathscr{M}$ is computable as a finite dimensional vector space and we obtain the following proposition.

Proposition 6.2. *Procedure* 1.8 *is correct.*

The heart of Procedure 1.8 is the truncation of a resolution with respect to a filtration defined by the weight vector w by a root of b-function [33]. Let us briefly explain the idea. Let $\mathscr{M}$ be a holonomic $\mathscr{D}$-module and $b(s;x)$ be the b-function (or indicial polynomial) along $x_1 = 0$. Then, $x_1 \cdot : gr_{k+1}(\mathscr{M})_p \to gr_k(\mathscr{M})_p$ is bijective if $b(k;p) \neq 0$ (see [32, Section 5]). Here $gr(\mathscr{M})$ is the graded module associated with the weight vector $w = (-1, 0, \ldots, 0; 1, 0, \ldots, 0)$. Hence, in order to obtain the kernel and the image of the map x_1, we may truncate the high degree part and the low degree part of the filtration of $\mathscr{M}$ with respect to the weight vector w. In order to obtain all the cohomology groups of the restriction, we need a diagram chase to determine the degree of the truncation. As to details, see [32, Section 5] and [33].

By Propositions 3.10, 4.2, Theorem 5.1 and Proposition 6.2, we obtain the following theorem and complete our proof of Theorems 1.1 and 2.1.

Theorem 6.3. *Algorithms* 1.2 *and* 2.3 *are correct.*

We close this section with the following theorem, which generalizes Theorem 2.1 when the coefficient sheaf $\mathscr{V}$ is expressed in terms of regular holonomic $\mathscr{D}$-module $\mathscr{D}M$ as $\mathscr{V} = \mathscr{H}om_{\mathscr{D}_U}(\mathscr{O}_U, \mathscr{D}_U^{an} \otimes_{A_n} M)$. (As to definitions of regular holonomic systems, see, e.g., [6, p. 302, p. 305], [17, pp. 94–100], [22].) Note that it is a difficult problem in general to reconstruct M from a given $\mathscr{V}$, which is called the Riemann–Hilbert problem.

Theorem 6.4. *Let M be an A_n-module $(A_n)^p/I$ where I is a left submodule of $(A_n)^p$. We assume that $\mathscr{M} := \mathscr{D}_X \otimes_{A_n} M$ is regular holonomic* [6, Definition 11.3] *and that the singular locus of $\mathscr{M}$ on X is given by $f = 0$ with a polynomial $f \in K[x]$, where K is a computable subfield of $\mathbf{C}$. Put $U = \mathbf{C}^n \setminus V(f)$. Then the cohomology groups $H^k(U, \mathscr{H}om_{\mathscr{D}_U}(\mathscr{O}_U, \mathscr{D}_U^{an} \otimes_{A_n} M))$ are computable.*

Proof. An algorithm to compute $M[1/f]$ as a left A_n-module is given in [33, Section 6] under the condition that M is holonomic. Since $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module on U, $\mathscr{M}[1/f]$ is a locally free $\mathscr{O}_X[1/f]$-module on X. Hence any point of X has an affine open neighborhood W so that

$$\mathscr{M}[1/f] = \mathscr{O}_X[1/f]u_1 \oplus \cdots \oplus \mathscr{O}_X[1/f]u_m$$

holds on W as $\mathcal{O}_X[1/f]$-module with sections $u_1,\dots,u_m$ of $\mathcal{M}[1/f]$ on W. There exist $a_{ijk}\in\Gamma(W,\mathcal{O}_X[1/f])$ such that

$$\partial_i u_j=\sum_{k=1}^m a_{ijk}u_k,\quad (1\le i\le n,\ 1\le j\le m). \tag{6.2}$$

Since $\Omega_X^\nu\otimes_{\mathcal{O}_X}\mathcal{M}[1/f]=\Omega_X[1/f]^\nu\otimes_{\mathcal{O}_X[1/f]}\mathcal{M}[1/f]$ and

$$\partial_i\sum_{j=1}^m\varphi_j u_j=\sum_{j=1}^m\left((\partial_i\varphi_j)u_j+\varphi_j\sum_{k=1}^m a_{ijk}u_k\right),$$

the de Rham complex $\mathrm{DR}(\mathcal{M}[1/f])$ is isomorphic to the complex

$$0\to\Omega_X^0[1/f]^m\xrightarrow{\nabla}\Omega_X^1[1/f]^m\xrightarrow{\nabla}\cdots\xrightarrow{\nabla}\Omega_X^n[1/f]^m\to 0$$

on W; here ∇ is defined by

$$\nabla(\varphi dx_{\mu_1}\wedge\cdots\wedge dx_{i\mu_\nu})=\sum_{i=1}^n\left(\frac{\partial\varphi}{\partial x_i}+{}^t a_i\varphi\right)dx_i\wedge dx_{\mu_1}\wedge\cdots\wedge dx_{\mu_\nu},$$

where a_i is the $m\times m$ matrix whose (j,k)-component is a_{ijk}, and $\varphi\in\mathcal{O}_X[1/f]^m$ is regarded as a column vector. Thus we see that $\mathrm{DR}(\mathcal{M}[1/f])$ coincides with the integrable connection on U associated with the locally constant sheaf

$$\mathcal{V}:=\{\varphi\in(\mathcal{O}_U^{\mathrm{an}})^m\mid(\partial_i+{}^t a_i)\varphi=0\quad(i=1,\dots,n)\}.$$

On the other hand, in view of Eq. (6.2), there is an isomorphism

$$\mathcal{M}\simeq\mathcal{D}_U^m/(\mathcal{D}_U^m(\partial_1-a_1)+\cdots+\mathcal{D}_U^m(\partial_n-a_n))$$

on W. Let φ be an element of $\mathcal{H}om_{\mathcal{D}_U}(\mathcal{O}_U,\mathcal{D}_U^{\mathrm{an}}\otimes_{\mathcal{D}_U}\mathcal{M})$. Then $\varphi(1)=\varphi_1u_1+\cdots+\varphi_mu_m$ satisfies

$$0=\partial_i\varphi(1)=\sum_{j=1}^m\left((\partial_i\varphi_j)u_j+\varphi_j\sum_{k=1}^m a_{ijk}u_k\right).$$

Hence the correspondence $\varphi\leftrightarrow(\varphi_1,\dots,\varphi_m)$ defines an isomorphism

$$\mathcal{H}om_{\mathcal{D}_U}(\mathcal{O}_U,\mathcal{D}_U^{\mathrm{an}}\otimes_{\mathcal{D}_U}\mathcal{M})\simeq\mathcal{V}.$$

Finally we can apply the comparison theorem [10, Théorème 6.2] because $\mathcal{M}|_U$ can be regarded as a regular connection in the sense of Deligne [10] ([17, p. 98], [22, Theorem 2.3.2]). In conclusion, we have proved

$$\int_X\mathcal{M}[1/f]=R\Gamma(X,\mathrm{DR}(\mathcal{M}[1/f])=R\Gamma(U,\mathcal{V})[-n].\qquad\square$$

Example 6.5. If $\mathcal{M}$ is not regular holonomic, then the comparison theorem no longer holds. For example, put

$$\mathcal{M}=\mathcal{D}_X/\langle\partial_x+2x\rangle,\quad X=U=\mathbf{C}.$$

The operator ∂_x+2x is not regular at $x=\infty$. One can verify that

$$H^0(X,\mathrm{DR}(\mathcal{M}))=\mathbf{C},\qquad H^{-1}(X,\mathrm{DR}(\mathcal{M}))=0$$

by applying our integration algorithm. Now, take $\varphi \in \mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}^{an})$. We may assume that $f = \varphi(1)$ belongs to $\mathscr{O}^{an}$ since $\partial_x = -2x$ in $\mathscr{M}^{an}$. We have $\partial_x f = 0$ in $\mathscr{M}^{an}$, which means that $f\partial_x + f' \in \langle \partial_x + 2x \rangle$. Then, we have $f'/f = 2x$ and hence $f = \varphi(1) = ce^{x^2} \in \mathscr{M}^{an}$ for a constant c. Therefore, we have $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}^{an}) \simeq \mathbf{C}$. On the other hand,

$$H^1(X,\mathbf{C}) = 0 \neq H^0(X, \mathrm{DR}(\mathscr{M})) \quad \text{and} \quad H^0(X,\mathbf{C}) = \mathbf{C} \neq H^{-1}(X, \mathrm{DR}(\mathscr{M})).$$

7. Computation of cohomology groups on the complement of an algebraic set when its algebraic local cohomology group vanishes except for one degree

The purpose of this section is to establish a connection between the de Rham cohomology of $\mathbf{C}^n$ with an algebraic set removed, and the integration of modules over the Weyl algebra. We use the algebraic local cohomology groups lying in between these two objects. The contents of this section except the last theorem should be well known to specialists ([20, 23, 24, 27]).

Let X be an n-dimensional non-singular algebraic variety over $\mathbf{C}$ and let Y be an arbitrary algebraic set of X. For an $\mathscr{O}_X$-module $\mathscr{F}$, the algebraic local cohomology group $\mathscr{H}^i_{[Y]}(\mathscr{F})$ with support Y (in the sense of Grothendieck) is the ith derived functor of the functor

$$\Gamma_{[Y]}(\mathscr{F}) = \lim_{k\to\infty} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{J}_Y^k, \mathscr{F}),$$

where $\mathscr{J}_Y$ denotes the defining ideal of Y. For any $i \geq 0$, $\mathscr{H}^i_{[Y]}(\mathscr{O}_X)$ is a holonomic $\mathscr{D}_X$-module (Theorem 1.4 of [19]). Note that this is a sheaf and the set of its global sections on the affine space $\mathbf{C}^n$ agrees with the local cohomology module $H^i_Y(\mathbf{C}[x_1,\ldots,x_n])$. When $X = \mathbf{C}^n$, algorithms for computing the algebraic local cohomology groups have been given in [32] for the case where Y is of codimension one and [33,42] for the general case.

In general, for a bounded complex $\mathscr{M}$ of left $\mathscr{D}_X$-modules, the algebraic and the analytic de Rham functors are defined by

$$\mathrm{DR}(\mathscr{M}) := \Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M} = R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M})[n],$$

$$\mathrm{DR}^{an}(\mathscr{M}^{an}) := \Omega_X^{an} \otimes^L_{\mathscr{D}_X^{an}} \mathscr{M}^{an} = R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}^{an})[n]$$

in the derived category of $\mathbf{C}_X$-modules (cf. [20,27]), where Ω_X^{an} denotes the sheaf of holomorphic n-forms and $\mathscr{M}^{an} := \mathscr{D}_X^{an} \otimes_{\mathscr{D}_X} \mathscr{M}$.

Lemma 7.1. *For a quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$ and an algebraic set Y of X, we have $R\Gamma_Y(\mathscr{M}) = R\Gamma_{[Y]}(\mathscr{M})$, where Γ_Y denotes the functor of taking the sections with support contained in Y in the Zariski topology.*

Proof Since $\mathscr{M}$ is quasi-coherent, the Hilbert Nullstellensatz implies $\Gamma_Y(\mathscr{M}) = \Gamma_{[Y]}(\mathscr{M})$. First, suppose that Y is the zeros of a polynomial $f \in \mathbf{C}[x]$. Let p be an arbitrary point of X. Note that

$$\mathscr{H}^i_Y(\mathscr{M})_p = \varinjlim H^i_Y(U,\mathscr{M}) = \varinjlim H^{i-1}(U\setminus Y,\mathscr{M}) = 0$$

for $i \geq 2$, where U runs through the affine open neighborhoods of p, since $U\setminus Y$ is also affine. For $i=1$, we get

$$\begin{aligned}\mathscr{H}^1_Y(\mathscr{M})_p &= \varinjlim \Gamma(U\setminus Y,\mathscr{M})/\Gamma(U,\mathscr{M})\\ &= \varinjlim \Gamma(U,\mathscr{M}[1/f])/\Gamma(U,\mathscr{M})\\ &= \mathscr{H}^1_{[Y]}(\mathscr{M})_p.\end{aligned}$$

Thus we see that $\mathscr{H}^i_Y(\mathscr{M}) = \mathscr{H}^i_{[Y]}(\mathscr{M})$ for any i. For the general case where $Y = \{f_1 = \cdots = f_d = 0\}$ with polynomials $f_1,\ldots,f_d$, we can prove the lemma by expressing $\mathscr{H}^i_{[Y]}(\mathscr{M})$ and $\mathscr{H}^i_Y(\mathscr{M})$ as the Čech cohomology groups with respect to the affine covering $\{X\setminus V(f_i)\}_{i=1}^d$ of $X\setminus Y$ (see [42]). □

Proposition 7.2. *For any coherent $\mathscr{D}_X$-module $\mathscr{M}$ and any algebraic set Y of X, we have $R\Gamma_Y \mathrm{DR}(\mathscr{M}) = \mathrm{DR}(R\Gamma_{[Y]}(\mathscr{M}))$.*

Proof Since $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module, we have

$$\begin{aligned}R\Gamma_Y \mathrm{DR}(\mathscr{M}) &= R\Gamma_Y(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{M}))[n]\\ &= R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, R\Gamma_Y(\mathscr{M}))[n]\\ &= R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, R\Gamma_{[Y]}(\mathscr{M}))[n]\\ &= \mathrm{DR}(R\Gamma_{[Y]}(\mathscr{M}))\end{aligned}$$

in view of the preceding lemma. □

Proposition 7.3. *Let $\mathbf{C}_X$ be the constant sheaf on X with stalk $\mathbf{C}$. Then there is an isomorphism*

$$R\Gamma_Y(X,\mathbf{C}_X) \simeq \int_X R\Gamma_{[Y]}(\mathscr{O}_X)[-n],$$

where $[-n]$ denotes the shift operator in the derived category. In particular, if $\mathscr{H}^i_{[Y]}(\mathscr{O}_X) = 0$ for $i \neq d$, then for any $i \in \mathbf{Z}$, there is an isomorphism

$$H^i_Y(X;\mathbf{C}_X) \simeq \int_X^{i-n-d} \mathscr{H}^d_{[Y]}(\mathscr{O}_X).$$

Proof By Proposition 7.2 we have

$$\begin{aligned}\int_X R\Gamma_{[Y]}(\mathscr{O}_X) &= R\Gamma(X;\mathrm{DR}(R\Gamma_{[Y]}(\mathscr{O}_X)))\\ &= R\Gamma(X;R\Gamma_Y(\mathrm{DR}(\mathscr{O}_X)))\\ &= R\Gamma_Y(X;\mathrm{DR}(\mathscr{O}_X)).\end{aligned}$$

On the other hand, there are two distinguished triangles and a morphism between them:

$$\begin{array}{ccccc} R\Gamma_Y(X;\mathrm{DR}(\mathcal{O}_X)) & \to & R\Gamma(X;\mathrm{DR}(\mathcal{O}_X)) & \stackrel{+1}{\to} & R\Gamma(X\setminus Y;\mathrm{DR}(\mathcal{O}_X)) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_Y(X;\mathrm{DR}^{\mathrm{an}}(\mathcal{O}_X^{\mathrm{an}})) & \to & R\Gamma(X;\mathrm{DR}^{\mathrm{an}}(\mathcal{O}_X^{\mathrm{an}})) & \stackrel{+1}{\to} & R\Gamma(X\setminus Y;\mathrm{DR}^{\mathrm{an}}(\mathcal{O}_X^{\mathrm{an}})). \end{array}$$

Here the vertical homomorphisms except the leftmost one are isomorphisms by virtue of the comparison theorem of Grothendieck [15]. Hence the leftmost vertical homomorphism is also an isomorphism. Moreover, the complex de Rham lemma implies $\mathrm{DR}^{\mathrm{an}}(\mathcal{O}_X^{\mathrm{an}}) = \mathbf{C}_X[n]$. Consequently, we get

$$R\Gamma_Y(X;\mathrm{DR}(\mathcal{O}_X)) = R\Gamma_Y(X;\mathrm{DR}^{\mathrm{an}}(\mathcal{O}_X^{\mathrm{an}})) = R\Gamma_Y(X;\mathbf{C}_X)[n].$$

This completes the proof. □

From the above proposition and the isomorphism $H^i(X\setminus Y;\mathbf{C}) \simeq H_Y^{i+1}(X;\mathbf{C})$ (see, e.g., [16, p. 212, Exercises 2.3]), we obtain

Corollary 7.4. *Assume $\mathcal{H}^i_{[Y]}(\mathcal{O}_X)=0$ for $i \neq d$. Then for any $i \geq 1$, we have an isomorphism*

$$H^i(X\setminus Y;\mathbf{C}_X) \simeq \int_X^{i-n-d+1} \mathcal{H}^d_{[Y]}(\mathcal{O}_X).$$

If Y is non-singular, we can also relate the de Rham cohomology of $X\setminus Y$ to that of Y itself:

Corollary 7.5. *Assume that Y is non-singular and of codimension d. Then, for any $i \geq 1$, there exists an isomorphism*

$$H^i(X\setminus Y;\mathbf{C}_X) \simeq H^{i+1-2d}(Y;\mathbf{C}_Y).$$

Hence, $H^i(Y;\mathbf{C}_Y)$ is computable for any $i \geq 0$.

Proof Let $\iota : Y \to X$ be the embedding. Then by the Kashiwara equivalence (cf. [17, p. 34, Theorem 1.6.1] and [19]), we have an isomorphism $\mathcal{H}^d_{[Y]}(\mathcal{O}_X) = \iota_+\mathcal{O}_Y$. Thus by using Proposition 7.3, we obtain

$$H^i_Y(X;\mathbf{C}_X) \simeq \int_X^{i-n-d} \mathcal{H}^d_{[Y]}(\mathcal{O}_X) \simeq \int_X^{i-n-d} \iota_+\mathcal{O}_Y \simeq \int_Y^{i-n-d} \mathcal{O}_Y \simeq H^{i-2d}(Y;\mathbf{C}_Y).$$

Combining this with the preceding corollary, we are done. □

In [42], Walther gave an algorithm to compute the local cohomology groups $\mathcal{H}^k_{[Y]}(\mathcal{M})$ with a Čech complex under the condition that $\mathcal{M}$ is $(f_1 \cdots f_d)$-saturated. Since $\mathcal{O}_X$ satisfies this condition, we can compute algebraic local cohomology groups $\mathcal{H}^i_{[Y]}(\mathcal{O}_X)$ for any $i \geq 0$ where $Y := \{f_1 = \cdots = f_d = 0\}$. Another approach to compute algebraic local cohomology groups of $\mathcal{M}$ with a resolution and without the condition of saturation is given in [33]. Thus, we have two algorithms for the next theorem.

Theorem 7.6. *The cohomology groups* $H^i(X \setminus Y; \mathbf{C}_X)$ *for any* $i \geq 0$ *is computable if* $Y = V(f_1, \dots, f_d)$, $f_i \in \mathbf{Q}[x_1, \dots, x_n]$ *and if* $\mathscr{H}^j_{[Y]}(\mathscr{O}_X)$ *vanishes except for one* j.

This theorem generalizes Theorem 1.1 under the condition on vanishing of the local cohomology groups $\mathscr{H}^j_{[Y]}(\mathscr{O}_X)$. Note that if $d = 1$, then this condition always holds.

References

[1] W.W. Adams, P. Loustaunau, An Introduction to Gröbner Bases, American Mathematical Society, Providence, RI, 1994.

[2] K. Aomoto, N. Kita, Theory of Hypergeometric Functions, Springer, Tokyo, 1994 (in Japanese).

[3] A. Assi, F.J. Castro-Jiménez, J.M. Granger, How to calculate the slopes of a $\mathscr{D}$-module, Composit. Math. 104 (1996) 107–123.

[4] I.N. Bernstein, The analytic continuation of generalized functions with respect to a parameter, Funct. Anal. Appl. 6 (1972) 26–40.

[5] J.E. Björk, Rings of Differential Operators, North-Holland, Amsterdam, 1979.

[6] A. Borel et al., Algebraic D-modules, Academic Press, Boston, 1987.

[7] F. Castro, Théorème de division pour les opérateurs differentiels et calcul des multiplicités, These 3ème cycle, Université Paris VII, 1984.

[8] F. Castro, Calculs effectifs pour les idéaux d'opérateurs différentiels, Travaux en Cours, vol. 24, Hermann, Paris, 1987, pp. 1–19.

[9] D. Cox, J. Little, D. O'Shea, Ideals, Varieties and Algorithms, Springer, New York, 1991.

[10] P. Deligne, Équations Différentielles à Points Singuliers Réguliers, Lecture Notes in Math., vol. 163, Springer, Berlin, 1970.

[11] A. Dimca, Singularities and topology of hypersurfaces, Universitext. Springer, New York, 1992.

[12] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Springer, New York, 1995.

[13] D. Eisenbud, Computation of cohomology, in: W.V. Vasconcelos et al. (Eds.), Computational Methods in Commutative Algebra and Algebraic Geometry, Springer, New York, 1997, pp. 209–216.

[14] A. Galligo, Some algorithmic questions on ideals of differential operators, Lecture Notes in Computer Science, vol. 204, Springer, Berlin, 1985, pp. 413–421.

[15] A. Grothendieck, On the de Rham cohomology of algebraic varieties, Publication Mathématique IHES 29 (1966) 95–103.

[16] R. Hartshorne, Algebraic Geometry, Springer, New York, 1977.

[17] R. Hotta, T. Tanisaki, $\mathscr{D}$-modules and algebraic groups, Springer, Tokyo, 1995 (in Japanese).

[18] M. Kashiwara, B-functions and holonomic systems – rationality of roots of b-functions, Invent. Math. 38 (1976) 33–53.

[19] M. Kashiwara, On the holonomic systems of linear differential equations, II, Invent. Math. 49 (1978) 121–135.

[20] M. Kashiwara, The Riemann–Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci., Kyoto Univ. 20 (2) (1984) 319–365.

[21] M. Kashiwara, T. Kawai, On the characteristic variety of a holonomic system with regular singularities, Adv. Math. 34 (1979) 163–184.

[22] M. Kashiwara, T. Kawai, On holonomic systems of microdifferential equations, III. Publ. Res. Inst. Math. Sci., Kyoto Univ. 17 (1981) 813–979.

[23] Ph. Maisonobe, C. Sabbah (Eds.), D-modules cohérents et holonomes, Les cours du CIMPA, Travaux en cours 45, Hermann, Paris, 1993.

[24] Ph. Maisonobe, C. Sabbah (Eds.), Images directes et constructibilité, Les cours du CIMPA, Travaux en cours 46, Hermann, Paris, 1993.

[25] Ph. Maisonobe, $\mathscr{D}$-modules: an overview towards effectivity, in: Computer Algebra and Differential Equations, London Mathematical Society Lecture Note Series, vol. 193, London Mathematical Society, London, 1994, pp. 21–56.

[26] B. Malgrange, Le polynome de Bernstein d'une singularité isolée, Lecture Notes in Mathematics, vol. 459, Springer, Berlin, 1975, pp. 98–119.
[27] Z. Mebkhout, Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}_X$-modules cohérents, Travaux en cours 35, Hermann, Paris, 1989.
[28] T. Oaku, Algorithmic methods for Fuchsian systems of linear partial differential equations, J. Math. Soc. Japan 47 (1995) 297–328.
[29] T. Oaku, Gröbner bases and systems of linear partial differential equations – an introduction to computational algebraic analysis, Sophia Kokyuroku in Mathematics 38, Department of Mathematics, Sophia University, Tokyo, 1994, (in Japanese).
[30] T. Oaku, An algorithm of computing b-functions, Duke Math. J. 87 (1997) 115–132.
[31] T. Oaku, Algorithms for the b-function and D-modules associated with a polynomial, J. Pure Appl. Algebra 117 & 118 (1997) 495–518.
[32] T. Oaku, Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules, Adv. Appl. Math. 19 (1997) 61–105.
[33] T. Oaku, N. Takayama, Algorithms for $\mathscr{D}$-modules – restriction, tensor product, localization, and algebraic local cohomology groups, Preprint, 1998, math.AG/9805006 at http://xxx.lanl.gov.
[34] P. Orlik, H. Terao, Arrangements of Hyperplanes, Springer, Berlin, 1992.
[35] C. Sabbah, Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques, Compositio Math. 64 (1987) 213–241.
[36] J.P. Serre, Faisceaux algébriques cohérents, Ann. Math. 61 (1955) 197–278.
[37] R. La Scala, M. Stillman, Strategies for computing minimal free resolutions, J. Symbol. Comput. 26 (1998) 409–431.
[38] B. Sturmfels, N. Takayama, Gröbner bases and hypergeometric functions, in: B. Buchberger, F. Winkler (Eds.), Gröbner Bases and Applications, Cambridge University Press, Cambridge, 1998, pp. 246–258.
[39] N. Takayama, Gröbner basis and the problem of contiguous relations, Jpn. J. Appl. Math. 6 (1989) 147–160.
[40] N. Takayama, An algorithm of constructing the integral of a module – an infinite dimensional analog of Gröbner basis, in: S. Watanabe, M. Nagata (Eds.), Proc. of Internat. Symp. on Symbolic and Algebraic Computation, ACM, New York, 1990, pp. 206–211.
[41] N. Takayama, Kan: A system for computation in algebraic analysis, 1991—, Source code available for Unix computers from ftp.math.kobe-u.ac.jp via anonymous ftp. See also www.math.kobe-u.ac.jp/KAN/.
[42] U. Walther, Algorithmic computation of local cohomology modules and the cohomological dimension of algebraic varieties, J. Pure Appl. Algebra 139 (1999) 303–321.