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Construction of the integral closure of an affine domain in a finite field extension of its quotient field

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Abstract

The construction of the normalization of an affine domain over a field is a classical problem solved since sixteen's by Stolzenberg (1968) and Seidenberg (1970–1975) thanks to classical algebraic methods and more recently by Vasconcelos (1991–1998) and de Jong (1998) thanks to homological methods. The aim of this paper is to explain how to use such a construction to obtain effectively the integral closure of such a domain in any finite extension of its quotient field, thanks to Dieudonné characterization of such an integral closure. As application of our construction, we explain how to obtain an effective decomposition of a quasi-finite and dominant morphism from a normal affine irreducible variety to an affine irreducible variety as a product of an open immersion and a finite morphism, conformly to the classical Grothendieck's version of Zariski's main theorem.

MSC: 14Q99; 13P10

1. Introduction

In this paper, we present an algorithm for computing the integral closure of a finite integral domain over a field in a finite field extension of its quotient field. We start

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by recalling the problem of integral closure in the affine case. Let k be a computable commutative field. Let us give next a prime ideal $\mathfrak{B} := \langle \mathfrak{p}_1, \ldots, \mathfrak{p}_r \rangle$ in a polynomial ring $k[X_1, \ldots, X_n]$. We then form the quotient ring $A := k[X_1, \ldots, X_n]/\mathfrak{B} = k[x_1, \ldots, x_n]$ and consider a finite extension L of its quotient field $Q(A) := k(x_1, \ldots, x_n)$. The integral closure B of A in L is then formed by the elements of L which are integral over A. The problem is to represent B as a finite A-module: $B = x_{n+1}A + \cdots + x_N A$, and as a k-affine domain: $B = k[x_1, \ldots, x_n, x_{n+1}, \ldots, x_N] = k[X_1, \ldots, X_N]/\mathfrak{D}$ where $\mathfrak{D}$ is a finitely generated ideal. In geometric terms, we recall that the normalization of an affine algebraic irreducible variety $\mathscr{V}$ in a finite field extension L of its function field $k(\mathscr{V})$ is given by a pair $(\mathscr{W}, f)$ where $\mathscr{W}$ is an affine algebraic irreducible variety and $f : \mathscr{W} \to \mathscr{V}$ is a morphism determined uniquely (up to an isomorphism) by the following properties: $(\star)$ $\mathscr{W}$ is normal, $(\star\star)$ f is finite, and $(\star\star\star)$ $k(\mathscr{W}) = L$ ($f^\star : k(\mathscr{V}) \to k(\mathscr{W})$ determines the inclusion of $k(\mathscr{V})$ in L). Let us recall a theoretical classical example [15, p. 279]: the normalization of the surface $\mathscr{X} : x_3^2 = x_1 x_2$ of $\mathbb{A}^3$ in the finite field extension $L = k(\sqrt{x_1}, x_3)$ of its function field $k(\mathscr{X})$ is given by the pair $(\mathbb{A}^2, f)$ where f maps $(u, v) \in \mathbb{A}^2$ to $(u^2, v^2, uv) \in \mathscr{X}$ while the normalization of $\mathscr{X}$ in its function field is $\mathscr{X}$ (i.e. $\mathscr{X}$ is a normal variety).

The classical Nœther normalization Theorem (Theorem 2.1) guarantees that B may be represented as an A-module and as a k-affine domain. Unfortunately, this classical theorem does not provide any method to find generators of B (as a finitely generated A-module) nor generators of the defining ideal of B (as an affine domain). Our goal is to present an effective method to determine generators of B as well as an ideal defining B as a k-affine domain. Our approach is based on a characterization of the integral closure coming from Dieudonné [5]. This characterization allows us to reduce our problem to the computation of the normalization of another affine domain constructed from A.

This last problem is a very basic construction in commutative algebra and it is a canonical way of removing singularities in codimension one. The problem of normalization of affine domains was visited by various authors since the 1960s. Stolzenberg [20] gave a construction, based on classical algebraic methods, when the base domain is separably generated. His construction was generalized by Seidenberg in a series of papers [17,18] to affine domains over fields. Seidenberg introduced a condition, denoted (P), on the base field. The construction of Seidenberg was revisited by Traverso [21]. He used computational techniques based on Gröbner basis to improve Seidenberg's construction and proposed the first effective algorithm to compute the normalization of an affine domain. Vasconcelos proposed in [22] a different construction, based on homological effective methods, when the characteristic of the base field is zero or high. More recently, de Jong [14] has adopted the same point of view as Vasconcelos; he proposed an algorithm to compute the integral closure of a reduced Nœtherian ring which integral closure is finitely generated as a module. In particular, his algorithm is valid for affine domains over perfect fields and has been implemented in `Macaulay2` [8] and in `Singular` [11]. In this paper, in order not to restrict the choice of the algorithm computing the normalization, we assume that the base field fulfills the condition introduced by Seidenberg. This condition is fulfilled by perfect fields and thus makes it possible to use also the algorithm of de Jong.

To reduce our problem of integral closure to a problem of normalization, we need crucial algorithmic statements exposed in Section 3. We first expose an algorithm to compute the ideal of algebraic relations between a finite number of elements of the quotient field of an affine domain over a field (Theorem 3.1). This result is a generalization of earlier results given by Shannon and Sweedler [19], Cox et al. [3, p. 131] and van den Essen [7, p. 288]. Next, we present an algorithm to compute the minimal polynomial of an element of an algebraic extension of the quotient field of an affine domain (Proposition 3.2). A less general case of Proposition 3.2 is considered by Adams and Loustaunau in [1, p. 97]. At the end of Section 3, we expose an algorithm to compute a primitive element of a finite and separable extension field (Proposition 3.3). This statement is an improvement of an earlier result given in [2, p. 386]. We point out that all the preceding statements may be also useful to improve a few steps of Seidenberg's algorithm [17,18] in addition to the improvements given by Traverso [21].

All the results of Section 3 together with an effective construction of the normalization of an affine domain over a field in the most general context, i.e. when the base field satisfies the condition (P) introduced by Seidenberg, leads to an effective version of the Nœther normalization Theorem (Theorem 4.2). The proof of this theorem is constructive and provides an algorithm to compute B (Algorithm 4.3).

As an application of our construction, we explain in Section 5 how to obtain an effective decomposition of a quasi-finite and dominant morphism from a normal affine irreducible variety to an affine irreducible variety as a product of an open immersion and a finite morphism, conformly to the theoretical Grothendieck's version of Zariski's main Theorem. We point out that Vasconcelos [23] obtained such a decomposition through effective homological methods when the characteristic of the base field is zero.

Through this paper, we adopt the following notation: k is a computable commutative field. For a k-affine domain $A=k[x_1,\dots,x_n]$, $Q(A)=k(x_1,\dots,x_n)$ is its quotient field and Spec(A) is the set of all prime ideals of A topologized by the Zariski topology.

2. Background

In this subsection, we recall two important classical results. The first result is the celebrated Nœther normalization theorem [16, p. 132]

Theorem 2.1 (Emmy Nœther). *Let A be an affine domain over a commutative field k. Let L be a finite field extension of $Q(A)$. Then, the integral closure of A in L is a finitely generated A-module and an affine domain over k.*

The second statement is a characterization of the integral closure of an affine domain in a finite field extension of its quotient field. This characterization, due to Dieudonné [5, p. 129], expresses in a geometric context the normalization of an affine irreducible variety over a field in a finite extension of its function field.

Proposition 2.2. *Let k be any commutative field. Let A be a k-affine domain and L an finite field extension of $Q(A)$. Then, the integral closure of A in L is the unique* (*up to an isomorphism*) *finite A-sub-algebra of L which is a normal domain and which L is its quotient field.*

3. Some tools

In this section, we present results which are helpful to our main result. We first describe a computational method to determine a generating set of the ideal of relations between elements of the quotient field of an affine domain (Theorem 3.1). In [19], Shannon and Sweedler gave an algorithm to compute a generating set of the ideal of relations between elements of a polynomial ring. Next, this algorithm was extended, firstly, by Cox et al. [3, p. 131] to elements of the quotient field of a polynomial rings and, secondly, by van den Essen [7, p. 288] to elements of an affine domain. We next expose two applications of Theorem 3.1. The first one is a computational method to determine the minimal polynomial of an element of an algebraic extension of the quotient field of an affine domain (Proposition 3.2). The second one consists in a computational method to determine the primitive element of a finite separable extension field (Proposition 3.3). All the results exposed below are straightforward adaptations of results available in the literature. The proofs are hence omitted.

Theorem 3.1. *Let* $A=k[X_1,\dots,X_n]/\mathfrak{B}$ *with* $\mathfrak{B}=\langle \mathfrak{p}_1,\dots,\mathfrak{p}_r\rangle$ *a prime ideal of* $k[X_1,\dots,X_n]$. *Let* $P_1,\dots,P_m$, $Q_1,\dots,Q_m\in k[X_1,\dots,X_n]$ *be such that, for each* $i\in\{1,\dots,m\}$, $Q_i\notin\mathfrak{B}$. *Set* $\bar{P}_i := P_i+\mathfrak{B}$ *and* $\bar{Q}_i := Q_i+\mathfrak{B}$ *for each* $i\in\{1,\dots,m\}$. *Denote by* $\mathfrak{D}\subset k[Y_1,\dots,Y_m]$ *the ideal of algebraic relations over* k *between* $\bar{F}_1 := \dfrac{\bar{P}_1}{\bar{Q}_1},\dots,\bar{F}_m := \dfrac{\bar{P}_m}{\bar{Q}_m}$. *Let G be a Gröbner basis* (*resp. the reduced Gröbner basis*) *of*

$$\mathscr{J} := \left\langle Y_1Q_1-P_1,\dots,Y_mQ_m-P_m, 1-Z\prod_{i=1}^{m} Q_i, \mathfrak{p}_1,\dots,\mathfrak{p}_r \right\rangle \subset k[Z,X_1,\dots,X_n,Y_1,\dots,Y_m]$$

with respect to an admissible order "$>$" *such that* $Z>X_1>\cdots>X_n>Y_1>\cdots>Y_m$. *Then a Gröbner basis* (resp. *the reduced Gröbner basis*) *of* $\mathfrak{D}$ *is* $G\cap k[Y_1,\dots,Y_m]$.

An application of Theorem 3.1 given by the proposition below consists in answering to the following question: given an algebraic extension field $Q(A)$ of $k(f_1,\dots,f_m)$ where $Q(A)$ is the quotient field of an affine domain A over k and $k(f_1,\dots,f_m)$ denotes the quotient field generated by elements $f_1,\dots,f_m$ of $Q(A)$, how to compute the minimal polynomial of an element f of $Q(A)$ over $k(f_1,\dots,f_m)$?

We can find in the literature various answers to this question in cases less general than ours. For instance, in [1, p. 97–100], Adams and Loustaunau gave a method to compute the minimal polynomial of an element of $Q(A)$ over k.

Proposition 3.2. *Let* $A=k[x_1,\dots,x_n]$ *be a k-affine domain. Let* f, $f_1,\dots,f_m$ *be elements of* $k(x_1,\dots,x_n)$ *such that, for each* $i\in\{1,\dots,n\}$, *the ideal of algebraic relations* *over k between x_i, $f_1,\ldots,f_m$ is not equal to* (0). *Then, the minimal polynomial $P(Y)$ of f over $k(f_1,\ldots,f_m)$ may be obtained as*

$$P(Y)=\frac{\tilde{R}(Y)}{\operatorname{Lc}_Y \tilde{R}},$$

where

$$\tilde{R}=R(Y,Y_1,\ldots,Y_m)|_{Y_1=f_1,\ldots,Y_m=f_m},$$

$\operatorname{Lc}_Y \tilde{R}$ *denotes the leading coefficient of $\tilde{R}$ w.r.t. Y*

$R(Y,Y_1,\ldots,Y_m)$ *is a polynomial of G with minimal degree w.r.t. Y*

and G is the reduced Gröbner basis of the ideal $\mathfrak{D}\subset k[Y,Y_1,\ldots,Y_m]$ of algebraic relations between f, $f_1,\ldots,f_m$ with an admissible term ordering "$>$" such that $Y>Y_1>\cdots>Y_m$.

Finally we expose an effective construction of a primitive element of a finite and separable extension field (see [24]).

Proposition 3.3. *Let $L:=K(\eta_1,\ldots,\eta_n)$ be an algebraic extension finitely generated by elements $\eta_1,\ldots,\eta_n$ of L over a commutative field K containing a infinite subset K_0. If $R(Y)\in K(X_1,\ldots,X_n)[Y]$ denotes the minimal polynomial of $\xi:=\eta_1X_1+\cdots+\eta_nX_n\in K(X_1,\ldots,X_n)(\eta_1,\ldots,\eta_n)$ over $K(X_1,\ldots,X_n)$ (where $X_1,\ldots,X_n$ denotes indeterminates over K), $S(X_1,\ldots,X_n,Y):=D(X_1,\ldots,X_n)R(Y)$ denotes a polynomial of $k[X_1,\ldots,X_n][Y]$ deduced from $R(Y)$ by clearing all denominators.*

Then $\lambda:=\sum_{i=1}^{n}\lambda_i\eta_i$ is a primitive element of L/K where $\lambda_1,\ldots,\lambda_n\in K_0$ are chosen such that

$$\left.\frac{\partial S}{\partial Y}(X_1,\ldots,X_n,Y)\right|_{Y=\sum_{i=1}^{n}\lambda_i\eta_i}\neq 0.$$

Remark 3.4. Note that there is another version of the previous proposition given in [2, p. 386]. But it requires that K be a perfect infinite field and needs to compute the minimal polynomial of each η_i instead of only one as in our version.

Remark 3.5. All the mentioned statements of this section may be helpful to improve a few steps of Seidenberg's algorithm [17,18]. Indeed, in this algorithm, the construction of minimal polynomials and of primitive elements of a field extension are considered. These constructions may be performed through Grœbner basis using Propositions 3.2 and 3.3. Moreover, in the last step of this algorithm, Seidenberg deduced from a generating set of the normalization as a finitely generated module, a generating set of such a normalization as a finitely generated algebra over the base field. This step is done by the author through an inefficient way and may be improved using Theorem 3.1.

4. Effective version of the Nœther normalization Theorem

In this section, we present a computational method for computing the integral closure of an affine domain in a finite field extension of its quotient field. We use in this section the notation of the introduction. We require that the base field k satisfies the condition (P) introduced by Seidenberg:

(P) The characteristic of k is 0 or, whenever the characteristic p of k is positive, given a finite system of linear homogeneous equations $\sum a_{ij}X_j = 0$ with $a_{ij} \in k$, we may decide whether this system has a non-trivial solution in k^p and, if it does, we may find one.

With the aim to do explicit calculations, it is quite natural to give an effective representation of L. Set $d = [L : Q(A)]$. Assume that $d \in \mathbb{N}^\star$, i.e. that $L \neq Q(A)$. Introduce new indeterminates $X_{n+1}, \dots, X_{n+d}$. Then we may build a prime ideal $\mathfrak{B}'$ in the polynomial ring $k[X_1, \dots, X_{n+d}]$ such that

$$\mathfrak{B}' \cap k[X_1, \dots, X_n] = \mathfrak{B}.$$

and an isomorphism between k-algebras from L to

$$Q(A') := Q(k[X_1, \dots, X_{n+d}]/\mathfrak{B}') = k(x'_1, \dots, x'_{n+d})$$

which maps x_i to x'_i for each $i \in \{1, \dots, n\}$. Denote by ϕ the canonical morphism from A to A' which maps x_i to x'_i for each $i \in \{1, \dots, n\}$. We then use the isomorphism between L and $Q(A')$ and we compute the integral closure B' of $\phi(A)$ in $Q(A')$. The representation of the integral closure B as an A-module may be deduced from the representation of B' as a $\phi(A)$-module. Indeed the generators of B as an A-module are the preimages of the generators of B' as a $\phi(A)$-module under the isomorphism between L and $Q(A')$.

From now on, all that follows is related to the problem of the integral closure B' of $\phi(A)$ in $L' := Q(A')$. Theorem 2.1 ensures the finiteness of B' as a $\phi(A)$-module as well as a k-affine domain. We now explain how to compute B'; more precisely, we present an algorithm to determine generators $x_{n+1}, \dots, x_N$ of B' as a $\phi(A)$-module. Furthermore, we present a method to determine the defining ideal $\mathfrak{D}' = \langle \mathfrak{q}'_1, \dots, \mathfrak{q}'_s \rangle \subset k[X_1, \dots, X_N]$ of B' as a k-affine domain. Our approach is to reduce the computation of B' to the computation of the normalization of a suitable affine domain R' (constructed from $\phi(A)$). Various authors proposed algorithms to compute the normalization of an affine domain. We choose not to present them and send the reader to [14,17,18,21,22]. In the following proposition, we do not describe in detail the algorithm exposed in [21]; we only state the outputs of this algorithm.

Proposition 4.1. *Let k be a commutative computable field satisfying Condition* (P). *Let $R := k[y_1, \dots, y_p]$ be an affine domain; let us denote by $\bar{R}$ the normalization of R. Then, we can compute elements $c_1, \dots, c_m$ of $Q(R) = k(y_1, \dots, y_p)$ such that*

1. $k(y_1, \dots, y_p, c_1, \dots, c_m) = k(y_1, \dots, y_p)$.
2. $\bar{R} = \sum_{j=1}^{m} c_j k[y_1, \dots, y_p] = k[y_1, \dots, y_p, c_1, \dots, c_m]$.

We now state our main result which shows that the known constructions of normalization work in a more general context.

Theorem 4.2. *Introduce new indeterminates* $Y_1,\dots,Y_m$ *over* $k[X_1,\dots,X_n]$. *We can compute elements* $b'_1,\dots,b'_m$ *of* $k(x'_1,\dots,x'_{n+d})$ *and elements* $\mathfrak{q}'_1,\dots,\mathfrak{q}'_s$ *of* $k[X_1,\dots,X_n,Y_1,\dots,Y_m]$ *such that*

1. $k(x'_1,\dots,x'_n,b'_1,\dots,b'_m)=k(x'_1,\dots,x'_{n+d})$.
2. $B'=\sum_{j=1}^m b'_jk[x'_1,\dots,x'_n]=k[x'_1,\dots,x'_n,b'_1,\dots,b'_m]$.
3. $\mathfrak{D}'=\langle\mathfrak{q}'_1,\dots,\mathfrak{q}'_s\rangle$ *is a prime ideal of* $k[X_1,\dots,X_n,Y_1,\dots,Y_m]$.
4. $\mathfrak{D}'\cap k[X_1,\dots,X_n]=\mathfrak{B}$.
5. $B'\xrightarrow{\psi}k[X_1,\dots,X_n,Y_1,\dots,Y_m]/\mathfrak{D}'$ *where* ψ *is the isomorphism of* k*-algebras such that* $\psi^{-1}(X_i+\mathfrak{D}')=x'_i$ *for each* $i\in\{1,\dots,n\}$ *and* $\psi^{-1}(Y_j+\mathfrak{D}')=b'_j$ *for each* $j\in\{1,\dots,m\}$.

Proof. $k(x'_1,\dots,x'_{n+d})$ is an algebraic extension of the quotient field of $\phi(A)=k[x'_1,\dots,x'_n]$. Hence, according to Proposition 3.2, we can compute non-zero elements $a'_{n+1},\dots,a'_{n+d}$ of $\phi(A)$ such that, for each $i\in\{n+1,\dots,n+d\}$, $a'_ix'_i$ is integral over $\phi(A)$. Set

$$R':=k[x'_1,\dots,x'_n,a'_{n+1}x'_{n+1},\dots,a'_{n+d}x'_{n+d}].$$

We compute, using Proposition 4.1, elements $b'_1,\dots,b'_m$ of $k(x'_1,\dots,x'_{n+d})$ which generate the normalization $\bar{R}'$ of R' as a $k[x'_1,\dots,x'_n]$-module:

$$\bar{R}'=\sum_{i=1}^m b'_ik[x'_1,\dots,x'_n]=k[x'_1,\dots,x'_n,b'_1,\dots,b'_m]\subset Q(A').$$

Seeing that R' is integral over $\phi(A)$ and that R' has the same quotient field as A', it holds that $Q(\bar{R}')=k(x'_1,\dots,x'_n,b'_1,\dots,b'_m)=Q(A')$ and that $\bar{R}'$ is integral over $\phi(A)$. Hence, according to Proposition 2.2, we can choose $B'=\bar{R}'$. Now, let $\mathfrak{D}'\subset k[X_1,\dots,X_n,Y_1,\dots,Y_m]$ be the ideal of algebraic relations over k between elements $x'_1,\dots,x'_n$, $b'_1,\dots,b'_m$. A generating set $\{\mathfrak{q}'_1,\dots,\mathfrak{q}'_s\}$ of $\mathfrak{D}'$ can be computed by Theorem 3.1. Next, note that the homomorphism between k-algebras from B' to $k[X_1,\dots,X_n,Y_1,\dots,Y_m]$ which maps x'_i to X_i for each $i\in\{1,\dots,n\}$ and b'_j to Y_j for each $j\in\{1,\dots,m\}$ induces an isomorphism ψ from B' to $k[X_1,\dots,X_n,Y_1,\dots,Y_m]/\mathfrak{D}'$. Moreover, since $\phi(A)\subset B'$, we have $\mathfrak{D}'\cap k[X_1,\dots,X_n]=\mathfrak{B}$. □

We now provide an algorithm to compute the integral closure on an affine domain in a finite field extension of its quotient field together with an example.

Algorithm 4.3. *Inputs*: Let k be a computable field satisfying the condition (P), $d\in\mathbb{N}^\star$.

Let $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ be elements of $k[X_1,\dots,X_n]$ such that $\langle\mathfrak{p}_1,\dots,\mathfrak{p}_r\rangle$ is a prime ideal of $k[X_1,\dots,X_n]$.

Let $\mathfrak{p}'_1,\dots,\mathfrak{p}'_{r'}\in k[X_1,\dots,X_n,X_{n+1},\dots,X_{n+d}]$ be such that

- $\langle\mathfrak{p}'_1,\dots,\mathfrak{p}'_{r'}\rangle$ is a prime ideal of $k[X_1,\dots,X_n,X_{n+1},\dots,X_{n+d}]$.

- $\langle \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{r'} \rangle \cap k[X_1, \ldots, X_n] = \langle \mathfrak{p}_1, \ldots, \mathfrak{p}_r \rangle$.
- $k(x'_1, \ldots, x'_{n+d})$ is a finite field extension of $k(x'_1, \ldots, x'_n)$ with degree d where $k(x'_1, \ldots, x'_{n+d})$ and $k(x'_1, \ldots, x'_n)$ denote, respectively, the quotient field of

$$k[x'_1, \ldots, x'_{n+d}] := k[X_1, \ldots, X_{n+d}]/\langle \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{r'} \rangle$$

and

$$k[x'_1, \ldots, x'_n] := \phi(k[X_1, \ldots, X_n]/\langle \mathfrak{p}_1, \ldots, \mathfrak{p}_r \rangle).$$

Here ϕ is the canonical map from $k[X_1, \ldots, X_n]/\langle \mathfrak{p}_1, \ldots, \mathfrak{p}_r \rangle$ to $k[X_1, \ldots, X_{n+d}]/\langle \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{r'} \rangle$ such that $\phi(X_i + \langle \mathfrak{p}_1, \ldots, \mathfrak{p}_r \rangle) = X_i + \langle \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{r'} \rangle$ for each $i \in \{1, \ldots, n\}$.

Output: $B'=$ integral closure of $k[x'_1, \ldots x'_n]$ in $k(x'_1, \ldots, x'_{n+d})$

First step (*construction of a finitely generated k-algebra R integral over* $k[x'_1, \ldots, x'_n]$ *which quotient field is* $k(x'_1, \ldots, x'_n, x'_{n+1}, \ldots, x'_{n+d})$):

- We compute, thanks to Proposition 3.2, the minimal polynomial $P_i \in k(x'_1, \ldots, x'_n)[Y_i]$ of $x'_i \in k(x'_1, \ldots, x'_{n+d})$ over $k(x'_1, \ldots, x'_n)$ for each $i \in \{n+1, \ldots, n+d\}$.
- For each $i \in \{n+1, \ldots, n+d\}$, set $P_i(Y_i) := Y_i^{s_i} + \sum_{j=0}^{s_i-1} \gamma_{i,j} Y_i^j$ where $\gamma_{i,j} \in k(x'_1, \ldots, x'_n)$ denotes the coefficient of P_i with respect to Y_i^j and $s_i := \deg_{Y_i} P_i$. For all $i \in \{n+1, \ldots, n+d\}$, if $a_i \in k[x'_1, \ldots, x'_n]$ denotes a common denominator of γ_{ij}, then we set

$$R := k[x'_1, \ldots, x'_n, a_{n+1}x'_{n+1}, \ldots, a_{n+d}x'_{n+d}].$$

Second step (*computation of the normalization* $\bar{R}$ *of* R *as a* $k[x'_1, \ldots, x'_n]$*-module*):

We compute, thanks to Proposition 4.1, the normalization $\bar{R}$ of R as a R-module. Denote by $\{l_1, \ldots, l_t\}$ a generating set of $\bar{R}$ as R-module where $l_i \in k(x'_1, \ldots, x'_{n+d})$ for each $i \in \{1, \ldots, t\}$. We set

$$B' := \bar{R}.$$

Third step (*computation of* B'):

($\star$) *Representation of* B' *as a finitely generated* $k[x'_1, \ldots, x'_n]$*-module*: We compute the set $\{\bar{b}_1, \ldots, \bar{b}_m\} \subset k(x'_1, \ldots, x'_{n+d})$ of all the products $l_k \prod_{j=n+1}^{n+d} (a_j x'_j)^{\omega_j}$ with $k \in \{1, \ldots, t\}$ and where, for each $j \in \{n+1, \ldots, n+d\}$, $s_j = \deg_{Y_j} P_j$ and $\omega_j \in \{0, \ldots, s_j - 1\}$. Thus

$$B' = \sum_{j=1}^{m} \bar{b}_j k[x'_1, \ldots, x'_n].$$

($\star\star$) *Representation of* B' *as a finitely generated k-algebra*: We compute, thanks to Theorem 3.1, the ideal $\mathfrak{D}' \subset k[X_1, \ldots, X_n, Y_{n+1}, \ldots, Y_{n+m}]$ of algebraic relations over k between $x'_1, \ldots, x'_n$, $\bar{b}_1, \ldots, \bar{b}_m$

$$B' \simeq k[Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+m}]/\mathfrak{D}'.$$

Remark 4.4. The first step of this algorithm can be simplified in the particular case where $k(x'_1, \ldots, x'_{n+d})$ is a separable extension field of $k(x'_1, \ldots, x'_n)$. In this case, we first compute a primitive element x' of $k(x'_1,\dots,x'_{n+d})/k(x'_1,\dots,x'_n)$ thanks to Proposition 3.3. Next, as in the first step of our algorithm, we compute $a' \in k(x'_1,\dots,x'_{n+d})$ so that $a'x'$ is integral over $k[x'_1,\dots,x'_n]$. Then, we set $R := k[x'_1,\dots,x'_n,a'x']$.

Example 4.5. In order to illustrate our algorithm, we compute the integral closure B' of $\phi(A) := \phi(\mathbb{Q}[X_1,X_2,X_3]/\langle X_3^2 - X_1X_2\rangle) = k[x'_1,x'_2,x'_3]$ in the finite field extension $Q(A')$ of $Q(\phi(A))$ with $A' = \mathbb{Q}[X_1,X_2,X_3,X_4,X_5]/\langle X_3^2 - X_1X_2, X_3^2X_4^2X_5 + 1, X_3^2X_4^4 + X_5^2\rangle := k[x'_1,x'_2,x'_3,x'_4,x'_5]$. Throughout this example, we use the Computer Algebra system Singular [9,11] and we adopt the following typographical conventions: text in `typewriter` denotes Singular input and output (moreover, we add an arrow `->` to specify Singular output). The software Singular provides a procedure `primeClosure` [13] for computing the normalization of an affine domain which is based on de Jong's algorithm [4,14].

First step:

(⋆) we compute the minimal polynomials of x'_4 and x'_5 over $\mathbb{Q}(x'_1,x'_2,x'_3)$:

```
ring R=0,(X(1..5),Y(4),Y(1..3)),lp;
option(redSB);
ideal I=Y(4)-X(4),Y(3)-X(3),Y(2)-X(2),
      Y(1)-X(1),X(1)*X(2)-X(3)^2,X(3)^ 2*X(4)^2*X(5)+1,
      X(3)^2*X(4)^4+X(5) ^ 2;
eliminate(normalize(groebner(I)),X(1)*X(2)*X(3)*X(4)*X(5));
-> _[1]=Y(1)*Y(2)-Y(3)^2
-> _[2]=Y(4)^8*Y(3)^6+1
ring R=0,(X(1..5),Y(5),Y(1..3)),lp;
ideal I=Y(5)-X(5),Y(3)-X(3),Y(2)-X(2),Y(1)-X(1),
         X(1)*X(2)-X(3)^2,X(3)^2*X(4)^2*X(5)+1,
         X(3)^2*X(4)^4+X(5)^2;
eliminate(normalize(groebner(I)),X(1)*X(2)*X(3)*X(4)*X(5));
-> _[1]=Y(1)*Y(2)-Y(3)^2
-> _[2]=Y(5)^4*Y(3)^2+1
```

The above calculations show that the minimal polynomials of x'_4 and x'_5 over $\mathbb{Q}(x'_1,x'_2,x'_3)$ are, respectively, $(x'_3)^6T^8 + 1$ and $(x'_3)^2T^4 + 1$.

(⋆⋆) we set

$$R = \mathbb{Q}[x'_1,x'_2,x'_3,(x'_3)^6x'_4,(x'_3)^2x'_5].$$

In order to use the function `primeClosure` of Singular, we require to represent R as a $\mathbb{Q}$-algebra. To this end, we compute the ideal of polynomial relations between x'_1, x'_2, x'_3, $(x'_3)^6x'_4$, $(x'_3)^2x'_5$ thanks to Theorem 3.1.

```
ring R=0,(X(1..5),Y(1..5)),lp;
ideal I=
   X(3)^2-X(1)*X(2),X(3)^2*X(4)^ 2*X(5)+1,
   X(3)^2*X(4)^4+X(5)^2,
   Y(1..3)-X(1..3),
```

```
    Y(4)-X(3)^6*X(4),
    Y(5)-X(3)^2*X(5);
    eliminate(groebner(I),X(1)*X(2)*X(3)*X(4)*X(5));
  ->I[1]=Y(4)^2+Y(5)^7
  ->I[2]=Y(3)^6+Y(5)^4
  ->I[3]=Y(1)*Y(2)-Y(3)^2
```

Second step: We compute the normalization $\bar{R}$ of R

```
ring R=0,(Y(1..5)),dp;
ideal ker=Y(4)^2+Y(5)^7,Y(3)^6+Y(5)^4,Y(1)*Y(2)-Y(3)^2;
LIB "reesclos.lib";
list nor=primeClosure(R);
```

The function `primeClosure` creates a list `nor` consisting of rings such that the first element is a copy of R and the last element is the normalization $\bar{R}$ of R. The output of `nor` has the form:

```
[1]:
//  characteristic     : 0
//  number of vars     : 5
//     block 1         : ordering dp
//                     : names Y(1) Y(2) Y(3) Y(4) Y(5)
//     block 2         : ordering C
...
[15]:
//  characteristic     : 0
//  number of vars     : 4
//     block 1         : ordering dp
//                     : names T(1) T(2) T(3) T(4)
//     block 2         : ordering C.
```

Above, we uses ellipses to indicate that more is printed but we have removed it for the sake of clarity.

Third step:

The integral closure B' of $\phi(A)$ in $Q(A')$ is $\bar{R}$ for which we compute a generating set as a $\phi(A)$-module and a representation as a $\mathbb{Q}$-algebra.

($\star$) `Singular` provides several auxiliary functions for computing the fractions which are integral over R and which generate $\bar{R}$ as a finite algebra over R:

```
closureRingtower(nor);
setring R(15); poly f=T(1); closureFrac(nor); setring R(1);
fraction;
-> [1]:
->   -16*Y(1)*Y(3)^2*Y(4)^3*Y(5)^2
-> [2]:
->   -16*Y(3)^2*Y(4)^3*Y(5)^2
setring R(15); poly f=T(2); closureFrac(nor); setring R(1);
```

```
fraction;
-> [1]:
->   -16*Y(2)*Y(3)^2*Y(4)^3*Y(5)^2
-> [2]:
->   -16*Y(3)^2*Y(4)^3*Y(5)^2
setring R(15); poly f=T(3); closureFrac(nor); setring R(1);
fraction;
->[1]:
->   16*Y(3)^5*Y(4)^3
->[2]:
->   -16*Y(3)^2*Y(4)^3*Y(5)^2
setring R(15); poly f=T(4); closureFrac(nor); setring R(1);
fraction;
-> [1]:
->   Y(4)^4
-> [2]:
->   -16*Y(3)^2*Y(4)^3*Y(5)^2
```

The fractions we are looking for are obtained by dividing [1] by [2] and by substituting x'_1, x'_2, x'_3, $(x'_3)^6x'_4$ and $(x'_3)^2x'_5$ to Y(1), Y(2), Y(3), Y(4) and Y(5). Thus

$$\bar{R} = R\left[\frac{1}{x'_3(x'_5)^2}, \frac{x'_4}{(x'_5)^2}\right] = \mathbb{Q}\left[x'_1, x'_2, x'_3, (x'_3)^6x'_4, (x'_3)^2x'_5, \frac{1}{x'_3(x'_5)^2}, \frac{x'_4}{(x'_5)^2}\right].$$

We next compute the minimal polynomials of $1/x'_3(x'_5)^2$ and $x'_4/(x'_5)^2$ over $\mathbb{Q}[x'_1, x'_2, x'_3]$

```
ring R=0,(X(1..5),T,Y(1..3),Z),lp;
ideal i=T*X(3)*X(5)^2-1,1-Z*X(3)*X(5)^2,Y(3)-X(3),Y(2)-X(2),
         Y(1)-X(1),X(1)*X(2)-X(3)^2,X(3)^2*X(4)^2*X(5)+1,
         X(3)^2*X(4)^4+X(5)^2;
eliminate(normalize(groebner(i)),
X(1)*X(2)*X(3)*X(4)*X(5)*Z);
-> _[1]=Y(1)*Y(2)-Y(3)^2
-> _[2]=T^2+1
ideal i=T*X(5)^2-X(4), 1-Z*X(5)^2,Y(3)-X(3),Y(2)-X(2),Y(1)-X(1),
   X(1)*X(2)-X(3)^2,X(3)^2*X(4)^2*X(5)+1,X(3)^2*X(4)^4+X(5)^2;
eliminate(normalize(groebner(i)),X(1)*X(2)*X(3)*X(4)*X(5)*Z);
-> _[1]=Y(1)*Y(2)-Y(3)^2
-> _[2]=T^8+Y(3)^2.
```

The only useful information above is the degree of the minimal polynomials. A generating set of B' as a $\phi(A)$-module is then the set formed of all products

$$((x'_3)^6x'_4)^i((x'_3)^2x'_5)^j\left(\frac{1}{x'_3(x'_5)^2}\right)^k\left(\frac{x'_4}{(x'_5)^2}\right)^l$$

with $i,\ l \in \{0,1,2,3,4,5,6,7\}$, $j \in \{0,1,2,3\}$ and $k \in \{0,1\}$.

($\star\star$) : We now compute the ideal of polynomial relations between the generators of $\bar{R}$ as a $\mathbb{Q}$-algebra

```
ring R=0,(X(1..5),Y(1..7),Z),lp;
ideal i=X(1)*X(2)-X(3)^2,X(3)^2*X(4)^2*X(5)+1,
            X(3)^2*X(4)^4+X(5)^2,
            Y(1)-X(1),Y(2)-X(2),Y(3)-X(3),Y(4)-X(3)^6*X(4),
            Y(5)-X(3)^2*X(5),Y(6)*X(3)*X(5)^2-1,
            Y(7)*X(5)^2-X(4),1-X(3)*X(5)^4*Z;
eliminate(normalize(groebner(i)),X(1)*X(2)*X(3)*X(4)*X(5)*Z);
-> _[1]=Y(6)^2+1
-> _[2]=Y(5)+Y(7)^6
-> _[3]=Y(4)+Y(7)^21
-> _[4]=Y(3)+Y(6)*Y(7)^4
-> _[5]=Y(1)*Y(2)+Y(7)^8.
```

Thus

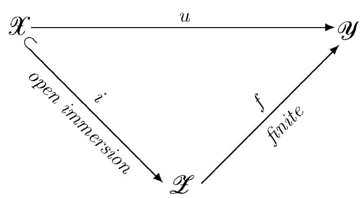
$$B' \cong \mathbb{Q}[Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7] \\ /\langle Y_6^2 + 1, Y_5 + Y_7^6, Y_4 + Y_7^{21}, Y_3 + Y_6 Y_7^4, Y_1 Y_2 + Y_7^8\rangle.$$

5. Application: effective version of the main Zariski Theorem in the normal case

Let k be a field satisfying the condition (P) of Seidenberg. Set $k[X] := k[X_1,\dots,X_n]$ (resp. $k[Y] := k[Y_1,\dots,Y_m]$) the polynomial ring in n (resp. m) variables over k. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ (resp. $\mathfrak{q}_1,\dots,\mathfrak{q}_s$) belong to $k[X]$ (resp. $k[Y]$) such that the ideal $\mathfrak{B}$ (resp. $\mathfrak{B}'$) generated by $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ (resp. $\mathfrak{q}_1,\dots,\mathfrak{q}_s$) is prime. Set $A := k[X]/\mathfrak{B}$ (resp. $A' := k[Y]/\mathfrak{B}'$), $x_i := X_i + \mathfrak{B}$ (resp. $y_j = Y_j + \mathfrak{B}'$) for each $i \in \{1,\dots,n\}$ (resp. $j \in \{1,\dots,m\}$). Let $Q(A) := k(x_1,\dots,x_n)$ (resp. $Q(A') := k(y_1,\dots,y_m)$) be the quotient field of A (resp. A'). Let f be a k-algebra homomorphism from A to A' such that $f(x_i) = F_i(y_1,\dots,y_m) =: f_i'$ with $F_i \in k[Y]$ for each $i \in \{1,\dots,n\}$. Finally, we let $\mathrm{spec}(f)\colon \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ be the comorphism of f. A classical theorem, that we shall call the Grothendieck's version of the main Zariski Theorem, states that every quasi-finite morphism between irreducible affine varieties can be written as the product of an open immersion and of a finite morphism [12, p. 48] and [15, p. 209].

Theorem 5.1 (Grothendieck's version of the main Zariski Theorem). *Let $u\colon \mathscr{X} \mapsto \mathscr{Y}$ be a quasi-finite morphism between irreducible affine varieties $\mathscr{X}$ and $\mathscr{Y}$ over a field.*

1. *There exists an irreducible affine variety $\mathscr{Z}$ such that u is the product of an open immersion i and a finite morphism f:*



2. *Moreover, if* $\mathscr{X}$ *is normal and* u *is a dominant morphism, then we can choose* $\mathscr{Z}$ *as the normalization of* $\mathscr{X}$ *in the function field of* $\mathscr{Y}$.

Basing on the preceding theorem and in particular on the second assertion, we establish the following version of the main Zariski Theorem for dominant and quasi-finite morphisms from normal irreducible affine varieties to irreducible affine varieties.

Theorem 5.2. *Assume that the morphism* $\mathrm{spec}(f)$ *is quasi-finite and dominant. Assume that* A' *is a normal domain. Then, we can compute elements* $b_1,\dots,b_l$ *of* $k(y_1,\dots,y_l)$ *and polynomials* $\mathfrak{g}_1,\dots,\mathfrak{g}_t$ *of the polynomial ring* $k[X_1,\dots,X_n,Z_1,\dots,Z_l]$ (*where* $Z_1,\dots,Z_l$ *are new indeterminates over* k) *such that*

1. $k(f_1',\dots,f_n',b_1,\dots,b_l)=k(y_1,\dots,y_m)$.
2. b_i *is integral over* $k[f_1',\dots,f_n']$ *for each* $i\in\{1,\dots,l\}$.
3. $A'':=\sum_{i=1}^{l} k[f_1',\dots,f_n']b_i=k[f_1',\dots,f_n',b_1,\dots,b_l]$ *is a normal domain.*
4. $\mathfrak{D}:=\langle\mathfrak{g}_1,\dots,\mathfrak{g}_t\rangle$ *is a prime ideal of* $k[X_1,\dots,X_n,Z_1,\dots,Z_l]$.
5. $\mathfrak{D}\cap k[X_1,\dots,X_n]=\mathfrak{B}$.
6. *There exists an isomorphism of* k*-algebra* ϕ *from* A'' *to* $k[X_1,\dots,X_n,Z_1,\dots,Z_l]/\mathfrak{D}$ *such that* $\phi^{-1}(X_i+\mathfrak{D})=f_i'$ *for each* $i\in\{1,\dots,n\}$ *and* $\phi^{-1}(Z_j+\mathfrak{D})=b_j$ *for each* $j\in\{1,\dots,l\}$

and so

1. *The morphism* $\mathrm{Spec}(A'')\to\mathrm{Spec}(A)$ *induced by* f *and by the inclusion of* $k[f_1',\dots,f_n']$ *in* A'' *is finite.*
2. *The morphism* $\mathrm{Spec}(A')\to\mathrm{Spec}(A'')$ *induced by the inclusion of* A'' *in* A' *is an open immersion.*
3. *The morphism* $\mathrm{Spec}(A')\to\mathrm{Spec}(A)$ *induced by* f *is the product of the two preceding morphisms.*

Proof. The hypothesis on $\mathrm{spec}(f)$ imply that the k-affine domain A' is a quasi-finite extension of the k-affine domain $f(A)=k[f_1',\dots,f_n']$. The quotient field $k(y_1,\dots,y_m)$ of A' is then finite over the quotient field $k(f_1',\dots,f_n')$ of $f(A)$. Now, we denote by A'' the integral closure of $f(A)$ in $k(y_1,\dots,y_m)$. A' is assumed to be a normal domain. So, we have $f(A)\subset A''\subset A'$. Theorem 4.2 allows us to construct elements $b_1,\dots,b_l$ of $k(y_1,\dots,y_l)$ and polynomials $\mathfrak{g}_1,\dots,\mathfrak{g}_t$ of $k[X_1,\dots,X_n,Z_1,\dots,Z_l]$ satisfying conditions (1)–(7) and (9). Furthermore, the morphism $\mathrm{Spec}(A')\to\mathrm{Spec}(A'')$ of (8) is birational and $\mathrm{Spec}(A'')$ is normal (by conditions (1) and (3)). On the other hand, under the hypothesis of quasi-finiteness of $\mathrm{spec}(f)$, we deduce then (8) from the classical Grothendieck's version of Zariski main Theorem [12, Corollary 8-2-10 p. 48] and [15, p. 209]. □

Remark 5.3. The quasi-finiteness condition may be checked using an algorithm given by van den Essen in [6].

Example 5.4. To illustrate Theorem 5.2, we consider the following morphism

$$\begin{array}{rcccl} F & : & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & & (x_1,x_2) & \mapsto & (F_1(x_1,x_2),F_2(x_1,x_2)), \end{array}$$

where $(F_1,F_2)\in(\mathbb{C}[X_1,X_2])^2$ are defined as

$$(F_1,F_2)=(X_1-2(X_1X_2+1)-(X_1X_2+1)^2X_2,-1-(X_1X_2+1)X_2).$$

This example is taken from the book of van den Essen [7, Example D.2.3 p. 293]. In this example, we use the function `normal` [10] which computes the normalization of an affine variety. Moreover, we use below the same typographical conventions as in Example 4.5.

This morphism F is quasi-finite and dominant but it is not finite (indeed: suppose F is finite, then it is onto contradicting $(0,0)\notin \operatorname{Im} F$). Hence, by Theorem 5.2, we can write F as the product of an open immersion with a finite morphism. Conformly to Theorem 5.2, we need to compute the integral closure of $\mathbb{C}[F_1,F_2]$ in $\mathbb{C}(X_1,X_2)$ thanks to Algorithm 4.3. We first compute the minimal polynomials of X_1 and X_2 over $\mathbb{C}(F_1,F_2)$:

```
option(redSB);
ring r=0,(X(1..2),Y(1..3)),lp;
ideal i=
   Y(1)-X(1),
   Y(2)-X(1)+2*(X(1)*X(2)+1)+(X(1)*X(2)+1)^2*X(2),
   Y(3)+1+(X(1)*X(2)+1)*X(2);
eliminate(groebner(i),X(1)*X(2));
-> _[1]=Y(1)^2-2*Y(1)*Y(2)
+Y(1)*Y(3)^3-Y(1)*Y(3)^2+Y(2)^2-Y(2)*Y(3)+Y(2)
ring r=0,(X(1..2),Y(1..3)),lp;
ideal i=
   Y(1)-X(2),
   Y(2)-X(1)+2*(X(1)*X(2)+1)+(X(1)*X(2)+1)^2*X(2),
   Y(3)+1+(X(1)*X(2)+1)*X(2);
eliminate(groebner(i),X(1)*X(2));
-> _[1]=Y(1)^2*Y(2)+Y(1)*Y(3)^2+Y(3)+1.
```

The minimal polynomials of X_1 and X_2 over $\mathbb{C}(F_1,F_2)$ are therefore, respectively,

$$T^2-2TF_1+TF_2^3-TF_2^2+F_1^2-F_1F_2+F_1$$

and

$$T^2F_1+TF_2^2+F_2+1.$$

We next set

$$R=\mathbb{C}[F_1,F_2,X_1,F_1X_2].$$

To represent R as a k-algebra, we compute a generating set of the ideal of polynomial relations between F_1, F_2, X_1 and F_1X_2.

```
ring r=0,(X(1..2),Y(1..4)),lp;
ideal i=
   Y(1)-X(1)+2*(X(1)*X(2)+1)+(X(1)*X(2)+1)^2*X(2),
   Y(2)+1+(X(1)*X(2)+1)*X(2),
   Y(3)-X(1),
   Y(4)-X(1)*X(2)+2*(X(1)*X(2)+1)*X(2)+(X(1)*X(2)+1)^2*X(2)^2;
eliminate(groebner(i),X(1)*X(2));
-> _[1]=Y(2)^4+2*Y(2)^2*Y(4)-Y(2)^2+Y(2)*Y(3)+Y(3)+Y(4)^2-Y(4)
-> _[2]=Y(1)-Y(2)^3+Y(2)^2-Y(2)*Y(4)-Y(3)+Y(4).
```

Hence

$$R \simeq \mathbb{C}[Y_1, Y_2, Y_3, Y_4]/ \\ \langle Y_2^4 + 2Y_2^2 Y_4 - Y_2^2 + Y_2 Y_3 + Y_3 + Y_4^2 - Y_4, Y_1 - Y_2^3 + Y_2^2 - Y_2 Y_4 - Y_3 + Y_4\rangle.$$

We then compute the normalization of R

```
ring r=0,(Y(1..4)),lp;
ideal i=
    Y(2)^4+2*Y(2)^2*Y(4)-Y(2)^2+Y(2)*Y(3)+Y(3)+Y(4)^2-Y(4),
    Y(1)-Y(2)^3+Y(2)^2-Y(2)*Y(4)-Y(3)+Y(4);
LIB "normal.lib";
list nor=normal(i);
// list, 1 element(s):
// [1]:
// ring: (0),(T(1),T(2),T(3)),(dp(3),C);
// minpoly = 0
// objects belonging to this ring:
// normap  [0] ideal, 4 generator(s)
// norid   [0] ideal, 1 generator(s)
def R = nor[1]; setring R; norid; normap;
-> norid[1]=T(1)4+2T(1)2T(3)-T(1)2+T(1)T(2)+T(3)2+T(2)-T(3)
-> normap[1]=T(1)3-T(1)2+T(1)T(3)+T(2)-T(3)
-> normap[2]=T(1)
-> normap[3]=T(2)
-> normap[4]=T(3).
```

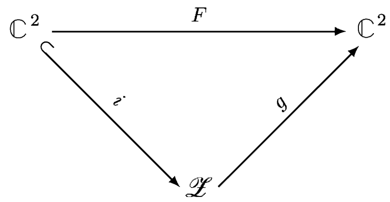
The normalization of $\mathbb{C}[F_1, F_2]$ in $\mathbb{C}(X_1, X_2)$ is therefore $(\mathscr{Z}, g)$ where

$$\mathscr{Z} = \mathrm{Spec}(\mathbb{C}[T_1, T_2, T_3]/\langle T_1^4 + 2T_1^2 T_3 - T_1^2 + T_1 T_2 + T_3^2 + T_2 - T_3\rangle)$$

and g is the normalization map

$$\begin{array}{ccc} \mathscr{Z} & \xrightarrow{g} & \mathbb{C}^2, \\ (t_1, t_2, t_3) & \mapsto & (t_1^3 - t_1^2 + t_1 t_3 + t_2 - t_3, t_1). \end{array}$$

We deduce in particular the following diagram



where i is the open immersion from $\mathbb{C}^2$ in $\mathscr{X} \subset \mathbb{C}^3$:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{i} & \mathscr{X}, \\ (x_1, x_2) & \mapsto & (F_2(x_1,x_2), x_1, x_2F_1(x_1,x_2)). \end{array}$$

Remark 5.5. Theorem 5.2 is valid whenever A' is assumed to be a normal domain. If this latter condition fails then, according to Dieudonné [5, p. 134], we have to take A'' as the intersection of A' and of the integral closure of $f(A)$ in $Q(A')$. So, to obtain an effective decomposition of a quasi-finite and dominant morphism between the two varieties $\mathscr{X}'$ and $\mathscr{X}$ (conformly to the theoretical Grothendieck's version of main Zariski Theorem), we have to compute such an intersection as a finitely generated module over $f(A)$ or as a finitely generated algebra over k.

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