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Symbolic computation and signal processing

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Abstract

Many problems in digital signal processing can be converted to algebraic problems over polynomial and Laurent polynomial rings, and can be solved using the existing methods of algebraic and symbolic computation. This paper aims to establish this connection in a systematic manner, and demonstrate how it can be used to solve various problems arising from multidimensional signal processing. The method of Gröbner bases is used as a main computational tool.

Keywords: Signal processing; Laurent polynomial rings

1. Introduction

This paper aims to show how the processing of discrete-time signals is related to linear algebra over polynomial rings and how the methods of computational algebra can be used naturally for various problems of multidimensional signal processing.

We start by reviewing basic concepts from signal processing, and relate the processing of discrete-time signals to linear algebra over Laurent polynomial rings. Then, we show how to efficiently convert problems over Laurent polynomial rings to the ones over (regular) polynomial rings. Emphasis is given on the problem of unimodular completion of Laurent polynomial matrices, and it is explained how this problem is related to the problem of parametrizing the synthesis of perfect reconstruction (PR) finite impulse response (FIR) systems. Some of these results appeared in Kalker et al. (1995), Park et al. (1997) and Park (1999), whose full proofs are given in this paper.

It should be noted that many researchers, Faugère et al. (1998), Selesnick (1999), Lebrun and Vetterli (1998), Lin (1999) and Charoenlarpnopparut and Bose (1999), have

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successfully used computational algebra for multidimensional systems and signal processing. This is made possible essentially because many signal processing problems can be modeled in the form of polynomial equations, which can then be solved by the methods of computational algebra, notably Gröbner bases.

2. Basic concepts from signal processing

2.1. 1D discrete-time signals

Definition 2.1. 1. A one-dimensional (1D) discrete-time signal is a sequence of real numbers, i.e. $(a_n)_{n\in\mathbb{Z}} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$, where $a_n \in \mathbb{R}$ and there exists $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$.
2. The set of 1D discrete-time signals is denoted by $\mathcal{S}$.

Discrete-time signals arise naturally, for example, by sampling continuous-time signals: for a continuous-time signal $f(t)$, define a_n to be $f(nT)$ where T is a preset sampling period.

Remark. The above definition is a formal one. In practice, a 1D discrete-time signal often means a square-summable sequence. The set of such square-summable sequences is denoted by $l_2(\mathbb{Z})$.

Remark. In this paper, a 1D signal $(a_n)_{n\in\mathbb{Z}}$ will be abbreviated as (a_n).

The set $\mathcal{S}$ of 1D discrete-time signals naturally forms an $\mathbb{R}$-vector space with the well-defined operations of the superposition and the scalar multiplication of sequences.

Definition 2.2. Convolution of discrete-time signals: for two given signals (a_n) and (c_n), their convolution $(b_n) := (a_n) * (c_n)$ is defined by $b_n := \sum_{i+j=n} a_i c_j$.

Definition 2.3. For a fixed $(c_n) \in \mathcal{S}$, the operator $L_{(c_n)}$ on the set $\mathcal{S}$ of discrete-time signals is defined by $L_{(c_n)}((a_n)) := (a_n) * (c_n)$.

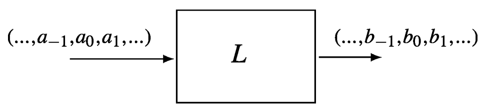
Trivially, the map $L_{(c_n)} : \mathcal{S} \longrightarrow \mathcal{S}$ is a linear map of $\mathbb{R}$-vector spaces. And the set $\mathcal{S}$ of discrete-time signals equipped with the two operations of **superposition** and **convolution** forms a commutative ring with identity $(\delta_{n,0})$, where $\delta_{0,0} = 1$ and $\delta_{n,0} = 0, \forall n \neq 0$.

2.2. Linear time-invariant systems

Definition 2.4. Then an $\mathbb{R}$-linear map $L : \mathcal{S} \longrightarrow \mathcal{S}$ is said to be **time-invariant** if, for any fixed integer i,

$$L((a_n)) = (b_n) \text{ implies } L((a_{n+i})) = (b_{n+i}).$$

Such an operator can be described by the following single-input single-output (SISO) system.



Lemma 2.1. *Let $\mathcal{S}$ be the $\mathbb{R}$-vector space of discrete-time signals. Then a map $L : \mathcal{S} \longrightarrow \mathcal{S}$ is $\mathbb{R}$-linear and time-invariant if and only if L is $\mathcal{S}$-linear.*

Proof. An easy exercise. □

An immediate consequence of this lemma is:

Corollary 2.1. *Let $\mathcal{S}$ be the $\mathbb{R}$-vector space of discrete-time signals. If a map $L : \mathcal{S} \longrightarrow \mathcal{S}$ is linear and time-invariant, then it can be represented by a convolution, i.e. there exists a unique discrete-time signal $(c_n) \in \mathcal{S}$ such that $L = L_{(c_n)}$.*

In such a case, (c_n) is called the **modulating signal** for L or the **impulse response** for L.

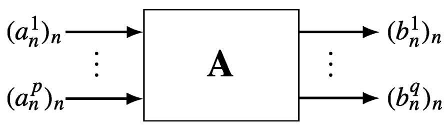
If $L = L_{(c_n)}$ with $c_n = 0, \forall n < 0$, then L is called a **causal** system. In this case, one checks easily that b_n is determined completely by a_i's with $i \leq n$. Loosely speaking, this means that the present value in the output signal does not depend on the future values in the input signal.

If $L = L_{(c_n)}$ and (c_n) is a discrete-time signal of finite duration, i.e. a finite sequence, then L is called an **FIR** system.

Definition 2.5. Let $\mathcal{S}$ be the ring of discrete-time signals, and $p, q \in \mathbb{N}$. Then an $\mathcal{S}$-module homomorphism $\mathbf{A} : \mathcal{S}^p \longrightarrow \mathcal{S}^q$ is called a linear time-invariant multi-input multi-output (MIMO) system.

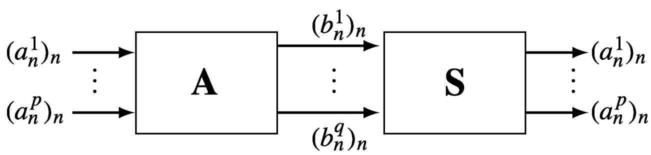
Remark. To understand this definition, consider a map $\mathbf{A} : \mathcal{S}^p \longrightarrow \mathcal{S}^q$, which can be viewed as a map between $\mathbb{R}$-vector spaces. One can show that, if $\mathbf{A}$ is $\mathbb{R}$-linear and time-invariant, then it is actually an $\mathcal{S}$-module homomorphism.

A MIMO system $\mathbf{A} : \mathcal{S}^p \longrightarrow \mathcal{S}^q$ can be described by the following picture:



In this case, such a p-input q-output linear time-invariant system is an operator from the module $\mathcal{S}^p$ to the module $\mathcal{S}^q$ defined by convolutions with various fixed signals.

2.3. Perfect reconstruction of signals



Let $\mathbf{A}$ and $\mathbf{S}$ be a p-input q-output MIMO system and a q-input p-output MIMO system, respectively. Suppose that, when an incoming signal goes into $\mathbf{A}$ and the subsequent output is fed into $\mathbf{S}$, the resulting output of $\mathbf{S}$ is identical to the original input signal of $\mathbf{A}$. If this is true for any input, then the combined effect of the overall system made of $\mathbf{A}$ and $\mathbf{S}$ is complete preservation of inputs.

For a given p-input q-output MIMO system **A**, if there exists a q-input p-output MIMO system **S** such that the overall system (made of **A** and **S**) preserves inputs completely, then **A** is said to have the **perfect reconstruction** property. In this case, **A** and **S** are said to make a PR system, and **A** (**S**, respectively) is called the analysis (synthesis, respectively) part of the overall system.

3. Algebraic formulation

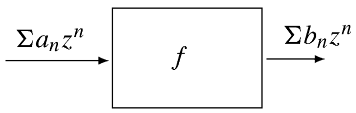
3.1. Z-transform

In the previous section, it was established that the set $\mathcal{S}$ of 1D discrete-time signals equipped with the operations of **superposition** and **convolution** forms a commutative ring. This ring $\mathcal{S}$ is isomorphic to the ring $\mathbb{C}[[z^{-1}]]_{z^{-1}}$, a localization of the formal power series ring $\mathbb{C}[[z^{-1}]]$, via the following correspondence:

$$(a_n) \longmapsto \sum_{n=-\infty}^{\infty} a_n z^{-n}.$$

This mapping is usually called the Z**-transform** in signal processing literature.

A SISO system can be viewed as an operator on $\mathbb{C}[[z^{-1}]]_{z^{-1}}$.



If f is a linear time-invariant system, then it is a multiplication by a power series in $\mathbb{C}[[z^{-1}]]_{z^{-1}}$, and the causal system is a multiplication by a power series in $\mathbb{C}[[z^{-1}]]$.

If f is an FIR system, then it is a multiplication by a Laurent polynomial in $\mathbb{C}[z^{-1}]_{z^{-1}} = \mathbb{C}[z, z^{-1}]$, and therefore, a causal FIR system is a multiplication by a polynomial in $\mathbb{C}[z^{-1}]$.

This is readily generalized to a (linear time-invariant) **multi-input multi-output** system, that is, a linear time-invariant p-input q-output FIR system $\mathbf{A} : (\mathbb{C}[z^{\pm 1}])^p \to (\mathbb{C}[z^{\pm 1}])^q$ is a multiplication by a matrix, i.e.

$$\mathbf{A} \in M_{qp}(\mathbb{C}[z^{\pm 1}]).$$

This matrix **A** is sometimes called **the transfer matrix** of the underlying MIMO system.

Remark. Various signal processing problems can be understood in terms of MIMO systems which are characterized by their transfer matrices Vetterli (1986), Janssen (1989), Vaidyanathan (1993) and Vetterli and Herley (1992). For example, by using the method of polyphase decomposition, the design of PR oversampled filter bank can be reduced to the design of a PR MIMO system Park (1999).

3.2. Perfect reconstruction in the Z-transform domain

Consider a given p-input q-output MIMO system whose Z-transform representation is a $q \times p$ matrix $\mathbf{A}$. Then clearly, this MIMO system has the PR property if and only if $\mathbf{A}$ has a left inverse $\mathbf{S}$ such that

$$\mathbf{SA} = \mathbf{I}_p,$$

where $\mathbf{I}_p$ is the $p \times p$ identity matrix. In this case, the overall system made of $\mathbf{A}$ and $\mathbf{S}$ makes a PR system, and $\mathbf{A}$ ($\mathbf{S}$, respectively) is the analysis (synthesis, respectively) part of the overall system.

Remark. In signal processing literature, the MIMO system represented by a $q \times p$ Laurent polynomial matrix $\mathbf{A}$, $q \geq p$, is often said to have the PR property if there is a $p \times q$ Laurent polynomial matrix $\mathbf{S}$ and an integer d such that

$$\mathbf{SA} = z^d \mathbf{I}_p.$$

In this context, the integer $|d|$ is called a delay if d is negative, and is called an advance if d is positive.

Note that these two definitions of PR are actually identical: that is, if $\mathbf{SA} = z^d \mathbf{I}_p$, then $z^{-d}\mathbf{S}$ is the left inverse of $\mathbf{A}$.

4. Extensions to higher dimensions

Definition 4.1. An m-D discrete-time signal is a multiply indexed sequence of real numbers, i.e. $(a_{i_1 \cdots i_m})_{(i_1 \cdots i_m) \in \mathbb{Z}^m}$, or an infinite m-dimensional array of numbers, where each $a_{i_1 \cdots i_m} \in \mathbb{R}$ and there exists $N \in \mathbb{Z}$ such that $a_{i_1 \cdots i_m} = 0$ if $i_i < N$ for some i.

One can define superposition and convolution of m-D discrete-time signals as in the 1D case. Linear time-invariant m-D systems are defined in the same way. It is easy to check that the set of m-D discrete-time signals forms a commutative ring with these two operations. This set is naturally isomorphic to the ring $\mathbb{C}[[z_1^{-1}, \ldots, z_m^{-1}]]_{z_1^{-1} \cdots z_m^{-1}}$, a localization of the multivariate formal power series ring $\mathbb{C}[[z_1^{-1}, \ldots, z_m^{-1}]]$, via the Z-transform

$$(a_{i_1 \cdots i_m})_{(i_1 \cdots i_m) \in \mathbb{Z}^m} \longmapsto \sum_{(i_1 \cdots i_m) \in \mathbb{Z}^m} a_{i_1 \cdots i_m} z_1^{-i_1} \cdots z_m^{-i_m}.$$

All the concepts introduced for 1D signals in the preceding sections can be readily extended to the m-D signals. For example, in the Z-transform domain, an m-D FIR MIMO system is described by a matrix whose entries are Laurent polynomials in m variables, i.e. elements of $\mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm}]$. The method of polyphase representation can be extended to multidimensional filter banks. In this case, the delay chain is replaced by cosets of a fixed sampling lattice (see Kalker and Shah, 1996).

5. Unimodularity and perfect reconstruction

Definition 5.1. Let R be a commutative ring.

1. Let $\mathbf{v} = (v_1, \dots, v_n)^t \in R^n$ for some $n \in \mathbb{N}$. Then $\mathbf{v}$ is called a **unimodular column vector** if its components generate R, i.e. if there exist $g_1, \dots, g_n \in R$ such that $v_1 g_1 + \cdots + v_n g_n = 1$.
2. A matrix $\mathbf{A} \in M_{pq}(R)$ is called a **unimodular matrix** if its maximal minors generate the unit ideal in R.

Theorem 5.1. *A $q \times p$ Laurent polynomial matrix, $q \geq p$, has a left inverse if and only if it is unimodular.*

A proof of this assertion in the case of polynomial matrices can be found in Logar and Sturmfels (1992), and this result was extended to the case of Laurent polynomial matrices in Park (1995). An immediate corollary of this theorem is

Corollary 5.1. *A p-input q-output FIR MIMO system can be the analysis portion of a PR FIR MIMO system if and only if its Z-transform representation is a unimodular Laurent polynomial matrix.*

This corollary allows us to see the study of PR FIR linear time-invariant MIMO systems as the study of unimodular matrices over Laurent polynomial rings.

Example 5.2. Consider an FIR MIMO system whose Z-transform representation is given by

$$\mathbf{U} = \begin{pmatrix} \frac{3}{z} - 2 - 2z + 2z^2 & \frac{6}{z} + 25 - 23z - 16z^2 + 20z^3 \\ \frac{3}{z} - 2z & \frac{6}{z} + 29 - 4z - 20z^2 \\ 2z & 2 + 4z + 20z^2 \end{pmatrix}.$$

Determine whether this system allows PR of arbitrary input signals.

Solution. The three maximal minors of $\mathbf{U}$ are -1, $-4 + 6/z - 2z + 2z^2$, $6/z - 2z$. These three Laurent polynomials do not have a common zero in $\mathbb{C}^*$, and by a Laurent polynomial analogue of Nullstellensatz, generate the unit ideal. Hence the given system allows PR of arbitrary input signals. $\square$

6. Construction of the synthesis matrix

Consider a unimodular $q \times p$ matrix $\mathbf{A}$, $q \geq p$, with Laurent polynomial entries. By Theorem 5.1, $\mathbf{A}$ represents a PR MIMO system, and there exists a $p \times q$ matrix $\mathbf{S}$ such that $\mathbf{SA} = \mathbf{I}_p$.

In the 1D case, such a matrix $\mathbf{S}$ (not unique unless $p = q$) can be easily computed by using a Laurent polynomial analogue of the Euclidean Division Algorithm.

Example 6.1. Consider again the FIR MIMO system in Example 5.2. It was determined that this system allows PR of arbitrary input signals. Let us explicitly construct a synthesis system which will reconstruct the original inputs.

Using the Laurent polynomial analogue of Euclidean Division Algorithm, we can successively apply elementary row operations to reduce $\mathbf{U}$ to its row echelon form:

$$\mathbf{EU} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where the 3×3 matrix $\mathbf{E}$ is found as

$$\begin{pmatrix} \frac{z}{18}(-18 - 125z - 188z^2 + 252z^3 - 215z^4 + 178z^5 + 6z^6) & \frac{z}{3}(-2 - 27z + 30z^2 + z^3) & \frac{(-12-89z+51z^2-60z^3-2z^4)}{6} \\ \frac{z}{6}(3 + 19z - 32z^2 + 23z^3 - 9z^4 - 8z^5 + 6z^6) & z(4 - 3z - z^2 + z^3) & 9/2 - 4z + 3z^2/2 + z^3 - z^4 \\ z(-4z + 23\frac{z^2}{3} - 5z^3 + z^4 + 8\frac{z^5}{3} - 2z^6) & 2z(-3 + 2z + z^2 - z^3) & -6 + 6z - z^2 - 2z^3 + 2z^4 \end{pmatrix}.$$

Here the 3×3 matrix $\mathbf{E}$ represents the series of elementary row operations applied to $\mathbf{U}$, and the first two rows of $\mathbf{U}$ make a left inverse of $\mathbf{U}$. $\square$

In m-D case, however, this method for the univariate case is no longer applicable as the Euclidean Division Algorithm is not available any more, and computing $\mathbf{S}$ is substantially harder. For example, consider the 2-D linear time-invariant system whose Z-transform representation is given by

$$\mathbf{A} = \begin{pmatrix} \frac{1}{z_2} + \frac{z_1}{z_2} + z_1 \\ \frac{z_1}{z_2^2} + 1 + z_2 + z_1 z_2 \end{pmatrix} \in (\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}])^2.$$

7. Working over Laurent polynomial rings

Many of the known methods for unimodular matrices are developed mainly over polynomial rings, i.e. when the matrices involved are unimodular **polynomial** matrices rather than Laurent polynomial matrices. For example, for a polynomial matrix $\mathbf{A} \in M_{pq}(k[x_1, \ldots, x_n])$, determining its unimodularity over $k[x_1, \ldots, x_n]$ is equivalent to determining the ideal membership of $1 \in k[x_1, \ldots, x_n]$ to the ideal generated by the maximal minors of $\mathbf{A}$. And the resulting problem can be effectively solved by a Gröbner bases computation (Kalker et al. (1995) and Park et al. (1997)).

In system theoretic terminology, causal-invertibility of causal filters is therefore covered by these methods.

Remark. It may occur to the reader that, to deal with Laurent polynomial entries in a matrix, one could just multiply all the entries by a common monomial and then work with the resulting polynomial matrix.

The situation, however, is not as simple as this scenario. For an example, consider the polynomial vector $\binom{z}{z^2} \in (k[z])^2$. While the relation $(1/2z) \cdot z + (1/2z^2) \cdot z^2 = 1$ clearly shows the FIR-invertibility of this vector, it is not causal-invertible since there are no polynomials $f(z), g(z) \in k[z]$ satisfying

$$f(z) \cdot z + g(z) \cdot z^2 = 1$$

as we can see easily by evaluating both sides at $z = 0$.

Therefore, any polynomial-based method will incorrectly conclude that this MIMO system does not have the PR property.

In order to extend any affine results (i.e. causal cases) to general FIR systems, we need an effective process of converting a given Laurent polynomial vector to a polynomial vector while preserving unimodularity. One immediate solution would be to use the ring isomorphism (see Park, 1999, for this approach),

$$k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \cong k[z_1, \ldots, z_n, w]/(z_1 z_2 \cdots z_n \cdot w - 1).$$

However, this process increases the complexity of the problem by introducing an extra variable. To remedy the situation, an alternative systematic process for the same purpose was developed in Park (1995). This process uses a change of variables, but keeps the number of variables the same. After the process is applied to a given Laurent polynomial vector, we get a polynomial vector of the same size in the same number of variables, and we determine the unimodularity of this polynomial vector by a Gröbner bases computation and find its left inverses by tracing the details of the Gröbner bases construction. The original Laurent polynomial vector is unimodular (as a Laurent polynomial vector) if and only if the converted polynomial vector is unimodular (as a polynomial vector).

8. A special 1-input p-output case

Consider a 1-input p-output ($p > 1$) multidimensional FIR system whose Z-transform representation is a $p \times 1$ Laurent polynomial matrix (or a p-dimensional Laurent polynomial vector)

$$\mathbf{v}(z_1, \ldots, z_n) = \begin{pmatrix} f_1(z_1, \ldots, z_n) \\ \vdots \\ f_p(z_1, \ldots, z_n) \end{pmatrix}.$$

Suppose there is an invertible change of variables,

$$(z_1, \ldots, z_n) \longrightarrow (z'_1, \ldots, z'_n),$$

such that $\mathbf{v}$, expressed in terms of the new variables $z'_1, \ldots, z'_n$, represents a **causal** system with **causal inverse**. This means that all f_i's become **polynomials** in $z_1'^{-1}, \ldots, z_n'^{-1}$ and there is a synthesis vector

$$\mathbf{w} = (g_1(z'_1, \ldots, z'_n), \ldots, g_p(z'_1, \ldots, z'_n))$$

such that g_i's are polynomials in $z_1'^{-1}, \ldots, z_n'^{-1}$ and $g_1 f_1 + \cdots + g_p f_p = 1$.

Let us recall an algorithm introduced in Park and Woodburn (1995, Theorem 4.5). It was named the Algorithm for Elementary Column Property which will now be called the Elementary Reduction Algorithm. In this algorithm, an $n \times n$ elementary matrix $\mathbf{E}_{ij}(f)$ is a matrix which has 1's on the diagonal, and 0's elsewhere except that its (i, j)th entry is a polynomial f. A schematic description of this algorithm is given in Table 1.

Note that this Elementary Reduction Algorithm offers an analogue of Gaussian Elimination for multivariate polynomial vectors.

By applying this algorithm to the polynomial vector $\mathbf{v}(z'_1, \ldots, z'_n)$, we find elementary matrices $\mathbf{E}_1, \ldots, \mathbf{E}_l$ such that

$$\mathbf{E}_1 \cdots \mathbf{E}_l \mathbf{v} = (1, 0, \ldots, 0)^t. \quad (1)$$

Table 1
Elementary Reduction Algorithm

Input:	$\mathbf{u} \in (k[x_1, \ldots, x_n])^p$, a unimodular polynomial vector.
Output:	$\mathbf{E}_1, \ldots, \mathbf{E}_l$, elementary polynomial matrices.
Specification:	The matrices $\mathbf{E}_1, \ldots, \mathbf{E}_l$ represent the elementary row operations reducing $\mathbf{u}$ to $(1, 0, \ldots, 0)^t$, i.e. $\mathbf{E}_1 \cdots \mathbf{E}_l \mathbf{u} = (1, 0, \ldots, 0)^t$.

Denoting the product $\mathbf{E}_1 \cdots \mathbf{E}_l$ by $\mathbf{E}$, and expressing the above reduction relation of Eq. (1) in terms of the old variables $z_1, \ldots, z_m$, we get the relation

$$\mathbf{E}\mathbf{v} = (1, 0, \ldots, 0)^t.$$

It is immediate that the first row vector $\mathbf{w}_1(z_1, \ldots, z_m)$ of $\mathbf{E}$ satisfies $\mathbf{w}_1\mathbf{v} = 1$, and defines a synthesis system that, together with the analysis system defined by $\mathbf{v}$, makes a PR system. A natural question regarding the role of the other row vectors $\mathbf{w}_2, \ldots, \mathbf{w}_p$ of $\mathbf{E}$ arises here. One notes that $\mathbf{E}\mathbf{v} = \mathbf{e}_p$ implies

$$\mathbf{w}_2\mathbf{v} = \cdots = \mathbf{w}_p\mathbf{v} = 0.$$

Therefore, for any Laurent polynomials $t_2, \ldots, t_p$,

$$(\mathbf{w}_1 + t_2\mathbf{w}_2 + \cdots + t_p\mathbf{w}_p)\mathbf{v} = 1,$$

and this formula gives a parametrized family

$$\mathbf{w} := \mathbf{w}_1 + t_2\mathbf{w}_2 + \cdots + t_p\mathbf{w}_p \tag{2}$$

of left inverses of $\mathbf{v}$ in terms of the $p - 1$ Laurent polynomial parameters $t_2, \ldots, t_p$.

The parametrization in Eq. (2) is **complete** in the sense that any left inverse of $\mathbf{v}$ can be written in such a form, and is **canonical** in the sense that the expression of a synthesis system in terms of the above parameters is unique. The proof of the completeness and canonicalness of a general version of this parametrization can be found in Park (1995).

9. General 1-input p-output systems

9.1. Overview: Causal Conversion Algorithm

The results in the preceding section work only for the very special 1-input p-output systems, namely, the systems for which their Z-transforms become invertible polynomial vectors in terms of the new variables. In this section, we develop an algorithm that transforms a given Laurent polynomial column vector to a polynomial column vector while preserving unimodularity.

A schematic description of this algorithm is presented in Table 2, where $p \geq 2$ is a nonzero integer and the shorthand notation $\mathbf{x} := x_1, \ldots, x_m$ is used.

It should be noted that this process converts the unimodularity of the Laurent polynomial vector $\mathbf{v}(\mathbf{x}) \in (k[\mathbf{x}^{\pm 1}])^p$ to the unimodularity of the polynomial vector $\hat{\mathbf{v}}(\mathbf{y}) \in (k[\mathbf{x}])^p$. A graphical demonstration of this process is shown in Fig. 1.

Table 2
Causal Conversion Algorithm

Input:	$\mathbf{v}$, a p-dimensional Laurent polynomial column vector in the variables $x_1, \ldots, x_m$.
Output:	$\mathbf{x} \to \mathbf{y}$, an invertible change of variables $\mathbf{T}$, a $p \times p$ unimodular (or invertible) Laurent polynomial matrix in the variables $x_1, \ldots, x_m$.
Specification:	(1) $\hat{\mathbf{v}} := \mathbf{T}\mathbf{v}$ is a **polynomial** column vector in the new variables $y_1, \ldots, y_m$. (2) $\mathbf{v}$ is FIR-invertible in the old variables $x_1, \ldots, x_m$ (i.e. unimodular as a Laurent polynomial vector) if and only if $\hat{\mathbf{v}}$ is causal-FIR-invertible in the new variables $y_1, \ldots, y_m$ (i.e. unimodular as a polynomial vector).

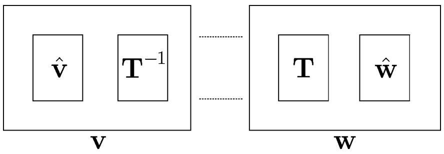


Fig. 1. Conversion of an FIR system $\mathbf{v}$ to a causal FIR system $\hat{\mathbf{v}}$.

Finding an FIR inverse $\mathbf{w}$ to the given FIR analysis $\mathbf{v}$ is equivalent to finding a causal inverse $\hat{\mathbf{w}}$ to the causal system $\hat{\mathbf{v}}$. The Elementary Column Reduction Algorithm of Section 8 produces a completely parametrized family of left inverses of $\hat{\mathbf{v}}$. Once a left inverse $\hat{\mathbf{w}}$ of $\hat{\mathbf{v}}$ is found, $\mathbf{w} := \hat{\mathbf{w}}\mathbf{T}$ is a (not necessarily causal) left inverse of $\mathbf{v}$, producing a complete parametrization of PR FIR pairs for the given analysis $\mathbf{v}$.

In order to describe the details of the Causal Conversion Algorithm, we start by generalizing the Noether Normalization Lemma to the case of Laurent polynomials.

9.2. *Normalization of Laurent polynomials*

Let $f \in k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ be a Laurent polynomial. Since f is a finite sum of monomials,

$$f = \sum_{(i_1,\ldots,i_m) \in I} a_{i_1,\ldots,i_m} x_1^{i_1} \cdots x_m^{i_m},$$

where $I \subset \mathbb{Z}^m$ is a finite index set.

Defining new variables $y_1, \ldots, y_m$ by

$$x_1 = y_1, \qquad x_2 = y_2 y_1^{l}, \ldots, x_m = y_m y_1^{l^{m-1}},$$

Table 3
Laurent Normalization Algorithm

Input:	f, a Laurent polynomial in the variables $x_1, \ldots, x_m$.
Output:	$\mathbf{x} \to \mathbf{y}$, an invertible change of variables.
Specification:	f, when viewed as a Laurent polynomial in the new variables $y_1, \ldots, y_m$, can be written with respect to the first variable y_1 in the following form: $f = b_s y_1^s + b_{s+1} y_1^{s+1} + \cdots + b_t y_1^t$, where $b_s, \ldots, b_t$ are Laurent polynomials in $y_2, \ldots, y_m$, and in particular, b_s and b_t are monomials in $y_2, \ldots, y_m$.

and letting $\mathbf{i} = (i_1, \ldots, i_m)$ and $\mathbf{l} = (1, l, l^2, \ldots, l^{m-1})$, one has

$$\begin{aligned} f &= \sum_{\mathbf{i} \in I} a_{\mathbf{i}} x_1^{i_1} \cdots x_m^{i_m} \\ &= \sum_{\mathbf{i} \in I} a_{\mathbf{i}} y_1^{i_1} (y_2^{i_2} y_1^{i_2 l}) \cdots (y_m^{i_m} y_1^{i_m l^{m-1}}) \\ &= \sum_{\mathbf{i} \in I} a_{\mathbf{i}} y_1^{i_1 + i_2 l + \cdots + i_m l^{m-1}} y_2^{i_2} \cdots y_m^{i_m} \\ &= \sum_{\mathbf{i} \in I} a_{\mathbf{i}} y_1^{\mathbf{i} \cdot \mathbf{l}} y_2^{i_2} \cdots y_m^{i_m}. \end{aligned}$$

By choosing a sufficiently large l, the integers $\mathbf{i} \cdot \mathbf{l}$ for $\mathbf{i} \in I$ can be made all distinct. Let

$$\begin{aligned} s &= \min\{\mathbf{i} \cdot \mathbf{l} \mid \mathbf{i} \in I\} \\ t &= \max\{\mathbf{i} \cdot \mathbf{l} \mid \mathbf{i} \in I\}. \end{aligned}$$

Then

$$f = b_s y_1^s + b_{s+1} y_1^{s+1} + \cdots + b_t y_1^t$$

where all the b_i's are units of $k[y_2^{\pm 1}, \ldots, y_m^{\pm 1}]$, i.e. monomials (Table 3).

9.3. *Description of the Causal Conversion Algorithm*

Let $\mathbf{v} = (v_1, \ldots, v_p)^t$ be a Laurent polynomial vector in the variables $x_1, \ldots, x_m$, where $p \geq 2$. By using the Laurent Normalization Algorithm, one may assume that the leading and the lowest coefficients of v_1 w.r.t. x_1 are monomials in $x_2, \ldots, x_m$. Write

$$v_1 = a_p x_1^p + a_{p+1} x_1^{p+1} + \cdots + a_q x_1^q$$

where a_p and a_q are monomials in $x_2, \ldots, x_m$.

- **Step 1.** Define a $p \times p$ matrix $\mathbf{D}$ and a p-dimensional column vector $\mathbf{v}'$ by

$$\mathbf{D} := \begin{pmatrix} a_p^{-1} x_1^{-p} & 0 & \\ 0 & a_p x_1^p & \\ & & \mathbf{I}_{n-2} \end{pmatrix}$$

$$\mathbf{v}' = (v_1', \ldots, v_n')^t := \mathbf{D}\mathbf{v}.$$

Note here that the matrix

$$\mathbf{D} = \mathbf{E}_{21}(a_p x_1^p)\mathbf{E}_{12}(1 - a_p^{-1} x_1^{-p})\mathbf{E}_{21}(1)\mathbf{E}_{12}(1 - a_p x_1^p)$$

is a product of elementary matrices with Laurent polynomial entries, and

$$v_1' = a_p^{-1} x_1^{-p} v_1 = 1 + a_{p+1}/a_p x_1 + \cdots + a_q/a_p x_1^{q-p} \tag{3}$$

has no negative powers of x_1.

- **Step 2.** Write the Laurent polynomials $v_1', \ldots, v_p'$ w.r.t. x_1. Then

$$v_i' = b_{s_i} x_1^{s_i} + b_{s_1+1} x_1^{s_i+1} + \cdots + b_{t_i} x_1^{t_i}$$

becomes a Laurent polynomial in x_1 with coefficients being Laurent polynomials in $x_2, \ldots, x_m$. Eq. (3) is such an expression for v_1', and since the smallest degree term of v_1' in this expression is 1, by adding suitable multiples of v_1' to v_i''s, $i = 2, \ldots, p$, we can make $v_2', \ldots, v_p'$ have only positive powers of x_1 (with constant terms being zero), i.e. find $\mathbf{E}$, a product of elementary matrices, such that the entries of

$$\hat{\mathbf{v}} := \mathbf{E}\mathbf{v}' = \begin{pmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_p \end{pmatrix}$$

are **polynomials** in x_1 (i.e. have no negative powers of x_1) with coefficients being Laurent polynomials in $x_2, \ldots, x_m$, and

$$\hat{v}_1 \equiv 1 \bmod x_1$$
$$\hat{v}_i \equiv 0 \bmod x_1, \qquad \forall i = 2, \ldots, n.$$

- **Step 3.** Choose a sufficiently large number $l \in \mathbb{N}$ that, with the following change of variables,

$$\begin{aligned} x_1 &= y_1 \cdot (y_2 \cdots y_m)^l \\ x_2 &= y_2 \\ &\vdots \\ x_m &= y_m, \end{aligned}$$

all the $\hat{v}_i$'s become polynomials in the new variables $y_1, \ldots, y_m$. Then $\hat{v}_1 \equiv 1 \bmod y_1 \cdots y_m$. Return the transformation matrix $\mathbf{T} := \mathbf{ED}$ as the output.

It still remains to prove the following theorem.

Theorem 9.1. *With the notation as in the above,* $\mathbf{v}(\mathbf{x})$ *is unimodular over* $k[\mathbf{x}^{\pm 1}]$ *if and only if* $\hat{\mathbf{v}}(\mathbf{y})$ *is unimodular over* $k[\mathbf{y}]$.

Proof. ($\Longleftarrow$) The unimodularity of $\hat{\mathbf{v}}(\mathbf{y})$ over $k[\mathbf{y}]$ trivially implies the unimodularity of $\hat{\mathbf{v}}(\mathbf{x})$ over $k[\mathbf{x}^{\pm 1}]$. This, together with the unimodularity of $\mathbf{T} \in M_n(k[\mathbf{x}^{\pm 1}])$, immediately implies the unimodularity of $\mathbf{v}(\mathbf{x}) = \mathbf{T}^{-1}\hat{\mathbf{v}}(\mathbf{x})$ over $k[\mathbf{x}^{\pm 1}]$.
($\Longrightarrow$) $\hat{\mathbf{v}}$ is unimodular over $k[\mathbf{y}^{\pm 1}]$ and there exist $h_1, \ldots, h_n \in k[\mathbf{y}]$ and $k \in \mathbb{N}$ such that

$$h_1 \hat{v}_1 + \cdots + h_n \hat{v}_n = (y_1 \cdots y_m)^k.$$

Since $\hat{v}_1 \equiv 1 \bmod y_1 \cdots y_m$, there exists $g \in k[\mathbf{y}]$ such that

$$\hat{v}_1 = 1 + g \cdot (y_1 \cdots y_m).$$

Define recursively a sequence of polynomials $\{ f_i \in k[\mathbf{y}] \mid i \in \mathbb{N}\}$ in the following way:

$$f_1 = 1 - g \cdot (y_1 \cdots y_m)$$
$$f_{i+1} = (1 - g^{2^i} \cdot (y_1 \cdots y_m)^{2^i}) \cdot f_i.$$

Then the f_i's defined in this way satisfy the following property:

$$\begin{aligned}
f_1 \hat{v}_1 &= (1 - g \cdot (y_1 \cdots y_m)) \cdot (1 + g \cdot (y_1 \cdots y_m)) = 1 - g^2 \cdot (y_1 \cdots y_m)^2 \\
f_2 \hat{v}_1 &= (1 - g^2 \cdot (y_1 \cdots y_m)^2) \cdot f_1 \hat{v}_1 = 1 - g^4 \cdot (y_1 \cdots y_m)^4 \\
&\vdots \\
f_i \hat{v}_1 &= 1 - g^{2^i} (y_1 \cdots y_m)^{2^i}.
\end{aligned}$$

Let $r \in \mathbb{N}$ be the smallest number such that $2^r \geq k$, and define $h \in k[\mathbf{y}]$ by $h = g^{2^r} (y_1 \cdots y_m)^{2^r - k}$. Then,

$$\begin{aligned}
1 &= f_r \hat{v}_1 + g^{2^r} (y_1 \cdots y_m)^{2^r} \\
&= f_r \hat{v}_1 + g^{2^r} (y_1 \cdots y_m)^{2^r - k} \cdot (h_1 \hat{v}_1 + \cdots + h_n \hat{v}_n) \\
&= f_r \hat{v}_1 + h(h_1 \hat{v}_1 + \cdots + h_n \hat{v}_n) \\
&= (f_r + h h_1) \hat{v}_1 + h h_2 \hat{v}_2 + \cdots + h h_n \hat{v}_n.
\end{aligned}$$

This gives a required unimodular relation. □

10. General q-input p-output systems

Now we have enough tools to deal with general multidimensional FIR systems.

- **Step 1.** Consider a multidimensional FIR system whose Z-transform representation is a $p \times q$ Laurent polynomial matrix $\mathbf{A}(z_1, \ldots, z_n)$, and let the Laurent polynomial vector

  $$\mathbf{v}(z_1, \ldots, z_n) = \begin{pmatrix} f_1(z_1, \ldots, z_n) \\ \vdots \\ f_p(z_1, \ldots, z_n) \end{pmatrix}$$

  be the first column vector of $\mathbf{A}$.
  Apply the Causal Conversion Algorithm to $\mathbf{v}$ in order to find a $p \times p$ unimodular matrix $\mathbf{T}$, and a new set of variables $z'_1, \ldots, z'_m$ such that

  $$\hat{\mathbf{v}} := \mathbf{T}\mathbf{v}$$

  is a polynomial vector in the new variables $z_1'^{-1}, \ldots, z_m'^{-1}$.
- **Step 2.** Use the Elementary Reduction Algorithm of Section 8 to find $\mathbf{U}$, a product of elementary polynomial matrices in the new variables $z_1'^{-1}, \ldots, z_m'^{-1}$, such that

  $$\mathbf{U}\hat{\mathbf{v}} = \mathbf{e}_p.$$

If such $\mathbf{U}$ does not exist, then the given multidimensional FIR system does **not** constitute the analysis part of a PR FIR system; terminate the process.
If such $\mathbf{U}$ exists, denote $\mathbf{UT}$ by $\hat{\mathbf{U}}$; then $\hat{\mathbf{U}}$ is a unimodular Laurent polynomial matrix in the old variables $z_1, \ldots, z_n$ such that

$$\hat{\mathbf{U}}\mathbf{v} = \mathbf{e}_p.$$

- **Step 3.** From $\hat{\mathbf{U}}\mathbf{v} = \mathbf{e}_p$, one deduces

$$\hat{\mathbf{U}}\mathbf{A} = \begin{pmatrix} 1 & h_{12} & \cdots & h_{1q} \\ 0 & & & \\ \vdots & & \mathbf{C} & \\ 0 & & & \end{pmatrix},$$

where $h_{12}, \ldots, h_{1q}$ are Laurent polynomials in $z_1, \ldots, z_n$, and $\mathbf{C}$ is a $(p-1)\times(q-1)$ Laurent polynomial matrix.

Now go back to **Step 1** with $\mathbf{A}$ replaced by $\mathbf{C}$, a matrix of strictly smaller size.

If the given multidimensional FIR system $\mathbf{A} \in M_{p\times q}$ can constitute the analysis of a PR FIR system, then the above algorithm should produce unimodular Laurent polynomial matrices $\mathbf{S}_1, \ldots, \mathbf{S}_q$ such that

$$\mathbf{S}_q \cdots \mathbf{S}_1\mathbf{A} = \begin{pmatrix} 1 & h_{12} & \cdots & h_{1q} \\ 0 & 1 & \cdots & h_{2q} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{p\times q}. \tag{4}$$

One can get rid of all h_{ij}'s by performing elementary row operations on this matrix, i.e. by finding $\mathbf{E}$, a product of $p \times p$ elementary matrices, such that

$$\mathbf{E}\mathbf{S}_q \cdots \mathbf{S}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{p\times q}.$$

Define a $p \times p$ Laurent polynomial matrix $\hat{\mathbf{S}}$ by

$$\hat{\mathbf{S}} = \mathbf{E}\mathbf{S}_q \cdots \mathbf{S}_1.$$

Denote by $\mathbf{S}_{\text{part}}$ the submatrix of $\hat{\mathbf{S}}$ consisting of its first q rows. Then

$$\mathbf{S}_{\text{part}}\mathbf{A} = \mathbf{I}_q.$$

That is, the $q \times p$ matrix $\mathbf{S}_{\text{part}}$ defines a *particular* synthesis FIR system that, together with the given analysis $\mathbf{A}$, makes a PR system. Eq. (4) implies that the $p \times q$ matrix consisting of the first q columns of $\hat{\mathbf{S}}^{-1}$ is precisely the given matrix $\mathbf{A}$, i.e. $\hat{\mathbf{S}}^{-1}$ is a **unimodular completion** of $\mathbf{A}$.

What is the role played by the remaining $p - q$ rows of $\hat{\mathbf{S}}$? Again, they parametrize all possible synthesis systems with the same PR property: if $\mathbf{S} \in M_{qp}(k[\mathbf{x}^{\pm 1}])$ is an arbitrary left inverse of $\mathbf{A}$, then

$$\begin{aligned}\mathbf{SA} = \mathbf{I}_q &\Longrightarrow \mathbf{S}\hat{\mathbf{S}}^{-1} = (\mathbf{I}_q, \mathbf{u}_1, \ldots, \mathbf{u}_{p-q}) \\ &\Longrightarrow \mathbf{S} = (\mathbf{I}_q, \mathbf{u}_1, \ldots, \mathbf{u}_{p-q})\hat{\mathbf{S}},\end{aligned}$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_{p-q}$ are Laurent polynomial column vectors in $(k[\mathbf{x}^{\pm 1}])^q$. Now regarding $\mathbf{u}_1, \ldots, \mathbf{u}_{p-q}$ as free vector parameters ranging over $(k[\mathbf{x}^{\pm 1}])^q$, we get a parametrized family of left inverses involving $q \times (p - q)$ free (Laurent polynomial) parameters. More explicitly,

$$\begin{aligned}\mathbf{S} &= (\mathbf{I}_q, \mathbf{u}_1, \ldots, \mathbf{u}_{p-q})\hat{\mathbf{S}} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 & u_{11} & \cdots & u_{1(p-q)} \\ 0 & 1 & \cdots & 0 & u_{21} & \cdots & u_{2(p-q)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & u_{q1} & \cdots & u_{q(p-q)} \end{pmatrix}\hat{\mathbf{S}},\end{aligned} \tag{5}$$

where u_{ij}'s are arbitrary Laurent polynomials.

The parametrization in the above Eq. (5) is **complete** in the sense that any left inverse of $\mathbf{S}$ can be written in that form, and is **canonical** in the sense that the expression of a synthesis filter in terms of the above parameters is unique. The number of free parameters, $q \times (p-q)$, is an invariant for the given matrix $\mathbf{A}$ and represents the **degree of freedom** in obtaining its left inverses. The proof of the completeness and canonicalness of this parametrization can be found in Park (1995).

Example 10.1. Determine whether $\mathbf{v} := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{y} + \frac{x}{y} + x \\ \frac{x}{y^2} + 1 + y + xy \end{pmatrix}$, a Laurent polynomial vector in the two variables x and y, is FIR-invertible, i.e. whether it has an FIR inverse.

- **Step 1.** Write f_1 in terms of x: $f_1 = (1/y) + ((1/y) + 1)x$. The leading coefficient of f_1 w.r.t. x is $(1/y) + 1$, not a monomial in y.

  So the Causal Conversion Algorithm has to be applied to f_1: define a new variable z by putting $y = zx^l$ where the integer l is to be determined. With respect to the new variables x and z, f_1 becomes $v_1 = (1/zx^l) + (1/zx^{l-1}) + x$. Let $l = 1$. Then $f_1 = (1/zx) + (1/z) + x$ in which the leading and the lowest coefficients w.r.t. x are monomials in z.

- **Step 2.**

$$\mathbf{v} = \begin{pmatrix} \frac{1}{zx} + \frac{1}{z} + x \\ \frac{1}{z^2x} + 1 + zx + zx^2 \end{pmatrix}$$
$$\Longrightarrow \mathbf{v}_1 := \begin{pmatrix} zx & 0 \\ 0 & \frac{1}{zx} \end{pmatrix} \mathbf{v} = \begin{pmatrix} 1 + x + x^2z \\ \frac{1}{z^3x^2} + \frac{1}{zx} + 1 + x \end{pmatrix}.$$

Apply elementary operations to $\mathbf{v}_1$ to make its second component a polynomial in x whose constant term is zero.

1.
$$\mathbf{v}_2 := \mathbf{E}_{21}\left(-\frac{1}{z^3x^2}\right)\mathbf{v}_1 = \begin{pmatrix} 1 + x + x^2z \\ -\frac{1}{z^3x} + \frac{1}{zx} + 1 - z^{-2} + x \end{pmatrix}.$$

2.
$$\mathbf{v}_3 := \mathbf{E}_{21}\left(\left(\frac{1}{z^3} - \frac{1}{z}\right)x\right)\mathbf{v}_2 = \begin{pmatrix} 1 + x + x^2z \\ \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} + 1 + \frac{x}{z^2} \end{pmatrix}.$$

3.
$$\begin{aligned}\hat{\mathbf{v}} &:= \mathbf{E}_{21}\left(-\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} - 1\right)\mathbf{v}_3 \\ &= \begin{pmatrix} 1 + x + x^2z \\ \frac{x(-1+2z-xz+z^2+xz^2-z^3+xz^3-xz^4)}{z^3} \end{pmatrix}.\end{aligned}$$

The transformation matrix is

$$\begin{aligned}\mathbf{T} &:= \mathbf{E}_{21}\left(-\frac{1 - z - z^2 + z^3}{z^3}\right)\mathbf{E}_{21}\left(\left(\frac{1}{z^3} - \frac{1}{z}\right)x\right) \\ &\quad \times \mathbf{E}_{21}\left(-\frac{1}{z^3x^2}\right)\begin{pmatrix} zx & 0 \\ 0 & \frac{1}{zx} \end{pmatrix},\end{aligned}$$

and the converted vector is

$$\hat{\mathbf{v}} = \mathbf{T}\mathbf{v} = \begin{pmatrix} 1 + x + x^2z \\ \frac{x(-1+2z-xz+z^2+xz^2-z^3+xz^3-xz^4)}{z^3} \end{pmatrix}.$$

- **Step 3.** Make another change of variables. Define a new variable w by $x = w \cdot z^l$ where the integer l is to be determined.

Then w.r.t. the new variables z, w, $\hat{\mathbf{v}}$ becomes

$$\hat{\mathbf{v}} = \begin{pmatrix} 1 + wz^l + w^2z^{2l+1} \\ wz^{l-3}(-1 + 2z - wz^{l+1} + z^2 + wz^{l+2} - z^3 + wz^{l+3} - wz^{l+4}) \end{pmatrix}.$$

Let $l = 3$ as it is the smallest integer that makes the components of $\hat{\mathbf{v}}$ polynomials in z and w. Then

$$\hat{\mathbf{v}} := \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix} = \begin{pmatrix} 1 + wz^3 + w^2z^7 \\ w(-1 + 2z + z^2 - z^3 - wz^4 + wz^5 + wz^6 - wz^7) \end{pmatrix}.$$

The unimodularity of $\mathbf{v}$ over $k[x^{\pm 1}, y^{\pm 1}]$ is equivalent to the unimodularity of $\hat{\mathbf{v}}$ as a polynomial vector in $k[z, w]$. And the unimodularity of a polynomial vector can be easily checked by a Gröbner basis computation: for $\hat{\mathbf{v}}$ to be unimodular, the reduced Gröbner basis of $\{\hat{v}_1, \hat{v}_2\} \subset k[w, z]$ w.r.t. an arbitrary term order must be $\{1\}$.

A computation with the computer algebra package SINGULAR Greuel et al. (2001) shows that the reduced Gröbner basis of $\{\hat{v}_1, \hat{v}_2\} \subset k[w, z]$ w.r.t. the reverse degree lexicographic order is

$$\begin{aligned}&\{-z^2 + 81w + 17z - 11, -21wz - 4z^2 + 9w + 5z - 2,\\ &\quad - 567w^2 - 116wz - z^2 + 77w - 2z + 4\}.\end{aligned}$$

Therefore $\hat{\mathbf{v}}$ is not unimodular over $k[w, z]$, and neither is $\mathbf{v}$ over $k[x^{\pm 1}, y^{\pm 1}]$, i.e. $\mathbf{v}$ is **not** FIR-invertible. $\square$

Example 10.2. Suppose $\mathbf{A} := (f_1, f_2, f_3, f_4)^t$ where

$$\begin{aligned}f_1 &= 1 - xy - 2z - 4xz - x^2z - 2xyz + 2x^2y^2z - 2xz^2 - 2x^2z^2 + 2x^2yz^2\\ f_2 &= 2 + 4x + x^2 + 2xy - 2x^2y^2 + 2xz + 2x^2z - 2x^2yz\\ f_3 &= 1 + 2x + xy - x^2y^2 + xz + x^2z - x^2yz\\ f_4 &= 2 + x + y - xy^2 + z - xyz.\end{aligned}$$

Let us consider the problem of finding a complete parametrization for all the left inverses of the 4×1 matrix $\mathbf{A}$:

The unimodularity of the matrix $\mathbf{A}$ can be shown using the method of Gröbner bases again, i.e. the reduced Gröbner basis of $\{f_1, f_2, f_3, f_4\}$ w.r.t an arbitrary term order is $\{1\}$. The Elementary Reduction Algorithm of Section 8 produces

$$\hat{\mathbf{S}} := \begin{pmatrix} 0 & -z+1 & 2z-1 & -x \\ -y-z & xz-yz-z^2-x+2z-2 & -2xz+x-4z+2 & x^2+2x+1 \\ -y^2-yz+1 & -y^2z-yz^2+2yz-2y+2z-1 & -4yz+2y-2z+1 & 2xy+x+y \\ xy+xz & xyz+xz^2-2xz+2x+1 & 4xz-2x-2 & -2x^2-x \end{pmatrix}.$$

Therefore, an arbitrary left inverse $\mathbf{S}$ of $\mathbf{A}$ is of the form

$$\begin{aligned}\mathbf{S} &= (1, u_1, u_2, u_3)\hat{\mathbf{S}}\\ &= (1, u_1, u_2, u_3)\\ &\quad\times \begin{pmatrix} 0 & -z+1 & 2z-1 & -x \\ -y-z & xz-yz-z^2-x+2z-2 & -2xz+x-4z+2 & x^2+2x+1 \\ -y^2-yz+1 & -y^2z-yz^2+2yz-2y+2z-1 & -4yz+2y-2z+1 & 2xy+x+y \\ xy+xz & xyz+xz^2-2xz+2x+1 & 4xz-2x-2 & -2x^2-x \end{pmatrix}\\ &= (0, -z+1, 2z-1, -x)\\ &\quad + u_1(-y-z, xz-yz-z^2-x+2z-2, -2xz+x-4z+2, x^2+2x+1)\\ &\quad + u_2(-y^2-yz+1, -y^2z-yz^2+2yz-2y+2z-1, -4yz\\ &\quad + 2y-2z+1, 2xy+x+y)\\ &\quad + u_3(xy+xz, xyz+xz^2-2xz+2x+1, 4xz-2x-2, -2x^2-x).\end{aligned}$$

where u_1, u_2, u_3 are arbitrary Laurent polynomials in the variables x, y, z.

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