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PLANT/CONTROLLER DESIGN INTEGRATION FOR $\mathcal{H}_2$ CONTROL BASED ON SYMBOLIC-NUMERIC HYBRID OPTIMIZATION*

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Abstract. This paper proposes a new plant/controller design integration framework that seeks the optimal pair of the plant and the controller achieving the best possible closed-loop performance in $\mathcal{H}_2$ control. The framework is further equipped with a symbolic-numeric hybrid optimization approach to effectively search the optimum. The first step of the suggested approach relies on an algebraic approach to parametric polynomial spectral factorization. The paper first reviews an algebraic approach in the continuous-time case and then generalizes the approach to the δ domain so that the suggested hybrid approach may deal with digital control systems, allowing the sampling period to be treated explicitly as a parameter. Then it is indicated that the optimal cost in the $\mathcal{H}_2$ control problem may be characterized in the presence of parameters. It is further discussed that the obtained expression relating the achievable performance level and parameters can be utilized for numerical optimization over the admissible parameter range to find the best parameter values. Two design examples are used to demonstrate the suggested approaches.

Key words: Parametric polynomial spectral factorization, Gröbner bases, digital control, δ domain, Newton's method

1. Introduction. A typical scenario in control system design assumes a fixed plant to be given and demands that a controller should be synthesized so that the resulting closed-loop system may satisfy some required performance specifications. However a more realistic system design (*not* controller design) scenario supposes some freedom in the plant that is to be controlled [9]. Indeed the performance of the final closed-loop system depends not only on the synthesized controller but also heavily on the plant, and one would desire the best combination of the plant and the controller. Thus systematic integrated design of the plant and the controller is truly anticipated. Nevertheless few approaches have emerged to deal with such problems [8, 16]. This is because treating plant parameters and controller parameters simultaneously is a difficult task for numerical computation.

In order to overcome the computational difficulty and accomplish an effective

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approach towards plant/controller design integration, the paper attempts to make the best use of two disciplines in computation, namely, symbolic computation and numerical computation. For a plant with tuning parameters, the paper develops a symbolic-numeric hybrid computation approach that minimizes the optimal cost in $\mathcal{H}_2$ control (i.e., finds the 'best of the best') over a given parameter range, which constitutes the first contribution of the paper. The crucial point is that an algebraic approach is employed to derive an expression of the optimal cost that is amenable to the subsequent optimization and that afterwards a numerical method is utilized to seek the minimum of the optimal cost. To this end an algebraic solution method to polynomial spectral factorization based on the the *Sum of Roots* (SoR) [1, 13] and its variants is utilized to characterize the optimal cost in $\mathcal{H}_2$ control, yielding a result suitable for a conventional optimization approach that follows. It is stressed that the computational burden of symbolic computation/algebraic methods is typically large and thus that structural properties of the problem have to be exploited for practical applications. The algebraic approaches employed here indeed achieve this and immediately compute a Gröbner basis instead of relying on a black-box routine. This is made possible by discovering and focusing on effective quantities such as the SoR.

Most control systems are realized on digital computers these days and the choice of the sampling period is of practical significance. In order for the suggested hybrid optimization approach to be applicable to digital control systems, the paper further extends the algebraic approach to polynomial spectral factorization to the δ domain, which is the second contribution of the paper. This extension allows the sampling period to be treated as a parameter and, as a consequence, physical parameters in the plant and the sampling period can be designed simultaneously to achieve a better performance. The significance of the developed approach is that simple manipulation of equations obtained from the problem formulation yields a Gröbner basis and further that the existence of the so-called shape basis is indicated. Moreover it is pointed out that a quantity called the *Product of Roots* (PoR) plays an important role, just as in the SoR in the continuous-time case, allowing easy characterization of the performance limitation.

The current paper is a full version of a conference paper [12] by the same authors, giving more details of the proposed approach and also developing polynomial spectral factorization in the δ domain so that parameter optimization including the sampling period may be possible for a digital control system. The rest of the paper is organized as follows. Section 2 formally establishes the suggested integrated design framework through an extensive discussion over some existing frameworks and the proposed design framework. Then Section 3 first reviews the algebraic approach to polynomial spectral factorization for continuous-time systems and further develops polynomial spectral factorization in the δ domain. Section 4 formally proposes the

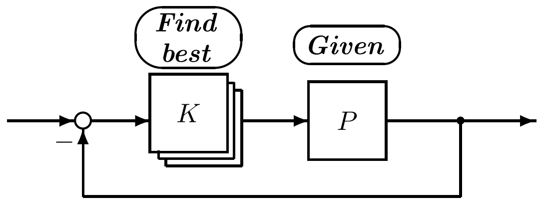

FIG. 2.1. *Conventional sequential design approach.*

hybrid optimization approach. In Section 5, the proposed hybrid optimization approach is applied to a realistic $\mathcal{H}_2$ control problem in the continuous-time framework for demonstration purposes of the first contribution. In Section 6, another simple $\mathcal{H}_2$ control problem is considered for a digital control system to demonstrate how the achievable performance level is related to plant parameters and also the sampling period by means of polynomial spectral factorization in the δ domain. Some concluding remarks are made in Section 7.

2. Plant/Controller Integrated Design Framework.

2.1. Conventional Design Approaches. A conventional 'textbook' framework of control system design is that, given a fixed plant P, one is requested to design an optimal/robust controller K for the particular plant (Figure 2.1):

$$\Phi(P) := \inf_{K \in \mathcal{K}} J(P,\ K)\ , \tag{2.1}$$

where $\mathcal{K}$ is the set of all feasible controllers and J the cost function expressing the performance level of the closed-loop system. *Sequential* design is typically assumed where a plant is first designed under some requirements but without taking into account the performance of the final controlled system, and a controller is then synthesized specifically for that plant.

The obtained controller may be the most preferable one for the already fixed plant, but it is most likely that even better performance can be achieved when another admissible plant is chosen and a suitable controller is synthesized. The conventional approach is thus not fully helpful when there is some room in *plant* design and one can choose a preferable plant from a set of admissible plants.

This is indeed the motivation for introducing another approach called *simultaneous* optimization, where optimization is performed for plant/controller joint design in the hope of exploiting the freedom in the plant and in pursuit of the best combination of the plant and the controller (Figure 2.2):

$$\inf_{P \in \mathcal{P},\ K \in \mathcal{K}} J(P,\ K)\ ,$$

where $\mathcal{P}$ is the set of all admissible plants. In this approach each and every possible combination of plants and controllers is in essence examined. There are some research

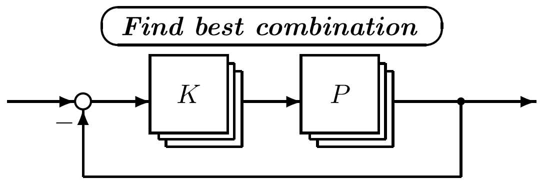

FIG. 2.2. *Simultaneous optimization approach.*

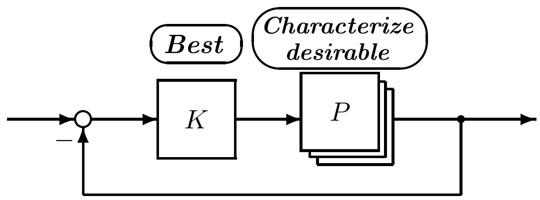

FIG. 2.3. *Characterization of easily controllable plants.*

results along the line, which are mainly based on simultaneous optimization of both plant and controller parameters via numerical optimization (see, e.g., [8, 16]). Whilst the optimization problem may be well formulated, the problem usually falls into the category of non-convex optimization with a number of decision variables and is intrinsically difficult to tackle. This urges a new approach to be pursued.

2.2. Integrated Design Approach. A different direction of plant design methodology emerged that avoids optimizing plant/controller parameters simultaneously. The idea proposed in [9, 11] is to characterize easily controllable plants by expressing in terms of plant properties the best performance levels in some $\mathcal{H}_2$ or $\mathcal{H}_\infty$ control problems [3, 4, 19, 20], or by identifying desirable gain properties such as the low frequency positive real property [11] (Figure 2.3). Namely it is aimed to obtain expressions for $\Phi(P)$ explicitly in terms of plant characteristics. Such expressions elucidate which plant characteristics hinder good performance from being achieved, where plant characteristics typically used are unstable poles, non-minimum phase zeros, gain, and time-delay. The results would yield a plant design scheme that guarantees the existence of a controller achieving a desired closed-loop performance.

In this approach of plant design, the closed-loop performance is explicitly taken into account, which is in contrast to the conventional sequential design approach mentioned in the preceding subsection. It is however stressed that such results can only give some guidelines as to what sort of plants are easy/difficult to control in system-theoretic terms. It is not necessarily suited when it comes to optimization of plant physical parameters. Hence a different approach needs to be devised that overcomes the difficulty.

2.3. Symbolic-numeric Hybrid Integrated Design Approach. To this end a new framework for plant/controller design integration is proposed. The approach

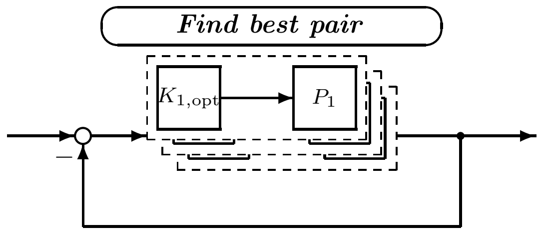

FIG. 2.4. *Proposed approach framework.*

aims to seek

$$\inf_{P\in\mathcal{P}} \Phi(P) = \inf_{P\in\mathcal{P}} \inf_{K\in\mathcal{K}} J(P,\ K)\ . \tag{2.2}$$

In a typical scenario based on numerical computation, $\Phi(P)$ is obtained as a result of numerical solution of nonlinear equations or numerical optimization. Therefore systematic optimization is inconceivable and the optimization in (2.2) could be at best executed in a heuristic way.

A promising approach thus may be an appropriate combination of symbolic and numerical computation. The framework is based on a two-step approach:

1. Parametric Controller Optimization (by symbolic computation)
   — '$\inf_{K\in\mathcal{K}}$' part in (2.2);
2. Optimization over Parameter Ranges (by numerical computation)
   — '$\inf_{P\in\mathcal{P}}$' part in (2.2).

There is some similarity between these steps and the conventional sequential design approach, but the main difference is the use of *parametric optimization*. In this parametric optimization, given a plant with tuning parameters, the optimal controller is designed *with parameters as they are* and the optimal performance level $\Phi(P)$ is expressed in the presence of plant parameters. In that way the pair of the plant and the corresponding optimal controller is made. Once an expression for $\Phi(P)$ is obtained, the optimal performance level is optimized over the admissible range of parameters. Or one can see the approach as optimization over all pairs of plants and their corresponding optimal controllers (Figure 2.4). Notice the difference from the simultaneous optimization discussed in Subsection 2.1.

The parametric optimization in the first step needs to be carried out so that the resulting expression may be tractable in the optimization of plant parameters in the second step. This requires that the expression should not contain functions such as 'inf' and the integral sign. One plausible approach is to employ direct symbolic computation to derive closed-form expressions for the optimal controller and the optimal cost. However this approach is not attractive in that solution of high order nonlinear equations is usually required (which is not in general exactly solvable) and thus the approach is only applicable to a very small problem.

Instead the approach suggested in this paper employs an algebraic method to polynomial spectral factorization that is applicable in the presence of parameters. The approach allows one to derive an algebraic relationship between plant parameters and the optimal controller/optimal cost, which indicates the effectiveness for the problem of optimal design in mind. In the continuous-time system case, a quantity called the *Sum of Roots* [1] is introduced and plant parameters and the optimal controller/optimal cost are related in the form of algebraic expressions. Write the vector of plant parameters as $\mathbf{q}$, and suppose that the plant coefficients are expressed algebraically in $\mathbf{q}$. Then parametric polynomial spectral factorization yields an expression that algebraically relates $\mathbf{q}$ and the SoR σ_c: $S_f(\mathbf{q};\sigma_c) = 0$. The SoR plays an important role as a medium connecting plant parameters and the optimal controller/optimal cost. A variant suitable for the δ domain is proposed in Section 3, where a quantity called the *Product of Roots*, denoted by σ_δ, also plays a significant role.

Once such a relationship is obtained, various algebraic manipulations are possible and such an advantage is exploited. Indeed this fact is a valuable feature and is a stark contrast to the existing approaches mentioned so far. With exact algebraic expressions in hand, one may for instance immediately deduce how parameters should be changed to improve the best performance level achieved by the corresponding optimal controller. Alternatively explicit optimality conditions may be computable. The proposed approach suggested here computes the 'derivatives' of the optimal cost with respect to plant parameters and optimizes the optimal cost numerically via Newton's method. By taking the partial derivative of $S_f(\mathbf{q};\sigma_\bullet) = 0$ with respect to $\mathbf{q}$, the sensitivity of $\sigma_\bullet$ to $\mathbf{q}$, $\nabla\sigma_\bullet = (\frac{\partial\sigma_\bullet}{\partial q_i})$, can be computed. Furthermore the gradient vector $\nabla\Phi(P)$ can be computed, which is obtained as rational functions in $\mathbf{q}$ and $\sigma_\bullet$. That is, an algebraic method is employed in the first step to relate plant parameters and the optimal cost, and the second step utilizes a typical numerical optimization method to find the best parameter values. Section 4 discusses in detail how it can be achieved.

3. Algebraic Polynomial Spectral Factorization. This section states the algebraic approach to polynomial spectral factorization which is a crucial mathematical tool to accomplish the symbolic-numeric hybrid integrated design approach. The result for the continuous-time case [1] is firstly reviewed. Since modern control systems are more often than not realized by computer systems, it is beneficial to analyze and synthesize feedback systems in the digital control framework. In order to adopt the digital control implementation while keeping the continuous-time nature, this section further develops an algebraic approach to polynomial spectral factorization in the δ domain, which is one of the contributions of the paper. The essential points in the approach are that a Gröbner basis is computed in a straightforward manner and that the existence of the shape basis is shown, allowing simple computation of the shape basis.

3.1. Continuous-time Case. In the continuous-time framework, the differential operator s is used to describe system transfer functions. The task of polynomial spectral factorization in the continuous-time domain is stated as follows. Given a $2n$-th order even polynomial $f(s)$ in s with no roots on the imaginary axis,

$$f(s) = (-1)^n s^{2n} + a_{2n-2}s^{2n-2} + \cdots + a_2 s^2 + a_0 \ , \tag{3.1}$$

the task is to find a unique polynomial

$$g(s) = s^n + \sigma_c s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0 \tag{3.2}$$

that satisfies the relationship

$$f(s) = g(s)g(-s) \tag{3.3}$$

and moreover has roots in the open left half plane only (i.e., is stable). The polynomial $g(s)$ that is sought is called the spectral factor. A number of numerical approaches have been suggested [6, 18]. As is usually the case such numerical approaches are not applicable for parametric spectral factorization. Recently an algebraic approach was developed by the authors [1], which can be utilized for the parametric case where a_i in (3.1) are expressed as polynomials/rational functions of parameters. More specifically the problem of polynomial spectral factorization reduces to finding the algebraic relationship between parameters and the quantity σ_c called the *Sum of Roots*. Then the coefficients of the spectral factor $g(s)$ are expressed in polynomial form in the SoR and rational form in parameters.

THEOREM 3.1 ([1]). *Given $f(s)$ and $g(s)$ as in* (3.1) *and* (3.2)*, respectively, consider σ_c, b_i, $i = 0, \ldots, n-2$, as variables. A system of algebraic equations in terms of σ_c and b_i's is obtained by comparing the coefficients of* (3.3). *Then the set $\mathcal{G}$ of the polynomials obtained from the polynomial parts of the equations forms the reduced Gröbner basis of the ideal generated by itself with respect to the graded reverse lexicographic order $\sigma_c \succ b_{n-2} \succ \cdots \succ b_0$. Moreover, in the generic case, σ_c is a separating element, and one can get a special Gröbner basis called* shape basis *with respect to any elimination ordering $\{b_0, \ldots, b_{n-2}\} \succ\succ \sigma_c$:*

$$\left\{S_f(\sigma_c),\, b_{n-2} - h_{n-2}(\sigma_c),\, \ldots,\, b_0 - h_0(\sigma_c)\right\} \ ,$$

where S_f is a polynomial of degree exactly 2^n and h_i's are polynomials of degree strictly less than 2^n.

For Gröbner bases and associated ideas such as the graded reverse lexicographic order, readers are referred to standard textbooks, e.g., [5].

It is noted that, in this case, the *shape basis* is effectively obtained by means of the basis conversion (change-of-order) technique [5, Appendix D, §2], [7], since all is needed is a conversion from a particular Gröbner basis to another Gröbner basis. It is then straightforward to compute the spectral factor, and one has to find the largest real root of $S_f(\sigma_c)$ and then to substitute it into $h_i(\sigma_c)$ [1]. In the parametric case, S_f is a polynomial in σ_c and parameters, whilst h_i's are polynomials in σ_c but in general rational functions in parameters. The result indicates that all the coefficients of the spectral factor can be related with the SoR and parameters in an algebraic manner.

3.2. Discrete-time Case in the δ Domain: Problem Formulation and General Properties. In the δ domain framework, system descriptions are based on the forward difference [15], rather than the forward shift operator used in the z domain which is typically employed in discrete-time control theory. The approach developed here is in fact a generalization and unification of polynomial spectral factorization in the s and z domains [1, 14]. The s, z and δ operators are related in the following way:

$$z = e^{Ts} \ , \quad \delta = \frac{z-1}{T} = \frac{e^{Ts}-1}{T} \ , \tag{3.4}$$

where T is the sampling period. The stability region in the δ domain is the inside of the circle of radius $\frac{1}{T}$ centred at $\delta = -\frac{1}{T}$ in the complex plane:

$$\mathbb{D}_T := \left\{ \delta \in \mathbb{C} \ \middle| \ |T\delta + 1| < 1 \right\} \ .$$

The boundary of the stable and anti-stable regions is

$$\partial\mathbb{D}_T := \left\{ \delta \in \mathbb{C} \ \middle| \ |T\delta + 1| = 1 \right\} \ .$$

It is noted that, as $T \to 0$, the stability region $\mathbb{D}_T$ tends to the left half plane of the complex plane which is the stability region for continuous-time systems, and $\partial\mathbb{D}_T$ to the imaginary axis, the stability and anti-stability boundary. It is thus known that the δ operator provides an expression for a discretized system that preserves the 'flavour' of the underlying continuous-time system [15].

The $2n$-th order polynomial[1] in δ to be factorized is written as

$$f(\delta) = \frac{1}{(T\delta+1)^n} \left(a_{2n}\delta^{2n} + a_{2n-1}\delta^{2n-1} + a_{2n-2}\delta^{2n-2} + \cdots + a_2\delta^2 + a_1\delta + a_0 \right) \ .$$

Under the assumptions that

$$f(\delta) = f\Big(\frac{-\,\delta}{T\delta+1} \Big) \tag{3.5}$$

[1] It is noted that $f(\delta)$ is *not* in fact a polynomial. However it can easily converted to a polynomial by multiplying $(T\delta+1)^n$, and it is regarded as a polynomial for the sake of brevity of notation.

holds and further that $f(\delta)$ has no roots in $\partial\mathbb{D}_T$ (which holds true in typical control problems), there are exactly n roots in $\mathbb{D}_T$ (and n roots in its compliment). Then the task is to decompose $f(\delta)$ as

$$f(\delta) = g(\delta)\, g\Big(\frac{-\,\delta}{T\delta+1}\Big)\ , \tag{3.6}$$

where $g(\delta)$ is a polynomial (in the strict sense) of degree n having all the roots in $\mathbb{D}_T$ only, i.e. a stable polynomial in the δ domain. It is noted here that, due to the requirement (3.5), of $2n+1$ coefficients a_i of $f(\delta)$, there are only $n+1$ independent coefficients: $a_0, a_2, \ldots, a_{2n}$. The remaining coefficients can be expressed as

$$a_{2j+1} = T \times (\text{linear combination of } a_0, a_2, \ldots, a_{2j}) \quad (\, j = 0, 1, \ldots n-1\,)\ .$$

Write $g(\delta)$ as

$$g(\delta) = \sigma_\delta \delta^n + b_{n-1}\delta^{n-1} + \cdots + b_1\delta + b_0 \quad (\,\sigma_\delta > 0\,)\ .$$

Also let $\mathbf{B} := \big\{b_0, \ldots, b_{n-1}, \sigma_\delta\big\}$. Then the problem of polynomial spectral factorization in the δ domain can be stated as follows: given independent coefficients a_{2j} of $f(\delta)$, find $\mathbf{B}$ that yields a stable $g(\delta)$. Firstly some properties of the solution are investigated that can be used to develop an algebraic approach. Denote the roots of $f(\delta)$ in the stability region $\mathbb{D}_T$ by α_i $(i = 1, 2, \ldots, n)$. Then the anti-stable roots corresponding to them can be written as $\frac{-\alpha_i}{T\alpha_i+1}$. By using α_i, two polynomials $f(\delta)$ and $g(\delta)$ can be expressed as follows:

$$f(\delta) = \frac{a_{2n}}{(T\delta+1)^n}\prod_{i=1}^{n}\big(\delta-\alpha_i\big)\Big(\delta+\frac{\alpha_i}{T\alpha_i+1}\Big)\ ,$$
$$g(\delta) = \sigma_\delta\prod_{i=1}^{n}\big(\delta-\alpha_i\big)\ .$$

Moreover,

$$g\Big(\frac{-\,\delta}{T\delta+1}\Big) = \sigma_\delta\prod_{i=1}^{n}\Big(\frac{-\,\delta}{T\delta+1}-\alpha_i\Big) = \frac{(-1)^n}{(T\delta+1)^n}\,\sigma_\delta\prod_{i=1}^{n}\big(T\alpha_i+1\big)\Big(\delta+\frac{\alpha_i}{T\alpha_i+1}\Big)\ .$$

From (3.6), the following relationship holds:

$$\sigma_\delta^2\prod_{i=1}^{n}\big(T\alpha_i+1\big) = (-1)^n a_{2n} \quad \text{or} \quad \sigma_\delta^2 = \frac{(-1)^n a_{2n}}{\prod_{i=1}^{n}\big(T\alpha_i+1\big)}\ .$$

As is seen below, σ_δ plays a significant role. Observe that $\prod_{i=1}^{n}\big(T\alpha_i+1\big)$ is the product of stable roots (in the z domain) in $f(\delta)$. The quantity is called the *Product of Roots* (PoR) for simplicity, although it is in fact the square root of the reciprocal of the product of stable roots (with sign and gain adjustment).

The facts that $T\alpha_i+1$ is a root in the z domain from (3.4) and that $\left|T\alpha_i+1\right|<1$ facilitate the characterization of σ_δ. Firstly a polynomial which has σ_δ as one of its roots is defined.

DEFINITION 3.2. *Define* $\mathcal{P}=\left\{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\mid \varepsilon_i\in\{1,-1\}\right\}$ *and*

$$C(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)=(T\alpha_1+1)^{\varepsilon_1}\cdot(T\alpha_2+1)^{\varepsilon_2}\cdots(T\alpha_n+1)^{\varepsilon_n}$$

for each $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\mathcal{P}$. *The* characteristic polynomial $S_f(y)$ *of* σ_δ *is defined as*

$$S_f(y)=\prod_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\mathcal{P}}\left(y^2-(-1)^n a_{2n}C(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\right)\ . \tag{3.7}$$

For $T>0$, the following lemma holds.

LEMMA 3.3. *The quantity* σ_δ *coincides with the largest real root of* $S_f(y)$. *Moreover, under the assumption that* $f(\delta)$ *does not have roots in* $\partial\mathbb{D}_T$, σ_δ *is always a simple root.*

3.3. Algebraic Solution Approach in the δ Domain Case. Since $g(\delta)$ should satisfy (3.6), one may find $\mathbf{B}$ by solving a set of algebraic equations derived by comparing the coefficients of both the right and left hand sides of (3.6). Thus write

$$\begin{aligned}g(\delta)g^{\sim}(\delta)-f(\delta)=\frac{1}{(T\delta+1)^n}\times\big((-1)^n g_n\delta^{2n}+(-1)^{n-1}T\tilde{g}_{n-1}\delta^{2(n-1)+1}\\ +(-1)^{n-1}g_{n-1}\delta^{2(n-1)}+\cdots-T\tilde{g}_1\delta^3-g_1\delta^2+T\tilde{g}_0\delta+g_0\big)\ ,\end{aligned} \tag{3.8}$$

where g_i, $\tilde{g}_j$ are quadratic polynomials in $\mathbf{B}$. In order to find $g(\delta)$, one has to compute solutions to the following set of algebraic equations:

$$\begin{cases} g_0=0\,,\ g_1=0\,,\ \ldots\,,\ g_{n-1}=0\,,\ g_n=0\,,\\ \tilde{g}_0=0\,,\ \tilde{g}_1=0\,,\ \ldots\,,\ \tilde{g}_{n-1}=0\,. \end{cases} \tag{3.9}$$

To solve a set of algebraic equations, one may employ a generic algorithm to compute Gröbner bases [5], but the following fact allows a simplified approach to be taken.

THEOREM 3.4. *For the polynomials* g_i, $\tilde{g}_j$ *in* $\mathbf{B}$ *obtained as above,* $\tilde{g}_j$*'s belong to the ideal in* $\mathbb{R}[\mathbf{B}]$ *generated by* $\left\{g_0,g_1,\ldots,g_n\right\}$. *Moreover construct* $\bar{g}_i$ $(i=0,1,\ldots,n)$ *as follows:*

$$\bar{g}_i=g_i-\sum_{j=1}^{\min(i,n-i)}(-1)^j\binom{n-i+j}{n-i-j}T^{2j}\,\bar{g}_{i-j} \tag{3.10}$$

Then the set of polynomials

$$\mathcal{G}:=\left\{\bar{g}_0,\bar{g}_1,\ldots,\bar{g}_n\right\}$$

forms the reduced Gröbner basis of the ideal generated by $\{g_0, g_1, \ldots, g_n\}$ *in* $\mathbb{R}[\mathbf{B}]$ *with respect to the graded reverse lexicographic order* $\sigma_\delta \succ b_{n-1} \succ \cdots \succ b_0$*, with* σ_δ^2 *or* b_i^2 *being the leading monomial of* $\bar{g}_i$ *(and the leading coefficient is* 1*).*

The proof is straightforward yet lengthy, and is thus omitted.

Call the ideal $\langle \mathcal{G} \rangle$ the ideal of spectral factorization. The properties similar to those that hold true for the z domain case [14] also hold. Firstly the set of the leading monomials of the elements of $\mathcal{G}$ is $\{b_0^2, , \ldots, b_{n-1}^2, \sigma_\delta^2\}$ and, therefore, $\mathcal{LB} := \{b_0^{d_0} \cdots b_{n-1}^{d_{n-1}} \sigma_\delta^{d_n} \mid d_i \in \{0,1\}\}$ forms a basis of the residue class ring $\mathbb{R}[\mathbf{B}]/\langle G \rangle$ as an $\mathbb{R}$-linear space. Moreover, $\dim_{\mathbb{R}} \mathbb{R}[\mathbf{B}]/\langle G \rangle = \#\mathcal{LB} = 2^{n+1}$. The following lemma thus holds.

LEMMA 3.5. *The ideal* $\langle \mathcal{G} \rangle$ *of spectral factorization is* 0*-dimensional, and the number of its zeros with multiplicities counted is* 2^{n+1}*.*

When $f(\delta)$ does not have multiple roots, there are exactly 2^n different combinations of choosing roots of $g(\delta)$ if the stability condition of $g(\delta)$ is ignored. Furthermore, when the requirement of the positivity of the leading coefficient σ_δ of $g(\delta)$ is neglected, there are 2^{n+1} different $g(\delta)$ satisfying (3.6). Namely the number of zeros of the ideal $\langle \mathcal{G} \rangle$ is exactly 2^{n+1}. The true $g(\delta)$ that has roots in $\mathbb{D}_T$ only, that is, the stable $g(\delta)$ with a positive leading coefficient, corresponds to the zero with the largest real σ_δ. By computing the shape basis of $\mathcal{G}$, this zero can be easily computed.

When distinct $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathcal{P}$ yield distinct $C(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, it is called a 'generic case'. Otherwise it is called a 'singular case'. In either case, $S_f(y)$ in (3.7) coincides with the characteristic polynomial of σ_δ modulo $\langle \mathcal{G} \rangle$ [22]. In the generic case, $S_f(y)$ is a square-free polynomial of degree 2^{n+1} and moreover the number of distinct zeros of $\langle \mathcal{G} \rangle$ are at most 2^{n+1}, which implies that σ_δ is a separating element [22]. As a consequence a special Gröbner basis called the *shape basis* can be obtained.

THEOREM 3.6. *In the generic case, when* $T > 0$*, the spectral factorization ideal* $\langle \mathcal{G} \rangle$ *has a Gröbner basis so-called* shape basis *with respect to any elimination ordering* $\{b_0, b_1, \ldots, b_{n-1}\} \succ\succ \sigma_\delta$*:*

$$\mathcal{F} := \{S_f(\sigma_\delta), b_{n-1} - h_{n-1}(\sigma_\delta), \ldots, b_0 - h_0(\sigma_\delta)\} \ ,$$

where S_f *is a polynomial of degree exactly* 2^{n+1} *and* h_i*'s are polynomials of degree strictly less than* 2^{n+1}*.*

Again one can rely on the basis conversion technique to obtain the shape basis.

As is mentioned in Subsection 3.2, the δ operator provides a unified approach to both the continuous-time and discrete-time systems; as $T \to 0$, a discretized system model expressed in the δ domain tends to the underlying continuous-time system model [15]. Notice that, as T tends to 0, $g(\frac{-\delta}{T\delta+1}) \to g(-\delta)$, which indicates that (3.6) tends to (3.3). Moreover, (3.10) reduces to $\bar{g}_i = g_i$, suggesting that a Gröbner basis is immediately obtained, which in turn agrees with the result in [1]. It is thus observed that the proposed approach is an extension of polynomial spectral factorization to digital control systems.

Before closing this section, the relationship between the result in the z domain [14] and the one proposed here is clarified. In the approach to the polynomial spectral factorization problem in the z domain suggested in [14], the stable polynomial to be found is parameterized as

$$g(z) = \beta_n(z-1)^n + \beta_{n-1}(z-1)^{n-1} + \cdots + \beta_1(z-1) + \beta_0 \ .$$

What is sought is a particular zero of a set of polynomials in β_i's obtained as coefficients of $f(z) - g(z)g(\frac{1}{z})$. It is shown that linear combinations of the polynomials give a Gröbner basis [14]. The relation (3.4) indicates that the algebraic approach in the δ domain in essence expresses $f(\delta)$ as a polynomial in $(z-1)$ (plus scaling of the coefficients with powers of T). In a similar manner, a Gröbner basis is derived from simple linear combinations of polynomials that are directly obtained from (3.6). Thus the algebraic approach is also a generalization of the approach derived in [14].

4. Symbolic-numeric Hybrid Optimization. Now the suggested symbolic-numeric hybrid optimization approach for plant/controller design integration is formally presented. The important point is that appropriate computation tools (namely symbolic computation and numerical computation) are used for their suitable purposes and that the strength of each tool is fully exploited. Algebraic method/symbolic computation is employed for parametric optimization to obtain the optimal cost $\Phi(P)$ in the presence of parameters and further to manipulate the obtained results; numerical computation is then utilized to carry out orthodox optimization for finding desired parameter values. What is crucial in this approach is that the gradient vector/Hessian matrix of the optimal cost with respect to parameters can be explicitly constructed and evaluated since the algebraic relationship between parameters and the optimal cost is available. Therefore standard optimization methods such as the steepest descent method and Newton's method are applicable to the optimization over the admissible range of parameters.

The suggested approach consists of 3 steps:

1. Relate parameters and the optimal cost;
2. Compute the gradient vector/Hessian matrix etc. required in the optimization in the subsequent step; and
3. Execute optimization over the admissible range of parameters using the results obtained in Step 2.

The first two steps are carried out by way of symbolic computation, whilst the last step employs numerical computation. For most $\mathcal{H}_2$ problems, the optimal controller and the optimal cost can be found from the result of polynomial spectral factorization [17]. The third step is nothing but a simple application of an orthodox optimization method that can be found in standard optimization textbooks, e.g. [2].

The second step is executed in the following way. Firstly the derivative of $\sigma_\bullet$ with respect to a parameter is computed as follows:

1. Obtain an algebraic relationship

$$S_f(\sigma_\bullet;\, \mathbf{q}) = 0 \tag{4.1}$$

that relates $\sigma_\bullet$ and parameters $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ in polynomial form by way of algebraic polynomial spectral factorization reviewed in Section 3.

2. Consider $\sigma_\bullet$ as a function of $\mathbf{q}$, and take the partial derivative of (4.1) with respect to q_i. The result can be written as

$$T_1(\sigma_\bullet;\, \mathbf{q}) + T_2(\sigma_\bullet;\, \mathbf{q})\, \frac{\partial \sigma_\bullet}{\partial q_i}(\mathbf{q}) = 0 \ ,$$

where T_1 and T_2 are polynomials in $\sigma_\bullet$ and $\mathbf{q}$. Further solve the above equation for $\frac{\partial \sigma_\bullet}{\partial q_i}$:

$$\frac{\partial \sigma_\bullet}{\partial q_i}(\mathbf{q}) = -\, \frac{T_1(\sigma_\bullet;\, \mathbf{q})}{T_2(\sigma_\bullet;\, \mathbf{q})} \ . \tag{4.2}$$

This procedure is executed for all parameters.

The above result yields the sensitivity of $\sigma_\bullet$ with respect to each parameter. The gradient vector of $\Phi(P)$ can now be computed in the following way. Suppose that $\Phi(P)$ is a rational function of parameters and $\sigma_\bullet$ (which holds true for most $\mathcal{H}_2$ control problems). First take the partial derivative of $\Phi(P)$ with respect to q_i, assuming that $\sigma_\bullet$ is a function of $\mathbf{q}$, which gives $\frac{\partial \Phi(P)}{\partial q_i}$ as a rational function in $\mathbf{q}$, $\sigma_\bullet$, and $\frac{\partial \sigma_\bullet}{\partial q_i}$. Then use (4.2) to get an expression in $\mathbf{q}$ and $\sigma_\bullet$ only. By computing $\frac{\partial \Phi(P)}{\partial q_i}$ for all parameters, the gradient vector $\nabla \Phi(P)$ can be computed. The actual value of $\frac{\partial \Phi(P)}{\partial q_i}$ can be obtained by

- fixing parameter values;
- computing the largest real root of (4.1); and
- substituting those values into the expression of $\frac{\partial \Phi(P)}{\partial q_i}$.

Higher order derivatives can be computed in exactly the same way since $\frac{\partial \Phi(P)}{\partial q_i}$ is now expressed as a rational function in $\mathbf{q}$ and $\sigma_\bullet$.

It is emphasized that the computed gradient vector is exact, and one can immediately employ a simple optimization method and directly aim at finding the optimum over the parameter range. This is a striking contrast to the existing approaches, where $\Phi(P)$ is obtained as a result of numerical computation (in which case only approximated gradient vector is available and a heuristic approach is unavoidable), or where parameters of the plant and of the controller are simultaneously optimized (in which case the optimization problem has more decision variables than the optimization problem solved here and often exhibits multi-modality making the true optimum difficult to achieve). It should be mentioned that $\Phi(P)$ is in general a non-convex function of parameters and that optimization methods such as the steepest descent method

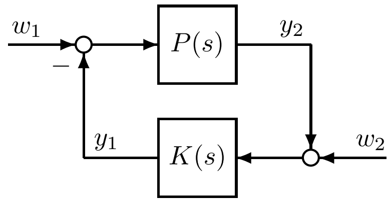

FIG. 5.1. *Feedback system configuration.*

may not always achieve the global optimum. However, in the suggested approach, the number of decision variables is reduced owing to parametric controller design which eliminates controller parameters from decision variables. It is thus expected that there are less peaks or valleys so that the global optimum may be achieved if one employs the multi-start approach in the optimization over parameters.

Before closing this section, it is stated that, even if polynomial spectral factorization should be carried out for multiple times and there are two or more Sums of Roots, for example, this approach can also be applied. All that has to be done is to compute the partial derivative of each and every SoR with respect to all parameters; see also Section 5.

5. Continuous-time Design Example: Weighted LQG Problem. Expressions in the δ domain are in general more involved than those in the continuous-time domain. In order to focus on the demonstration of the proposed hybrid optimization approach presented in Section 4, this section employs a realistic $\mathcal{H}_2$ control problem, called the weighted LQG problem [17], in the continuous-time framework.

5.1. Problem Formulation. In Figure 5.1, $P(s)$ is a linear, time-invariant continuous-time plant, $K(s)$ a continuous-time controller to be designed, w_i exogenous inputs (disturbances), and y_i system outputs which are to be made small. The feedback performance used is

$$(5.1)\quad J_{\rho,\mu}(P,\,K) := \left(\|y_1(t)\|_2^2 + \rho^2\,\|y_2(t)\|_2^2\right)\Big|_{\substack{w_1(t)=\mu\delta(t)\\ w_2(t)=0}} + \left(\|y_1(t)\|_2^2 + \rho^2\,\|y_2(t)\|_2^2\right)\Big|_{\substack{w_1(t)=0\\ w_2(t)=\delta(t)}},$$

where $\delta(t)$ is the unit impulse function, ρ and μ are positive numbers, and ρ (resp., μ) specifies the relative weight between $y_1(t)$ and $y_2(t)$ (resp., $w_1(t)$ and $w_2(t)$). The task is to minimize $J_{\rho,\mu}(P,\,K)$ over all feasible (i.e., stabilizing) controllers. In particular the paper focuses on the optimal value of $\inf_{K\in\mathcal{K}} J_{\rho,\mu}(P,\,K)$, namely, $\Phi(P)$ in (2.1), and attempts to minimize $\Phi(P)$ over all admissible plants, i.e., execute optimization (2.2), when the plant has some real parameters that can be utilized for tuning.

The problem under consideration is a particular case of general $\mathcal{H}_2$ control problems, and standard solution approaches are applicable. By writing down the feedback system in Figure 5.1 in generalized plant form, a textbook formula solution [23] may be used. Namely solution of two algebraic Riccati equations yields the optimal cost and the optimal controller. An alternative approach is to formulate the problem as an linear matrix inequality (LMI) optimization problem and then to solve it numerically.

Solution of Riccati equations or LMI optimization is executed numerically, and either approach based on numerical computation solves the problem efficiently when P does not contain any parameters. For a plant with parameters, such an approach requires parameter values to be fixed before executing a solution procedure. Therefore little information is given from such an approach on how parameter values affect the best achievable performance level. One would thus have to resort to some heuristic optimization approach when solving the optimization problem in (2.2).

5.2. Sum of Roots Characterization of the Best Performance Level. As discussed above typical numerical treatment does not lead to a systematic approach to the problem of plant/controller integrated design. This subsection discusses how parametric treatment can exhibit its strength. To this end it is indicated that the best performance level is characterized in terms of the SoR [1].

Polynomial spectral factorization can be used to solve the weighted LQG problem [17]. The single-input-single-output (SISO) plant case admits the subsequent algorithm which yields the desired SoR characterization:

1. Write the n-th order (strictly proper) plant $P(s)$ as a ratio of two coprime polynomials:

   $$P(s) = \frac{P_N(s)}{P_D(s)} \ .$$

   Without loss of generality it is assumed that $P_D(s)$ is a monic polynomial of degree n.
2. Relate parameters and two SoRs σ_ρ and σ_μ in the following way. Construct two even polynomials $f_\rho(s)$ and $f_\mu(s)$ from $P_N(s)$ and $P_D(s)$ and carry out polynomial spectral factorization for both polynomials:

   $$f_\rho(s) := \rho^2 P_N(s)P_N(-s) + P_D(s)P_D(-s) = g_\rho(s)g_\rho(-s) \ , \tag{5.2}$$

   $$f_\mu(s) := \mu^2 P_N(s)P_N(-s) + P_D(s)P_D(-s) = g_\mu(s)g_\mu(-s) \ , \tag{5.3}$$

   where $g_\rho(s)$ and $g_\mu(s)$ are the spectral factors for $f_\rho(s)$ and $f_\mu(s)$, respectively. This also gives expressions of the coefficients of $g_\rho(s)$ and $g_\mu(s)$ in terms of parameters and the SoRs.
3. Get an expression of the optimal controller in terms of parameters and the SoRs. It can be achieved as follows. Find a polynomial $K_N(s)$ of degree up

to $n-1$ and a monic polynomial $K_D(s)$ of degree exactly n that satisfy the following identity (Diophantine equation):

$$P_N(s)K_N(s) + P_D(s)K_D(s) = g_\rho(s)g_\mu(s) \ . \tag{5.4}$$

It is noted that, under the assumption that $P_N(s)$ and $P_D(s)$ are coprime, $K_N(s)$ and $K_D(s)$ are uniquely determined by solving a set of linear equations (in terms of the coefficients of $K_N(s)$ and $K_D(s)$). Furthermore the coefficients of $K_N(s)$ and $K_D(s)$ are rational functions of the coefficients of $P_N(s)$, $P_D(s)$, $g_\rho(s)$ and $g_\mu(s)$. The optimal controller $K_{\rm opt}(s)$ that achieves the optimum, i.e., $\Phi(P)$, can then be written as

$$K_{\rm opt}(s) = \frac{K_N(s)}{K_D(s)} \ .$$

Now the coefficients of $K_N(s)$ and $K_D(s)$ are expressed in terms of parameters and the SoRs.

4. Express the optimal cost $\Phi(P)$ in terms of parameters and the SoRs by

$$\Phi(P) = \mu^2 \left\| \frac{g_\rho(s) - P_D(s)}{g_\rho(s)} \right\|_2^2 + \rho^2\mu^2 \left\| \frac{P_N(s)}{g_\rho(s)} \right\|_2^2 + \mu^2 \left\| \frac{g_\mu(s) - K_D(s)}{g_\mu(s)} \right\|_2^2 + \left\| \frac{K_N(s)}{g_\mu(s)} \right\|_2^2 \ . \tag{5.5}$$

Since the square of the $\mathcal{H}_2$-norm can be expressed as a rational function of the system coefficients, $\Phi(P)$ above can also be expressed as a rational function of the coefficients of $P_N(s)$, $P_D(s)$, $g_\rho(s)$, and $g_\mu(s)$. Consequently it yields the desired expression and $\Phi(P)$ is characterized in terms of the SoR.

5.3. Magnetic Levitation System. Now the proposed hybrid optimization method is applied to the so-called magnetic levitation system. In this subsection the system description is given. In the magnetic levitation system consisting of an electromagnet and a steel ball depicted in Figure 5.2, the aim is to control the vertical position y of the ball by changing the current i going through the coil of the electromagnet manipulated by the control input, namely, the input voltage v. Here only the vertical movement of the ball is considered. Let the equilibrium point be (y_0, i_0, v_0). By linearizing the dynamical equation around the equilibrium point, the transfer function from the input to the output is obtained as

$$P(s) = \frac{-2\beta}{(Ls+R)(s^2-\alpha^2)} \ ,$$

$$\alpha = \frac{g}{y_0} \ , \quad \beta = \alpha\sqrt{\frac{K}{M}} \ , \quad g: \text{acceleration of gravity.}$$

Set $L = 1$ and $\alpha = 1$. Further define two parameters as

$$q_1 = R \ , \quad q_2 = \frac{1}{R}\sqrt{\frac{K}{M}} \ .$$

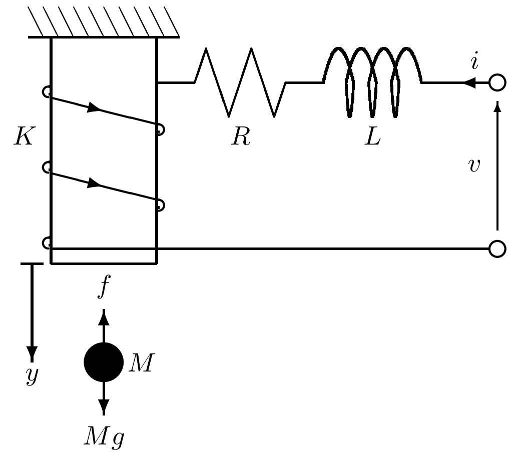

FIG. 5.2. *Magnetic levitation system.*

Then the transfer function becomes

$$P(s) = \frac{-2q_1q_2}{(s+q_1)(s^2-1)} \; .$$

Once the values for the parameters q_1 and q_2 are fixed, there exist established numerical methods for finding the optimal controller that minimizes the cost function (5.1). The task here is to find parameter values such that the *optimal cost is minimized*, i.e., to carry out optimization over the range of parameters for the best possible performance among all admissible plants and feasible controllers. Consider the case where $\rho = 2$ and $\mu = 1$, and let the admissible range of parameters be

$$\mathcal{Q} := \{\mathbf{q} = (q_1, q_2) \mid 5 \le q_1 \le 20\, ,\; 0.5 \le q_2 \le 2\} \; .$$

5.4. Optimization Result. The symbolic-numeric optimization approach proposed in Section 4 is applied to the problem stated above. Firstly the SoR approach to polynomial spectral factorization expounded in Subsection 3.1 is employed. Given an even polynomial

$$f(s) = \,-\, s^6 + a_4 s^4 + a_2 s^2 + a_0 \; ,$$

the coefficients of its spectral factor

$$g(s) = s^3 + \sigma_c s^2 + b_1 s + b_0$$

satisfy the following relationship:

$$
(5.6)\quad \begin{cases} S_f(\sigma_c) := \sigma_c^8 - 4a_4\,\sigma_c^6 + 2(3a_4^2 + 4a_2)\,\sigma_c^4 \\ \qquad\qquad - 4(a_4^3 + 4a_2a_4 + 16a_0)\,\sigma_c^2 + (a_4^2 + 4a_2)^2 = 0\ , \\ b_1 = \dfrac{1}{2}(\sigma_c^2 - a_4)\ , \\ b_0 = -\dfrac{\sigma_c}{8}\dfrac{\sigma_c^6 - 4a_4\sigma_c^4 + (5a_4^2 + 4a_2)\sigma_c^2 - 2(a_4^3 + 4a_2a_4 + 32a_0)}{a_4^2 + 4a_2}\ . \end{cases}
$$

It is noted that the relationship is in fact pre-computable and can be used as formulae. Namely, given the expressions of the coefficients of $f(s)$, one can relate the coefficients of $f(s)$ and those of $g(s)$ by substituting the actual coefficients of $f(s)$ into (5.6).

For the problem under consideration, $f_\rho(s)$ in (5.2) is

$$
f_\rho(s) = -s^6 + (q_1^2 + 2)\,s^4 - (2q_1^2 + 1)\,s^2 + 16q_1^2q_2^2 + q_1^2\ .
$$

The result in (5.6) shows that σ_ρ is the largest real root of

$$
(5.7)\quad \sigma_\rho^8 - 4(q_1^2 + 2)\,\sigma_\rho^6 + 2(3q_1^4 + 4q_1^2 + 8)\,\sigma_\rho^4 \\ - 4(q_1^6 - 2q_1^4 + 256q_1^2q_2^2 + 8q_1^2)\,\sigma_\rho^2 + q_1^4(q_1 - 2)^2(q_1 + 2)^2\ .
$$

Other coefficients of the spectral factor $g_\rho(s)$ can be expressed in terms of the parameters and the SoR σ_ρ as

$$
\begin{cases} b_{1\rho} = \dfrac{1}{2}(\sigma_\rho^2 - q_1^2 - 2)\ , \\ b_{0\rho} = -\dfrac{\sigma_\rho}{8} \\ \qquad \times \dfrac{\sigma_\rho^6 - 4(q_1^2 + 2)\sigma_\rho^4 + (5q_1^4 + 12q_1^2 + 16)\sigma_\rho^2 - 2(q_1^4 - 2q_1^2 + 512q_1^2 + 24)q_1^2}{q_1^2(q_1^2 - 4)}\ . \end{cases}
$$

Polynomial spectral factorization for $f_\rho(s)$ is thus carried out algebraically. The other polynomial $f_\mu(s)$ in (5.3) can be dealt with in the exactly identical way.

Once polynomial spectral factorization is completed, the optimal cost (5.5) can be expressed as a rational function of parameters and the two SoRs σ_ρ and σ_μ. The expression of $\Phi(P)$, the relationship (5.7) and the corresponding relationship for σ_μ allow one to compute the gradient vector $\nabla\Phi(P)$ of $\Phi(P)$ with respect to parameters q_1 and q_2. In this way optimization of $\Phi(P)$ over parameters becomes amenable. For instance, by equating (5.7) with 0 and taking its partial derivative with respect to q_1, one gets

$$
\underbrace{-8q_1\big(\sigma_\rho^6 - (3q_1^2 + 2)\sigma_\rho^4 + (3q_1^4 - 4q_1^2 + 256q_2^2 + 8)\sigma_\rho^2 - q_1^2(q_1^2 - 4)(q_1^2 - 2)\big)}_{T_1(\sigma_\rho;q_1,q_2)} \\ + \underbrace{8\sigma_\rho\big(\sigma_\rho^6 - 3(q_1^2 + 2)\sigma_\rho^4 + (3q_1^4 + 4q_1^2 + 8)\sigma_\rho^2 - q_1^2(q_1^4 - 2q_1^2 + 256q_2^2 + 8)\big)}_{T_2(\sigma_\rho;q_1,q_2)}\frac{\partial\sigma_\rho}{\partial q_1} \\ = 0\ ,
$$

which further leads to

$$
\begin{aligned}
\frac{\partial\sigma_\rho}{\partial q_1} &= -\frac{T_1(\sigma_\rho; q_1, q_2)}{T_2(\sigma_\rho; q_1, q_2)} \\
&= \frac{q_1\big(\sigma_\rho^6 - (3q_1^2+2)\sigma_\rho^4 + (3q_1^4 - 4q_1^2 + 256q_2^2 + 8)\sigma_\rho^2 - q_1^2(q_1^2-4)(q_1^2-2)\big)}{\sigma_\rho\big(\sigma_\rho^6 - 3(q_1^2+2)\sigma_\rho^4 + (3q_1^4 + 4q_1^2 + 8)\sigma_\rho^2 - q_1^2(q_1^4 - 2q_1^2 + 256q_2^2 + 8)\big)} \ .
\end{aligned} \tag{5.8}
$$

Also,

$$
\left\| \frac{P_N(s)}{g_\rho(s)} \right\|_2^2 = \frac{32q_1^2q_2^2\sigma_\rho}{\sigma_\rho^6 - 3(q_1^2+2)\sigma_\rho^4 + (3q_1^4 + 4q_1^2 + 8)\sigma_\rho^2 - q_1^2(q_1^4 - 2q_1^2 + 256q_2^2 + 8)} \ .
$$

By taking the partial derivatives of such costs with respect to q_1 and q_2 and using the relationship like (5.8), the partial derivatives of $\Phi(P)$ with respect to q_1 and q_2 are computed. It is also straightforward to compute the Hessian matrix of $\Phi(P)$.

Now the preparation for optimization over parameters is accomplished. Here Newton's method is used. The result is shown in Figure 5.3. Taking the starting point $\mathbf{q} = (10,\ 1.0)$, the optimization terminated after 8 iterations, reaching the optimum:

$$
\inf_{\mathbf{q}\in\mathcal{Q}} \Phi(P) = 65.905 \ , \quad \mathbf{q}_{\rm opt} = (20,\ 1.368) \ . \tag{5.9}
$$

Figure 5.3 confirms that the global optimum is achieved.

The computation was executed in Maple 12 running on a 1.2GHz PC with Intel Pentium M, and it took about 45 seconds to complete all the computation. For comparison purposes a naive brute-force conventional approach was implemented using MATLAB on the same PC. The optimal controllers and the corresponding optimal costs for evenly spaced 22,801 points in the admissible parameter space $\mathcal{Q}$ were computed, and the optimum which is essentially identical to (5.9) was attained. However the computation took approximately 260 seconds. A heuristic optimization approach would reduce the computation time, but it would not be clear whether the global optimum is achieved. It is considered that the proposed approach gives a systematic approach to the problem of plant/controller integrated design.

Before closing the section the smoothness of the plot of $\Phi(P)$ in Figure 5.3 is pointed out. When $\Phi(P)$ is given implicitly, it is in general difficult to plot (Figure 5.3 was created by gridding the parameter space and computing the optimal cost for each point by way of solving Riccati equations repetitively.) The smoothness property of $\Phi(P)$ does not hold in general, but it is expected to hold for many practical systems. This is because the difficulty in control does not usually change drastically and repetitively as physical parameters in the plant change and hence $\Phi(P)$ does not show a multi-modal property. If this holds true and one can compute the derivative of $\Phi(P)$, optimization for such a well-behaving function tends to be an easy task. The suggested symbolic-numeric hybrid approach precisely achieves this by giving a method to compute the derivative of $\Phi(P)$ explicitly.

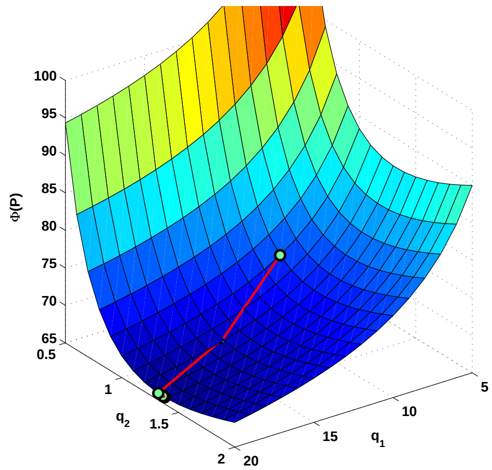

FIG. 5.3. *Plot of $\Phi(P)$ against q_1 and q_2 and optimization result by Newton's method.*

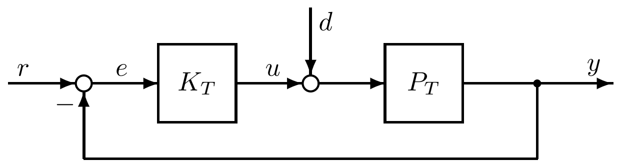

FIG. 6.1. *Unity feedback system configuration.*

6. Digital Control Design Example: $\mathcal{H}_2$ Optimal Regulation Problem. This section takes up a simpler $\mathcal{H}_2$ control problem to show how the optimal performance can be related to parameters in the plant and also the sampling period in the digital control framework. The main focus is on the demonstration of polynomial spectral factorization in the δ domain developed in Section 3. Once the relationship between the optimal performance level and parameters and the sampling period is obtained, the same hybrid optimization approach proposed in Section 4 and demonstrated in Section 5 is applicable.

6.1. Problem Formulation and Expressions for Optimal Performances. The formulation of the problem considered here is firstly stated. In the SISO unity feedback control system depicted in Figure 6.1, given $P_T(\delta)$ which is obtained by discretizing continuous-time plant $P_c(s)$ with sampling period T, the task is to minimize the performance level

$$E_T(P_T) := T \sum_{k=0}^{\infty} \left(|y(k)|^2 + |u(k)|^2 \right)$$

under the assumption that the disturbance signal input is the unit pulse signal and that the there is no reference signal, i.e., $r(k) \equiv 0$. It is noted that T is not fixed but is considered as a parameter in this problem.

For the digital system with sampling period T and for a strictly proper, minimum phase plant $P_T(\delta)$, the optimal $E_T(P_T)$, denoted by $E_T^\star(P_T)$, can be written as follows [21]. Write the plant $P_T(\delta)$ of degree n as

$$P_T(\delta) = \frac{\eta_{n-1}\delta^{n-1} + \cdots + \eta_0}{\delta^n + \zeta_{n-1}\delta^{n-1} + \cdots + \zeta_0} =: \frac{P_{TN}(\delta)}{P_{TD}(\delta)} \ ,$$

where $P_{TN}(\delta)$ and $P_{TD}(\delta)$ are coprime polynomials. Let

$$M_{TD}(\delta) := \sigma_\delta \delta^n + b_{n-1}\delta^{n-1} + b_{n-2}\delta^{n-2} + \cdots + b_1\delta + b_0$$

be the spectral factor of

$$P_{TN}(\delta)P_{TN}\Big(\frac{-\delta}{T\delta+1}\Big) + P_{TD}(\delta)P_{TD}\Big(\frac{-\delta}{T\delta+1}\Big) \ .$$

Then it is deduced that [21]

$$E_T^\star(P_T) := \inf_{K\in\mathcal{K}} E_T(P_T) = \frac{\sigma_\delta^2 - 1}{T} \ . \tag{6.1}$$

Notice that $E_T^\star(P_T)$ is related to parameters in $P_c(s)$ and also sampling period T, or, equivalently, parameters in $P_T(\delta)$.

For comparison purposes the result for the continuous-time counterpart is also stated. The optimal cost $E_c^\star(P_c)$ in the case of strictly proper minimum phase $P_c(s)$ is given as follows [10]. Write the plant $P_c(s)$ of degree n as

$$P_c(s) = \frac{\bar{\eta}_{n-1}s^{n-1} + \cdots + \bar{\eta}_0}{s^n + \bar{\zeta}_{n-1}s^{n-1} + \cdots + \bar{\zeta}_0} =: \frac{P_{cN}(s)}{P_{cD}(s)} \ ,$$

where $P_{cN}(s)$ and $P_{cD}(s)$ are coprime polynomials. Let

$$M_{cD}(s) := s^n + \sigma_c s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0 \tag{6.2}$$

be the spectral factor of

$$P_{cN}(s)P_{cN}(-s) + P_{cD}(s)P_{cD}(-s) \ .$$

Then it is deduced that [10]

$$E_c^\star(P_c) := \inf_{K\in\mathcal{K}} \int_0^\infty \big(|y(t)|^2 + |u(t)|^2\big)dt = \sigma_c - \bar{\zeta}_{n-1} \ . \tag{6.3}$$

It is noted that σ_c is the SoR in Subsection 3.1 and that σ_c and, consequently, $E_c^\star(P_c)$ are related to parameters in $P_c(s)$, as in the digital control case.

6.2. Numerical Example. Consider the following continuous-time plant

$$P_c(s) = \frac{s+5}{s^2 + s - q - 2} \ ,$$

where q is a parameter which is supposed to vary around 0. By means of parametric polynomial spectral factorization and the expression for $E_c^\star(P_c)$ in (6.3), it is derived that

$$E_c^\star(P_c) = \sigma_c - 1 \ ,$$

where σ_c is the largest real root of

$$S_f(\sigma_c; q) := \sigma_c^4 - 4(q+3)\sigma_c^2 + 8(q-10) \ .$$

For this particular case, an exact expression for σ_c can be obtained:

$$E_c^\star(P_c) = \sqrt{2q + 6 + 2\sqrt{q^2 + 4q + 29}} - 1 \ . \tag{6.4}$$

When $P_c(s)$ is discretized assuming a zero-order hold input with sampling period T, the following discrete-time plant $P_T(\delta)$ is obtained:

$$P_T(\delta) = \frac{\eta_1\delta + \eta_0}{\delta^2 + \zeta_1\delta + \zeta_0} =: \frac{P_{TN}(\delta)}{P_{TD}(\delta)} \ ,$$

where

$$\begin{aligned}
\zeta_1 &= -\frac{1}{T}\Big(e^{-\frac{1}{2}(1+\gamma)T} + e^{-\frac{1}{2}(1-\gamma)T} - 2\Big) \ , \\
\zeta_0 &= -\frac{1}{T^2}\Big(e^{-\frac{1}{2}(1+\gamma)T} + e^{-\frac{1}{2}(1-\gamma)T} - 1 - e^{-T}\Big) \ , \\
\eta_1 &= \frac{1}{2(q+2)\gamma T}\Big(-10\gamma + (5\gamma - 2q - 9)e^{-\frac{1}{2}(1+\gamma)T} + (5\gamma + 2q + 9)e^{-\frac{1}{2}(1-\gamma)T}\Big) \ , \\
\eta_0 &= \frac{5}{(q+2)T^2}\Big(e^{-\frac{1}{2}(1+\gamma)T} + e^{-\frac{1}{2}(1-\gamma)T} - 1 - e^{-T}\Big) \ ,
\end{aligned}$$

with $\gamma = \sqrt{4q+9}$. It can be confirmed that, as T tends to 0,

$$\zeta_1 \to 1 \ , \quad \zeta_0 \to -q-2 \ , \quad \eta_1 \to 1 \ , \quad \eta_0 \to 5 \ ,$$

and $P_c(s)$ is recovered.

Now polynomial spectral factorization is carried out for

$$\begin{aligned}
&P_{TN}(\delta)P_{TN}\Big(\frac{-\delta}{T\delta+1}\Big) + P_{TD}(\delta)P_{TD}\Big(\frac{-\delta}{T\delta+1}\Big) \\
&\quad = \frac{1}{(T\delta+1)^2}\Big((\zeta_0T^2 - \zeta_1T + 1)\delta^4 + T\big((\zeta_0\zeta_1 + \eta_0\eta_1)T + (2\zeta_0 - \zeta_1^2 - \eta_1^2)\big)\delta^3 \\
&\qquad\qquad + \big((\zeta_0^2 + \eta_0^2)T^2 + (\zeta_0\zeta_1 + \eta_0\eta_1)T + 2\zeta_0 - \zeta_1^2 - \eta_1^2\big)\delta^2 \\
&\qquad\qquad\qquad + 2T\big(\zeta_0^2 + \eta_0^2\big)\delta + \zeta_0^2 + \eta_0^2\Big) \ .
\end{aligned}$$

Write the spectral factor as

$$M_{TD}(\delta) := \sigma_\delta\delta^2 + b_1\delta + b_0 \ .$$

The algebraic polynomial spectral factorization approach developed in Subsection 3.3 yields a polynomial relating q, T and σ_δ:

$$\begin{aligned}
&\sigma_\delta^8 + \big(-(\zeta_0^2+\eta_0^2)T^4 + 2(\zeta_0\zeta_1+\eta_1\eta_0)T^3 - 2(\zeta_1^2+\eta_1^2)T^2 + 4\zeta_1 T - 4\big)\sigma_\delta^6 \\
&+ \big((2\zeta_1\zeta_0\eta_0\eta_1 + \eta_0^2\eta_1^2 - 2\eta_0^2\zeta_0 - 2\zeta_0^3 + \zeta_1^2\zeta_0^2)T^6 \\
&+ 2(\zeta_0^2\zeta_1 - \zeta_1\zeta_0\eta_1^2 + \eta_0^2\zeta_1 - \zeta_1^3\zeta_0 - \zeta_1^2\eta_0\eta_1 - \eta_1^3\eta_0)T^5 \\
&+ (-2\eta_0^2 + \zeta_1^4 + \eta_1^4 + 4\zeta_1\eta_0\eta_1 + 4\zeta_1^2\zeta_0 + 2\zeta_1^2\eta_1^2)T^4 - 4(\zeta_1^3 + \eta_1\eta_0 + 2\zeta_0\zeta_1 + \zeta_1\eta_1^2)T^3 \\
&+ 2(2\eta_1^2 + 2\zeta_0 + 5\zeta_1^2)T^2 - 12\zeta_1 T + 6\big)\sigma_\delta^4 + \big(-\zeta_0^2(\zeta_0^2+\eta_0^2)T^8 + 2\zeta_0(\zeta_0\eta_0\eta_1 + \eta_0^2\zeta_1 + 2\zeta_0^2\zeta_1)T^7 \\
&- (7\zeta_1^2\zeta_0^2 + \zeta_1^2\eta_0^2 + 2\zeta_0^3 + 2\eta_0^2\zeta_0 + 4\zeta_1\zeta_0\eta_0\eta_1 + 2\zeta_0^2\eta_1^2)T^6 \\
&+ 2(5\zeta_0^2\zeta_1 + \eta_0^2\zeta_1 + 3\zeta_1^3\zeta_0 + \zeta_1^2\eta_0\eta_1 + 2\zeta_0\eta_0\eta_1 + 2\zeta_1\zeta_0\eta_1^2)T^5 \\
&- (5\zeta_0^2 + \eta_0^2 + 16\zeta_1^2\zeta_0 + 4\zeta_1\eta_0\eta_1 + 2\zeta_1^2\eta_1^2 + 2\zeta_1^4 + 4\zeta_0\eta_1^2)T^4 \\
&+ 2(9\zeta_0\zeta_1 + \eta_1\eta_0 + 2\zeta_1\eta_1^2 + 4\zeta_1^3)T^3 - 2(\eta_1^2 + 7\zeta_1^2 + 4\zeta_0)T^2 + 12\zeta_1 T - 4\big)\sigma_\delta^2 \\
&+ \zeta_0^4 T^8 - 4\zeta_0^3\zeta_1 T^7 + 2\zeta_0^2(2\zeta_0 + 3\zeta_1^2)T^6 - 4\zeta_0\zeta_1(\zeta_1^2 + 3\zeta_0)T^5 \\
&+ (12\zeta_1^2\zeta_0 + \zeta_1^4 + 6\zeta_0^2)T^4 - 4\zeta_1(\zeta_1^2 + 3\zeta_0)T^3 + 2(2\zeta_0 + 3\zeta_1^2)T^2 - 4\zeta_1 T + 1 = 0
\end{aligned}$$

The above polynomial explicitly relates σ_δ and parameters q and T. Also, $E_T^\star(P_T)$ and σ_δ are related as in (6.1). Thus optimization of $E_T^\star(P_T)$ over parameters is now amenable in the same way demonstrated in Section 5. It is emphasized that the sampling period T is also considered to be a parameter and that, as an explicit relationship is derived, one can analyze how change in T affects the optimal choice of q.

For completeness it is shown that the optimal cost in the continuous-time case is recovered. From the above polynomial and (6.1), a 4th order polynomial which relates q, T and $E_T^\star(P_T)$ and whose largest real root is $E_T^\star(P_T)$ is computed, which is too lengthy to include here. As T tends to 0, the polynomial tends to

$$x^4 + 4x^3 - 2(2q+3)x^2 - 4(2q+5)x + 4q - 91 = 0\ .$$

It can be confirmed that that its largest real root is indeed identical to (6.4).

7. Concluding Remarks. The paper has established a new framework for plant/controller design integration that aims to find the best pair of the plant and the controller. The framework admits a hybrid solution approach that utilizes symbolic computation and numerical computation and makes the best use of the strengths of both computation methods. The symbolic part exploits the algebraic approach to polynomial spectral factorization to characterize the optimal cost in the presence of parameters. The resulting expression allows efficient numerical optimization for finding the best pair of the plant and the controller. Furthermore an algebraic approach to polynomial spectral factorization in the δ domain is developed to adopt the digital control implementation in practical applications, which can also be seen as a unified approach to both the continuous-time and discrete-time systems. This tool allows the same hybrid approach to be executed for the design integration problem in digital control. The suggested approaches are demonstrated on two design examples.

The current situation of computational burden is mentioned. At this moment the suggested approach can deal with up to 4th order system with 3 parameters. This limitation comes from the computational complexity of the basis conversion in polynomial spectral factorization and also from complicated expressions obtained as the result. It is expected that more structural properties in polynomial spectral factorization will be exploited in the computation of the shape basis. Moreover it may be sensible to introduce intermediate variables to relate plant parameters and $\sigma_\bullet$ and the achievable performance level $\Phi(P)$ rather than to get an explicit expression of $\Phi(P)$ in terms of plant parameters and $\sigma_\bullet$.

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