

EFFICIENT COMPUTATION OF CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. In this paper, we introduce the notion of a homogeneous ideal in *quasi stable position* (QSP); a new definition for the notion of generic coordinates to compute *efficiently* the Castelnuovo-Mumford regularity of a homogeneous ideal. This definition is simple to check, because it is tested on the initial ideal for the degree reverse lexicographic ordering. It is explicit, because we provide an algorithm to decide whether a monomial ideal is in QSP or not. The main result of this paper is that *the Castelnuovo-Mumford regularity of an ideal in QSP is the maximal degree of the elements of its reduced Gröbner basis with respect to the reverse lexicographic ordering*. We have implemented an algorithm in (the distributed library `noether.lib` of) SINGULAR based on the above results for computing the Castelnuovo-Mumford regularity of a general ideal, and we evaluate its performance via some examples.

1. INTRODUCTION

Let us consider a homogeneous ideal I over the ring $R = K[x_0, \dots, x_n]$ where K is an arbitrary field. A good measure to estimate the complexity of the computation of the Gröbner basis of I is the maximal degree of the polynomials which appear during the computation (see [11, 14, 15]). On the other hand, let

$$0 \longrightarrow F_\ell \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

be a minimal graded free resolution of I and e_i the maximal degree of the generators of F_i . Then the Castelnuovo-Mumford regularity of I is defined as $\text{reg}(I) = \max\{e_i - i \mid i = 0, \dots, \ell\}$ (see [3, 8]). By a well-known result, if coordinates are generic and the monomial ordering is the reverse lexicographic ordering, then $\text{reg}(I)$ is an upper bound for the degree of the Gröbner basis elements. This upper bound is reached if the characteristic of K is zero (see [3]).

It follows from these remarks and other similar properties that all computations are much easier if the coordinates are *sufficiently generic* in order that the above relations are satisfied. More specifically, it is easy to compute the Castelnuovo-Mumford regularity from a Gröbner basis if the coordinates are sufficiently generic and if one knows that they are. Therefore, one needs a testable definition of sufficiently generic. This definition has to be as large as possible in order to avoid as much as possible a costly change of coordinates.

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Bayer and Stillman in [3] have already introduced an explicit definition of generic coordinates for a homogeneous ideal. Bermejo and Gimenez in [5] have defined the nice notion of nested type (with a test) for a monomial ideal. But, all of these definitions of generic coordinates are not sufficiently generic such that the degree of the Gröbner basis elements is equal to $\text{reg}(I)$. On the other hand, if I is a Borel monomial ideal and the characteristic of K is zero, then $\text{reg}(I)$ is the maximal degree of the elements of the reduced Gröbner basis of the ideal with respect to the reverse lexicographic ordering (see [2, 3, 7, 10]).

In this paper, we introduce the notion of a homogeneous ideal in *quasi stable position* (QSP); a new definition for the notion of generic coordinates for computing efficiently the Castelnuovo-Mumford regularity of a homogeneous ideal. This definition is simple to check, because it is tested on the initial ideal for the degree reverse lexicographic ordering. It is explicit, because we provide an algorithm to decide whether a monomial ideal is in QSP or not within a quadratic complexity in input size. We show that, (if the base field is large enough) almost all linear change of variables transform an ideal into QSP. *We prove that the Castelnuovo-Mumford regularity of an ideal in QSP is the maximal degree of the elements of its reduced Gröbner basis with respect to the reverse lexicographic ordering.* Thus, QSP is a generalization of Borel.

From the computational point of view, there are three important ways to compute the Castelnuovo-Mumford regularity of an ideal.

- Computing the minimal free resolution which is infeasible in practice.
- Making a random linear change of variables in $K^{n(n+1)/2}$ to put the input ideal in Borel position and computing the Gröbner basis of the changed ideal for testing this position and reading the regularity ([3, 10]).
- Making a random linear change of variables in $K^{nd-d(d-1)/2}$ where d is the dimension of input ideal to put the input ideal in nested type position, computing the Gröbner basis of the changed ideal to test nested type and then computing the saturations of some monomial ideals associated to this ideal for computing the regularity ([5]).

Roughly speaking, QSP allows for having a method (to compute the regularity of an ideal) with the advantages of the second and third approach, making an incremental random linear change of variables in K^N with $N = n(n+1)/2 - (n-d-1)(n-d)/2$ (which is smaller than $K^{n(n+1)/2}$) then computing the Gröbner basis of the changed ideal for testing QSP (which is less expensive than Borel test and nested type test) and reading the regularity without further computation.

We have implemented an algorithm in the distributed library `noether.lib` [1] of SINGULAR [13] based on the above results to compute the Castelnuovo-Mumford regularity of an ideal. It makes an incremental (one variable after the other) random linear change of variables to put the input ideal in QSP. We compare its efficiency with the algorithm `regIdeal` (from library `mregular.lib`) of SINGULAR via some examples.

The main results of this paper are the following, in which $\text{in}(I)$ is the initial ideal of I , i.e., the ideal generated by the leading terms of the elements of I with respect to the reverse lexicographic ordering \prec with $x_n \prec x_{n-1} \prec \cdots \prec x_0$.

- QSP can be tested on the monomial ideal $\text{in}(I)$ instead of I .
- An algorithm (with a polynomial complexity) for determining whether a monomial ideal is in QSP is given.

- If the characteristic of K is zero, or sufficiently large, almost all linear changes of variables transform a homogeneous ideal in QSP.
- If I is in QSP, then $\text{reg}(I) = \text{reg}(\text{in}(I))$.
- If I is in QSP, then $\text{reg}(I)$ is the maximal degree of the elements of $\text{in}(I)$.
- If the characteristic of K is zero, or sufficiently large, one can compute efficiently the regularity of an ideal by putting it in QSP.

Now, we give the structure of the paper. In Sections 2, we describe an algorithm to compute the satiety of an ideal. In Section 3, we define the notion of a homogeneous ideal in QSP. In Section 4, we prove our main result on the Castelnuovo-Mumford regularity of an ideal in QSP, and we describe also an algorithm to compute the Castelnuovo-Mumford regularity of a homogeneous ideal. In Section 5, we show the performance of our algorithms with the existing algorithms of SINGULAR via some examples. Section 6 presents our conclusions.

2. SATIETY OF AN IDEAL

In this section we first introduce a new notion of generic coordinates (sat-generic position) to compute the satiety of an ideal. Then, we describe an algorithm to compute the satiety of a general ideal.

We recall first the definition of the saturation and the satiety of a homogeneous ideal. Let I be a homogeneous ideal of the polynomial ring $R = K[x_0, \dots, x_n]$ where K is an arbitrary field. If $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$ is the unique maximal homogeneous ideal of R , then the ideal $I : \mathfrak{m}^i$ is defined for any positive integer i as

$$I : \mathfrak{m}^i = \{F \in R \mid \forall G \in \mathfrak{m}^i, GF \in I\}.$$

The ideal $I : \mathfrak{m}^\infty$ is defined $\bigcup_{i=1}^\infty I : \mathfrak{m}^i$. Denote by I_ℓ the set of homogeneous elements of degree ℓ of I .

Proposition 2.1 (Definition). *The ideal $I^{\text{sat}} = I : \mathfrak{m}^\infty$ is called the saturation of I and it is equal to the unique largest ideal $J \subset R$ having the following property:*

$$\exists m \text{ s.t. } \forall \ell \geq m \quad J_\ell = I_\ell.$$

Proof. One can see easily that the saturation of I satisfies this property. □

Definition 2.1. The *satiety* of I , denoted by $\text{sat}(I)$, is the smallest positive integer m such that $I_\ell = I_\ell^{\text{sat}}$ for all $\ell \geq m$.

Now, we show that if the last variable is *generic* for an ideal, then its satiety is the same as that of its initial ideal with respect to the reverse lexicographical ordering. For this, we need the following lemmas from [3] in which a linear form $y \in R$ is generic for a homogeneous ideal $I \subset R$ if y is a nonzero divisor in R/I^{sat} .

Lemma 2.1. *Let $I \subset R$ be a homogeneous ideal and let $y \in R$ be a linear form. Then the following conditions are equivalent:*

- (1) $(I : y)_\ell = I_\ell$ for any $\ell \geq m$.
- (2) $\text{sat}(I) \leq m$ and y is generic for I .

Lemma 2.2. *Let $I \subset R$ be a homogeneous ideal and m a natural number. Then*

$$(I : x_n)_m = I_m \Leftrightarrow (\text{in}(I) : x_n)_m = \text{in}(I)_m.$$

Proposition 2.2. *Let x_n be generic for a homogeneous ideal $I \subset R$. Then $\text{sat}(I) = \text{sat}(\text{in}(I))$.*

Proof. We have $(I : x_n)_\ell = I_\ell$ for any $\ell \geq \text{sat}(I)$ by Lemma 2.1. Thus, $(\text{in}(I) : x_n)_\ell = \text{in}(I)_\ell$ for any $\ell \geq \text{sat}(I)$ by Lemma 2.2, and this implies that $\text{sat}(\text{in}(I)) \leq \text{sat}(I)$ using Lemma 2.1. To prove the other inequality, the same proof works using the fact that x_n is also generic for $\text{in}(I)$ by Lemma 2.2. \square

The following corollary was inspired by [5], Proposition 2.1.

Corollary 2.1. *Let x_n be generic for a homogeneous ideal $I \subset R$. Then*

$$\text{sat}(I) = \max_{m \in (\text{in}(I) : x_n) \setminus \text{in}(I)} \{\deg(m)\} + 1.$$

Proof. We have $\text{sat}(I)$ is equal to $\text{sat}(\text{in}(I))$ by Proposition 2.2. On the other hand, from hypothesis x_n is also generic for $\text{in}(I)$ (Lemma 2.2). Thus, $(\text{in}(I) : x_n)_\ell = \text{in}(I)_\ell$ for any $\ell \geq \text{sat}(\text{in}(I))$ by Lemma 2.1, and the satiety of $\text{in}(I)$ is the smallest degree from which $\text{in}(I) : x_n$ and $\text{in}(I)$ are equal. This is equal to the maximum degree of the elements of $(\text{in}(I) : x_n) \setminus \text{in}(I)$ plus one which proves the assertion. \square

We introduce now a new notion to check whether the last variable is *sufficiently* generic for an ideal to compute its satiety. A homogeneous ideal I is in *sat-generic position* if the quotient of $\text{in}(I)$ with respect to the maximal homogeneous ideal is equal to its quotient with respect to the last variable. The following corollary results easily from Proposition 2.2 and Corollary 2.1.

Corollary 2.2. *Let $I \subset R$ be a homogeneous ideal in sat-generic position. Then $\text{sat}(I) = \text{sat}(\text{in}(I))$ and this is equal to the maximal degree of the elements of $\text{in}(I)$ containing x_n .*

We describe also an algorithm to check this definition for a monomial ideal.

Algorithm 1 sat-generic-test

Input: $J \subset R$ a monomial ideal
Output: The answer of “Is J in sat-generic position?”
 $G :=$ a minimal system of generators for ideal J
for each $x_0^{e_0} \cdots x_n^{e_n} \in G$ with $e_n > 0$ **do**
 for $j = 0, \dots, n-1$ **do**
 if $x_0^{e_0} \cdots x_{n-1}^{e_{n-1}} x_n^{e_n-1} x_j \notin G$ **then**
 return “No”
 end if
 end for
end for
return “Yes”

Remark 2.1. The complexity of this algorithm is quadratic in input size.

In the rest of this section, we suppose that the characteristic of K is zero, or sufficiently large. We will prove that almost all linear changes of the last variable transform a homogeneous ideal into sat-generic position. For this we recall some preliminaries from [7], page 352. Let R_m be the K -vector space consisting of all forms of degree m . If V is a t -dimensional subspace of R_m and f_1, \dots, f_t is a K -basis of V , then the t -exterior power $\wedge^t V$ of V is equal to Kf where $f = f_1 \wedge \cdots \wedge f_t$. If f_1, \dots, f_t and g_1, \dots, g_t are linearly independent systems in R^m , then generate

the same K -subspace iff $f_1 \wedge \cdots \wedge f_t = \lambda g_1 \wedge \cdots \wedge g_t$ for some nonzero $\lambda \in K$. If m_1, \dots, m_t are distinct monomials in R_m , then $m_1 \wedge \cdots \wedge m_t$ is called a *monomial* of $\wedge^t R_m$. Such a monomial is called *normal expression* if $m_t \prec \cdots \prec m_1$. We order the monomials of $\wedge^t R_m$ by ordering their normal expressions lexicographically. That is, $m_1 \wedge \cdots \wedge m_t < m'_1 \wedge \cdots \wedge m'_t$ iff $m_i \prec m'_i$ for the smallest index i such that $m'_i \neq m_i$. Then for each $f \in \wedge^t R_m$, we can consider its initial monomial $\text{in}(f)$.

Let $I \subset R$ be a homogeneous ideal. For all $\Lambda = (a_0, \dots, a_{n-1})$ in K^n , denote by $\Lambda(I)$ the result of the substitution $x_n \mapsto x_n + a_{n-1}x_{n-1} + \cdots + a_0x_0$ in I .

Recall that the *Zariski topology* on K^n is the topology which has the algebraic sets as closed sets, i.e., a set V is closed if and only if there exists a (finite) family of polynomials $f_1, \dots, f_k \in K[a_0, \dots, a_{n-1}]$ such that

$$V = \{\Lambda \in K^n \mid f_1(\Lambda) = \cdots = f_k(\Lambda) = 0\}.$$

It is well known that a nonempty Zariski open set is dense. Under these assumptions, by a restriction to the last variable of Galligo's theorem [10] (see also [7], Theorem 15.18) we have the following theorem.

Theorem 2.1. *Let K be an infinite field. Then, there is a monomial ideal $J \subset R$ and a Zariski open set $U \subset K^n$ such that $\text{in}(\Lambda(I)) = J$ for each $\Lambda \in U$. Moreover, for each $m \geq 0$, if J_m has dimension t , $\wedge^t J_m$ is generated by $\max\{\text{in}(f) \mid f \in \wedge^t \Lambda(I)_m, \Lambda \in K^n\}$.*

Proof. Let f_1, \dots, f_t be a basis for I_m and $f = f_1 \wedge \cdots \wedge f_t$. For any $\Lambda = (a_0, \dots, a_{n-1})$ we have $\Lambda(f) = \Lambda(f_1) \wedge \cdots \wedge \Lambda(f_t)$ is a linear combination of monomials in $\wedge^t R_m$. Let $p(a_0, \dots, a_{n-1}) \in K[a_0, \dots, a_{n-1}]$ be the coefficient of $\text{in}(\Lambda(f))$. Now set

$$U_m = \{\Lambda \in K^n \mid p(a_0, \dots, a_{n-1}) \neq 0\}.$$

Let J_m be the subspace of R_m generated by m_1, \dots, m_t where $\text{in}(\Lambda(f)) = m_1 \wedge \cdots \wedge m_t$. Set $J = \sum_{m \geq 1} J_m$ and $U = \bigcap_{m=1}^\infty U_m$. We now show that J is an ideal and U is a Zariski open set. To show that J is an ideal, it suffices to prove that $R_i J_m \subset R_{m+i}$ for any i and m . Note that U_m and U_{m+i} are dense, therefore $U_m \cap U_{m+i} \neq \emptyset$. Pick $\Lambda \in U_m \cap U_{m+i}$, it follows that $\text{in}(\Lambda(I))_m = J_m$ and $\text{in}(\Lambda(I))_{m+i} = J_{m+i}$. Since $\text{in}(\Lambda(I))$ is an ideal, we can conclude the desired inclusion. Also, the ideal J satisfies the last statement of the theorem.

We will show that U is a Zariski open set of K^n . Since each U_m is Zariski open, it is enough to show that U is equal to finite intersection of the U_m . Suppose that J is generated by m_1, \dots, m_s . Let e be the highest degree of these monomials. We claim that $U = \bigcap_{m=1}^e U_m$. Take $\Lambda \in \bigcap_{m=1}^e U_m$, $\text{in}(\Lambda(I))_m = J_m$ for any $m \leq e$, hence $J \subset \text{in}(\Lambda(I))$. Since $\dim_K J_m = \dim_K I_m = \dim_K \text{in}(\Lambda(I))_m$ for every m we have $J = \text{in}(\Lambda(I))$ and thus $\Lambda \in U$. The other inclusion is trivial. This argument was a rewriting of the proof of [7], Theorem 15.18. \square

Corollary 2.3. *Let the characteristic of K be zero, or sufficiently large. With the notation of Theorem 2.1, J is in sat-generic position (and therefore almost all linear changes of the last variable transform I in sat-generic position).*

Proof. We have to show that if $x_n m \in J$, then $x_i m \in J$. Let $\Lambda_i = (0, \dots, 0, 1, 0, \dots, 0) \in K^n$ such that the i th element is one and the others are zero. So, it is enough to prove that $\Lambda_i(J_s) = J_s$ with $s = \deg(m) + 1$. Also, let Λ be an element of K^n such that $\text{in}(\Lambda(I)) = J$. Choose a basis f_1, \dots, f_t for $\Lambda(I)_s$. Set $f = f_1 \wedge \cdots \wedge f_t$. Thus $\text{in}(f) = \text{in}(f_1) \wedge \cdots \wedge \text{in}(f_t)$.

If $\Lambda_i(J_s) \neq J_s$, then $\Lambda_i(\text{in}(f)) \neq \text{in}(f)$. By definition of $\Lambda_i(\text{in}(f))$, its terms (other than $\text{in}(f)$) are strictly greater than $\text{in}(f)$. We will show that for a suitable $\Lambda' \in K^n$, a monomial (with nonzero coefficient) strictly greater than $\text{in}(f)$ appears in $\Lambda'(f)$. This contradicts the last statement of Theorem 2.1, proving the assertion.

Let Λ' be the vector Λ_i by replacing 1 by c where c is a generic element of K . Let $f_i = c_i x_n^{p_i} f'_i$ for any i where p_i is a nonnegative integer and $f'_i \in R$ is a polynomial whose leading coefficient is 1. Let m be the normal expression form of $x_i^{p_i} \text{in}(f'_1) \wedge \cdots \wedge x_i^{p_t} \text{in}(f'_t)$. It is trivial that m appears in $\Lambda'(f)$. We claim that m is strictly greater than $\text{in}(f)$ and it has a nonzero coefficient if $c \in K$ is sufficiently generic. From supposition and definition of orderings, it is clear that m is strictly greater than $\text{in}(f)$. To prove the other part of claim, let $p(c)$ be the coefficient of m in f . One can see easily that $c_1 \cdots c_t c^{p_1 + \cdots + p_t}$ is the leading coefficient of $p(c)$ (which is nonzero). By the assumption on the characteristic of K , it follows that for a sufficiently generic value of c , the value of $p(c)$ is nonzero and this ends the proof. This argument was inspired by the proof of [7], Theorem 15.20. \square

Example 2.1. If K is finite and its characteristic is not sufficiently large, Corollary 2.3 may be false. For example, let the characteristic of K be two and $I = \langle x^2, y^2 \rangle \subset K[x, y]$. Then, any linear change of coordinates transforms I to itself which is not in sat-generic position, because $I : y = \langle x^2, y \rangle$ is not equal to $I : \langle x, y \rangle = \langle x^2, xy, y^2 \rangle$.

Finally, we describe our algorithm to compute the satiety of a general ideal. For this, we denote by $D(n, J)$ the maximal degree of the elements of a minimal system of generators of a monomial ideal J containing x_n . The correctness and termination of this algorithm follow, respectively, from Corollaries 2.2 and 2.3. We have implemented this algorithm in SINGULAR and we compare its running time with `satiety` and `satiety(., 1)` (from the library `mregular.lib`) of SINGULAR in Section 5.

Algorithm 2 sat

Input: $I \subset R$ a homogeneous ideal

Output: The satiety of I

$J := \text{in}(I)$

while sat-generic-test(J)="No" **do**

$\psi :=$ The map $x_n \mapsto x_n + a_{n-1}x_{n-1} + \cdots + a_0x_0$ with $a_i \in K$ a random element

$J := \text{in}(\psi(I))$

end while

return $D(n, J)$

3. QUASI STABLE POSITION

In this section, we define the new notion of a homogeneous ideal in quasi stable position (QSP). We show also the relationship between this notion and the other existing notions of generic coordinates for computing the Castelnuovo-Mumford regularity of a homogeneous ideal.

To define the notion QSP, we need the following sequence of ideals associated to an ideal in which J is a monomial ideal of the ring $R = K[x_0, \dots, x_n]$ and $\mathfrak{m}_i = \langle x_0, \dots, x_i \rangle$ is the unique maximal ideal of the ring $R_i = K[x_0, \dots, x_i]$.

Notation 3.1. Let $\overline{\text{sec}}(J, 0) = J$ and $\overline{\text{sec}}(J, i) = J|_{x_{n-i+1}=\dots=x_n=0} \cap R_{n-i}$ for $i = 1, \dots, n$.

Definition 3.1. A monomial ideal J is in *quasi stable position* (QSP) if we have $\overline{\text{sec}}(J, i) : \mathfrak{m}_{n-i} = \overline{\text{sec}}(J, i) : x_{n-i}$ for $i = 0, \dots, \dim(J)$.

Definition 3.2. A homogeneous ideal $I \subset R$ is in *quasi stable position* (QSP) if $\text{in}(I)$ is in QSP.

We note an immediate consequence of these definitions.

Corollary 3.1 (QSP-test). *Let J be a monomial ideal and $w(m)$ the maximal integer ℓ such that x_ℓ divides a monomial m . The following conditions are equivalent:*

- J is in QSP.
- For all monomial $m \in J$, if $w(m) \geq n - \dim(J)$ we have $x_i(m/x_{w(m)}) \in J$ for all $i < w(m)$.

Using this test, we give an algorithm to decide whether a monomial ideal is in QSP.

Algorithm 3 QSP-test

Input: $J \subset R$ a monomial ideal
Output: The answer of “Is J in QSP?”
 $G := \{m_1, \dots, m_k\}$ a minimal system of generators for J
 $\text{Deg} := \max\{\text{deg}(m_1), \dots, \text{deg}(m_k)\}$
 $d :=$ smallest ℓ such that $x_i^{\text{Deg}} \in J$ for $i = 0, \dots, n - \ell$
for each $x_0^{e_0} \cdots x_h^{e_h} \in G$ with $h \geq n - d$ and $e_h > 0$ **do**
 for $j = 1, \dots, h - 1$ **do**
 if $x_0^{e_0} \cdots x_{h-1}^{e_{h-1}} x_h^{e_h-1} x_j \notin G$ **then**
 return “No”
 end if
 end for
end for
return “Yes”

Proof of algorithm QSP-test. The termination of the algorithm is obvious. Let us show its correctness. For this, we have to prove that J is in QSP if and only if the response of the algorithm is “Yes”. Suppose that J is in QSP and $x_0^{e_0} \cdots x_h^{e_h} \in G$ for some $h \geq n - d$ with $e_h > 0$. This implies that

$$x_0^{e_0} \cdots x_{h-1}^{e_{h-1}} x_h^{e_h-1} \in J : x_h = J : \langle x_0, \dots, x_h \rangle.$$

Thus, $x_0^{e_0} \cdots x_{h-1}^{e_{h-1}} x_j$ must be in J for any $j = 0, \dots, h - 1$, and the answer is “Yes”. In this case, d is the dimension of J by the Noether normalization test (see [4], Lemma 3.1). Conversely, we can conclude that $J : x_h \subset J : \langle x_0, \dots, x_h \rangle$, and therefore J is in QSP by definition of the notation. \square

Remark 3.1. In this algorithm, the integer d is the dimension of J if the answer is “Yes” (see the proof of algorithm).

Proposition 3.1. *The complexity of this algorithm is quadratic in kn .*

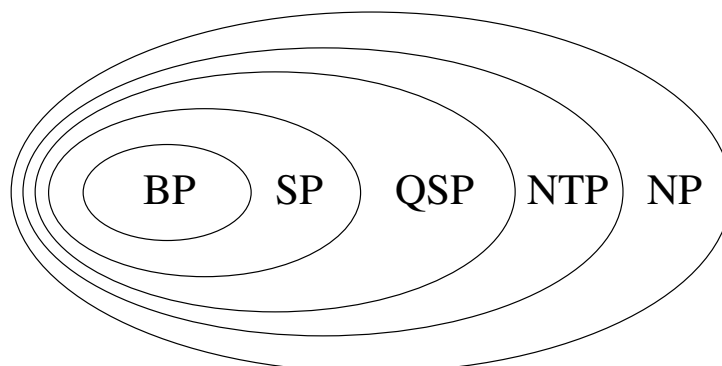


FIGURE 1. Classes of ideals

Proof. One can easily see that the number of operations in the loop “For” is kn^2 . Thus the complexity of the algorithm is polynomial in kn . \square

Example 3.1. There is a nontrivial homogeneous ideal in QSP. Set $R = \mathbb{Q}[x, y, z, t, u, v, h]$. Let I be the homogenization of the 5-th Katsura ideal with respect to h , i.e., the homogenization of the ideal generated by $2x^2 + 2y^2 + 2z^2 + 2t^2 + 2u^2 + v^2 - v, xy + yz + 2zt + 2tu + 2uv - u, 2xz + 2yt + 2zu + u^2 + 2tv - t, 2xt + 2yu + 2tu + 2zv - z, t^2 + 2xv + 2yv + 2zv - y$ and $2x + 2y + 2z + 2t + 2u + v - 1$. Its dimension is 1, and its initial ideal, $\text{in}(I)$, is generated by $x, t^2, zt, yt, yz, y^2, u^3, tu^2, zu^2, yu^2, z^2u, z^3, u^2v^2, tuv^2, zuv^2, yuv^2, z^2v^2, uv^4, tv^4, zv^4, yv^4, v^6$. One can easily check that this initial ideal is in QSP by the above algorithm, and therefore I is in QSP by definition.

We now compare QSP with some other notions of generic coordinates. Recall that a monomial ideal $J \subset R$ is in *nested type position* (NTP) if $\overline{\text{sec}}(J, i)^{\text{sat}} = \overline{\text{sec}}(J, i) : x_{n-i}^\infty$ for $i = 0, \dots, \dim(J) - 1$ (see [5], Proposition 3.2).

Lemma 3.1. Any monomial ideal in QSP is in NTP.

Proof. Let J be a monomial ideal in QSP. We have to show that any monomial $m \in \overline{\text{sec}}(\text{in}(J), i) : x_{n-i}^\infty$ belongs to $\overline{\text{sec}}(\text{in}(J), i)^{\text{sat}}$ for any $i < \dim(J)$. Let $x_{n-i}^k m \in \overline{\text{sec}}(\text{in}(J), i)$ for some integers k and i . From hypothesis we have $x_j^k m \in \overline{\text{sec}}(\text{in}(J), i)$ for any $j < n - i$, and this follows the assertion. \square

Recall that a monomial ideal $J \subset R$ is *Borel* if it is fixed under the action of the upper triangular invertible matrices in $GL(n + 1, K)$. Provided the characteristic of K is zero, a monomial ideal $J \subset R$ is Borel if and only if $x_j m \in J$ implies that $x_i m \in J$ for all $i < j$. Also, there is another class of monomial ideals called *stable ideals* (see [9]). A monomial ideal $J \subset R$ is stable if the equality of Definition 3.1 is true for $i = 0, \dots, n - 1$. Thus, in characteristic zero a Borel monomial ideal is a stable ideal, and in any characteristic a stable ideal is in QSP. These imply that QSP is a notion between stable and NTP. In Figure 1, we compare the different notions of generic coordinates in which BP (resp. SP, NTP and NP) shows the class of the ideals whose initial ideals are in Borel position (resp. stable position, NTP and Noether position). Recall that a monomial ideal J is in Noether position if a pure power of x_i belongs to J for $i = 0, \dots, n - \dim(J)$ (see [4]). Note that QSP and NTP are not equivalent. For example, the ideal $\langle x^2, y^2 \rangle \subset \mathbb{Q}[x, y]$ is in NTP,

but it is not in QSP. Also, QSP is not equivalent to SP, because, for example, the initial ideal of Example 3.1 is an ideal in QSP but it is not in SP (this is also true for the initial ideal of the 6, 7, 8, 9 and 10-th Katsura ideals).

In the rest of this section we suppose that the characteristic of K is zero, or sufficiently large. We show that for almost all linear change of variables an ideal is in QSP. Let $I \subset R$ be a homogeneous ideal of dimension d . For all $\Lambda = (a_1, \dots, a_N) \in K^N$ with $N = n(n + 1)/2 - (n - d - 1)(n - d)/2$, denote by $\Lambda(I)$ the result of the following substitution in I :

$$\begin{aligned} x_n &\mapsto x_n + a_1x_{n-1} + a_2x_{n-2} + \dots + a_nx_0, \\ x_{n-1} &\mapsto x_{n-1} + a_{n+1}x_{n-2} + \dots + a_{2n-1}x_0, \\ &\vdots \\ x_{n-d} &\mapsto x_{n-d} + a_{N-n+d+1}x_{n-d-1} + \dots + a_Nx_0. \end{aligned}$$

Under these assumptions, we can conclude the following.

Corollary 3.2. *Let the characteristic of K be zero, or sufficiently large. There is a monomial ideal $J \subset R$ and a Zariski open set $U \subset K^N$ such that $\text{in}(\Lambda(I)) = J$ for each $\Lambda \in U$. Moreover, J is in QSP (and thus almost all linear changes of the variables x_n, \dots, x_{n-d} put I in QSP).*

Proof. The assertion follows by applying Corollary 2.3 to the variables x_n, \dots, x_{n-d} . □

For finite fields, this is not always true. For instance, in characteristic $p > 0$, there is no linear change of variables to put the ideal $\langle x^p, y^p \rangle$ in QSP. Note that the fact that the characteristic of K is small causes this problem.

4. CASTELNUOVO-MUMFORD REGULARITY OF AN IDEAL IN QSP

In this section, we prove the main result of this paper: the Castelnuovo-Mumford regularity of an ideal in QSP is the maximal degree of the elements of its reduced Gröbner basis with respect to the reverse lexicographic ordering. We describe also an algorithm based on this result to compute the Castelnuovo-Mumford regularity of a general homogeneous ideal.

We prove first that the maximal degree of the elements of the reduced Gröbner basis of an ideal in QSP is less than or equal to its Castelnuovo-Mumford regularity (Proposition 4.2).

Lemma 4.1 ([3]). *Let $I \subset R$ be a homogeneous ideal, and $y \in R$ a generic linear form for I . The following conditions are equivalent:*

- $\text{reg}(I) \leq m$.
- $\text{sat}(I) \leq m$ and $\text{reg}(I + \langle y \rangle) \leq m$.

Lemma 4.2 ([3]). *Let $I \subset R$ be a zero-dimensional homogeneous ideal. The following conditions are equivalent:*

- $\text{sat}(I) \leq m$.
- $\text{reg}(I) \leq m$.
- $I_m = R_m$.

Corollary 4.1. *With the same hypothesis $\text{reg}(I) = \text{sat}(I)$.*

Lemma 4.3 ([12]). *Let $I \subset R$ homogeneous ideal. Then*

$$\operatorname{reg}(I) = \max\{\operatorname{sat}(I), \operatorname{reg}(I^{\operatorname{sat}})\}.$$

We need some more notation as follows: We define $J \subset R$ to be a monomial ideal. Let $\operatorname{sec}(J, 0) = J$ and $\operatorname{sec}(J, i) = J + \langle x_{n-i+1}, \dots, x_n \rangle$ for $i = 1, \dots, n$.

Proposition 4.1. *Let $J \subset R$ be a monomial ideal in QSP and let $d = \dim(J)$, then*

$$\begin{aligned} (1) \quad \operatorname{reg}(J) &= \max_{0 \leq i \leq d} \{\operatorname{sat}(\operatorname{sec}(J, i))\} \\ (2) \quad &= \max_{0 \leq i \leq d} \{\operatorname{sat}(\overline{\operatorname{sec}}(J, i))\}. \end{aligned}$$

Proof. To prove the equality (1), from Lemma 4.1 we have

$$\operatorname{reg}(J) = \max\{\operatorname{sat}(J), \operatorname{reg}(\operatorname{sec}(J, 1))\}.$$

By reusing this formula for the ideal $\operatorname{sec}(J, 1)$ and using the fact that x_{n-1} is generic for $\operatorname{sec}(J, 1)$ we obtain

$$\begin{aligned} \operatorname{reg}(J) &= \max\{\operatorname{sat}(J), \max\{\operatorname{sat}(\operatorname{sec}(J, 1)), \operatorname{reg}(\operatorname{sec}(J, 2))\}\} \\ &= \max\{\operatorname{sat}(J), \operatorname{sat}(\operatorname{sec}(J, 1)), \operatorname{reg}(\operatorname{sec}(J, 2))\}. \end{aligned}$$

So by an induction, we can conclude that $\operatorname{reg}(J)$ is equal to

$$\max\{\operatorname{sat}(J), \operatorname{sat}(\operatorname{sec}(J, 1)), \dots, \operatorname{sat}(\operatorname{sec}(J, d-1)), \operatorname{reg}(\operatorname{sec}(J, d))\}.$$

Since $\operatorname{sec}(J, d)$ is zero-dimensional, then $\operatorname{reg}(\operatorname{sec}(J, d)) = \operatorname{sat}(\operatorname{sec}(J, d))$ (Corollary 4.1), and this proves the assertion.

Let us prove (2). It is enough to show that $\operatorname{sat}(\overline{\operatorname{sec}}(J, i)) = \operatorname{sat}(\operatorname{sec}(J, i))$ for any i . By the membership $x_{n-i+1}, \dots, x_n \in \operatorname{sec}(J, i)$ we have

$$\overline{\operatorname{sec}}(J, i)^{\operatorname{sat}} = \operatorname{sec}(J, i)^{\operatorname{sat}}|_{x_{n-i+1}=\dots=x_n=0} \cap R_{n-i}$$

by definition of the saturation of an ideal, and this proves the assertion. \square

Lemma 4.4. *If $I \subset R$ is a homogeneous ideal in QSP then $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I))$.*

Proof. Following [3], a sequence $y_1, \dots, y_i \in R$ of linear forms is called generic for an ideal $I \subset R$ if y_j is generic for $I + \langle y_1, \dots, y_{j-1} \rangle$ for any $j = 1, \dots, i$. By hypothesis the sequence $x_n, \dots, x_{n-\dim(I)+1}$ is generic for $\operatorname{in}(I)$. Thus the assertion follows from [3], Theorem 2.4. \square

Proposition 4.2. *Let $I \subset R$ be a homogeneous ideal in QSP. The maximal degree of the elements of the reduced Gröbner basis of I is less than or equal to $\operatorname{reg}(I)$.*

Proof. Using Lemma 4.4, it is enough to prove the assertion for $J = \operatorname{in}(I)$ which is a monomial ideal. Let us consider $G = \{m_1, \dots, m_s\}$ a minimal system of generators of J and a monomial m_j in G . Let x_i be the greatest variable which divides m_j . Two cases are possible: If $i \in \{n-d+1, \dots, n\}$ with $d = \dim(J)$, these assumptions imply that $m_j/x_i \in \overline{\operatorname{sec}}(J, n-i) : x_i$ and $m_j/x_i \notin \overline{\operatorname{sec}}(J, n-i)$. In addition, $\overline{\operatorname{sec}}(J, n-i)^{\operatorname{sat}} = \overline{\operatorname{sec}}(J, n-i) : x_i^\infty$ because J is in QSP. This implies that $m_j/x_i \in \overline{\operatorname{sec}}(J, n-i)^{\operatorname{sat}}$. The facts that $m_j/x_i \in \overline{\operatorname{sec}}(J, n-i)^{\operatorname{sat}}$ and $m_j/x_i \notin \overline{\operatorname{sec}}(J, n-i)$ result in $\deg(m_j) \leq \operatorname{sat}(\operatorname{sec}(J, n-i))$ by definition. Thus $\deg(m_j)$ is less than or equal to $\operatorname{reg}(J)$ by Proposition 4.1.

In the other case, i.e., $i \in \{0, \dots, n-d\}$, we use the facts that $\overline{\operatorname{sec}}(J, d)$ is zero-dimensional and its regularity is less than or equal to $\operatorname{reg}(J)$ (Lemma 4.1 and the

fact that J is in QSP). Suppose ad absurdum that $\deg(m_j) > \text{reg}(J)$ for some j . Let m be a monomial of degree $\text{reg}(J)$ which divides m_j . Thus $m \in \overline{\text{sec}}(J, d)$ by Lemma 4.2, and $m \in J$ by supposition. This is a contradiction with minimality of G which follows that $\deg(m_j) \leq \text{reg}(J)$, and this proves the assertion. \square

Bayer and Stillman [3] have proved that if the characteristic of K is zero, then the Castelnuovo-Mumford regularity of a Borel monomial ideal is the maximal degree of the elements of its reduced Gröbner basis with respect to the reverse lexicographic ordering. In the following, we generalize this result to any characteristic and the ideals in QSP.

Theorem 4.1. *The Castelnuovo-Mumford regularity of an ideal in QSP is the maximal degree of the elements of its reduced Gröbner basis with respect to the reverse lexicographic ordering.*

Proof. Let $I \subset R$ be an ideal in QSP. Also, let D be the maximal degree of the elements of $\text{in}(I)$. From Proposition 4.2, we have $D \leq \text{reg}(I)$. To prove the other inequality, from Proposition 4.1 (2) and Lemma 4.4 we have $\text{reg}(I) = \text{reg}(\text{in}(I)) \leq \text{sat}(\overline{\text{sec}}(\text{in}(I), i))$ for some i with $0 \leq i \leq \dim(I)$. From hypothesis, $\overline{\text{sec}}(\text{in}(I), i)$ is in QSP, and its satiety is equal to the maximal degree of the elements of $\text{in}(I)$ containing x_{n-i} by Corollary 2.2. Thus, $\text{reg}(I) \leq D$. \square

Algorithm 4 reg

Input: $I \subset R$ a homogeneous ideal
Output: The Castelnuovo-Mumford regularity of I
 $J := \text{in}(I)$
if QSP-test(J)="Yes" **then**
 return Maximal degree of the elements of J
end if
while sat-generic-test(J)="No" **do**
 $\psi :=$ The map $x_n \mapsto x_n + a_{n-1}x_{n-1} + \cdots + a_0x_0$ with $a_i \in K$ a random element
 $J := \text{in}(\psi(I))$
end while
 $r := D(n, J)$
 $d := \dim(I)$
for $i = n, \dots, n - d + 1$ **do**
 $I := I|_{x_i=0} \subset K[x_0, \dots, x_{i-1}]$
 $J := \text{in}(I)$
 while sat-generic-test(J)="No" **do**
 $\psi :=$ The map $x_{i-1} \mapsto x_{i-1} + a_{i-2}x_{i-2} + \cdots + a_0x_0$ with $a_i \in K$ a random element
 $J := \text{in}(\psi(I))$
 end while
 $r := \max\{r, D(i - 1, J)\}$
end for
return r

This algorithm is based on the above theorem for computing the Castelnuovo-Mumford regularity of a general ideal by an incremental (one variable after the

other) random linear change of variables to put the input ideal in QSP. We denote by $D(i, J)$ the maximal degree of the elements of a minimal system of generators of a monomial ideal J containing x_i . The correctness and termination of this algorithm follow, respectively, from Theorem 4.1 and Corollary 3.2. We have implemented this algorithm in the distributed library `noether.lib` [1] of SINGULAR and we compare its running time with the function `regIdeal` (from the library `mregular.lin`) of SINGULAR in Section 5.

5. EXPERIMENTS AND REMARKS

We have implemented `sat` and `reg` algorithms with the computer algebra system SINGULAR (version 3-0-1). We have also implemented the `msat` (resp. `mreg`) algorithm which is a modular version of `sat` (resp. `reg`) algorithm, i.e., it computes the initial ideal modulo 2147483647 (the biggest prime number less than 2^{31}) with a high probability. All of these algorithms are contained in the distributed library `noether.lib` [1].

We choose SINGULAR since it has many functionalities needed which are related to a multivariate polynomial ring. Also, we want to be able to compare our algorithms with the existing functions: `satiety`, `satiety(.,1)` and `regIdeal` of SINGULAR.

For this experiment, we use some examples from [6], the Posso list¹ and [5]. For the polynomial systems which are not homogeneous, we use their homogenization. The results are shown in Table 1. All of the computations are done over \mathbb{Q} . The monomial ordering is always degree reverse lexicographical ordering. In this table, Example 1 is the ideal of the ring $\mathbb{Q}[x, y, z]$ generated by $8x^2y^2 + 5xy^3 + 3x^3z + x^2yz, x^5 + 2y^3z^2 + 13y^2z^3 + 5yz^4, 8x^3 + 12y^3 + xz^2, 7x^2y^4 + 18xy^3z^2 + y^3z^3$. Examples 2, 3, 5, 8, 9, 10, 12, 14, 15, 16, 17 and 18 are, respectively, Siebert, Macaulay, Schwarz, Cassou, Horrocks, Huneke, Amrhein2, Gerdt, Möller, Sturmfels/Eisenbud, `mat3`² and Butcher Examples (see [6]), and Examples 4, 6, 7 and 11 are, respectively, Bronstein2, Cyclic roots 5, Katsura4 and Katsura5 Examples from the Posso list. Examples 13 and 20 are respectively Examples 4.13 and 4.10 from [5]. Example 19 is Shimoyama/Yokoyama example and it is the ideal generated by the polynomials $-j^9 + 9j^8a - 36j^7a^2 + 84j^6a^3 - 126j^5a^4 + 126j^4a^5 - 84j^3a^6 + 36j^2a^7 - 9ja^8 + a^9, -bj^8 + 8bj^7a + j^8a - 28bj^6a^2 - 8j^7a^2 + 56bj^5a^3 + 28j^6a^3 - 70bj^4a^4 - 56j^5a^4 + 56bj^3a^5 + 70j^4a^5 - 28bj^2a^6 - 56j^3a^6 + 8bj^2a^7 + 28j^2a^7 - ba^8 - 8ja^8 + a^9, cj^7 - 7cj^6a - j^7a + 21cj^5a^2 + 7j^6a^2 - 35cj^4a^3 - 21j^5a^3 + 35cj^3a^4 + 35j^4a^4 - 21cj^2a^5 - 35j^3a^5 + 7cj^2a^6 + 21j^2a^6 - ca^7 - 7ja^7 + a^8, -dj^6 + 6dj^5a + j^6a - 15dj^4a^2 - 6j^5a^2 + 20dj^3a^3 + 15j^4a^3 - 15dj^2a^4 - 20j^3a^4 + 6dja^5 + 15j^2a^5 - da^6 - 6ja^6 + a^7, ej^5 - 5ej^4a - j^5a + 10ej^3a^2 + 5j^4a^2 - 10ej^2a^3 - 10j^3a^3 + 5eja^4 + 10j^2a^4 - ea^5 - 5ja^5 + a^6, -fj^4 + 4fj^3a + j^4a - 6fj^2a^2 - 4j^3a^2 + 4fja^3 + 6j^2a^3 - fa^4 - 4ja^4 + a^5, gj^3 - 3gj^2a - j^3a + 3gja^2 + 3j^2a^2 - ga^3 - 3ja^3 + a^4, -hj^2 + 2hja + j^2a - ha^2 - 2ja^2 + a^3, ij - ia - ja + a^2$ in the ring $\mathbb{Q}[a, b, c, d, e, f, g, h, i, j]$.

In Table 1, the column `var` (resp. `dim`, `sat`, `reg` and `ncv`) shows the number of variables of the ring (resp. the dimension, the satiety, the Castelnuovo-Mumford regularity of the ideal and the number of variables which are needed to change to put the ideal in QSP). The other columns show the timing of computing by the corresponding algorithm in which timing is measured in seconds (with a precision of hundredths of a second) and the symbol ∞ means more than 24 hours. The

¹<http://www-sop.inria.fr/saga/POL/BASE/3.posso/>

TABLE 1. Comparison of `msat`, `sat`, `satiety` and `satiety(.,1)` and of `mreg`, `reg` and `regIdeal` algorithms

Example	var	dim	sat	msat	sat	satiety	satiety(.,1)	reg	ncv	mreg	reg	regIdeal
1	3	1	8	0.01	0.01	0.06	0.01	8	0	0.01	0.01	0.01
2	5	2	9	0.03	0.11	0.06	8.62	9	2	0.06	*0.80	11.79
3	5	3	0	0.00	0.00	0.01	0.43	7	2	0.02	0.02	0.43
4	5	3	0	0.00	0.00	0.01	0.04	6	2	0.02	0.03	0.05
5	6	1	0	0.01	0.26	0.01	0.18	6	1	0.02	0.15	0.32
6	6	1	0	0.03	5.50	0.21	5.80	11	1	0.06	3.96	5.98
7	6	1	0	0.00	0.00	0.15	0.01	5	0	0.01	0.01	0.01
8	6	2	0	0.43	11.17	*0.37	∞	16	3	1.58	*2420.72	∞
9	6	3	0	0.03	0.11	0.02	1.16	5	3	0.11	0.34	1.28
10	6	3	0	0.02	0.03	0.66	∞	42	3	7.98	∞	∞
11	7	1	0	0.01	*0.05	4.59	0.06	6	0	0.02	0.06	0.07
12	8	1	0	0.01	*0.01	0.45	0.02	8	0	0.04	0.04	0.05
13	8	2	0	0.09	0.12	2.57	49.23	6	1	0.27	*3.14	46.07
14	8	3	7	0.08	0.32	0.23	1645.79	7	4	0.36	*32.32	1610.60
15	8	3	7	0.04	0.26	0.06	6.44	7	2	0.07	0.32	8.96
16	9	3	4	0.01	0.01	0.02	0.39	4	4	0.08	0.47	*0.40
17	9	4	4	0.01	0.00	0.02	1.66	4	3	0.05	0.11	1.63
18	10	5	0	1.42	6.71	2.20	∞	10	4	8.32	32502.42	∞
19	10	9	0	0.00	0.00	0.01	24.16	9	5	14.54	17.32	28.75
20	11	4	0	0.22	*0.21	7.66	2.19	29	?	∞	∞	*10.61

symbol * shows the timing which is important in comparison with the other timings. Timings were conducted on a personal computer with 3.2GHz, Intel(R)-Xeon(TM), 1024 MB memory and 64 bits under the Linux operating system.

The experiments we made seem to show that these first implementations are already very efficient. Especially, a comparison of the column time of `reg` and `regIdeal` in Table 1 shows that our algorithm to compute the Castelnuovo-Mumford regularity is *more efficient* than `regIdeal` (except for Examples 16 and 19). The timings of Examples 1, 7, 11 and 12 show that for the ideals which are already in QSP, our algorithm does not run faster than `regIdeal`. Note that `regIdeal` algorithm is based on [5] by using the notion of nested type for monomial ideals. It makes a random linear change of variables in $K^{nd-d(d-1)/2}$ where d is the dimension of input ideal to put the input ideal in this position. Next, it computes the Gröbner basis of the changed ideal (say I) to test nested type and then computes the satiety of $\widehat{\text{sec}}(\text{in}(I), i)$ for $i = 0, \dots, d$ for computing the regularity.

The efficiency of our algorithm comes from Theorem 4.1: By an *incremental* (one variable after the other) random linear change of variables it puts a general homogeneous ideal in QSP. Then by computing the Gröbner basis of the changed ideal it tests QSP (which is less expensive than the Borel test and the nested type test). Finally, it reads the regularity without further computation.

As a counterpart of its efficiency, it makes the linear change of variables in K^N with $N = n(n+1)/2 - (n-d-1)(n-d)/2$ which is larger than $K^{nd-d(d-1)/2}$ (that of [5]). But, in practice, since it makes this change of variables incrementally, the set which is used is less than K^N (see the column `ncv`).

Also, this incremental change of variables makes the computation of Gröbner bases less expensive than the case when we change all the last variables in one step and avoids destroying the possible sparsity and therefore transforming an easy problem into an intractable one (see Example 18 and 20).

This table shows also that `sat` algorithm is more efficient than `satiety(.,1)`. The last algorithm makes a random linear change of variables to find a monomial ideal of nested type with the same satiety. Once a monomial ideal is nested type,

one could compute its satiety from its Gröbner basis. The efficiency of our algorithm comes from the fact that we change only the last variable to put the ideal in generic position (see Section 2). But, it is not more efficient than `satiety`. The SINGULAR code of `satiety` is only the following line to compute the satiety of an ideal I :

```
maxdeg1(simplify(reduce(quotient(I,maxideal(1)),I),2))+1;
```

The efficiency of this algorithm is followed by using the commands `quotient` and `reduce`, which compute, respectively, the quotient ideal of I with respect to the maximal ideal and the normal form of this quotient with respect to I .

6. CONCLUSION

In this paper, we have introduced the new notion of a homogeneous ideal in QSP. A definition for the notion of generic coordinates to compute efficiently the Castelnuovo-Mumford regularity of a homogeneous ideal. We have provided an algorithm to decide whether a monomial ideal is in QSP or not which has a quadratic complexity in input (a monomial ideal) size. We have proved that the Castelnuovo-Mumford regularity of an ideal in QSP is the maximal degree of the elements of its reduced Gröbner basis with respect to the reverse lexicographic ordering. Then, we have presented two algorithms `sat` and `reg` (using this result) to compute, respectively, the satiety and the Castelnuovo-Mumford regularity of a general homogeneous ideal by an incremental random linear change of variables to put the ideal in a convenient generic position. Our experiments show the efficiency of our algorithms compared with the corresponding algorithms of SINGULAR.

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