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Finite lattices and Gröbner bases

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Gröbner bases of binomial ideals arising from finite lattices will be studied. In terms of Gröbner bases and initial ideals, a characterization of finite distributive lattices as well as planar distributive lattices will be given.

1 Introduction

Let L be a finite lattice and $K[L]$ the polynomial ring in $|L|$ variables over a field K whose variables are the elements of L. A binomial of $K[L]$ of the form $ab - (a \wedge b)(a \vee b)$ is called a basic binomial. Let I_L denote the ideal of $K[L]$ generated by basic binomials of $K[L]$. The ideal I_L was first introduced by [3]. It is shown in [3] that I_L is a prime ideal if and only if L is a distributive lattice, see also [2, Theorem 10.1.3]. When L is distributive, the set of basic binomials of $K[L]$ is a Gröbner basis with respect to any rank reverse lexicographic order.

In the present paper, by studying Gröbner bases of I_L, a Gröbner basis characterization of distributive lattices as well as planer distributive lattices will be obtained. Moreover, we discuss the problem when I_L has a quadratic Gröbner basis with respect to any monomial order.

2 Characterization of distributive lattices

We refer the reader to [6] for fundamental materials on finite lattices. Let $\hat{0}$ $\big($resp. $\hat{1}\big)$ denote a unique minimal (resp. maximal) element of a finite lattice. Recall that a finite lattice L is called *modular* if $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$ for all $x, a, b \in L$. A finite lattice is modular if and only if no sublattice of L is isomorphic the pentagon lattice of Figure 1. A finite lattice is called *distributive* if, for all $x, y, z \in L$, the distributive laws $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$ hold. Every distributive lattice is modular. A modular lattice is distributive if and only if no sublattice of L is isomorphic to the diamond lattice of Figure 1.

A finite lattice is called *pure* if all maximal chains between $\hat{0}$ and $\hat{1}$ have the same length. When a finite lattice is pure, then the *rank function* of L can be defined. More precisely, if L is a finite pure lattice and $a \in L$, then the *rank* of a in L, denoted by $\mathrm{rank}(a)$, is the largest integer r for which there exists a chain of L of the form

$$\hat{0} = a_0 < a_1 < \cdots < a_r = a.$$

If a finite lattice L is modular, then one has the equality

$$\mathrm{rank}(p) + \mathrm{rank}(q) = \mathrm{rank}(p \wedge q) + \mathrm{rank}(p \vee q) \tag{2.1}$$

for all $p, q \in L$.

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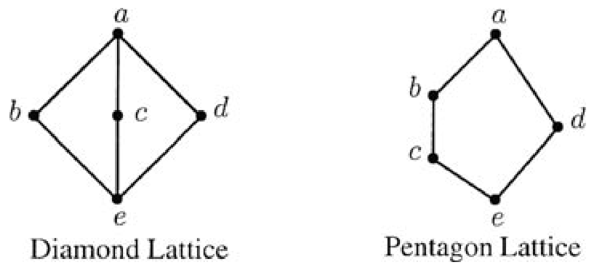

Fig. 1

Let K be field and L a finite lattice. We consider the polynomial ring $K[L]$ whose variables correspond to the elements of L, and define the ideal $I_L \subset K[L]$ as the binomial ideal whose generators are all basic binomials attached to L. As explained in the introduction, a binomial of the form $ab - cd$ with $c = a \vee b$ and $d = a \wedge b$ is called a *basic binomial*. The residue class ring $K[L]/I_L$ will be denoted by $R(L)$

We refer the reader to [2] for basic terminologies and notation on Gröbner bases. A *rank reverse lexicographic order* on $K[L]$ is the reverse lexicographic order with the property that $a > b$ if $\text{rank}(a) > \text{rank}(b)$.

Theorem 2.1 *Let L be a finite modular lattice. Then the following conditions are equivalent*:

(i) *L is a distributive lattice*;

(ii) *I_L has a squarefree Gröbner basis with respect to any rank reverse lexicographic order.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is well known, see [3] and [2, Theorem 10.1.3].

(ii) $\Rightarrow$ (i): Suppose that L is a finite modular lattice which is not distributive. Then L contains the diamond lattice of Figure 1. Since L is modular, one has the equality (2.1) for all $p, q \in L$. Hence if $g = pq - p'q'$ is a basic monomial of I_L, then

$$\text{rank}(p) + \text{rank}(q) = \text{rank}(p') + \text{rank}(q').$$

In particular the ranks of b, c and d coincide. We fix a rank reverse lexicographic order $<$ with the property that $d < q$ for all $q \in L$ with $\text{rank}(q) = \text{rank}(d)$. Our work is to show that $\text{in}_<(I_L)$ cannot be squarefree. Suppose, on the contrary, that $\text{in}_<(I_L)$ is squarefree.

First we claim $ad^2e \in \text{in}_<(I_L)$. In fact, $ad^2e - a^2e^2 \in I_L$, because

$$ad^2e - a^2e^2 = d(d(ae - bc) + c(bd - ae)) + ae(cd - ae).$$

Since $\text{in}_<(I_L)$ is squarefree and since $ad^2e \in \text{in}_<(I_L)$, it follows that $ade \in \text{in}_<(I_L)$. Hence there exists a binomial $f = ade - u \in I$, where u is a monomial of degree 3, with $\text{in}_<(f) = ade$.

Let $f = \sum_{i=1}^{N} x_i f_i$, where each x_i is a variable and where each $f_i = v_i - w_i$ is a basic binomial of I_L, such that $x_1 v_1 = ade$ and $x_i w_i = x_{i+1} v_{i+1}$ for all $1 \leq i < N$. A crucial fact is that each variable appearing in $x_i f_i$ belongs to the interval $[e, a]$ of L. To see why this is true, we observe that if $f_i = v_i - w_i$ is a basic binomial of I_L and if each variable appearing in v_i belongs to $[e, a]$, then each variable appearing in w_i must belong to $[e, a]$. Now, since $x_1 v_1 = ade$ and $x_i w_i = x_{i+1} v_{i+1}$ for all $1 \leq i < N$, this observation guarantees that each variable appearing in $x_i f_i$ belongs to the interval $[e, a]$ of L. In particular $u = x_N w_N$ consists of variables belonging to $[e, a]$, say $u = \ell mn$.

Now, one has $f = ade - \ell mn \in I$, where ℓ, m and n belong to $[e, a]$. Let $\ell \geq m \geq n$. Since we are working with a rank reverse lexicographic order, it follows that e is the smallest variable among all variables belonging to $[e, a]$. Since $\text{in}_<(f) = ade$, one has $n = e$.

On the other hand, since I_L is generated by basic binomials of L, it follows easily that if $g = p_1 p_2 \cdots p_r - q_1 q_2 \cdots q_r$ is a binomial belonging to I_L, then

$$\sum_{i=1}^{r} \text{rank}(p_i) = \sum_{i=1}^{r} \text{rank}(q_i).$$

Thus in particular one has

$$\operatorname{rank}(a) + \operatorname{rank}(d) = \operatorname{rank}(\ell) + \operatorname{rank}(m).$$

Since a is a unique maximal element of $[a, e]$, it follows that $\operatorname{rank}(a) \geq \operatorname{rank}(\ell)(\geq \operatorname{rank}(m))$. Hence $\operatorname{rank}(d) \leq \operatorname{rank}(m)$. If $\operatorname{rank}(d) = \operatorname{rank}(m)$, then $d < m$ by the given order of the variables. On the other hand, if $\operatorname{rank}(d) < \operatorname{rank}(m)$, then $d < m$, since we use a rank reverse lexicographic order. Thus in any case $d < m$, and this implies that $\operatorname{in}_<(f) = \ell mn$, a contradiction. Consequently, the monomial ade cannot belong to $\operatorname{in}_<(I_L)$. Hence ad^2e belongs to a unique minimal set of monomial generators of $\operatorname{in}_<(I_L)$. Thus $\operatorname{in}_<(I_L)$ cannot be squarefree. □

It can be easily checked that for any monomial order, $\operatorname{in}_<(I_{N_5})$ is squarefree where N_5 is the pentagon lattice, while $\operatorname{in}_<(I_{N_3})$ is not squarefree where N_3 is the diamond lattice.

Conjecture 2.2 (Squarefree conjecture) Let L be a modular lattice. Then for any monomial order $\operatorname{in}_<(I_L)$ is not squarefree, unless L is distributive.

3 Characterization of planar distributive lattices

Let N^2 denote the (infinite) distributive lattice consisting of all pairs (i, j) of nonnegative integers with the partial order $(i, j) \leq (k, l)$ if and only if $i \leq k$ and $j \leq l$. A *planar distributive lattice* is a finite sublattice L of N^2 with $(0, 0) \in D$.

Theorem 3.1 *Let L be a finite modular lattice. Then the following conditions are equivalent*:

(i) *L is a planar distributive lattice*;

(ii) *I_L has a squarefree initial ideal with respect to the lexicographic order for any order of the variables*;

(iii) *I_L has a squarefree initial ideal with respect to any monomial order.*

Proof. (i) ⇒ (iii): Let m and n be the smallest integers such that $L \subset [m] \times [n]$, and consider the polynomial ring $T = K[t_1, \dots, t_m, s_1, \dots, s_n]$. We define a K-algebra homomorphism $\varphi\colon R(L) \to T$ which assigns to $a = (i, j)$ the monomial $s_i t_j \in T$. The image A of φ is the edge ring of a bipartite graph G. The basic relations $ab = (a \vee b)(a \wedge b)$ of $R(L)$ are mapped under φ to relations of A corresponding to 4-cycles of the bipartite graph G. It is shown in [4, Theorem 1.2] that the defining ideal of the edge ring A is generated by the binomials corresponding to 4-cycles, if each even cycle of length ≥ 6 has a chord. That G satisfies this property is shown by Querishi [5]. It follows that $R(L) \cong A$. Hence we may identify I_L with the toric edge ideal J defining A. Next we use a result of Sturmfels (see [7, Chapter 9]) which says the universal Gröbner basis of the toric edge ideal of bipartite graph consists of the binomials corresponding to the even cycles with no chords. From this it follows that $\operatorname{in}_<(J)$ is squarefree for any monomial order.

(iii) ⇒ (ii) is trivial.

(ii) ⇒ (i): Suppose L is not planar, then L contains a sublattice which is isomorphic to the Boolean lattice B_3 of rank 3 as shown in Figure 2.

Let $<$ be the lexicographic order induced by an ordering such that $g < f < e < h < a < d < c < b$ and $b < q$ for any other $q \in L$. The initial ideal of I_{B_3} contains the monomial ah^2 in the minimal set of monomial

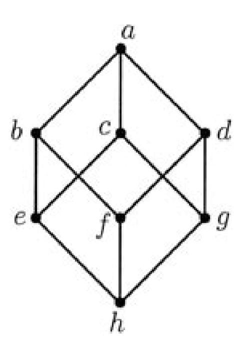

Fig. 2 Boolean lattice B_3.

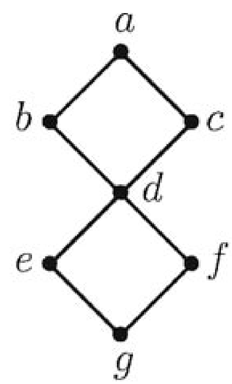

Fig. 3 Planar distributive lattice C_2.

generators. Since $<$ is an elimination order (see [2, Exercise 2.9]) it follows that ah^2 belongs to the minimal set of monomial generators of I_L. □

In contrast to the order given in the proof of the preceding theorem, there exist lexicographic orders such that I_{B_3} is quadratic, or not quadratic but squarefree. The question arises whether for any finite distributive lattice there exists a lexicographic monomial order such that $\mathrm{in}_<(I_L)$ is squarefree.

Recall that the *divisor lattice* of a positive integer n is the lattice D_n consisting of all divisors of n ordered by divisibility. Every divisor lattice is a distributive lattice.

Let, in general, L be a finite pure lattice. A *cut edge* of L is a pair (a, b) of elements of L with $\mathrm{rank}(b) = \mathrm{rank}(a) + 1$ such that

$$|\{c \in L: \ \mathrm{rank}(c) = \mathrm{rank}(a)\}| = |\{c \in L: \ \mathrm{rank}(c) = \mathrm{rank}(b)\}| = 1.$$

Theorem 3.2 *Let L be a finite lattice with no cut edges. Then the following conditions are equivalent*:

(i) *L is the divisor lattice of $2 \cdot 3^r$ for some $r \geq 1$;*

(ii) *I_L has a quadratic Gröbner basis with respect to any monomial order.*

Proof. (i) ⇒ (ii): The ideal I_L can be identified with the toric edge ideal of the complete bipartite graph of type $(2, r)$, since L has no cut edges. Each cycle in a bipartite graph of type $(2, r)$ is of length 4. Hence by using again [7, Chapter 9], the basic binomials of I_L form a universal Gröbner basis. This yields the desired conclusion.

(ii) ⇒ (i): As we have seen in the proofs of Theorem 2.1 and in Theorem 3.1 that the lattice L cannot contain as a sublattice the diamond lattice and the Boolean lattice B_3. It also cannot contain the pentagon lattice N_5 of Figure 1. Indeed, if we choose the lexicographic order induced by $c < e < a < d < b$, then bae is a minimal generator of $\mathrm{in}_<(I_{N_5})$. This implies that L is a planar distributive lattice. Finally, L cannot contain as a sublattice the planar distributive lattice C_2 of Figure 3.

In fact, if we choose the lexicographic order induced by $g < f < e < c < b < a < d$, then aef is a minimal generator of $\mathrm{in}_<(I_{N_5})$. Hence L must be the divisor lattice of $2 \cdot 3^r$ for some $r \geq 2$. □

As a strengthening of Theorem 3.2 we expect

Conjecture 3.3 (Quadratic conjecture) Let I be an ideal generated by binomials such that I has a quadratic Gröbner basis with respect to any monomial order. Then either the generators of I are binomials in pairwise different sets of variables or $I = I_L$ where L is the divisor lattice of $2 \cdot 3^r$ for some $r \geq 1$.

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