Computing the Stratification of Actions of Compact Lie Groups

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Abstract. We provide a constructive approach to the stratification of the representation- and the orbit space of linear actions of compact Lie groups contained in $GL_n(\mathbf{R})$ on $\mathbf{R}^n$. Strata of the representation space are described as differences of closed sets given by polynomial equations while d-dimensional strata of the orbit space are represented by means of polynomial equations and inequalities. All algorithms have been implemented in SINGULAR V2.0.

Introduction

In 1983 Abud and Sartori [1] pointed out the relation between symmetry breaking and stratifications of linear actions of compact Lie groups and presented several applications in physics. Let G be a compact Lie group which acts linearly on $\mathbf{R}^n$. Note that for all points $w \in \mathbf{R}^n$ which are sufficiently close to a point $v \in \mathbf{R}^n$, the stabilizer G_w is conjugated to some subgroup of G_v, i.e., the isotropy may jump, but if it jumps it only jumps up (loss of symmetry). Symmetry breaking can briefly be described as follows. Let $\phi_0 \in \mathbf{R}^n$ be the ground state of a physical system and $V_\gamma(z)$ be a G-invariant potential which determines ϕ_0 and depends on the parameter γ. Varying γ might change ϕ_0 and the stabilizer group G_0 of ϕ_0 which may be smaller than the previous stabilizer (loss of symmetry). Hence various patterns of spontaneous symmetry breaking, which correspond to distinct phases of the model, occur. Note that the orbit space of G and all strata are semialgebraic sets (see for instance [11]).

There are several approaches for constructing the stratification of the orbit space of a compact Lie group starting with Abud and Sartori, see [1]. Explicit algorithms for finite groups are given for instance in [3] and Gatermann [7] provides a systematic exposition for compact Lie groups.

These algorithms (except [3]) construct a stratification of the orbit space $\mathbf{R}^n/G$ of a compact Lie group G by using the matrix $\mathrm{grad}(z)$ which is defined on $\mathbf{R}^n/G$. We present a different approach, namely, we firstly compute a stratification of the representation space of G. Only then images of the strata are computed by means of elimination theory (equations) and results of Procesi and Schwarz (inequalities), see [11]. For several applications, like the construction of polynomial potentials on the orbit space, this approach may lead to easier computations. Namely inequalities need not be calculated since the Zariski-closure of a stratum Σ_x equals the zero set of the ideal obtained by computing the image $\pi(\Sigma_x)$ and primary decomposition is easier on the representation space than on the orbit space[1]. In addition, we show that each d-dimensional stratum can be presented by at most d strict inequalities up to generic equivalence, in contrast to the (at most) $2^n - 1$ inequalities obtained from the Theorem of Procesi and Schwarz.

[1] Equations for the (Zariski-closure) of strata are computed out of rank conditions on the matrix $\mathrm{grad}(z)$. The locus where $\mathrm{rank}\,(\mathrm{grad}(z)) \leq d$ contains all d-dimensional strata of the orbit space and must be decomposed in irreducible components in order to obtain equations defining these strata. On the representation space the corresponding locus is a finite union of vectorspaces which is not true for the orbit space.

1 On Invariant Theory of Compact Lie Groups and Orbit Spaces

We present some background on invariants of compact Lie groups and orbit spaces. In both sections we use fundamental facts from semialgebraic geometry like the Tarksi-Seidenberg principle, for which we refer to [5]. For short, an *basic open* (*basic closed*) *semialgebraic subset* of the algebraic set $V \subseteq \mathbf{R}^n$ is of the form $\{v \in V \mid g_i(v) > 0, 1 \leq i \leq r\}$, respectively, $\geq$ instead of $>$, where $g_1, g_2, \ldots, g_r \in \mathbf{R}[x_1, x_2, \ldots, x_n]$. An *open* (*closed*) *semialgebraic subset* of V is a finite union of basic open (basic closed) semialgebraic subsets of V.

1.1 Invariants of Lie Groups

Let G be a compact Lie group and $\rho : G \to GL_n(\mathbf{R})$ be a faithful representation. In the sequel we identify G and its image $\rho(G) \subset GL_n(\mathbf{R})$. It is well-known that $\mathbf{R}^n$ admits a G-invariant scalar product $(\_,\_)_G$ on $\mathbf{R}^n$ (see for instance [6]). By the Gram-Schmidt orthonormalization process there exists $A \in GL_n(n)$ such that $A \cdot G \cdot A^{-1} \subseteq O_n(\mathbf{R})$, i.e, the representation ρ is equivalent to an orthogonal representation. From now on we assume $G \subseteq O_n(\mathbf{R})$ and that G acts as usual on $\mathbf{R}^n$. In the sequel let $\mathbf{K}$ be on of the fields $\mathbf{R}$ or $\mathbf{C}$. For $X \subseteq \mathbf{K}^n$ we define $\mathcal{I}(X) := \{f \in \mathbf{K}[t_1, t_2, \ldots, t_n] \mid f(x) = 0 \text{ for all } x \in X\}$, the *ideal* of X and for an ideal $I \subseteq \mathbf{K}[x_1, x_2, \ldots, x_n]$ we define $\mathcal{V}(I) := \{x \in \mathbf{K}^n \mid f(x) = 0 \text{ for } f \in I\}$, the *variety* associated to I. A subset $U \subseteq \mathbf{K}^n$ is *closed* in the Zariski topology if and only if $U = \mathcal{V}(I)$ for some ideal $I \subseteq \mathbf{K}[x_1, x_2, \ldots, x_n]$. A polynomial $f \in \mathbf{K}[x_1, x_2, \ldots, x_n]$ is *invariant* w.r.t. G if $f(g^{-1} \cdot \mathbf{x}) = f(\mathbf{x})$ for all $g \in G$. The ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^G$, consisting of all invariant polynomials w.r.t. G, is called the *invariant ring* of G (ρ will be omitted). By Hilbert's Finiteness Theorem, the invariant ring is finitely generated as a $\mathbf{K}-$algebra. Homogeneous generators $\pi_1, \pi_2, \ldots, \pi_m$ of $\mathbf{K}[x_1, x_2, \ldots, x_n]^G$ are called *fundamental invariants* (i.e., each invariant polynomial is a polynomial in $\pi_1, \pi_2, \ldots, \pi_m$). Fundamental invariants define the projection

$$\begin{aligned} \pi : \mathbf{K}^n &\longrightarrow \mathbf{K}^n/G \subseteq \mathbf{K}^m \\ \mathbf{x} &\longmapsto (\pi_1(\mathbf{x}), \pi_2(\mathbf{x}), \ldots, \pi_m(\mathbf{x})) \end{aligned}$$

of $\mathbf{K}^n$ onto an embedding of the orbit space $\mathbf{K}^n/G \subseteq \mathbf{K}^m$, also called the *Hilbert map*. Note that π maps closed sets to closed sets[2] and that each fiber contains precisely one closed orbit (see for instance[9]). For $\mathbf{K} = \mathbf{C}$ the image of $\pi(\mathbf{C}^n) \subseteq \mathbf{C}^m$ equals the variety of the ideal of relations of $\pi_1, \pi_2, \ldots, \pi_m$ (see for instance[9]). Over $\mathbf{R}$ it is well-known that the image of π is a semialgebraic set.

Proposition 1. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group. The orbit space $\mathbf{R}^n/G$ of G is a semialgebraic set semialgebraically homeomorphic to $\pi(\mathbf{R}^n)$.*

Proof. It is well-known that the orbits of G can be separated by fundamental invariants of G (see for instance Theorem 3.4.3. in [10]). By the Tarski-Seidenberg principle (see for instance [5]) the real image of π is a semialgebraic set (it equals the projection of the graph, which is a real algebraic set). □

Note that the orbit space of an algebraic group parameterizes all closed orbits. Hence the orbit space of a compact Lie group G parameterizes all orbits of G since they are closed. Orbits which are not closed cannot be separated by polynomials so group actions having non-closed orbits cannot be stratified by using their invariant rings, see [12].

1.2 Inequalities Defining Orbit Spaces

Procesi and Schwarz have constructed polynomial inequalities which have to be added to the equations coming from the Hilbert map of a compact Lie group G, which need not be a subgroup of $O_n(\mathbf{R})$, in order to describe an embedding of the quotient $\mathbf{R}^n/G \subset \mathbf{R}^m$. Essential parts of

[2] Note that the map π is proper.

the proof are the existence of a closed orbit in each fiber of π (see for instance [9]) and the existence of a G-invariant inner product $(\_,\_)$ on $\mathbf{R}^n$, which is used to construct the $m \times m$ matrix $\mathrm{grad}(v) = (d\pi_i(v), d\pi_j(v))_{i,j=1,\ldots,m}$ for $v \in \mathbf{C}^n$ where $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is the Hilbert map. Here we have used the identification[3] of $\mathbf{R}^n$ with its dual $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$. They proved that a point $z \in \mathcal{V}(I)$, where $I \subset \mathbf{R}[y_1, y_2, \ldots, y_m]$ is the ideal of relations among $\pi_1, \pi_2, \ldots, \pi_m$, lies in $\mathbf{R}^n/G$ if and only if the matrix $\mathrm{grad}(z)$ is positive semidefinite. The constraint that $\mathrm{grad}(z)$ must be semidefinite yields inequalities for describing $\mathbf{R}^n/G$. Recall that the type of a real $m \times m$ Matrix M equals (p, q) where p, respectively, q denote the number of positive, respectively, negative eigenvalues counted with multiplicities. Obviously, $\mathrm{rank}\,(M) = p + q$.

Proposition 2. *An $m \times m$ matrix M over $\mathbf{R}$ is positive semidefinite (denoted by $M \geq 0$) iff all symmetric minors of M are non-negative. The matrix M is positive definite (denoted by $M > 0$) iff all principal minors of M are positive.*

Proof. We refer to, e.g., Section IX.72 in [14]. □

In order to define the matrix $\mathrm{grad}(z)$ on the orbit space we have to show that all entries are invariant w.r.t. G. By $d\pi(z)$ we denote the Jacobian matrix of π at z.

Proposition 3. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group. For $\sigma \in G_v$ the Jacobian of the Hilbert map $\pi : \mathbf{R}^n \to \mathbf{R}^m/G$ satisfies $d\pi(v) = d\pi(v) \circ \sigma$. In particular, the functions $v \mapsto \mathrm{grad}(v)_{ij}$ are invariant.*

Proof. Follows from $\pi(v) = \pi(\sigma \cdot v)$, the chain rule, and the fact that σ is linear. □

Therefore the matrix $\mathrm{grad}(v)$ is also defined on $\mathbf{C}^n/G$ and $z \in \pi(\mathbf{R}^n)$ implies that $\mathrm{grad}(z)$ is positive semidefinite. This can be checked by testing if the set of $2^n - 1$ symmetric minors of $\mathrm{grad}(z)$ are ≥ 0.

Theorem 1. *(Procesi-Schwarz [11]) Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group and let $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ be such that $\pi_1, \pi_2, \ldots, \pi_m$ generate $\mathbf{R}[x_1, x_2, \ldots, x_n]^G$. The quotient space is given by*

$$\mathbf{R}^m/G = \pi(\mathbf{R}^n) = \{z \in \mathbf{R}^m \mid \mathrm{grad}(z) \geq 0, z \in \mathcal{V}(I)\}$$

where $I \subset \mathbf{R}[y_1, y_2, \ldots, y_m]$ is the ideal of relations of $\pi_1, \pi_2, \ldots, \pi_m$.

Proof. We refer to [11]. □

In subsequent sections we use the theorem of Procesi and Schwarz to provide a finer description of the orbit space in terms of so called strata (defined in the following section), which are useful for several applications.

Example 1. Consider the compact Lie group $G = SO(1) \times \mathbf{Z}_2 \subset GL_3(\mathbf{R})$ (rotations around the z-axis and a reflection fixing the (x, y)-plane) and its complexification $G_{\mathbf{C}}$ (see Section 3.1). The invariant ring of G, respectively, $G_{\mathbf{C}}$ equals $\mathbf{K}[x, y, z]^G = \mathbf{K}[x^2 + y^2, z^2]$ where $\mathbf{K} = \mathbf{R}$, respectively, $\mathbf{K} = \mathbf{C}$. The Hilbert map is given by $\pi : \mathbf{K}^3 \to \mathbf{K}^2, (x, y, z) \mapsto (x^2 + y^2, z^2)$. Since the two invariants are algebraically independent, we obtain $\mathbf{C}^3/G_{\mathbf{C}} = \mathbf{C}^2 = \mathrm{im}(\pi)$. Over the reals, since $\mathrm{grad}(z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$, we have

$$\mathbf{R}^3/G = \mathrm{im}(\pi) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{R}^2 \mid z_1 \geq 0, z_2 \geq 0 \right\}$$

Remark 1. (a) For practical purposes the dependence on a G-invariant scalar product may be problematic.
(b) It is not necessary that $G \subseteq O_n(\mathbf{R})$ for computing inequalities if a G-invariant inner product is given in an effective form.

[3] Note that $d\pi_j$ is a differential form, so $d\pi_j(z) : \mathbf{R}^n \to \mathbf{R}$ is a linear form.

2 On the Stratification of the Representation and Orbit Space

Consider a compact Lie group $G \subset GL_n(\mathbf{R})$, the set of points having the same symmetry type w.r.t. G form a partition of $\mathbf{R}^n$ in finitely many distinct open sets, also called a stratification. We present underlying definitions and properties of of strata and their closures (Zariski- or metric topology). These properties will be used in subsequent sections to compute equations and inequalities for describing strata and semistrata.

2.1 Stratification of a Group-Action

We provide the definition of strata, respectively, stratifications and associated objects like orbit type, etc. In the sequel $G \subset GL_n(\mathbf{R})$ denotes a compact Lie group and $\overline{X}^z$, respectively, $\overline{X}^m$ denote the Zariski-, respectively, metric closure of the set X.

Definition 1. *Let $E \subseteq \mathbf{R}^n$ be a semialgebraic set. A stratification of E is a finite partition E_λ of E where each E_λ is a semialgebraically connected locally closed*[4] *equidimensional semialgebraic subset (or a finite set of points) of $\mathbf{R}^n$ such that $E_\lambda \cap \overline{E_\beta}^m \neq \emptyset$ and $\lambda \neq \beta$ implies $E_\lambda \subset E_\beta$ and* $\dim E_\lambda < \dim E_\beta$. *For $\lambda \in \Lambda$ the set E_λ is called a stratum and $\overline{E}_\lambda^m$ is called a semi-stratum of the stratification, and if $d = dim E_\lambda$ then E_λ is called a $d-$stratum.*

Given $x \in \mathbf{R}^n$, the set $G(x) = \{g \cdot x \mid g \in G\}$ is called the *orbit* of x and the group $G_x = \{g \in G \mid g \cdot x = x\}$ is called the *stabilizer* of x.

Proposition 4. *Let G be an algebraic group (defined over the field $\mathbf{K}$) which acts algebraically (via α) on $\mathbf{K}^n$. For $x \in \mathbf{K}^n$ the stabilizer G_x and the $X_d = \{x \in X \mid \dim G_x \geq d\}$ are closed.*

Proof. Let $\pi_2 : X \times X \to X$ be the projection onto the second component, $i_x : G \hookrightarrow G \times X, i_x(g) = (g, x)$ be an injection for $x \in X$ and define $\alpha' : G \times X \to X \times X$ by $\alpha'(g, x) = (\alpha(g, x), x)$. All maps are continuous (w.r.t. the Zariski-topology), hence the fibers of $\pi_2 \circ \alpha' \circ i$ are closed. The stabilizer of x is closed since G_x is isomorphic to $\alpha'^{-1}(x, x) = \{(g, x) \mid \alpha(g, x) = x\}$. We also obtain that $X_d = \{x \in X \mid \dim(\pi_2 \circ \alpha' \circ i)^{-1}(x) \geq d\}$ hence the claim follows from upper-continuity of the fiber dimension. □

Definition 2. *For a subgroup $H \subseteq G$ we denote the conjugacy class of H in G by by $[H] = \{gHg^{-1} \mid g \in G\}$. The orbit type of $x \in \mathbf{R}^n$ is $[x] := [G_x]$. For $u, v \in \mathbf{R}^n$ we define $[u] < [v]$ if $G_u \subset H$ for some $H \in [v]$. The associated stratum, respectively, semi-stratum of $[x]$ is $\Sigma_x := \{y \in \mathbf{R}^n \mid [x] = [y]\}$, respectively, $\overline{\Sigma_x}^m$.*

The orbit type is a measure for the symmetry of the points of $\mathbf{R}^n$. We have $[x] > [y]$ if the point x has more symmetries than the point y, i.e., $gG_yg^{-1} \subset G_x$ form some $g \in G$. The notation of strata is justified by the fact that these sets, respectively, their images under the Hilbert map form a stratification of the representation-, respectively, orbit space.

Proposition 5. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group.*

(a) There are only finitely many different orbit types, i.e,. the set $\{[G_x] \mid x \in \mathbf{R}^n\}$ is finite.
(b) The orbit types form a lattice. For $v \in \Sigma_p := \{x_0 \in \mathbf{R}^n \mid \operatorname{rank}(d\pi(x_0))$ is maximal$\}$ the orbit type $[v]$ is the least element.
(c) For each $v \in \mathbf{R}^n$ there exists a small neighborhood $U \subset \mathbf{R}^n$ of v such that $u \in U$ implies $[u] \leq [v]$.

Proof. (a) see for instance Ch. IV.10 in [6].
(b) Note that $\operatorname{rank}(d\pi(v))$ is maximal iff $\dim N_v^0$ is maximal (see Section 2.2) hence the stabilizer of v is contained in $[w]$ for all $w \in \mathbf{R}^n$.
(c) We refer, e.g., to [1].

□

[4] The set E_λ is open in its metric closure $\overline{E_\lambda}^m$.

The set Σ_p is called the *principal stratum* of G.

Proposition 6. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group.*

(a) For a subgroup $H \subseteq G$ of G the set $\mathbf{R}^n_H = \{x \in \mathbf{R}^n \mid H \subseteq G_x\}$ is a vectorspace. In particular, the set $\{x \in \mathbf{R}^n \mid G_x = H\}$ is Zariski-open in $\mathbf{R}^n_H$.
(b) For $0 \neq x \in \mathbf{R}^n$ each stratum Σ_x is open in its closure (both metric and Zariski) and $G(x)$ is a proper subset of Σ_x.

Proof. (a) Let $x, y \in \mathbf{R}^n_H$ and $g \in H$. Obviously, $g \cdot (x+y)$ and $g \cdot \lambda x, \lambda \in \mathbf{R}$, are contained in $\mathbf{R}^n_H$. The set $S = \{x \in \mathbf{R}^n_H \mid G_x \supset H\}$ is of dimension less than $\mathbf{R}^n_H$ and can be written as the union of all strata Σ_y with $[y] > [H]$ intersected with $\mathbf{R}^n_H$. By Proposition 2.1.5, the set S is closed, hence $\mathbf{R}^n_H \setminus S$ is Zariski-open.
(b) The first claim follows from Theorem 2.2.2. For the second claim note that $G(x)$ is compact, hence the set $\{\lambda x \mid \lambda \in \mathbf{R}, \lambda > 0\}$ is not contained in $G(x)$ but in Σ_x.

□

2.2 Properties of Strata

We describe properties of strata and semi-strata on the representation and orbit space. In the representation space semistrata, respectively, strata can be described by closed sets, respectively, differences of closed sets. For a description of the orbit space Procesi and Schwarz have derived inequalities (see Theorem 1.2.1). These inequalities, formed by the $2^n - 1$ symmetric minors of grad(z) where n equals the dimension of the representation space, may also be used to describe all semistrata on the orbit space and therefore also all strata by forming differences of closed sets. We show in addition that a d-dimensional stratum can be described up to generic equivalence by d inequalities and the ideal of its Zariski-closure in $\mathbf{R}^n/G$. In particular, we provide effective descriptions relying on equations and inequalities.

The stratification of the representation space of a compact Lie group is completely determined by the matrix $d\pi(v)$. Since $\mathbf{R}^n$ admits a G-invariant inner product $(\_,\_)_G$ we may define the orthogonal complement N_v to $T_v(G(v))$ and the decomposition N=$N^0_v \oplus N^1_v$, where $N^0_v = \{w \in N_v \mid w$ is G_v-invariant$\}$ and N^1_v is the orthogonal complement of N^0_v in N_v. Note that G need not be a subgroup of the orthogonal group.

Proposition 7. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group. We have*

$$\ker d\pi(x_0) = T_{x_0}G(x_0) \oplus N^1_{x_0} \text{ and } \mathrm{im} d\pi(x_0) \cong N^0_{x_0}.$$

Proof. Note that $v \in T_{x_0}G(x_0)$ implies $v \in \ker d\pi(x_0)$ since π is G-invariant. Let V be the the vectorspace generated by the gradients (considered as elements of $\mathbf{R}^n$) $d\pi_1(x_0), d\pi_2(x_0), \ldots, d\pi_m(x_0)$, i.e., $V = \mathrm{im}\ d\pi(x_0)$. Note that $v \in \ker d\pi(x_0)$ implies $d\pi_i(x_0) \cdot v = 0$ so $v \in N_{x_0}$. By Proposition 2.2.3 we have $d\pi_i(x_0) \circ \sigma = d\pi_i(x_0)$ for $\sigma \in G_v$, hence $V \subseteq N^0_v$. Now $v \in N^0_{x_0} \setminus V$ implies $v \in \ker d\pi(x_0)$. Hence the rank of the matrix $d\pi(x_0)$ augmented by the column v equals the rank of $d\pi(x_0)$ and so $v \in V$. □

Proposition 8. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group. We have*

$$T_{x_0}\Sigma_{x_0} = T_{x_0}G(x_0) \oplus N^0_{x_0}.$$

In particular, $T_{\pi(x_0)}\hat{\Sigma}_{x_0} \cong N^0_{x_0}$.

Proof. One has to show that any curve through x_0 and contained in Σ_{x_0} has a tangent vector at x_0 which is contained in $T_{x_0}G(x_0) \oplus N^0_{x_0}$. This prove can be found in Section V of [1]. □

Corollary 1. *We have* $\dim \Sigma_{x_0} = \dim T_{x_0} + \dim N^0_{x_0} = \dim G - \dim G_{x_0} + \dim N^0_{x_0}$ *and* $\dim \hat{\Sigma}_{\pi(x_0)} = \dim N^0_{x_0}$.

Theorem 2. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group and $\pi : \mathbf{R}^n \to \mathbf{R}^n/G \subseteq \mathbf{R}^m$ be the Hilbert map.*

(a) *The union $\Sigma^{(d)}$ of all strata whose image under π is of dimension d equals the open semi-algebraic set*

$$\Sigma^{(d)} = \{v \in \mathbf{R}^n \mid \operatorname{rank}(d\pi(v)) = d\}.$$

(b) *The union Σ^d of all strata whose image under π is of dimension at most d equals the closed semi-algebraic set*

$$\Sigma^d = \{v \in \mathbf{R}^n \mid \operatorname{rank}(d\pi(v)) \leq d\}$$

In addition, $\overline{\Sigma^{(d)}}^{Z} = \overline{\Sigma^{(d)}}^{m} = \Sigma^d$.

Proof. (a) Note that a stratum is a smooth semi-algebraic set, so by Proposition 2.2.8 we have $\operatorname{rank}(d\pi(v)) = \dim \operatorname{im} d\pi(v) = \dim T_{\pi(v)}\hat{\Sigma}_{\pi(v)} = \dim \hat{\Sigma}_{\pi(v)}$.

(b) The set Σ^d can be defined by the vanishing of all $(d+i) \times (d+i)$ minors of $\frac{\partial \pi}{\partial x}$ where $i \geq 1$. If $d \geq \min\{n, m\}$ then $\Sigma^d = \mathbf{R}^n$. Note that $\Sigma^{(d)} = \Sigma^d \setminus \Sigma^{d-1}$.

□

So far we have only considered semistrata, respectively, strata on the representation space. We now turn to a description of the orbit space (and its stratification) in terms of equations and inequalities obtained from Procesi and Schwarz (see [11]). These inequalities are not strict, i.e., they involve '$\geq$'. We start with a description of strata by means of differences of semi-algebraic closed sets.

Corollary 2. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group and $x \in \mathbf{R}^n$.*

(a) *Let $S_x \subseteq \mathbf{R}^n$ be a stratum. Then $\hat{S}_x = \pi(S_x) = \{z \in \mathbf{R}^m \mid \operatorname{grad}(z) \geq 0, z \in \mathcal{V}(J)\}$ where $J \subset \mathbf{R}[y_1, y_2, \ldots, y_m]$ is the ideal of the image of S_x under π.*

(b) *Let $S_x = \Sigma_x \cup B_x$ be a disjoint union (B_x is a finite union of semistrata). Then $\hat{\Sigma}_x = \pi(\Sigma_x) = \pi(S_x) - \pi(B_x)$, i.e.,*

$$\hat{\Sigma}_x = \{z \in \mathbf{R}^m \mid z \in \pi(S_x), z \notin \pi(B_x), \operatorname{grad}(z) \geq 0\}$$

Unfortunately, we need at most $2^n - 1$ inequalities, obtained from the symmetric minors of $\operatorname{grad}(z)$. A direct description of a d-dimensional stratum by means of equations and (strict) inequalities can be obtained from the constraint that the type of $\operatorname{grad}(z)$ equals $(d, 0)$.

Proposition 9. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group and d be the dimension of $\mathbf{R}^n/G$. Then, for $i > 0$, all $(d+i) \times (d+i)$-minors of $\operatorname{grad} z$ vanish identically.*

Proof. Note that $d = \max\{\operatorname{rank}(d)\,\pi(v) \mid v \in \mathbf{R}^n/G\}$. Hence all $(d+i) \times (d+i)$ minors of $\operatorname{grad} z$ vanish identically in v. □

Remark 2. By using Theorem 1.2.1 and Proposition 1.2.9 we obtain (at most) $2^d - 1$ inequalities, where $d = \dim \mathbf{R}^n/G$. Since each semistratum (and in particular the orbit space) is a basic closed set in $\mathcal{V}(I) \subseteq \mathbf{R}^m$ it can be defined by $\frac{1}{2}d(d+1)$ inequalities, see [13].

In case one is interested in 'generic' properties of a stratum Σ_y, respectively, semistratum $\hat{S}_y$ (or of the orbit space) a set of $dim(\hat{S}_y)$ inequalities suffice and can be easily obtained from the matrix $\operatorname{grad}(z)$. We call two semialgebraic sets $S, T \subset V$ *generically equivalent* in the algebraic set V if $\dim((S \setminus T) \cup (T \setminus S)) < \dim(V)$.

Proposition 10. *Let $x \in \mathbf{R}^n, d = \dim \hat{S}_{\pi(x)} = \pi(S_x)$ and let $\Delta_1, \Delta_2, \ldots, \Delta_{2^d-1}$ be all $k \times k$ symmetric minors of $\operatorname{grad}(z), 1 \leq k \leq d$. There exist $i_1, i_2, \ldots, i_d$ such that Δ_{i_k} is a $k \times k$-minor and the semialgebraic set $\hat{S}_{\pi(x)}$ is generically equivalent to $\{z \in \mathbf{R}^m \mid z \in \mathcal{V}(I), \Delta_{i_1}(z) > 0, \ldots, \Delta_{i_d}(z) > 0\}$ where $I = \mathcal{I}(\hat{S}_{\pi(x)})$. In particular, the orbit space $\mathbf{R}^n/G$ is generically equivalent to $\{z \in \mathbf{R}^m \mid z \in \mathcal{V}(I), \Delta_1(z) \geq 0, \ldots, \Delta_n(z) \geq 0\}$ where $I = \mathcal{I}(\pi(\mathbf{R}^n))$.*

Proof. Note that $v \in \hat{S}_x$ implies $\operatorname{rank}(\operatorname{grad}(z)) \leq d$, hence for $i > 0$ all $(d+i) \times (d+i)$-minors of $\operatorname{grad}(z)$ vanish identically on $\hat{S}_{\pi(x)}$. Suppose that $P(z)$ is so arranged that the first d principal minors of $P(z)$ do not vanish identically on $\hat{S}_{\pi(x)}$. Let $S = \{z \in \hat{S}_x \mid P(z) \geq 0\}$ and $T = \{z \in \hat{S}_x \mid \Delta_1(z) > 0, \ldots, \Delta_d(z) > 0\}$. Note that $z \in T$ implies that $P(z)$ has rank equal to d. Hence there exists a symmetric matrix $A_{i_1,i_2,\ldots,i_d}$ whose entry at position (r,s) equals $P(z)_{i_r,i_s}$, such that $\operatorname{rank}(A_{i_1,i_2,\ldots,i_d}) = d$ and all other submatrices of $P(z)$ of rank d are conjugated to $A_{i_1,i_2,\ldots,i_d}$ (see for instance Section IX in [14]). Since $A_{1,2,\ldots,d}$ is positive definite, we have $\operatorname{rank}(A_{1,2,\ldots,d}) = d$ and therefore $P(z)$ is of type $(d,0)$ whenever $z \in T$, hence $T \subset S$. Since the difference $S \setminus T$ is contained in the algebraic set $\{z \in \hat{S}_{\pi(x)} \mid \prod_{i=1}^{d} \Delta_i(z) = 0\}$ which is of dimension less than d (no minor vanishes identically and $\hat{S}_{\pi(x)}$ is irreducible), the claim follows. Note that strict inequalities may be relaxed (to inequalities of the form $f \geq 0$) without changing the generic equivalence class. □

3 Constructing the Stratification

As shown in Section 2.2 the d-dimensional components of the strata can be computed by conditions on the rank of the matrix $d\pi(v)$. In this section we provide an algorithm together with necessary tools for the construction of a stratification of the representation- and the orbit space.

More precisely, given a d-dimensional connected component C of a stratum (obtained from rank conditions), the corresponding stratum is given by the orbit of C. The same holds true for the associated semistrata. In this way we construct the stratification of the orbit space out of the stratification of the representation space by computing the image of π (recall Corollary 2.2.2). It remains to add a set of inequalities obtained from the Theorem of Procesi and Schwarz (Theorem 1.2.1, Corollary 2.2.2 and Proposition 3.2.12). We also present an algorithm for computing the stabilizer of a given vector subspace of $\mathbf{K}^n$, which may be used to distinguish the symmetry type of strata[5] of the same dimension.

All used algorithms but the computation of inequalities rely on algebraically closed ground fields. For this reason we present properties of complexifications of real varieties below.

3.1 On the Complexification of a Group-Action

We briefly mention some relations between a compact Lie group G and its complexification and the real- and complex orbit space. More precisely, given fundamental invariants $\pi_1, \pi_2, \ldots, \pi_m \in \mathbf{R}[x_1, x_2, \ldots, x_n]$ of G, in order to describe the orbit space we have to compute the image of the morphism π by Elimination Theory, i.e., one computes the ideal I of relations among $\pi_1, \pi_2, \ldots, \pi_m$, which requires an algebraically closed ground field. As we have already seen, the orbit space of G may be properly be contained in the real algebraic set $\mathcal{V}(I) \subseteq \mathbf{R}^m$. Therefore we have to take care if the computations performed over an algebraically closed field are valid over $\mathbf{R}$. Several important results are based on Kempf-Ness Theory. We refer, e.g., to [15].

Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group defined by the ideal[6] $I_G \subset \mathbf{R}[s_1, s_2, \ldots, s_m]$. The complexification of G is the zero set of I_G over the complex numbers, denoted by $G_{\mathbf{C}}$. Note that $G_{\mathbf{C}}$ is a complex reductive group with coordinate ring $\mathbf{C}[s_1, s_2, \ldots, s_m]/I_G = \mathbf{R}[s_1, s_2, \ldots, s_m]/I_G \otimes_{\mathbf{R}} \mathbf{C}$ and that G is Zariski-dense in $G_{\mathbf{C}}$. The ideals defining the (real) orbit and the stabilizer of a point $v \in \mathbf{R}^n$ can be computed by Elimination Theory from the ideal I_G and the necessary constructions.

By Hilbert's Finiteness Theorem the invariant ring of G is finitely generated, hence it follows $\mathbf{R}[t_1, t_2, \ldots, t_n]^G = \mathbf{R}[h_1, h_2, \ldots, h_m]$ for some homogeneous invariants $h_1, h_2, \ldots, h_m$. The action of G complexifies to an action of $G_{\mathbf{C}}$ on $\mathbf{C}^n$ and the invariant ring of $G_{\mathbf{C}}$ equals $\mathbf{C}[t_1, t_2, \ldots, t_n]^{G_{\mathbf{C}}} = \mathbf{R}[h_1, h_2, \ldots, h_m] \otimes_{\mathbf{R}} \mathbf{C}$. Hence the Hilbert map $\pi : \mathbf{R}^n \to \mathbf{R}^m$ complexifies to $\pi_{\mathbf{C}} : \mathbf{C}^n \to \mathbf{C}^m$ and $\pi_{\mathbf{C}}(\mathbf{C}^n) = \overline{\pi(\mathbf{R}^n)}^{z}$ (closure in $\mathbf{C}^m$). Let I be the ideal of relations of $h_1, h_2, \ldots, h_m$. Since $\mathcal{V}(I) = \overline{\pi(\mathbf{R}^m)}^{z}$ over $\mathbf{R}$, by Procesi and Schwarz (see Theorem 1.2.1) we have $\mathbf{R}^n/G = \{z \in \mathcal{V}(I) \cap \mathbf{R}^m \mid \operatorname{grad}(z) \geq 0\}$ where the latter closure is taken in $\mathbf{R}^m$.

[5] Strata of the same dimension may have different stabilizers of the same dimension but different number of connected components

[6] Compact Lie groups are algebraic groups, see for instance [10].

3.2 Stratification of the Representation Space

By using the results stated in Section 2 we are now able to provide an algorithm for computing a stratification $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ of the representation space of a compact Lie group G. The stratification of the orbit space $\mathbf{R}^m/G$ is obtained by computing the ideals of the images $\pi(\Sigma_1), \pi(\Sigma_2), \ldots, \pi(\Sigma_r)$ and adding appropriate inequalities to each set of equations.

Algorithm 1 *RepSpaceSemisstrata(I_G, ψ)*
In: *Ideal defining a compact Lie group $G \subset GLn\mathbf{R}$, ψ a list of polynomials in $\mathbf{R}[s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_n]$ defining the action of G.*
Out: *list of equations defining the semistrata $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ of G and their generic stabilizer .*
begin
$\pi = (\pi_1, \pi_2, \ldots, \pi_r)$; *// algebra generators of* $\mathbf{R}[t_1, t_2, \ldots, t_n]^G$;
$d = \dim \mathbf{R}^m/G$ *// dimension of the orbit space*
for $i = 1$ **to** d **do**
  $J_d = d \times d$ *minors of* $d\pi$; *// all $d \times d$ minors of the Jacobian*
  collectedSpaces = *primary decomposition of* $\sqrt{J_d}$.
  $c := 1$;
  for *each* $V \in$ *collectSpaces*$[i]$ **do**
    orbitV $= \psi(G, V)$; *// orbit of V*
    if *orbitV* $\notin \bigcup_{j=1}^{c-1}$ *Semistrata*$[d][j]$ **then begin**
      Semistrata$[d][c]$ = *Semistrata*$[d][c] \cup$ *orbitV*;
      stabilizer$[d][c]$ = *Stabilizer*(I_G, ψ, V); *// representative of the orbit-type*
      $c = c + 1$;
    end
  end-for;
end-for;
return([*Semistrata*, *stabilizer*]);
end *RepSpaceSemisstrata*.

A set of fundamental invariants for G may be computed by the algorithm given in [4], which works for all reductive groups. Algorithms restricted to compact Lie groups can be found in [7].

We are left with the problem of computing a representative of an orbit type $[v]$, i.e, given a semistratum S_x, find equations for the 'generic' stabilizer G_ξ of S_x. By computing a primary decomposition of the ideal of G_ξ we obtain the index $G_\xi/(G_\xi)_0$

Proposition 11. *Let G be an algebraic group defined by the ideal $I_G \subseteq \mathbf{K}[s_1, s_2, \ldots, s_m]$, let $\alpha : G \times \mathbf{K}^n \to \mathbf{K}^n$ be a linear action, let $V \subseteq \mathbf{K}^n$ be a subspace of dimension d and let $\phi = (\phi_1, \phi_2, \ldots, \phi_n), \phi_i \in \mathbf{K}[a_1, a_2, \ldots, a_d]$, be a parameterization of V. Define the ideals $I = \langle I_G, \alpha_i(s,t) - t_i, t_i - \phi_i : 1 \le i \le n\rangle \subset \mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n]$ and $J = I \cap \mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_m]$ and the (partial) substitution map $\varphi_{\mathbf{b}} : \mathbf{K}(a_1, a_2, \ldots, a_n) \to \mathbf{K}$, $a_i \longmapsto b_i$ for $(b_1, b_2, \ldots, b_n) \in \mathbf{K}^n$. There exists a non-empty Zariski-open set $U \subseteq V$ such that*

$$u \in U \Longrightarrow \varphi_u(J) \cong \mathcal{I}(G_u).$$

Proof. After a finite number of steps we obtain a Gröbner basis of I. In each step we collect the following data: If multiplication by a polynomial f occurs then let P_f be the set of all coefficients of monomials in f which contain some a_i. When computing $f - g$ then add all rational functions in $a_1, a_2, \ldots, a_n$ which are obtained from solving $f - g = 0$ by comparing coefficients. Exclude these sets from $\mathbf{K}^n$. □

Algorithm 2 *Stabilizer(I_G, ψ, I_V)*
In: *ideal I_G of a compact group G, ideal I_V of a vectorspace V.*
Out: *equations of the stabilizer*
Note: *Basering is* $\mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_n]$.
begin
$I = std(I_V)$; *// Gröbner Basis of V*

$c = 0$;
for $i = 1$ **to** n **do**
  if $\deg(NormalForm(t_i, I)) > 0$ *then* **begin**
    $c := c + 1$;
    $I = std(I \cup \{t_i - a_c\})$;
  end-if
end-for
$I = I \cup \{\psi_i - t_i : 1 \leq i \leq n\}$;
$J = std(I) \cap \mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_k]$;
return(J);
end *Stabilizer.*

Remark 3. An alternative way to compute the number of connected components of the stabilizer is as follows. Compute the generic orbit $G(\xi)$ of V and determine a primary decomposition and the multiplicity of $G(\xi)$ (see [3]).

3.3 Stratification of the Orbit Space

Given a (semi-)stratification of the representation space, the computation of the stratification of the orbit space is essentially the computation of the matrix $\mathrm{grad}(z)$ and its symmetric minors. If G is not finite then the dimension of the representation space is greater than the dimension of the orbit space, which yields a reduction on the number of inequalities.

Proposition 12. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group and $\pi_1, \pi_2, \ldots, \pi_m$ be homogeneous generators of the invariant ring of G. Suppose that $\pi_1, \pi_2, \ldots, \pi_d$ are algebraically independent and $d = \dim \mathbf{R}^n/G$. Inequalities defining $\mathbf{R}^n/G$ are given by the symmetric minors of $grad'(z)$, which is the matrix obtained from* $\mathrm{grad}(z)$ *by deleting all $d + i$-th rows and columns for $i \geq 1$.*

Proof. Note that $d = \dim \mathbf{R}^m/G = \mathrm{rank}\,(d\pi(v))$ hence any $(d+i) \times (d+i)$ minor, $i \geq 1$, of $d\pi(v)$ vanishes identically on $\mathbf{R}^m/G$. □

The algorithm returns a list of strata of the orbit space of G sorted by dimension. Each stratum $\hat{\Sigma}_{d,i}$ is described as a triple $[[f_1, f_2, \ldots, f_r], [g_1, g_2, \ldots, g_{2^d-1}], [h_1, h_2, \ldots, h_s]]$ where $\hat{\Sigma}_{d,i} = \{z \in \mathbf{R}^m \mid f_1(z) = 0, \ldots, f_r(z) = 0, g_1(z) > 0, \ldots, g_{2^d-1}(z) > 0, h_1(z) \neq 0, \ldots, h_s(z) \neq 0\}$.

Algorithm 3 *OrbitSpaceStrata(π, semistrata)*
In: $\pi = \pi_1, \pi_2, \ldots, \pi_m$ *fundamental invariants of $G \subseteq O_n(\mathbf{R})$, list of semistrata of the representation space. Assume that $\pi_1, \pi_2, \ldots, \pi_d$ are algebraically independent and $d = \dim \mathbf{R}^n/G$.*
Out: *list of strata of the orbit space (given by equations and inequalities)*
begin
$\mathrm{grad}(z) = (d\pi_i, d\pi_j)_{i=1..n}^{j=1..n}$;
$c = 0$;
for $d = 1$ **to** $|semistrata|$ **do**
  $I =$ *set of $d \times d$ minors of* $\mathrm{grad}'(z)$; *// see Proposition 3.2.12*
  $J =$ *set of $k \times k$ minors of* $\mathrm{grad}'(z), 1 \leq k < d$;
  for $i = 1$ **to** $|semistrata[d]|$ **do**
    $strata[d][i] = [semistratum[d][i], I, J]$;
  end-for
end-for
return$(strata)$;
end *OrbitSpaceStrata.*

Remark 4. A stratification up to generic equivalence can be obtained by replacing the line defining I by the line

$I =$ set of first $d \times d$ principal minors of $\mathrm{grad}'(z)$; // $\mathrm{grad}'(z)$ arranged as in Proposition 2.2.10

and defining $J = \emptyset$. In several occasions the sets defined by the modified algorithm (where the inequalities are taken to be strict) provide descriptions of the corresponding strata. In general, this is not true as can be seen by considering the action of $G = \mathbf{Z}_4 \times SO(3)$ on $\mathbf{R}^6$ (see example 3 in [1]).

Example 2. We consider the compact Lie group $G = O(1) \times \mathbf{Z}_2 \subset GL_2(\mathbf{R})$ ($O(1)$ acting on the first two coordinates, $\mathbf{Z}_2$ acting on the third coordinate) defined by the equations $s_1^2 + s_2^2 - 1, s_3^2 + s_4^2 - 1, s_1 s_3 + s_2 s_4, s_5^2 - 1$. The Jacobian of $\pi : \mathbf{R}^3 \to \mathbf{R}^2, (x, y, z) \mapsto (x^2 + y^2, z^2)$ equals $\begin{pmatrix} 2x & 2y & 0 \\ 0 & 0 & 2z \end{pmatrix}$, hence we have (all variables range over $\mathbf{R}$)

$$\begin{aligned} \Sigma_0 &= \{v = (a, b, c) \mid \operatorname{rank}(d\pi(v)) = 0\} = \{(0, 0, 0)\} \\ \Sigma_{1,1} \cup \Sigma_{1,2} &= \{v = (a, b, c) \mid \operatorname{rank}(d\pi(v)) = 1\} = \{(a, b, 0) \mid a \neq 0 \text{ or } b \neq 0\} \cup \{(0, 0, c) \mid c \neq 0\} \\ \Sigma_2 &= \{v = (a, b, c) \mid \operatorname{rank}(d\pi(v)) = 2\} = \{(a, b, c) \mid ac \neq 0 \text{ or } bc \neq 0\} \end{aligned}$$

By using the algorithm Stabilizer we obtain for the associated stabilizers the table

Stratum	Σ_0	$\Sigma_{1,1}$	$\Sigma_{1,2}$	Σ_2
Stabilizer	G	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$O(1)$	$\mathbf{Z}_2$

As an example, the ideal $I \subset \mathbf{R}(a_1, a_2)[s_1, s_2, \ldots, s_5]$ defining the generic stabilizer of $\Sigma_{1,1}$ is given by

$$\begin{aligned} I = \langle & a_1 s_3 + a_2 s_4 - a_2, a_1^3 s_2 + a_1 a_2^2 s_3 + a_1^2 a_2 + a_2^3 s_4 - a_1^2 a_2 - a_2^3, \\ & a_1 s_1 + a_2 s_2 - a_1, s_5^2 - 1, a_1^2 + a_2^2 s_4^2 + a_1 a_2 s_3 - a_2^2 s_4 - a_1^2 \rangle \end{aligned}$$

Substitution of $(a, b) \in \Sigma_{1,1}$ for (a_1, a_2) yields the the ideal of the stabilizer of the point (a, b). The matrix $\operatorname{grad}(z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ is as in Example 1.2.1, hence we obtain the following description of the strata:

$$\begin{aligned} \hat{\Sigma}_0 &= \{(0, 0)\} \\ \hat{\Sigma}_{1,1} \cup \Sigma_{1,2} &= \{(z_1, 0) \mid z_1 > 0\} \cup \{(0, z_2) \mid z_2 > 0\} \\ \hat{\Sigma}_2 &= \{(z_1, z_2) \mid z_1 > 0, z_1 z_2 > 0\} \end{aligned}$$

3.4 On the Construction of Smooth Potentials

As mentioned in the introduction, a stratification is useful for the modeling of spontaneous symmetry breaking. In many applications one has to construct a G-invariant potential[7] V (whence V defines a potential $\hat{V}$ on the orbit space $\mathbf{R}^n/G$) which assumes extrema on various strata[8].

If V must be smooth following observation due to Abud and Satori [1].

Proposition 13. *The condition that $V(x)$ has an extremum on a stratum σx is equivalent to the condition that $\hat{V}(z)$ has conditional extremum on $\hat{\Sigma}_z$.*

Note that a conditional extremum can be determined by Lagrange multipliers.

If V is a polynomial potential then we may omit inequalities by considering the Zariski-closure of $\hat{\Sigma}_z$.

Proposition 14. *Let $\Sigma_x \subseteq \mathbf{R}^n$ be a stratum of G. The (real) Zariski-closure of $\hat{\Sigma}_x$ is given by the (real) zero set of the ideal $I \subset \mathbf{R}[t_1, t_2, \ldots, t_m]$ which defines the semistratum $\hat{S}_x$.*

[7] The potential V often depends on additional parameters like temperature, time, etc.

[8] The strata where extrema of V must appear rely on the physics of the underlying model and may depend on the parameters of V, see for instance [18].

Conclusion and Future Work

We have presented an alternative approach for the computation of stratifications of compact Lie groups and have pointed out, that d inequalities suffice in order to describe a d-dimensional stratum of the orbit space up to generic equivalence. The advantage of the approach lies in the fact, that several applications (like the construction of polynomial potentials) do not necessarily need inequalities defining the orbit space and that primary decomposition is faster on the representation space than on the orbit space. Additionally, if the representation of G is not orthogonal, out approach may be used to compute the Zariski-closures of the strata of the orbit space. From a practical point of view, the dependence on orthogonal representations should be avoided. Also of interest is the study of an optimal lower- and upper bound for the number of inequalities needed to define strata and the orbit space.

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