# On Stability of Body Motions in Fluid

Valentin Irtegov and Tatyana Titorenko

Institute of Systems Dynamics and Control Theory, SB RAS,
134, Lermontov str., Irkutsk, 664033, Russia,
irteg@icc.ru

**Abstract.** The paper discusses some results of stability analysis of helical motions of rigid body in fluid. The research methods are based on Lyapunov's classical results [1]. For the purpose of computations, computer algebra systems "Mathematica 4.2", "Maple 8", and the software package [2] have been used.

## 1 Introduction

A broad class of physical phenomena can be described by Euler–Poisson type equations. Among these phenomena there are: motion of a rigid body having a fixed point, motion of systems of rigid bodies of a definite type, motion of rigid body in fluid, motion of weakly non-homogeneous ideal fluid inside an ellipsoid on a class of spatially linear fields of velocities and temperatures, etc. The diversity of applications of Euler–Poisson type equations allows one to hope that methods and results bound up with the analysis of these equations may be of interest for a wide group of specialists.

Recently there appeared many publications on the problems of integrability of various types of Euler–Poisson equations. Under different restrictions imposed on the parameters of the equations of motion, a number of cases, which assume additional first integrals, have been revealed. First results in this direction, which are concerned with rigid body motion in fluid, take back to Kirchhoff, Clebsch, Steklov, Lyapunov, Chaplygin, etc. A rather abundant bibliography on this question may be found in [3]. Recently another case [4], when the equations of a rigid body motion in fluid are quite integrable, has been found. A new additional algebraic first integral in this case has the fourth order.

Stability investigation of rigid body helical motions in ideal fluid without any restrictions on the body parameters was conducted for the first time in Lyapunov's paper [5].

In the present paper, the problem is considered with the values of parameters corresponding to Sokolov's case [4]. An attempt has been made to find steady-state motions of the body in this case and investigate their stability on the basis of Lyapunov's second method. Under the parameters indicated the differential equations of a rigid body motion in fluid are as follows:

$$\begin{aligned}
\dot{M}_1 &= -8\alpha^2\gamma_2\gamma_3 + \alpha(\gamma_2 M_1 + \gamma_1 M_2) + M_2 M_3,\\
\dot{M}_2 &= 4\alpha^2\gamma_1\gamma_3 - M_1 M_3 + 2\alpha(\gamma_3 M_3 - \gamma_1 M_1),\\
\dot{M}_3 &= 4\alpha^2\gamma_1\gamma_2 - \alpha(\gamma_3 M_2 + \gamma_2 M_3),\\
\dot{\gamma}_1 &= \alpha\gamma_1\gamma_2 - \gamma_3 M_2 + 2\gamma_2 M_3,\\
\dot{\gamma}_2 &= \alpha(\gamma_3^2 - \gamma_1^2) + \gamma_3 M_1 - 2\gamma_1 M_3,\\
\dot{\gamma}_3 &= -\alpha\gamma_2\gamma_3 - \gamma_2 M_1 + \gamma_1 M_2,
\end{aligned} \tag{1}$$

where $\bar{M} = \{M_1, M_2, M_3\}$ is the vector of "impulse moment", and $\bar{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$ is the vector of "impulse force" in the projections onto the axes rigidly bound up with the body, $\alpha$ is a constant.

The latter system assumes the four algebraic first integrals:

$$\begin{aligned}
2H &= 4\alpha^2\gamma_2^2 - 4\alpha^2\gamma_3^2 + 2\alpha\gamma_3 M_1 + M_1^2 + M_2^2 + 2\alpha\gamma_1 M_3 + 2M_3^2 = 2h,\\
V_1 &= \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = m,\\
V_2 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_1,\\
V_3 &= (3\alpha^2\gamma_2 M_2 + \alpha(\alpha\gamma_3 + M_1)(2\alpha\gamma_1 + M_3))^2 + \alpha^2(-\alpha\gamma_1 + M_3)^2(M_2^2\\
&\quad + (2\alpha\gamma_1 + M_3)^2) = c_2.
\end{aligned} \tag{2}$$

To the end of finding steady-state solutions and investigation of their stability, several techniques are currently available [6]. In the present problem, practically any of them leads to bulky computations. We will apply a rather natural classical method allowing one to perform a substantial part of computations with the aid of computer algebra systems.

## 2 Finding Steady-State Motions

Let us use the Routh–Lyapunov theorem [5]. Compose the following complete linear bundle of first integrals of the problem:

$$K = H - \lambda_1 V_1 - \frac{1}{2}\lambda_2 V_2 - \lambda_3 V_3 \tag{3}$$

and write out the steady-state conditions for it with respect to all the variables:

$$\begin{aligned}
\frac{\partial K}{\partial M_1} &= \alpha\gamma_3 - \lambda_1\gamma_1 + M_1 - 2\alpha^2\lambda_3(2\alpha\gamma_1 + M_3)[3\alpha\gamma_2 M_2 + (\alpha\gamma_3 + M_1) \\
&\quad \times(2\alpha\gamma_1 + M_3)] = 0, \\
\frac{\partial K}{\partial M_2} &= -\lambda_1\gamma_2 + M_2 - 2\alpha\lambda_3[M_2(-\alpha\gamma_1 + M_3) + 3\alpha\gamma_2(3\alpha\gamma_2 M_2 + (\alpha\gamma_3 + M_1)(2\alpha\gamma_1 \\
&\quad +M_3))] = 0, \\
\frac{\partial K}{\partial M_3} &= \alpha\gamma_1 - \lambda_1\gamma_3 + 2M_3 - 2\alpha\lambda_3(-\alpha\gamma_1 + M_3)^2(2\alpha\gamma_1 + M_3) + (\alpha\gamma_3 + M_1)[3\alpha\gamma_2 M_2 \\
&\quad +(\alpha\gamma_3 + M_1)(2\alpha\gamma_1 + M_3)] + (M_3 - \alpha\gamma_1)(M_2 + (2\alpha\gamma_1 + M_3)^2)) = 0, \\
\frac{\partial K}{\partial \gamma_1} &= -\lambda_2\gamma_1 - \lambda_1 M_1 + \alpha M_3 - 2\alpha\lambda_3[-2(M_3 + \alpha\gamma_1)(2\alpha\gamma_1 + M_3) + 2(M_1 + \alpha\gamma_3) \\
&\quad \times[3\alpha\gamma_2 M_2 + (\alpha\gamma_3 + M_1)(2\alpha\gamma_1 + M_3)] - (M_3 - \alpha\gamma_1)(M_2 + (2\alpha\gamma_1 + M_3)^2)) = 0, \\
\frac{\partial K}{\partial \gamma_2} &= 4\alpha\gamma_2 - \gamma_2\lambda_2 - \lambda_1 M_2 - 6\alpha\lambda_3 M_2(3\alpha\gamma_2 M_2 + (\alpha\gamma_3 + M_1)(2\alpha\gamma_1 + M_3)) = 0, \\
\frac{\partial K}{\partial \gamma_3} &= -4\alpha\gamma_3 - \lambda_2\gamma_3 + \alpha M_1 - \lambda_1 M_3 - 2\alpha\lambda_3(2\alpha\gamma_1 + M_3)(3\alpha\gamma_2 M_2 + (\alpha\gamma_3 + M_1) \\
&\quad \times(2\alpha\gamma_1 + M_3)) = 0.
\end{aligned} \tag{4}$$

It is known [5] that solutions of the latter system will define invariant manifolds of steady-state motions (IMSMs) of the system (1) (in particular, steady-state motions). In the general case, these steady-state solutions may contain the parameters $\lambda_i$, which are included into the family of integrals $K$, and so, represent families of steady-state solutions. Therefore, within the framework of the employed approach, to the end of reaching the formulated goal (finding out IMSMs of the system (1) corresponding to the family of first integrals $K$) we have come to necessity of investigating the solutions of six nonlinear algebraic equations involving three parameters. The method of Gröbner basis is one of contemporary methods for solving such problems. It is known that the form of Gröbner basis is substantially dependent on the technique of ordering of variables in the system under scrutiny. There is the need to use different orderings not only for obtaining the most "suitable" basis, but also, for example, for the purpose of finding free variables, estimating the number of solutions, etc. Note, a structure of expressions in Gröbner basis can also be used for finding transformations of the variables in which the initial problem could have a more compact form. Constructing the Gröbner bases and analysis of these bases for the system (4) have given evidence that it is comfortable to choose, for example, the following variables in the capacity of new ones:

$$\begin{aligned}
&x_1 = M_3 - \alpha\gamma_1,\ x_2 = M_1 - \alpha\gamma_3,\ x_3 = M_2, \\
&x_4 = -3\alpha\gamma_2,\ x_5 = 2\alpha\gamma_1 + M_3,\ x_6 = \gamma_3
\end{aligned} \tag{5}$$

(naturally, the choice under such an approach is not unequivocal). By the way, these variables coincide with those proposed on account of other considerations in [4].

In terms of these variables, the expression (3) for the bundle of first integrals has the form:

$$K = \frac{1}{18}(9x_2^2 + 9x_3^2 + 4x_4^2 + 2(2x_1 + x_5)(x_1 + 2x_5) - 45\alpha^2 x_6^2) - \lambda_1(\frac{x_4^2 + (x_1 - x_5)^2}{9\alpha^2} + x_6^2)$$
$$-\lambda_2(x_1x_6 - \frac{x_1x_2 + x_3x_4 - x_2x_5}{3\alpha}) - \alpha^2\lambda_3((x_3x_4 - x_2x_5)^2 + x_1^2(x_3^2 + x_5^2)). \quad (6)$$

The steady-state conditions for $K$ with respect to the variables $x_1, x_2, x_3, x_4, x_5, x_6$ writes as follows:

$$\begin{aligned}
&\frac{2}{9}\ (2 - \frac{\lambda_1}{\alpha^2})x_1 + \frac{\lambda_2}{3\alpha}x_2 - 2\alpha^2\lambda_3 x_1 x_3^2 + \frac{1}{9}\ (5 + \frac{2\lambda_1}{\alpha^2})x_5 - 2\alpha^2\lambda_3 x_1 x_5^2 - \lambda_2 x_6 = 0,\\
&\frac{\lambda_2}{3\alpha}x_1 + x_2 - \frac{\lambda_2}{3\alpha}x_5 + 2\alpha^2\lambda_3 x_3x_4x_5 - 2\alpha^2\lambda_3 x_2 x_5^2 = 0,\\
&x_3 - 2\alpha^2\lambda_3 x_1^2 x_3 + \frac{\lambda_2}{3\alpha}x_4 - 2\alpha^2\lambda_3 x_3 x_4^2 + 2\alpha^2\lambda_3 x_2x_4x_5 = 0,\\
&\frac{\lambda_2}{3\alpha}x_3 + \frac{2}{9}\ (2 - \frac{\lambda_1}{\alpha^2})x_4 - 2\alpha^2\lambda_3 x_3^2 x_4 + 2\alpha^2\lambda_3 x_2x_3x_5 = 0,\\
&\frac{1}{9}\ (5 + \frac{2\lambda_1}{\alpha^2})x_1 - \frac{\lambda_2}{3\alpha}x_2 + 2\alpha^2\lambda_3 x_2x_3x_4 + \frac{2}{9}\ (2 - \frac{\lambda_1}{\alpha^2})x_5 - 2\alpha^2\lambda_3 x_1^2 x_5 - 2\alpha^2\lambda_3 x_2^2 x_5 = 0,\\
&\lambda_2 x_1 + (5\alpha^2 + 2\lambda_1)x_6 = 0.
\end{aligned} \quad (7)$$

Let us conduct a preliminary analysis of a set of solutions for the system (7). For this purpose let us consider the problem first of all for the fixed (specially chosen) parameters $\lambda_i$:

$$\lambda_1 = -\frac{5}{2}\alpha^2,\ \lambda_2 = 3\alpha,\ \lambda_3 = \frac{1}{\alpha^2}.$$

Under such values of $\lambda_i$ equations (7) acquire a rather simple form:

$$\begin{aligned}
&x_1 + x_2 - 2x_1(x_3^2 + x_5^2) - 3\alpha x_6 = 0,\\
&x_1 + x_2 - x_5 + 2x_5(x_3x_4 - x_2x_5) = 0,\\
&x_3 + x_4 - 2x_3(x_1^2 + x_4^2) + 2x_2x_4x_5 = 0,\\
&x_3 + x_4 - 2x_3(x_3x_4 - x_2x_5) = 0,\\
&-x_2 + 2x_2x_3x_4 + x_5 - 2x_5(x_1^2 + x_2^2) = 0,\\
&3\alpha x_1 = 0,
\end{aligned} \quad (8)$$

what allows one to conduct complete analysis of the set of their solutions by the Gröbner basis method.

Hence, the last one of above equations gives $x_1 = 0$. System (8) – after substitution $x_1 = 0$ into it and excluding $x_6$ from the resulting equations – writes:

$$\begin{aligned}
&x_2 + x_5(2x_3x_4 - 2x_2x_5 - 1) = 0,\\
&x_3 - x_4(2x_3x_4 - 2x_2x_5 - 1) = 0,\\
&x_4 - x_3(2x_3x_4 - 2x_2x_5 - 1) = 0,\\
&x_5 + x_2(2x_3x_4 - 2x_2x_5 - 1) = 0.
\end{aligned} \quad (9)$$

For the system (9), Gröbner bases have been constructed for different lexicographical ranking of the variables $x_2, x_3, x_4, x_5$.

For example, in case of ordering of $x_2 > x_3 > x_4 > x_5$ one obtains a basis of the following form:

$$\begin{aligned}
&(x_4^2 + x_5^2 - 1)(x_4^2 + x_5^2)x_5,\\
&(x_4^2 + x_5^2 - 1)(x_4^2 + x_5^2)x_4,\\
&-x_3 - x_4 + 2x_4(x_4^2 + x_5^2),\\
&x_2 - x_5 + 2x_4^2x_5 + x_5^3.
\end{aligned}$$

Analysis of all the bases constructed allows, first of all, to draw a conclusion that the system under scrutiny is compatible – the bases do not contain a constant. Further on, the system has one free variable, because for any ranking of the variables there is no polynomial in the constructed bases, which is dependent on only one, the smallest variable, but there exist polynomials dependent at least on the two last smallest variables. According to [7], the system under consideration has one free variable and, consequently, an infinite set of solutions. The function "Solve" of the computer algebra system (CAS) "Mathematica" allows one to obtain a system solution in the general form. Below, a solution obtained with the aid of the function "Solve" for the above basis is given.

$$\begin{aligned}
&\{\{x_2 \to -ix_4, x_3 \to -x_4, x_5 \to -ix_4\}, \{x_2 \to ix_4, x_3 \to -x_4, x_5 \to ix_4\},\\
&\{x_2 \to -\sqrt{1-x_4^2}, x_3 \to x_4, x_5 \to \sqrt{1-x_4^2}\}, \{x_2 \to \sqrt{1-x_4^2}, x_3 \to x_4, x_5 \to -\sqrt{1-x_4^2}\},\\
&\{x_2 \to -1, x_3 \to 0, x_5 \to 1, x_4 \to 0\}, \{x_2 \to 0, x_3 \to -1, x_4 \to -1, x_5 \to 0\},\\
&\{x_2 \to 0, x_3 \to 0, x_4 \to 0, x_5 \to 0\}, \{x_2 \to 0, x_3 \to 1, x_4 \to 1, x_5 \to 0\},\\
&\{x_2 \to 1, x_3 \to 0, x_5 \to -1, x_4 \to 0\}\}.
\end{aligned} \tag{10}$$

Substitution of real solutions (10) into the equations of motions shows that $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = \dot{x}_5 = \dot{x}_6 = 0$ for all these solutions. In other words, for all the solutions, the variables $x_1, x_2, x_3, x_4, x_5, x_6$ are constants, and the presence of free variables in a number of the solutions means only that these constants are related by some dependences.

Consider the problem in the neighbourhood of above one by making one of the parameters, for example, $\lambda_2$, free, i.e. assuming

$$\lambda_1 = -\frac{5}{2}\alpha^2,\ \lambda_2,\ \lambda_3 = \frac{1}{\alpha^2}. \tag{11}$$

The scheme of investigation given above can also be applied here. As well as in the previous case, the construction of Gröbner's bases for the system (7) (where $\lambda_i$ has the form (11)) under different lexicographical ordering of the variables allows one to find a set of free unknowns (the system has one free variable), and so to draw a conclusion that the system under consideration has an infinite set of solutions. Using the function "Solve", it is possible to obtain a solution of the system in the general form.

Let us now make the previous problem more complex. Some relations between the parameters will be used:

$$\lambda_1 = -\frac{5}{18}\lambda_2^2,\ \lambda_2,\ \lambda_3 = \frac{9}{\lambda_2^2}. \tag{12}$$

For such values of $\lambda_i$, analysis of the set of system (7) solutions with the aid of above scheme appears to be rather complicated in the aspect of computations. Direct application of the method of Gröbner bases to the system under scrutiny did not allow us to obtain a result within a reasonable time. Simultaneous application of resultants and Gröbner bases has given the possibility to obtain a rather wide set of solutions for our system.

Consider the process of solving this problem in more detail.

The steady-state equations (7) for $\lambda_i$ (12) – after eliminating $x_1, x_6$ and substituting $a = \lambda_2/(3\alpha)$ – assume the form:

$$\begin{aligned}
&(-\frac{4}{9a} + \frac{4a}{9} - \frac{9a}{5-5a^2})x_2 + \frac{2}{a^3}x_2x_3^2 + (1 + \frac{9a^2}{5-5a^2})x_5 - \frac{2}{a^2}x_3^2x_5\\
&-\frac{2}{a}\ (\frac{4}{9a^2} + \frac{5}{9} + \frac{9}{5-5a^2})x_3x_4x_5 + \frac{4}{a^5}x_3^3x_4x_5 + \frac{2}{a}\ (\frac{13}{9a^2} + \frac{5}{9} + \frac{9}{5-5a^2})x_2x_5^2\\
&-\frac{4}{a^5}x_2x_3^2x_5^2 - \frac{2}{a^2}x_5^3 + \frac{4}{a^5}x_3x_4x_5^3 - \frac{4}{a^5}x_2x_5^4 = 0,\\
&x_3 - \frac{2}{a^4}x_2^2x_3 + ax_4 - \frac{2}{a^2}x_3x_4^2 + \frac{4}{a^3}x_2x_3x_5 + \frac{2}{a^2}x_2x_4x_5 - \frac{8}{a^6}x_2x_3^2x_4x_5 - \frac{2}{a^2}x_3x_5^2\\
&+\frac{8}{a^6}x_2^2x_3x_5^2 + \frac{8}{a^5}x_3^2x_4x_5^2 - \frac{8}{a^8}x_3^3x_4^2x_5^2 - \frac{8}{a^5}x_2x_3x_5^3 + \frac{16}{a^8}x_2x_3^2x_4x_5^3 - \frac{8}{a^8}x_2^2x_3x_5^4 = 0,\\
&ax_3 + \frac{1}{9}\ (4+5a^2)x_4 - \frac{2}{a^2}x_3^2x_4 + \frac{2}{a^2}x_2x_3x_5 = 0,
\end{aligned} \tag{13}$$

$$-\frac{1}{9}\,(\frac{5}{a}+4a)x_2+\frac{2}{a^2}x_2x_3x_4+x_5-\frac{2}{a^2}\,(\frac{1}{a^2}+1)x_2^2x_5+\frac{10}{9a}\,(1-\frac{1}{a^2})x_3x_4x_5$$
$$+\frac{2}{9a}\,(\frac{23}{a^2}-5)x_2x_5^2-\frac{8}{a^6}x_2x_3x_4x_5^2-\frac{2}{a^2}x_5^3+\frac{8}{a^6}x_2^2x_5^3+\frac{8}{a^5}x_3x_4x_5^3-\frac{8}{a^8}x_3^2x_4^2x_5^3$$
$$-\frac{8}{a^5}x_2x_5^4+\frac{16}{a^8}x_2x_3x_4x_5^4-\frac{8}{a^8}x_2^2x_5^5=0.$$

As a result, we have to investigate four nonlinear algebraic equations dependent on the variables $x_2, x_3, x_4, x_5$. Application of the method of Gröbner bases to the system (13) did not allow us to obtain a result within an admissible time. The computations have been conducted on Pentium 4 (2 GB RAM, 1400 MHz); CAS "Mathematica" and "Maple" have been used.

Let us eliminate the variable $x_4$ from (13). Using the function "Resultant" of CAS "Mathematica" for computing resultants, let us write down the compatibility condition with respect to $x_4$ for each pair of equations (13). As a result, we obtain a system of 6 nonlinear algebraic equations with respect to three variables, which after factorization assume the form:

$$\begin{aligned}
&x_3\ f_1(x_2,x_3,x_5)=0,\\
&(20a^2+41a^4+20a^6-90x_3^2+90a^2x_3^2-40x_5^2-10a^2x_5^2+50a^4x_5^2)\\
&\quad\times(4a^2x_2+5a^4x_2-18x_2x_3^2-9a^3x_5-18x_2x_5^2)=0,\\
&x_3^2x_5\ f_2(x_2,x_3,x_5)=0,\quad x_3\ f_3(x_2,x_3,x_5)=0,\\
&x_3^2\ f_4(x_2,x_3,x_5)=0,\quad f_5(x_2,x_3,x_5)=0,
\end{aligned}\tag{14}$$

where $f_1, f_2, f_3, f_4, f_5$ are polynomials with respect to the variables $x_2, x_3, x_5$. These polynomials have rather bulky expressions, and so are omitted here.

As obvious from (14), the above system of equations may be decomposed into several subsystems, whose analysis can be conducted separately, by writing out Gröbner bases for each one.

For all the subsystems of system (14), the Gröbner bases were constructed under lexicographical ordering of the variables as the most suitable for analysis of the system. The total degree, and then inverse lexicographical ordering, was used in the cases when application of lexicographical ordering was complicated in virtue of the bulky system's equations.

Analysis of the bases constructed has given the possibility to draw a conclusion that system (14) (and hence also (13)) has one free variable and, consequently, an infinite set of solutions.

Consider an example of investigation of one of subsystems of (14) and write down the group of solutions obtained.

The first equation (14) gives one of the solutions for $x_3$: $x_3=0$. After its substitution into the rest of equations we have:

$$\begin{aligned}
&(5a^2+4a^4+10(a^2-1)x_5^2)(a^2(4+5a^2)x_2-9a^3x_5-18x_2x_5^2)=0,\\
&a^7(5+4a^2)x_2-9a^8x_5+18a^4(a^2+1)x_2^2x_5+2a^5(5a^2-23)x_2x_5^2+18a^6x_5^3\\
&-72a^2x_2^2x_5^3+72a^3x_2x_5^4+72x_2^2x_5^5=0.
\end{aligned}\tag{15}$$

As obvious from (15), the obtained system of equations can be decomposed into two subsystems. For each of the subsystems a Gröbner basis has been constructed.

The Gröbner basis corresponding to the first subsystem is:

$$\begin{aligned}
&a^2(5+4a^2)+10(a^2-1)x_5^2,\\
&45a^4(a^2-1)-2(25+31a^2+25a^4)x_2^2+180a(a^2-1)x_2x_5,\\
&-18a^3(5+4a^2)x_2+45a^4(a^2-1)x_5-2(25+31a^2+25a^4)x_2^2x_5,\\
&-45a^{12}(5+4a^2)+2a^2(405+729a^2+729a^4+604a^6+349a^8+100a^{10})x_2^2\\
&-180a^9(5+4a^2)x_2x_5-1620x_2^2x_5^2.
\end{aligned}$$

The Gröbner basis for the second subsystem writes:

$$\begin{aligned}
&x_5(a^4(a^2-1)-4a^2(4+5a^2)x_5^2+36x_5^4),\\
&(25a+31a^3+25a^5)x_2-9a^2(4+5a^2)x_5+162x_5^3,
\end{aligned}$$

$$a^2(4+5a^2)x_2 - 9a^3x_5 - 18x_2x_5^2,$$
$$(25+31a^2+25a^4)x_2^2 - 81a^2x_5^2,$$
$$(25a^3+31a^5+25a^7)x_2 - 9a^4(4+5a^2)x_5 + (50+62a^2+50a^4)x_2^2x_5.$$

Analysis of above bases allows one to draw the conclusion that system (15) has a finite number of solutions. A solution obtained for the second basis is given below. It represents families of helical motions of a rigid body in fluid, which are parameterized by $a$. In the first case, the solution is rather bulky, and so it is omitted.

$$\begin{aligned}
&\{\{x_2 \to -\frac{3a\sqrt{a^2(4+5a^2-\sqrt{25+31a^2+25a^4})}}{\sqrt{50+62a^2+50a^4}},\ x_5 \to -\frac{\sqrt{a^2(4+5a^2-\sqrt{25+31a^2+25a^4})}}{3\sqrt{2}}\},\\
&\{x_2 \to \frac{3a\sqrt{a^2(4+5a^2-\sqrt{25+31a^2+25a^4})}}{\sqrt{50+62a^2+50a^4}},\ x_5 \to \frac{\sqrt{a^2(4+5a^2-\sqrt{25+31a^2+25a^4})}}{3\sqrt{2}}\},\\
&\{x_2 \to -\frac{3a\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{\sqrt{50+62a^2+50a^4}},\ x_5 \to \frac{\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{3\sqrt{2}}\},\\
&\{x_2 \to \frac{3a\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{\sqrt{50+62a^2+50a^4}},\ x_5 \to -\frac{\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{3\sqrt{2}}\},\\
&\{x_5 \to 0, x_2 \to 0\}\}.
\end{aligned} \tag{16}$$

Analysis of other subsystems of (14) has been conducted similarly, and a group of solutions, which are not given here for brevity, has been obtained for each one.

## 3 Investigation of Stability of Steady-State Motions

Let us conduct investigation of stability of the obtained steady-state solutions (helical motions of a rigid body in fluid). The method of Lyapunov functions [6] (in particular, the Routh–Lyapunov theorem [5] noted above) is one of well-known approaches to solving this problem. The technique of investigation of stability is practically reducible to the verification of sign definiteness of a variation of the integral $K$ (6) in the neighbourhood of the steady-state solution (which is of interest for us) on the manifold which is defined by variations of the several first integrals included into the bundle $K$.

Consider one of the families of solutions (16) given above, for example, a steady-state solution of the form:

$$x_{10} = A_1,\ x_{20} = A_2,\ x_{30} = 0, x_{40} = 0,\ x_{50} = A_5,\ x_{60} = A_6,$$

where

$$A_1 = \frac{5(a^2-1)\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{3\sqrt{50+62a^2+50a^4}}, A_2 = \frac{3a\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{\sqrt{50+62a^2+50a^4}},$$
$$A_5 = -\frac{\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{3\sqrt{2}}, A_6 = \frac{a\sqrt{a^2(4+5a^2+\sqrt{25+31a^2+25a^4})}}{\alpha\sqrt{50+62a^2+50a^4}}.$$

The expressions for $x_{10}, x_{40}, x_{60}$ have been obtained from equations (7) after substituting the corresponding values of $x_{20}, x_{30}, x_{50}$ into them.

The variation of $K$ in the neighbourhood of given steady-state solution written in terms of the deviations

$$y_1 = x_1 - x_{10},\ y_2 = x_2 - x_{20},\ y_3 = x_3,\ y_4 = x_4,\ y_5 = x_5 - x_{50},\ y_6 = x_6 - x_{60}$$

will have the form:

$$\begin{aligned}\varDelta K = &\frac{1}{18a^2}(-18(A_1^2+A_2^2)A_5^2+5a^4((A_1-A_5)^2+9A_6^2\alpha^2)+18a^3(-A_2A_5\\
&+A_1(A_2-3A_6\alpha))+a^2(2(2A_1+A_5)(A_1+2A_5)+9(A_2^2-5A_6^2\alpha^2)))\\
&+(\frac{2}{9}+\frac{5a^2}{18}-\frac{A_5^2}{a^2})y_1^2+ay_1y_2+(\frac{1}{2}-\frac{A_5^2}{a^2})y_2^2+(\frac{1}{2}-\frac{A_1^2}{a^2})y_3^2+(a+\frac{2A_2A_5}{a^2})y_3y_4\\
&+\frac{4+5a^2}{18}y_4^2+(\frac{5}{9}-\frac{5a^2}{9}-\frac{4A_1A_5}{a^2})y_1y_5-(a+\frac{4A_2A_5}{a^2})y_2y_5\\
&+\frac{4a^2+5a^4-18(A_1^2+A_2^2)}{18a^2}y_5^2-3a\alpha y_1y_6+\frac{5}{2}(a^2-1)\alpha^2y_6^2-\frac{2A_1}{a^2}y_1y_3^2\\
&-\frac{1}{a^2}y_1^2y_3^2+\frac{2A_5}{a^2}y_2y_3y_4-\frac{1}{a^2}y_3^2y_4^2-\frac{2A_5}{a^2}y_1^2y_5-\frac{2A_5}{a^2}y_2^2y_5+\frac{2A_2}{a^2}y_3y_4y_5\\
&+\frac{2}{a^2}y_2y_3y_4y_5-\frac{2A_1}{a^2}y_1y_5^2-\frac{1}{a^2}y_1^2y_5^2-\frac{2A_2}{a^2}y_2y_5^2-\frac{1}{a^2}y_2^2y_5^2,\end{aligned}$$

and the corresponding variations of the first integrals $H, V_1, V_2$ will write (with the precision up to the first order terms) as follows:

$$\begin{aligned}\delta H &= \frac{1}{9}(4A_1+5A_5)y_1+A_2y_2+\frac{1}{9}(5A_1+4A_5)y_5-5A_6\alpha^2y_6=0,\\
\delta V_1 &= \frac{2(A_1-A_5)}{9\alpha^2}y_1-\frac{2(A_1-A_5)}{9\alpha^2}y_5+2A_6y_6=0,\\
\delta V_2 &= (A_6-\frac{A_2}{3\alpha})y_1+\frac{A_5-A_1}{3\alpha}y_2+\frac{A_2}{3\alpha}y_5+A_1y_6=0.\end{aligned}$$

Having eliminated the variables $y_1, y_2$ from $\varDelta K$ with the aid of the latter equations (some of which are linear dependent), after substituting the corresponding expressions for $A_i$ into $\varDelta K$, we have:

$$\begin{aligned}\varDelta\tilde{K} = &\frac{125+354a^2+375a^4-125a^6-25(a^2-1)^2\sqrt{25+31a^2+25a^4}}{18(25+31a^2+25a^4)}y_3^2\\
&-\frac{a(4+5a^2)}{\sqrt{25+31a^2+25a^4}}y_3y_4+\frac{1}{18}(4+5a^2)y_4^2+\frac{1}{18}(21+15a^2+\sqrt{25+31a^2+25a^4})y_5^2\\
&+\frac{\alpha(-25-31a^2-25a^4+5(a^2-1)\sqrt{25+31a^2+25a^4})}{27a}y_5y_6\\
&+\frac{(25+31a^2+25a^4)\alpha^2}{2(5a^2-5+\sqrt{25+31a^2+25a^4})}y_6^2+K_n.\end{aligned}$$

Here $K_n$ denotes the nonquadratic part of $\varDelta\tilde{K}$.

It is known that conditions of sign definiteness for the quadratic part of $\varDelta\tilde{K}$ are the sufficient conditions of stability for the elements of the family of steady-state motions under consideration. Having written them in the form of Sylvester conditions, we have:

$$\begin{aligned}&1.\ 1-a^2>0,\\
&2.\ 125+354a^2+375a^4-125a^6-25(a^2-1)^2\sqrt{25+31a^2+25a^4}>0,\\
&3.\ (4+a^2-5a^4)(5\sqrt{2}\,(a^2-1)+\sqrt{50+62a^2+50a^4})>0,\\
&4.\ 4(4+a^2-5a^4)+\frac{-80+141a^2+39a^4-100a^6}{\sqrt{25+31a^2+25a^4}}>0.\end{aligned}\tag{17}$$

For the purpose of verification of compatibility of the above system of inequalities, the software package "Algebra‘ InequalitySolve‘" (included in the CAS "Mathematica") intended for solving system of inequalities has been used. Its application to (17) gives evidence that the inequalities are compatible under the conditions:

$$-1<a<0\vee 0<a<1.$$

In view of the fact that $a = \lambda_2/(3\alpha)$, we have:

$$-3\alpha < \lambda_2 < 0 \vee 0 < \lambda_2 < 3\alpha.$$

So, our steady-state solutions are stable for the value of the parameter $\lambda_2$, which satisfies the latter conditions. The given technique of stability investigation of steady-state solutions was used for practically all the obtained solutions of equations (7). For most of them the obtained sufficient conditions of stability (Sylvester conditions) turned out to be noncompatible for any values of $\lambda_2$. The question of instability of the latter motions needs additional investigations, and so it is not discussed herein.

## 4 Conclusion

The above investigations of steady-state solutions of motion equations for a rigid body in fluid give evidence that application of computer algebra allows one to obtain steady-state solutions of such systems rather efficiently and equally efficiently investigate their stability in cases, when there are not only quadratic ones among the first integrals.

Some part of the computations presented in the paper have been performed with the aid of the software package [2]. The latter has been designed and is currently under development by the authors on top of CAS "Mathematica". The software package is intended for the problems of modelling and qualitative investigation of the phase space for mechanical systems, including those which possess a large number of symmetries (first integrals). In particular, it includes the program implementation of the algorithms of obtaining steady-state conditions for the first integrals; verification of conditions of sign definiteness of algebraic forms; etc. Its application allows to adapt the CAS "Mathematica", which is a general-purpose system, to the problems of the investigated area, and so to increase the efficiency of its application in research.

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