Inequalities on Ranks and Cranks of Partitions

William Y.C. Chen

Center for Combinatorics, Nankai University, P. R. China

Center for Applied Mathematics, Tianjin University, P. R. China

(Joint work with Kathy Q. Ji and Wenston J. T. Zang)

Abstract

Let $N(\leq m, n)$ denote the number of partitions of n with rank not greater than m and let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Bringmann and Mahlburg observed that $N(\leq m, n) \leq M(\leq m, n) \leq N(\leq m+1, n)$ for $m < 0$ and $1 \leq n \leq 100$ and conjectured that these two inequalities may also be restated in terms of ordered lists of partitions.

Andrews, Dyson, and Rhoades showed that the conjectured inequality $N(\leq m, n) \leq M(\leq m, n)$ of Bringmann and Mahlburg is equivalent to their conjecture on the unimodal of spt-crank. We have proved the conjecture of Andrews, Dyson, and Rhoades by a purely combinatorial argument. Recently, we also proved that the inequality $M(\leq m, n) \leq N(\leq m+1, n)$ holds for $m < 0$ and $n \geq 1$. Based on these two inequalities, we are led to a bijection τ_n between the set of partitions of n and the set of partitions of n such that $|\text{crank}(\lambda)| - |\text{rank}(\tau_n(\lambda))| = 0,$ or 1. We then use this bijection to show that $spt(n) \leq \sqrt{2n}p(n)$, where $spt(n)$ counts the total number of smallest parts in all partitions of n.

Let $N(m, n)$ be the number of partitions of n with rank m. Recently, Chan and Mao showed that $N(m, n) \geq N(m, n-1)$ for $n \geq 12$ and $n \neq m+2$, and $N(m, n) \geq N(m+2, n)$ for $n \geq 0$ and $0 \leq m \leq n-2$. They raised the question of establishing similar inequalities for $M(m, n)$, the number of partitions of n with crank m. We establish two monotonicity properties of $M(m, n)$. More precisely, we show that $M(m, n) \geq M(m, n-1)$ for $n \geq 14$ and $0 \leq m \leq n-2$, and $M(m-1, n) \geq M(m, n)$ for $n \geq 43$ and $1 \leq m \leq n-1$. As a corollary, we deduce that $M(m, n) \geq M(m+2, n)$ for $n \geq 4$ and $0 \leq m \leq n-2$.