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The three-loop single-mass heavy flavor corrections to deep-inelastic scattering

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We report on the status of the calculation of the massive Wilson coefficients and operator matrix elements for deep-inelastic scattering to three-loop order. We discuss both the unpolarized and the polarized case, for which all the single-mass and nearly all two-mass contributions have been calculated. Numerical results on the structure function $F_2(x, Q^2)$ are presented. In the polarized case, we work in the Larin scheme and refer to parton distribution functions in this scheme. Furthermore, results on the three-loop variable flavor number scheme are presented.

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1. Introduction

The massive operator matrix elements (OMEs) [1, 2] and massive Wilson coefficient of deep-inelastic scattering [3–6] in the asymptotic region $Q^2 \gg m^2$, with m the heavy quark mass, have been calculated in the single-mass case to three-loop order in Quantum Chromodynamics (QCD). To two-loop order these OMEs have been computed in Refs. [1, 7–20]. At three-loop order seven unpolarized massive OMEs, $A_{qq,Q}^{\text{NS}}, A_{Qq}^{\text{PS}}, A_{qq,Q}^{\text{PS}}, A_{Qg}, A_{qg,Q}, A_{gq,Q}$ and $A_{gg,Q}$, and the corresponding polarized OMEs contribute. First a series of Mellin moments was calculated in Ref. [21]. The computation of these functions for general values of Mellin- N followed in Refs. [15, 16, 19, 22–31]. Two-mass corrections contribute starting from two-loop order, i.e. at next-to-leading-order (NLO), cf. [32], as factorizable terms. From three-loop order onward also irreducible two-mass terms contribute, cf. Refs. [2, 33–36]. The last missing term of this class will be published soon [37]. Also, for charged current structure functions a series of heavy-flavor corrections was calculated [38–41]. The massive Wilson coefficients depend on the massless three-loop unpolarized Wilson coefficients [42, 43] and the polarized ones [43]. The evolution of the massless parton densities to three-loop order depend on the unpolarized [25, 43–56] and polarized [57–59] three-loop anomalous dimensions.

The technical aspects of the calculation of these massive OMEs consist of a series of standard steps, described e.g. in Ref. [30]. The integration-by-parts reduction has been performed using `Reduze 2` [60, 61]. We used also more special analytic methods, such as summation and guessing methods applied to a very large number of moments [62–69], special higher transcendental function treatment of different kind [70–87], differential equation methods for first-order-factorizing systems [88] and non-first-order-factorizing systems [89, 90], including (general) semi-analytic solutions [91, 92]. In our calculations the use of the packages `Sigma` [62–64], `HarmonicSums` [70–87], `OreSys` [93–95], and others [96, 97] played an important role.

The present note is organized as follows. In Section 2 we present the three-loop single-mass contributions at next-to-next-to-leading order (NNLO) to the structure function $F_2(x, Q^2)$ for the first time. It is an important ingredient for precision QCD fits of the deep-inelastic World data to determine the strong coupling constant $a_s = \alpha_s/(4\pi)$, cf. [98–101] and of the parton distribution functions (PDFs), [102]. To obtain the same results for the polarized structure functions one needs also PDFs evolved in the Larin scheme. This is discussed in Section 3. The three-loop massive OMEs also allow one to derive the matching relations in the variable flavor number scheme (VFNS) at three-loop order, which is presented in Section 4. Section 5 contains the conclusions.

2. The Single-Mass Heavy-Flavor Contributions to $F_2(x, Q^2)$

The current results on the three-loop massive OMEs allow us to compute the structure function $F_2(x, Q^2)$, including the massless and single-mass heavy-flavor corrections due to c - and b -quarks at large enough scales Q^2 . The massive OMEs and massive asymptotic Wilson coefficients are calculated for quark masses in the on-shell scheme, $m_c = 1.59$ GeV, [104], and $m_b = 4.78$ GeV [105]. It has been shown in Ref. [1] that the criterion $Q^2 \gg m^2$, for which the asymptotic structure

function represents the full structure function $F_2(x, Q^2)$ at the 1%-level is fulfilled by $Q^2/m^2 \geq 10$, i.e. for $Q^2 \geq 25 \text{ GeV}^2$ in the case of charm at NLO.¹

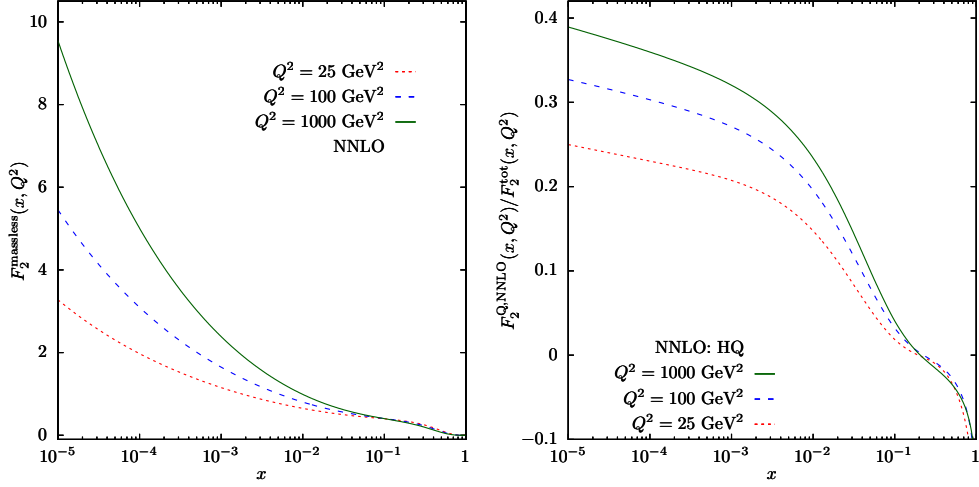


Figure 1: Left panel: the massless contributions to the structure function $F_2(x, Q^2)$ at NNLO using the PDFs of Ref. [103]. Right panel: The ratio of the NNLO single-mass charm and bottom contributions to $F_2(x, Q^2)$ to its total value. Dotted lines: $Q^2 = 25 \text{ GeV}^2$; dashed lines: $Q^2 = 100 \text{ GeV}^2$; full lines: $Q^2 = 10000 \text{ GeV}^2$.

In Figure 1 we present both the prediction for the massless contributions to the structure function $F_2(x, Q^2)$ as well as for the single-mass c and b -quark contributions at NNLO for a wide range in the kinematic variables Bjorken x and the virtuality Q^2 . The fraction of the (virtual and real) heavy quark contributions vary from $\sim 25\%$ to 40% at $x = 10^{-4}$ for Q^2 in the range between 25 GeV^2 and 10^4 GeV^2 and the contribution falls towards large values of x . Here five different massive Wilson coefficients contribute.

Already in 1990 the massive OME A_{Qg} has been investigated in its ultimate small x limit to any order in a_s , using methods of k_\perp -factorization [106]. In 1995 the respective expansion term of $O(a_s^2)$ has been confirmed by expanding the complete NLO result in Ref. [1]. After 34 years we have now confirmed also the $O(a_s^3)$ term for the first time in Refs. [30, 31]. One obtains

$$a_{Qg}^{(3),x \rightarrow 0}(x) = \frac{64}{243} C_A^2 T_F [1312 + 135\zeta_2 - 189\zeta_3] \frac{\ln(x)}{x} \quad (1)$$

for the constant part of the unrenormalized three-loop OME $A_{Qg}^{(3)}$. Here $C_A = N_c$, $T_F = 1/2$, $C_F = (N_c^2 - 1)/(2N_c)$ denote the color factors, with $N_c = 3$ for QCD, and ζ_k are the values of Riemann's ζ -function at integer argument, $k \geq 2$. However, this term does not describe the small x behaviour, neither of the massive OME nor of the structure function, due to very large sub-leading small x corrections, as the numerical analysis in Ref. [31] shows. This is a quite common observation for a long list of BFKL predictions.² Already in Ref. [25] we have computed the pure-singlet OME $A_{Qq}^{(3),PS}$ and derived the corresponding quantity $a_{Qq}^{(3),PS,x \rightarrow 0}(x)$ and its leading small x limit. It is related to (1) through rescaling by the factor C_F/C_A , as has been found by an explicit analytic calculation now.

¹This criterion may be different in the case of other structure functions.

²For a survey, see Ref. [107].

3. Polarized Parton Distributions in the Larin Scheme

The three-loop massless [43] and massive Wilson coefficients in the polarized case were calculated in the Larin scheme [108, 109]. Currently it is not possible to construct the transformation into the $\overline{\text{MS}}$ scheme at three-loop order for them. However, the polarized structure function $g_1(x, Q^2)$, as an observable, can be expressed in terms of the Wilson coefficients and parton distributions [110] in the Larin scheme. The scaling violations of the polarized massless parton densities are described in this scheme as well by using the corresponding anomalous dimensions [57, 59]. In Ref. [110] the polarized parton distribution functions have been provided in the Larin scheme up to NNLO recently.

Starting at NLO the scale evolution is different in the Larin and the $\overline{\text{MS}}$ scheme. We illustrate this in Figure 2 for the ratio $r = f^{\text{Larin}}/f^{\overline{\text{MS}}} - 1$ at NNLO. The effects are larger for the quarkonic distributions than for the gluon distribution. In the latter case they are caused by mixing effects with the quarkonic anomalous dimensions only, since $\Delta P_{gg}^{(1,2)}$ is the same in both schemes. In the large x limit the anomalous dimensions in both schemes approach each other.

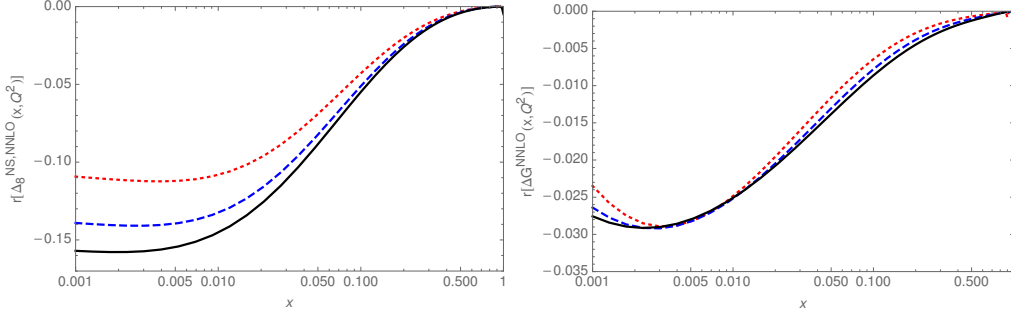


Figure 2: The relative change of the polarized parton distribution functions $\Delta g(x, Q^2) = \Delta u(x, Q^2) + \Delta d(x, Q^2)$ and $\Delta G(x, Q^2)$ comparing the evolution in the $\overline{\text{MS}}$ scheme and the Larin scheme. Dotted line: $Q^2 = 100$ GeV; dashed line: $Q^2 = 1000$ GeV; full line: $Q^2 = 10000$ GeV; from Ref. [110].

For the quark distributions the relative change in the small x region, $x \sim 0.001$, may reach 10–15%, while for the gluon distribution the corresponding effect amounts to $O(3\%)$. High precision QCD fits in the polarized case therefore require to use Larin-scheme PDFs.

4. The Single-Mass Variable Flavor Number Scheme

The single-mass three-loop massive OMEs allow one to construct the corresponding matching relations in the VFNS to three-loop order. The principal structure of the matching relations has been given in Ref. [10] and was corrected in Ref. [21]. The VFNS relates the massless parton densities with N_F massless quark flavors to the ones of $N_F + 1$ massless quark flavors in the region $Q^2 \gg m_Q^2$, where m_Q is the mass of the heavy quark becoming effectively massless. In course of this, one also obtains massive quark distributions, $f_Q(x, Q^2) + f_{\bar{Q}}(x, Q^2)$. The relations are derived from the structure functions at high scales Q^2 in the fixed flavor number scheme and are determined by the process-independent massive OMEs. It has been shown in Refs. [1, 9, 16–18] at two-loop order for the cases in which the complete heavy-quark-mass dependence is known analytically that

the effective massless approach in the case of charm and bottom quarks only applies at high scales Q^2 and neither at the scales m_c^2 nor m_b^2 . The new PDFs obtained in the VFNS are inserted into the massless representation of the corresponding structure functions. If one analytically expands the resulting expressions in the coupling constant a_s one obtains again the structure functions in the fixed flavor number scheme to the order one worked in. The differences in the VFNS to the result in the direct calculation are of higher order in a_s . Depending on the matching scale chosen, different size pile-up effects due to these terms are obtained.

An implementation of the three-loop matching relations will be given in Ref. [111]. In Figure 3 we illustrate the charm distribution $f_c(x, Q^2) + f_{\bar{c}}(x, Q^2)$ normalized to the singlet distribution for $N_F = 3$. The effects grow with Q^2 due to the logarithmic terms $\ln(m_Q^2/Q^2)$ in the matching relations.

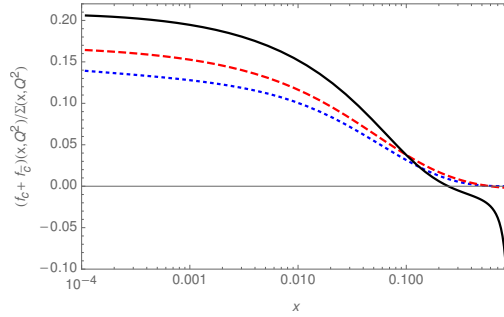


Figure 3: The distribution $f_c(x, Q^2) + f_{\bar{c}}(x, Q^2)$ normalized to $\Sigma^{\text{NF}=3}(x, Q^2)$. Dotted lines: $Q^2 = 30 \text{ GeV}^2$; dashed lines: $Q^2 = 100 \text{ GeV}^2$; full lines: $Q^2 = 10000 \text{ GeV}^2$; from Ref. [111].

5. Conclusions

We finished the calculation of all single-mass OMEs and asymptotic inclusive heavy-flavor Wilson coefficients to three-loop order and made numerical predictions for the structure function $F_2(x, Q^2)$. Very soon, also the two-mass corrections will also be finished both in the unpolarized and polarized cases. In the polarized case we worked in the Larin scheme and provided a first set of NNLO parton densities. Furthermore, the single-mass matching relations in the VFNS are now available both in the unpolarized and polarized cases.

The present results are of special importance for phenomenological predictions of the precision physics at future facilities such as the EIC [112] and LHeC [113, 114], but also for re-analysis of the HERA [115] and other World deep-inelastic data, as well as for inclusive measurements at the LHC in its high luminosity phase at CERN, and its future successor, the FCC [116]. In the flavor non-singlet case, the relations for a QCD-fit in the unpolarized and polarized cases were given in [117] in the scheme-invariant representation, which allows a direct fit of $a_s(M_Z)$ using measured input distributions at the starting scale Q_0^2 .

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References

- [1] M. Buza, Y. Matiounine, J. Smith, R. Migneron and W.L. van Neerven, *Heavy quark coefficient functions at asymptotic values $Q^2 \gg m^2$* , Nucl. Phys. B **472** (1996) 611–658 [hep-ph/9601302].
- [2] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, C. Schneider and F. Wißbrock, *Three Loop Massive Operator Matrix Elements and Asymptotic Wilson Coefficients with Two Different Masses*, Nucl. Phys. B **921** (2017) 585–688 [arXiv:1705.07030 [hep-ph]].
- [3] A.J. Buras, *Asymptotic Freedom in Deep Inelastic Processes in the Leading Order and Beyond*, Rev. Mod. Phys. **52** (1980) 199–276.
- [4] E. Reya, *Perturbative Quantum Chromodynamics*, Phys. Rept. **69** (1981) 195–353.
- [5] J. Blümlein, *The Theory of Deeply Inelastic Scattering*, Prog. Part. Nucl. Phys. **69** (2013) 28–84. [arXiv:1208.6087 [hep-ph]].
- [6] J. Blümlein, *Deep-Inelastic Scattering: What do we know?*, Int. J. Mod. Phys. A **39** (2024) 2441004 [arXiv:2306.01362 [hep-ph]].
- [7] E. Witten, *Heavy Quark Contributions to Deep Inelastic Scattering*, Nucl. Phys. B **104** (1976) 445–476.
- [8] A.D. Watson, *Spin Spin Asymmetries In Inclusive Muon Proton Charm Production*, Z. Phys. C **12** (1982) 123–125.
- [9] M. Buza, Y. Matiounine, J. Smith and W. L. van Neerven, *$O(\alpha_s^2)$ corrections to polarized heavy flavor production at $Q^2 \gg m^2$* , Nucl. Phys. B **485** (1997) 420–456 [arXiv:hep-ph/9608342 [hep-ph]].
- [10] M. Buza, Y. Matiounine, J. Smith and W.L. van Neerven, *Charm electroproduction viewed in the variable flavor number scheme versus fixed order perturbation theory*, Eur. Phys. J. C **1** (1998) 301–320 [arXiv:hep-ph/9612398 [hep-ph]].
- [11] I. Bierenbaum, J. Blümlein and S. Klein, *Two-Loop Massive Operator Matrix Elements and Unpolarized Heavy Flavor Production at Asymptotic Values $Q^2 \gg m^2$* , Nucl. Phys. B **780** (2007) 40–75 [hep-ph/0703285].
- [12] I. Bierenbaum, J. Blümlein, S. Klein and C. Schneider, *Two-Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$* , Nucl. Phys. B **803** (2008) 1–41 [arXiv:0803.0273 [hep-ph]].
- [13] I. Bierenbaum, J. Blümlein and S. Klein, *Two-loop massive operator matrix elements for polarized and unpolarized deep-inelastic scattering*, Proc. DIS 2007, p. 821–824, DESY-PROC-2007-01, Eds. G. Grindhammer and K. Sachs, [arXiv:0706.2738 [hep-ph]].
- [14] I. Bierenbaum, J. Blümlein and S. Klein, *The Gluonic Operator Matrix Elements at $O(\alpha_s^2)$ for DIS Heavy Flavor Production*, Phys. Lett. B **672** (2009) 401–406 [arXiv:0901.0669 [hep-ph]].
- [15] A. Behring, I. Bierenbaum, J. Blümlein, A. De Freitas, S. Klein and F. Wißbrock, *The logarithmic contributions to the $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering*, Eur. Phys. J. C **74** (2014) no.9, 3033 [arXiv:1403.6356 [hep-ph]].
- [16] J. Blümlein, G. Falcioni and A. De Freitas, *The Complete $O(\alpha_s^2)$ Non-Singlet Heavy Flavor Corrections to the Structure Functions $g_{1,2}^{eP}(x, Q^2)$, $F_{1,2,L}^{eP}(x, Q^2)$, $F_{1,2,3}^{\nu(\bar{\nu})}(x, Q^2)$ and the Associated Sum Rules*, Nucl. Phys. B **910** (2016) 568–617 [arXiv:1605.05541 [hep-ph]].
- [17] J. Blümlein, A. De Freitas, C.G. Raab and K. Schönwald, *The unpolarized two-loop massive pure singlet Wilson coefficients for deep-inelastic scattering*, Nucl. Phys. B **945** (2019) 114659 [arXiv:1903.06155 [hep-ph]].
- [18] J. Blümlein, C.G. Raab and K. Schönwald, *The Polarized Two-Loop Massive Pure Singlet Wilson Coefficient for Deep-Inelastic Scattering* Nucl. Phys. B **948** (2019) 114736 [arXiv:1904.08911 [hep-ph]].

- [19] J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider and K. Schönwald, *Logarithmic contributions to the polarized $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering*, Phys. Rev. D **104** (2021) no.3, 034030 [arXiv:2105.09572 [hep-ph]].
- [20] I. Bierenbaum, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein and K. Schönwald, *$O(\alpha_s^2)$ polarized heavy flavor corrections to deep-inelastic scattering at $Q^2 \gg m^2$* , Nucl. Phys. B **988** (2023) 116114 [arXiv:2211.15337 [hep-ph]].
- [21] I. Bierenbaum, J. Blümlein and S. Klein, *Mellin Moments of the $O(\alpha_s^3)$ Heavy Flavor Contributions to unpolarized Deep-Inelastic Scattering at $Q^2 \gg m^2$ and Anomalous Dimensions*, Nucl. Phys. B **820** (2009) 417–482 [arXiv:0904.3563 [hep-ph]].
- [22] J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, *The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(N_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity*, Nucl. Phys. B **844** (2011) 26–54 [arXiv:1008.3347 [hep-ph]].
- [23] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider and F. Wißbrock, *The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$* , Nucl. Phys. B **882** (2014) 263–288 [arXiv:1402.0359 [hep-ph]].
- [24] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, and F. Wißbrock, *The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity*, Nucl. Phys. B **886** (2014) 733–823 [arXiv:1406.4654 [hep-ph]].
- [25] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The 3-loop pure singlet heavy flavor contributions to the structure function $F_2(x, Q^2)$ and the anomalous dimension*, Nucl. Phys. B **890** (2014) 48–151 [arXiv:1409.1135 [hep-ph]].
- [26] A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer*, Nucl. Phys. B **897** (2015), 612-644 [arXiv:1504.08217 [hep-ph]].
- [27] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider and K. Schönwald, *The three-loop single mass polarized pure singlet operator matrix element*, Nucl. Phys. B **953** (2020) 114945 [arXiv:1912.02536 [hep-ph]].
- [28] A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, K. Schönwald and C. Schneider, *The polarized transition matrix element $A_{gq}(N)$ of the variable flavor number scheme at $O(\alpha_s^3)$* , Nucl. Phys. B **964** (2021) 115331 [arXiv:2101.05733 [hep-ph]].
- [29] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, A. von Manteuffel, C. Schneider and K. Schönwald, *The unpolarized and polarized single-mass three-loop heavy flavor operator matrix elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$* , JHEP **12** (2022) 134 [arXiv:2211.05462 [hep-ph]].
- [30] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider and K. Schönwald, *The first-order factorizable contributions to the three-loop massive operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$* , Nucl. Phys. B **999** (2024) 116427 [arXiv:2311.00644 [hep-ph]].
- [31] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider and K. Schönwald, *The non-first-order-factorizable contributions to the three-loop single-mass operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$* , Phys. Lett. B **854** (2024) 138713 [arXiv:2403.00513 [hep-ph]].
- [32] J. Blümlein, A. De Freitas, C. Schneider and K. Schönwald, *The Variable Flavor Number Scheme at Next-to-Leading Order*, Phys. Lett. B **782** (2018), 362-366 [arXiv:1804.03129 [hep-ph]].
- [33] J. Ablinger, J. Blümlein, A. De Freitas, C. Schneider and K. Schönwald, *The two-mass contribution to the three-loop pure singlet operator matrix element*, Nucl. Phys. B **927** (2018) 339–367 [arXiv:1711.06717 [hep-ph]].

- [34] J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, C. Schneider and K. Schönwald, *The Two-mass Contribution to the Three-Loop Gluonic Operator Matrix Element $A_{gg,Q}^{(3)}$* , Nucl. Phys. B **932** (2018) 129–240 [arXiv:1804.02226 [hep-ph]].
- [35] J. Ablinger, J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider and K. Schönwald, *The three-loop polarized pure singlet operator matrix element with two different masses*, Nucl. Phys. B **952** (2020), 114916 [arXiv:1911.11630 [hep-ph]].
- [36] J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, C. Schneider and K. Schönwald, *The two-mass contribution to the three-loop polarized gluonic operator matrix element $A_{gg,Q}^{(3)}$* , Nucl. Phys. B **955** (2020) 115059 [arXiv:2004.08916 [hep-ph]].
- [37] J. Ablinger, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider, and K. Schönwald, *The two-mass contributions to the three-loop massive operator matrix elements $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$* , DO-TH 23/16.
- [38] M. Buza and W.L. van Neerven, *$O(\alpha_s^2)$ contributions to charm production in charged current deep inelastic lepton-hadron scattering*, Nucl. Phys. B **500** (1997) 301–324 [arXiv:hep-ph/9702242 [hep-ph]].
- [39] J. Blümlein, A. Hasselhuhn and T. Pföh, *The $O(\alpha_s^2)$ heavy quark corrections to charged current deep-inelastic scattering at large virtualities*, Nucl. Phys. B **881** (2014) 1–41 [arXiv:1401.4352 [hep-ph]].
- [40] A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel and C. Schneider, *$O(\alpha_s^3)$ heavy flavor contributions to the charged current structure function $x F_3(x, Q^2)$ at large momentum transfer*, Phys. Rev. D **92** (2015) no.11, 114005 [arXiv:1508.01449 [hep-ph]].
- [41] A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel and C. Schneider, *Asymptotic 3-loop heavy flavor corrections to the charged current structure functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$* , Phys. Rev. D **94** (2016) no.11, 114006 [arXiv:1609.06255 [hep-ph]].
- [42] J.A.M. Vermaseren, A. Vogt and S. Moch, *The Third-order QCD corrections to deep-inelastic scattering by photon exchange*, Nucl. Phys. B **724** (2005) 3–182 [hep-ph/0504242].
- [43] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1* , JHEP **11** (2022) 156 [arXiv:2208.14325 [hep-ph]].
- [44] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, Nucl. Phys. B **688** (2004) 101–134 [hep-ph/0403192].
- [45] A. Vogt, S. Moch and J.A.M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case*, Nucl. Phys. B **691** (2004) 129–181 [hep-ph/0404111].
- [46] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, Phys. Rev. Lett. **114** (2015) 212001 [arXiv:1503.06056 [hep-ph]].
- [47] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,N_F)}$* , Nucl. Phys. B **922** (2017) 1–40 [arXiv:1705.01508 [hep-ph]].
- [48] B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, JHEP **05** (2018) 028 [arXiv:1802.00833 [hep-ph]].
- [49] M.X. Luo, T.Z. Yang, H.X. Zhu and Y.J. Zhu, *Quark Transverse Parton Distribution at the Next-to-Next-to-Next-to-Leading Order*, Phys. Rev. Lett. **124** (2020) no.9, 092001 [arXiv:1912.05778 [hep-ph]].
- [50] C. Duhr, F. Dulat and B. Mistlberger, *Drell-Yan Cross Section to Third Order in the Strong Coupling Constant*, Phys. Rev. Lett. **125** (2020) no.17 172001 [arXiv: 2001.07717 [hep-ph]].
- [51] M.A. Ebert, B. Mistlberger and G. Vita, *Transverse momentum dependent PDFs at N^3LO* , JHEP **09** (2020) 146 [arXiv: 2006.05329 [hep-ph]].

- [52] M.A. Ebert, B. Mistlberger and G. Vita, *N-jettiness beam functions at N^3LO* , JHEP **09** (2020) 143 [arXiv:2006.03056 [hep-ph]].
- [53] M.X. Luo, T.Z. Yang, H.X. Zhu and Y.J. Zhu, *Unpolarized quark and gluon TMD PDFs and FFs at N^3LO* , JHEP **06** (2021) 115 [arXiv:2012.03256 [hep-ph]].
- [54] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements*, Nucl. Phys. B **971** (2021) 115542 [arXiv:2107.06267 [hep-ph]].
- [55] D. Baranowski, A. Behring, K. Melnikov, L. Tancredi and C. Wever, *Beam functions for N-jettiness at N^3LO in perturbative QCD*, JHEP **02** (2023) 073 [arXiv:2211.05722 [hep-ph]].
- [56] T. Gehrmann, A. von Manteuffel and T.Z. Yang, *Renormalization of twist-two operators in covariant gauge to three loops in QCD*, JHEP **04** (2023) 041 [arXiv:2302.00022 [hep-ph]].
- [57] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case*, Nucl. Phys. B **889** (2014) 351–400 [arXiv:1409.5131 [hep-ph]].
- [58] A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, C. Schneider and K. Schönwald, *The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements*, Nucl. Phys. B **948** (2019) 114753 [arXiv:1908.03779 [hep-ph]].
- [59] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements*, JHEP **01** (2022) 193 [arXiv:2111.12401 [hep-ph]].
- [60] C. Studerus, *Reduze-Feynman Integral Reduction in C++*, Comput. Phys. Commun. **181** (2010) 1293–1300 [arXiv:0912.2546 [physics.comp-ph]].
- [61] A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, arXiv:1201.4330 [hep-ph].
- [62] C. Schneider, *Symbolic Summation Assists Combinatorics*, Sémin. Lothar. Combin. **56** (2007) 1–36 article B56b.
- [63] C. Schneider, *Simplifying Multiple Sums in Difference Fields*, in: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions Texts and Monographs in Symbolic Computation, eds. C. Schneider and J. Blümlein (Springer, Wien, 2013) 325–360 [arXiv:1304.4134 [cs.SC]].
- [64] C. Schneider, *Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation*. In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (eds.), Texts and Monographs in Symbolic Computation, pp. 423–485 (Springer, Berlin, 2021) [arXiv:2102.01471 [cs.SC]].
- [65] J. Blümlein and C. Schneider, *The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory*, Phys. Lett. B **771** (2017) 31–36 [arXiv:1701.04614 [hep-ph]].
- [66] M. Kauers, *Guessing Handbook*, JKU Linz, Technical Report RISC 09–07.
- [67] J. Blümlein, M. Kauers, S. Klein and C. Schneider, *Determining the closed forms of the $O(a_s^3)$ anomalous dimensions and Wilson coefficients from Mellin moments by means of computer algebra*, Comput. Phys. Commun. **180** (2009) 2143–2165 [arXiv:0902.4091 [hep-ph]].
- [68] S.A. Abramov, M. Bronstein, M. Petkovšek, and C. Schneider, *On Rational and Hypergeometric Solutions of Linear Ordinary Difference Equations in $\Pi\Sigma$ -field extensions*, J. Symb. Comput. **107** (2021) 23–66 [arXiv:2005.04944 [cs.SC]].
- [69] C. Schneider, *Refined telescoping algorithms in $\text{RII}\Sigma$ -extensions to reduce the degrees of the denominators*, In: ISSAC '23: Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation, G. Jeronimo (ed.), pp. 498–507. July 2023. ACM [arXiv:2302.03563 [cs.SC]].

- [70] J.A.M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, Int. J. Mod. Phys. A **14** (1999) 2037–2076 [hep-ph/9806280].
- [71] J. Blümlein and S. Kurth, *Harmonic sums and Mellin transforms up to two loop order*, Phys. Rev. D **60** (1999) 014018 [hep-ph/9810241].
- [72] J. Ablinger, J. Blümlein and C. Schneider, *Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms*, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378 [math-ph]].
- [73] J. Ablinger, J. Blümlein and C. Schneider, *Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials*, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063 [math-ph]].
- [74] J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider, *Iterated Binomial Sums and their Associated Iterated Integrals*, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822 [hep-th]].
- [75] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, Int. J. Mod. Phys. A **15** (2000) 725–754 [hep-ph/9905237].
- [76] J. Blümlein, *Algebraic relations between harmonic sums and associated quantities*, Comput. Phys. Commun. **159** (2004) 19–54 [hep-ph/0311046].
- [77] J. Blümlein, *Structural Relations of Harmonic Sums and Mellin Transforms up to Weight $w = 5$* , Comput. Phys. Commun. **180** (2009) 2218–2249 [arXiv:0901.3106 [hep-ph]].
- [78] J. Blümlein, *Structural Relations of Harmonic Sums and Mellin Transforms at Weight $w = 6$* , Clay Math. Proc. **12** (2010) 167–188 [arXiv:0901.0837 [math-ph]].
- [79] J. Ablinger, *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, Diploma Thesis, JKU Linz, 2009, arXiv:1011.1176[math-ph].
- [80] J. Ablinger, *Computer Algebra Algorithms for Special Functions in Particle Physics*, Ph.D. Thesis, Linz U. (2012) arXiv:1305.0687[math-ph].
- [81] J. Ablinger, *The package HarmonicSums: Computer Algebra and Analytic aspects of Nested Sums*, PoS (LL2014) 019 [arXiv:1407.6180 [cs.SC]].
- [82] J. Ablinger, *Discovering and Proving Infinite Binomial Sums Identities*, Exper. Math. **26** (2016) no.1, 62–71 [arXiv:1507.01703 [math.NT]].
- [83] J. Ablinger, *Inverse Mellin Transform of Holonomic Sequences*, PoS (LL2016) 067.
- [84] J. Ablinger, *An Improved Method to Compute the Inverse Mellin Transform of Holonomic Sequences*, PoS (LL2018) 063.
- [85] J. Ablinger, *Computing the Inverse Mellin Transform of Holonomic Sequences using Kovacic’s Algorithm*, PoS (RADCOR2017) 001 [arXiv:1801.01039 [cs.SC]].
- [86] J. Ablinger, *Discovering and Proving Infinite Pochhammer Sum Identities*, arXiv:1902.11001 [math.CO].
- [87] J. Ablinger, J. Blümlein and C. Schneider, *Iterated integrals over letters induced by quadratic forms*, Phys. Rev. D **103** (2021) no.9, 096025 [arXiv:2103.08330 [hep-th]].
- [88] J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider, *Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable*, Nucl. Phys. B **939** (2019) 253–291 [arXiv:1810.12261 [hep-ph]].
- [89] J. Ablinger, J. Blümlein, A. De Freitas, M. van Hoeij, E. Imamoglu, C.G. Raab, C.S. Radu and C. Schneider, *Iterated Elliptic and Hypergeometric Integrals for Feynman Diagrams*, J. Math. Phys. **59** (2018) no.6, 062305 [arXiv:1706.01299 [hep-th]].

- [90] A. Behring, J. Blümlein and K. Schönwald, *The inverse Mellin transform via analytic continuation*, JHEP **06** (2023) 062 [arXiv:2303.05943 [hep-ph]].
- [91] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, *A semi-analytic method to compute Feynman integrals applied to four-loop corrections to the $\overline{\text{MS}}$ -pole quark mass relation*, JHEP **09** (2021) 152 [arXiv:2106.05296 [hep-ph]].
- [92] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, *Singlet and nonsinglet three-loop massive form factors*, Phys. Rev. D **106** (2022) no.3, 034029 [arXiv:2207.00027 [hep-ph]].
- [93] B. Zürcher, *Rationale Normalformen von pseudo-linearen Abbildungen* Ph.D. Thesis Mathematik, ETH Zürich, 1994.
- [94] A. Bostan, F. Chyzak and de É. Panafieu, *Complexity estimates for two uncoupling algorithms* Proc. ISSAC'13 (2013), Boston, 85–92, Ed. M. Kauers, [arXiv:1301.5414 [cs.SC]].
- [95] S. Gerhold, *Uncoupling Systems of Linear Ore Operator Equations*. Master's thesis, RISC, J. Kepler University, Linz 2002.
- [96] J. Klappert and F. Lange, *Reconstructing rational functions with FireFly*, Comput. Phys. Commun. **247** (2020) 106951 [arXiv:1904.00009 [cs.SC]].
- [97] J. Klappert, S.Y. Klein and F. Lange, *Interpolation of dense and sparse rational functions and other improvements in FireFly*, Comput. Phys. Commun. **264** (2021) 107968 [arXiv:2004.01463 [cs.MS]].
- [98] S. Bethke *et al.*, *Workshop on Precision Measurements of α_s* , arXiv:1110.0016 [hep-ph].
- [99] S. Moch *et al.*, *High precision fundamental constants at the TeV scale*, arXiv:1405.4781 [hep-ph].
- [100] S. Alekhin, J. Blümlein and S.O. Moch, *α_s from global fits of parton distribution functions*, Mod. Phys. Lett. A **31** (2016) no.25, 1630023.
- [101] D. d'Enterria *et al.*, *The strong coupling constant: State of the art and the decade ahead*, J. Phys. G (2024) in print [arXiv:2203.08271 [hep-ph]].
- [102] A. Accardi *et al.*, *A Critical Appraisal and Evaluation of Modern PDFs*, Eur. Phys. J. C **76** (2016) no.8, 471 [arXiv:1603.08906 [hep-ph]].
- [103] S. Alekhin, J. Blümlein, S. Moch and R. Placakyte, *Parton distribution functions, α_s , and heavy-quark masses for LHC Run II*, Phys. Rev. D **96** (2017) no.1, 014011 [arXiv:1701.05838 [hep-ph]].
- [104] S. Alekhin, J. Blümlein, K. Daum, K. Lipka and S. Moch, *Precise charm-quark mass from deep-inelastic scattering*, Phys. Lett. B **720** (2013) 172–176 [arXiv:1212.2355 [hep-ph]].
- [105] K.A. Olive *et al.* [Particle Data Group], *Review of Particle Physics*, Chin. Phys. C **38** (2014) 090001.
- [106] S. Catani, M. Ciafaloni and F. Hautmann, *High-energy factorization and small x heavy flavor production*, Nucl. Phys. B **366** (1991) 135–188.
- [107] J. Blümlein, *QCD evolution of structure functions at small x* , Proc. New Trends in HERA Physics 1999, Lect. Notes Phys. **546** (2000) 42–57, (Springer, Berlin, 2001), Grindhammer, G., Kniehl, B.A., Kramer, G. (eds), [arXiv:hep-ph/9909449 [hep-ph]].
- [108] S.A. Larin, *The renormalization of the axial anomaly in dimensional regularization*, Phys. Lett. B **303** (1993) 113–118 [hep-ph/9302240].
- [109] Y. Matiounine, J. Smith and W.L. van Neerven, *Two loop operator matrix elements calculated up to finite terms for polarized deep inelastic lepton - hadron scattering* Phys. Rev. D **58** (1998) 076002 [arXiv:hep-ph/9803439 [hep-ph]].

- [110] J. Blümlein and M. Saragnese, *Next-to-Next-to-Leading Order Evolution of Polarized Parton Densities in the Larin Scheme*, Phys. Rev. D (2024) in print, [arXiv:2405.17252 [hep-ph]].
- [111] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider and K. Schönwald, *The Single Mass Variable Flavor Number Scheme at Next-to-Next-to-Leading Order*, DESY 24–037.
- [112] D. Boer *et al.*, *Gluons and the quark sea at high energies: Distributions, polarization, tomography*, arXiv: 1108.1713 [nucl-th].
- [113] J.L. Abelleira Fernandez *et al.*, *A Large Hadron Electron Collider at CERN: Report on the Physics and Design Concepts for Machine and Detector*, J. Phys. G **39** (2012) 075001 [arXiv:1206.2913 [physics.acc-ph]].
- [114] P. Agostini *et al.*, *The Large Hadron–Electron Collider at the HL-LHC*, J. Phys. G **48** (2021) 110501 [arXiv:2007.14491 [hep-ex]].
- [115] J. Blümlein, M. Klein, G. Ingelman and R. Rückl, *Testing QCD Scaling Violations in the HERA Energy Range*, Z. Phys. C **45** (1990) 501–513.
- [116] The FCC: https://en.wikipedia.org/wiki/Future_Circular_Collider.
- [117] J. Blümlein and M. Saragnese, *The N^3LO scheme-invariant QCD evolution of the non-singlet structure functions $F_2^{NS}(x, Q^2)$ and $g_1^{NS}(x, Q^2)$* , Phys. Lett. B **820** (2021) 136589 [arXiv:2107.01293 [hep-ph]].
- [118] S. Klein *et al.*, unpublished.
- [119] M. Steinhauser, *MATAD: A Program package for the computation of MAssive TADpoles*, Comput. Phys. Commun. **134** (2001) 335–364 [hep-ph/0009029].