Loops and Legs in QFT 2024
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Challenges of the large moment method

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Der Wissenschaftsfonds.

Task: Solve coupled systems
of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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given

A whole industry (for solutions of ε-expansions) started with

[J. Henn. Multiloop integrals in dimen- sional regularization made simple. Phys. Rev. Lett., 110:251601, 2013.]

Here we follow another successful tactic.

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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$$\Bigg\downarrow$$

uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

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$$\Big\downarrow \text{ uncoupling algorithms (Zürcher, Abramov/Zima, Gauss, ...)}$$

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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \mathsf{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \mathsf{LinComb}(..., D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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DE-solver
(see, e.g., [arXiv:1810.12261])

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DE-solver
(see, e.g., [arXiv:1810.12261])

REC-solver

Tactic: the DE-REC approach

DE system

$$\boxed{D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)}$$

Tactic: the DE-REC approach

$$\boxed{\begin{array}{c}\text{DE system}\\ D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)\end{array}} \xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}} \boxed{\begin{array}{c}\text{uncoupled DE system}\\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x)\\ \hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1\end{array}}$$

Tactic: the DE-REC approach

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$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

Tactic: the DE-REC approach

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$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$

uncoupled DE system

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$\downarrow$ holonomic closure prop.

linear recurrence

$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

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$\xleftarrow{\texttt{Sigma's} \text{ REC-solver}}$

closed form solutions of $I_1(n)$ in the class of nested sums and products – if this is possible

Tactic: the DE-REC approach (SolveCoupledSystem package)

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

$\xrightarrow{\text{OreSys package (S. Gerhold)}}$ uncoupling algorithm

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\mathbf{\hat{I}_k(x)} = \text{expr}_{\mathbf{k}}(\mathbf{\hat{I}_1(x)}), \mathbf{k > 1}$$

extract coefficient

SumProduction package

holonomic closure prop.

linear recurrence

$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

`Sigma's` REC-solver

closed form solutions of

$I_1(n), \mathbf{I_2(n)}, \dots, \mathbf{I_n(n)}$

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– if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$
- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

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- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$$

$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_0(n) + \ldots$$

General strategy: physical problem $\hat{P}(x)$

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$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

$$P(n) = \varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n) + \varepsilon^{0} P_0(n) + \ldots$$

Calculations based on this tactic:

- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(n)$ of the Variable Flavor number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, n. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSn 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor number Scheme at next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, n. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, A. von Manteuffel, CS, K. Schönwald. The Unpolarized and Polarized Single-Mass Three-Loop Heavy Flavor Operator Matrix Elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$. Journal of High Energy Physics 2022(12), pp. 1-55. 2022.

More flexible tactic: Compute large moments
and guessing recurrences

[coming, e.g., from IBP methods]

Tactic: the DE-REC approach (SolveCoupledSystem package)

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\mathbf{\hat{I}_k(x)} = \text{expr}_{\mathbf{k}}(\mathbf{\hat{I}_1(x)}), \mathbf{k > 1}$$

extract coefficient

SumProduction package

holonomic closure prop.

linear recurrence
$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

`Sigma's` REC-solver

closed form solutions of
$I_1(n), \mathbf{I_2(n)}, \dots, \mathbf{I_n(n)}$
in the class of nested
sums and products
– if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

More flexible tactic: compute large moments (SolveCoupledSystem package)

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold) → uncoupling algorithm

uncoupled DE system
$$\textstyle\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)),\, k > 1$$

extract coefficient
SumProduction package

holonomic closure prop.

linear recurrence
$$\textstyle\sum_i a_i'(n) I_1(n+i) = r'(n)$$

`Sigma's` REC-solver

closed form solutions of $I_1(n)$
in the class of nested sums and products
– if thi

Given $r'(0), r'(1), \ldots, r'(8000)$,
compute $I_1(0), I_1(1), \ldots, I_1(8000)$

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

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$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_0(n) + \ldots}_{\text{only numbers in } \mathbb{Q}}$$

$n = 0, 1, \ldots, 8000$

$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_0(n)}_{\text{numbers}} + \ldots$$

$n = 0, 1, \ldots, 8000$

$$\begin{array}{c} \text{coupled systems for } P(n) \\ \Big\downarrow{\small\texttt{SolveCoupledSystem.m}} \\ \text{large no. of moments} \end{array}$$

coupled systems for $P(n)$

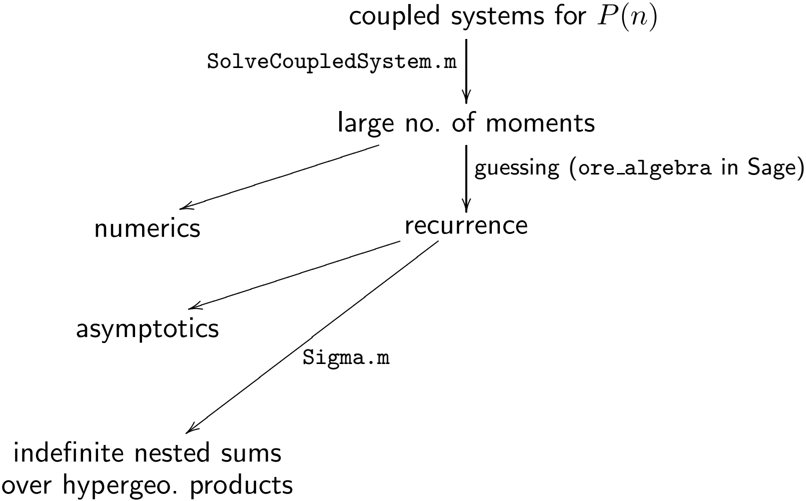
`SolveCoupledSystem.m`

large no. of moments

numerics

$$
\begin{tikzcd}
 & \text{coupled systems for } P(n) \arrow[d, "\texttt{SolveCoupledSystem.m}"'] \\
 & \text{large no. of moments} \arrow[dl] \arrow[d, "\text{guessing (\texttt{ore\_algebra} in Sage)}"] \\
\text{numerics} & \text{recurrence}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \text{coupled systems for } P(n) \arrow[d, "\texttt{SolveCoupledSystem.m}"'] \\
 & \text{large no. of moments} \arrow[dl] \arrow[d, "\text{guessing (}\texttt{ore\_algebra}\text{ in Sage)}"] \\
\text{numerics} & \text{recurrence} \arrow[dl] \\
\text{asymptotics} &
\end{tikzcd}
$$


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products

coupled systems for $P(n)$

`SolveCoupledSystem.m`

large no. of moments

guessing (`ore_algebra` in Sage)

numerics

recurrence

asymptotics

`Sigma.m`

indefinite nested sums
over hypergeo. products

indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

$$
\begin{tikzcd}
 & \text{coupled systems for } P(n) \arrow[d, "\texttt{SolveCoupledSystem.m}"'] \\
 & \text{large no. of moments} \arrow[d, "\text{guessing (}\texttt{ore\_algebra}\text{ in Sage)}"] \\
 & \text{recurrence} \arrow[dl, "\texttt{Sigma.m}"] \\
\begin{array}{c}\text{indefinite nested sums}\\ \text{over hypergeo. products}\end{array} &
\end{tikzcd}
$$

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial** $=<<$ **iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial =<< iFile16**

Out[2]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

52869784323778576751/80221899453696000,

49422862094045523994231/753773992230616156800,

33131879832907935920726113/509557943985299969760000,

5209274721836755168448777/80949984111854180459136,

56143711997344769021041145213/882589266383586456384353664,

453500433353845628194790025124807/7217228048879468556886950000000,

14061543374120479886110159898869387/226643167590350326435656036000000,

715586522666491903324905785178619936571168370307700222807811495895030000000,

16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,

1428729642632302467951426905844691837805299/23940759575034122827861315961573673600000,

498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000, • • •};

In[3]:= **rec =<< rFile16**

Out[3]=
$$
\begin{aligned}
&(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + \\
&2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + \\
&873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + \\
&1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + \\
&575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + \\
&69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + \\
&901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + \\
&7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + \\
&32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + \\
&86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + \\
&124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + \\
&87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + \\
&25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + \\
&1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]
\end{aligned}
$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 161315106167193 6n^{37} + \\
&101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
&2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
&9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
&7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
&356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
&9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
&170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
&1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
&13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
&59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
&167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
&284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
&279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + \\
&149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + \\
&38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + \\
&3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + \\
&101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\,\mathrm{P}[n+1]
\end{aligned}
$$

$$
\begin{aligned}
&+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\Big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+\\
&274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+\\
&25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+\\
&23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+\\
&6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+\\
&722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+\\
&28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+\\
&424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+\\
&1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+\\
&3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+\\
&4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+\\
&3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+\\
&1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+\\
&371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+\\
&6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\Big)\,\mathrm{P}[n+2]
\end{aligned}
$$

$$
\begin{aligned}
&-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+\\
&276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+\\
&25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+\\
&24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+\\
&7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+\\
&745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+\\
&29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+\\
&445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+\\
&1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+\\
&3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+\\
&4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+\\
&4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+\\
&1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+\\
&412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+\\
&7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)\,P[n+3]
\end{aligned}
$$

$$
\begin{aligned}
&+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38}+1546355730284544n^{37}+93441851805138944n^{36}+\\
&3636063211393908736n^{35}+102413434086873890816n^{34}+2225107112182077718528n^{33}+\\
&38808234188348931964928n^{32}+558299807912629375074304n^{31}+6755648626273815474733056n^{30}+\\
&69769132238801205785001984n^{29}+621900006220029229458259968n^{28}+4826558182244413850688946176n^{27}+\\
&32840774268722977511855751168n^{26}+196981883700048989849717882880n^{25}+\\
&1046061529031136798450810839040n^{24}+4934888224954929426023144030208n^{23}+\\
&20735286278224836075286873214976n^{22}+77745549200390911029444008457216n^{21}+\\
&260448286122609254214904458392064n^{20}+780087654447729149285799146869248n^{19}+\\
&2089276462852113795051294249728512n^{18}+5001455921015163002705347586646080n^{17}+\\
&10691068512696184477385875851523744n^{16}+20374769440121072185247660725156544n^{15}+\\
&34542976501702600883669655947085712n^{14}+51947527795197316142253213880200764n^{13}+\\
&69039779136078090572935768218052854n^{12}+80712286124402599779679594199103258n^{11}+\\
&82519759833385882007812859351392458n^{10}+73248127158607338722648198918322201n^{9}+\\
&55935262205790259307904762197107653n^{8}+36322355479155199114489624391144238n^{7}+\\
&19756597118002557191991191826327042n^{6}+8822212911433711339358062994077203n^{5}+\\
&3145597282374650512689680780380605n^{4}+859907105684964990690798899478888n^{3}+\\
&168963309995629650025632011492580n^{2}+21205680751316222158938757272000n+\\
&1274120732351744651125603886400\Big)\,\mathrm{P}[n+4]
\end{aligned}
$$

$$-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + 18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + 2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + 19313175036907229252501700n + 1248723341516324359641600\Big)\,\mathrm{P}[n+5]==0$$

In[4]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

In[4]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

Out[4]= $$\Big\{\Big\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\Big\}$$

$$\Big\{0, -\frac{(3+2n)\left(-8-9n+2n^2\right)}{(1+n)^2(1+2n)^2}\Big\}$$

$$\Big\{0, -\frac{(3+2n)\left(-5+8n^2\right)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big\}$$

$$\Big\{0, \frac{(3+2n)\left(-513-2184n-2416n^2+768n^4\right)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)\left(3+44n+48n^2\right)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)\left(3+44n+48n^2\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^2}{(1+n)(1+2n)}\Big\}$$

$$
\begin{aligned}
&\{0, \frac{1}{16(1+n)^4(1+2n)^4}\big(72519 + 572343n + 1814716n^2 + 2918100n^3 + 2442240n^4 + 912896n^5 + 24576n^6 - \\
&49152n^7\big) + \frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)} + \Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2} + \frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2} + \\
&\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3} + \frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \\
&\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \Big(- \\
&\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2} + \frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2 + \frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)} + \\
&\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)} - \frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2} + \\
&\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i} - \\
&\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2} + \frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)} - \\
&\frac{8(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{i}}{(1+n)(1+2n)} - \frac{16(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{-1+2i}}{(1+n)(1+2n)} \\
&\quad - \frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)} + \frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)} + \\
&\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{i}}{(1+n)(1+2n)} + \frac{64(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\}, \{1, 0\}\}
\end{aligned}
$$

In[5]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[5]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

Out[5]=

$$
\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-
$$

$$
\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+
$$

$$
\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+
$$

$$
\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+
$$

$$
\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-
$$

$$
\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+
$$

$$
\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+
$$

$$
\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-
$$

$$
\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+
$$

$$
\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}3(1+n)(1+2n)}{+}\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+
$$

$$
\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}
$$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= $\mathbf{sol = TransformToSSums[sol];}$

In[8]:= $\mathbf{sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;}$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

Out[8]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\left(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\right)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Bigg(-\frac{2(3+2n)\left(147+985n+1871n^2+1268n^3+212n^4\right)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Bigg)S[1,2n]-\frac{4(3+2n)\left(23+123n+114n^2\right)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\left(53+229n+190n^2\right)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\left(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\right)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\left(1+8n+8n^2\right)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[9]:= **SExpansion[sol, n, 2]**

Out[9]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[9]:= **SExpansion[sol, n, 2]**

Out[9]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Special function algorithms

- `HarmonicSums` package
  Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
  Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]
- `RICA` package
  Blümlein, Fadeev, CS. ACM Communications in Computer Algebra 57(2), pp. 31-34. 2023.

Calculations based on the large moment tactic:

- J. Blümlein, CS. The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory. Physics Letters B 771, pp. 31-36. 2017.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{n_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017.
- J. Blümlein, P. Marquard, n. Rana, CS. The Heavy Fermion Contributions to the Massive Three Loop Form Factors. Nuclear Physics B 949(114751), pp. 1-97. 2019.
- A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, CS, K. Schönwald. The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements. Nuclear Physics B 948(114753), pp. 1-41. 2019.
- J. Blümlein, A. Maier, P. Marquard, G. Schäfer, CS. From Momentum Expansions to Post-Minkowskian Hamiltonians by Computer Algebra Algorithms. Physics Letters B 801(135157), pp. 1-8. 2020.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS, K. Schönwald. The three-loop single mass polarized pure singlet operator matrix element. Nuclear Physics B 953(114945), pp. 1-25. 2020.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Polarized Operator Matrix Element $A_{gg,Q}^{(3)}$. Nuclear Physics B 955, pp. 1-70. 2020.
- A. Behring, J. Blm̈lein, A. De Freitas, A. von Manteuffel, K. Schönwald, CS. The Polarized Transition Matrix Element $A_{g,q}(n)$ of the Variable Flavor number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 964, pp. 115331-115356, 2021.
- J. Blümlein, A. De Freitas, M. Saragnese, K. Schönwald, CS. The Logarithmic Contributions to the Polarized $O(\alpha_s^3)$ Asymptotic Massive Wilson Coefficients and Operator Matrix Elements in Deeply Inelastic Scattering. Physical Review D 104(3), pp. 1-73. 2021.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements. Nucl. Phys. B 971, pp. 1-44. 2021.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements. Journal of High Energy Physics 2022(193), pp. 0-32. 2022.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The Two-Loop Massless Off-Shell QCD Operator Matrix Elements to Finite Terms. Nuclear Physics B 980(115794), pp. 1-131. 2022.
- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1. Journal of High Energy Physics. 1-83. 2022.
- J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

coupled systems for $P(n)$

`SolveCoupledSystem.m` ↓

large no. of moments

guessing (`ore_algebra` in Sage) ↓

numerics

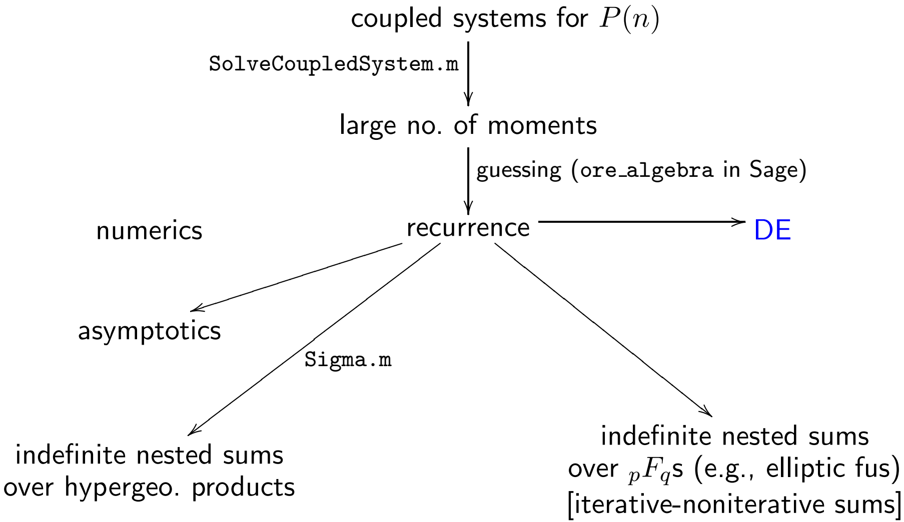
recurrence

asymptotics

`Sigma.m`

indefinite nested sums
over hypergeo. products

indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]


coupled systems for $P(n)$
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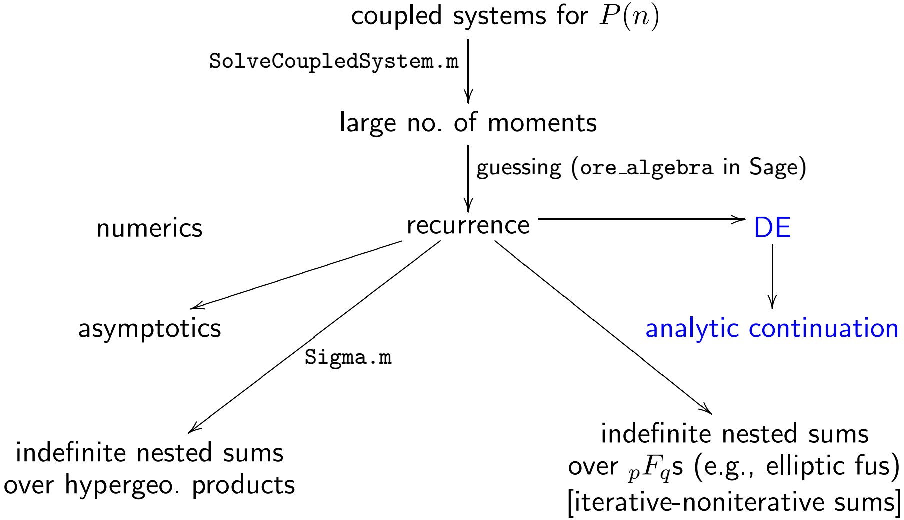
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indefinite nested sums over ${}_pF_q$s (e.g., elliptic fus) [iterative-noniterative sums]

Evaluate beyond 0 (compare A. De Freitas' talk)

$$\sum_{n=0}^{\infty} f_n (x+1)^n$$

given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

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convergency radius $r = 1$

$$\sum_{n=0}^{\infty} f_n (x+1)^n$$

given $f_n \in \mathbb{Q}$

$f_0 \ldots, f_{8000}$ with the LMM

$\downarrow$

a recurrence of order 42

$\downarrow$

arbitrary many f_n computable

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x (compare A. De Freitas' talk)

convergency radius $r = 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n \qquad\qquad \sum_{n=0}^{\infty} f_n (x+1)^n$$

find $g_{j,n} \in \mathbb{R}$

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Matching evaluations at a common point x (compare A. De Freitas' talk)

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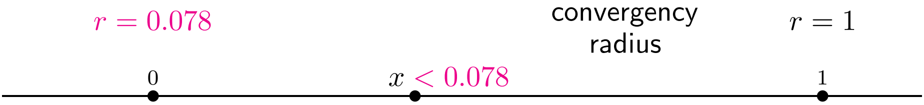

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0 $\quad x < 0.078 \quad$ 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n \qquad\qquad \sum_{n=0}^{500000} f_n (x+1)^n$$

find $g_{j,n} \in \mathbb{R}$ given $f_n \in \mathbb{Q}$

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$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800) $\displaystyle\sum_{n=0}^{500000} f_n (x+1)^n$

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Note:~ 10 available digits in

[M. Fael, F. Lange, K. Schönwald, M. Steinhauser, Phys. Rev. D 106, 034029 (2022)]

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$$\big\downarrow \text{PSLQ}$$

$g_{j,n} \in \mathbb{Q}(\pi, \zeta_3, \ldots)$

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Current status of the Form Factor project (Blümlein, de Freitas, Marquard)

1. quarkonic contributions: fully tackled in

   [J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Physical Review D 108(094003), pp. 1-73. 2023.]

   using 8000 moments

   - all symbolically solvable parts are computed
   - the remaining contributions are given in analytic series expansions (in terms of special constants using PSLQ)

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     - all $1/\varepsilon$- contributions are obtained in HPLs
     - the solvable ε^0-contributions are computed: $\zeta_5, \zeta_2\,\zeta_3$
     - recurrences are derived for the following contributions: $\text{Li}_4(1/2), \ln(2)^4, \ln(2)^2\zeta_2, \zeta_2^2, \zeta_2\zeta_3, \zeta_5$

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   - **ready for even more moments**

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   - the remaining contribu
     (in terms of special co

Biggest rational contribution of $1/\varepsilon$-term

vector case	no. moments	order	degree
quarkonic	850	19	266
gluonic	2000	27	364

2. gluonic contributions: work in progress

   - 4000 moments computed
     - all $1/\varepsilon$- contributions are obtained in HPLs
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Recent challenges of the Large Moment Method

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$
- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + q_i(x)\hat{I}_i(x) + \cdots + q_{2506}(x)\hat{I}_{2506}(x)$$

106 GB
in MMA

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + \boxed{q_i(x)\hat{I}_i(x)} + \cdots + q_{2506}(x)\hat{I}_{2506}(x)$$

106 GB
in MMA

$$\|$$

$$\frac{a_1(x)}{b_1(x)} + \cdots + \frac{a_{606}(x)}{b_{606}(x)} + \frac{a_{607}(x)}{b_{607}(x)} + \cdots + \frac{a_{1213}(x)}{b_{1213}(x)}$$

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + \boxed{q_i(x)\hat{I}_i(x)} + \cdots + q_{2506}(x)\hat{I}_{2506}(x)$$

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$$\| $$

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$$\| $$

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$$\downarrow \qquad\qquad \downarrow$$

$$\vdots \quad \text{MyTogether} \quad \vdots$$

Divide & Conquer

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{A_1(x)}{B_1(x)} + \frac{A_2(x)}{B_2(x)}$$

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + \boxed{q_i(x)\hat{I}_i(x)} + \cdots + q_{2506}(x)\hat{I}_{2506}(x)$$

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$$\downarrow \qquad \qquad \downarrow$$

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MyTogether

Divide & Conquer

$$\downarrow \qquad \qquad \downarrow$$

$$\frac{A_1(x)}{B_1(x)} + \frac{A_2(x)}{B_2(x)}$$

$$\| $$

$$\frac{A_1(x)B_2(x) + A_2(x)B_1(x)}{B_1(x)B_2(x)}$$

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$$\frac{A_1(x)}{B_1(x)} + \frac{A_2(x)}{B_2(x)}$$

$$\|$$

$$\frac{A_1(x)B_2(x) + A_2(x)B_1(x)}{B_1(x)B_2(x)}$$

$$\|$$ Together?

$$\boxed{\frac{A(x)}{B(x)}} \qquad \gcd(A, B) = 1$$

Example:

$$\frac{\overbrace{A_1(x,\varepsilon)B_2(x)+A_2(x,\varepsilon)B_1(x,\varepsilon)}^{A'(x,\varepsilon)=}}{\underbrace{B_1(x,\varepsilon)B_2(x,\varepsilon)}_{B'(x,\varepsilon)}}$$

$$\deg_x(A') \leq 1422$$
$$\deg_x(B') \leq 1405$$

Task: remove common factors in A' and B'

Example:

$$\frac{\overbrace{A_1(x,\varepsilon)B_2(x) + A_2(x,\varepsilon)B_1(x,\varepsilon)}^{A'(x,\varepsilon)=}}{\underbrace{B_1(x,\varepsilon)B_2(x,\varepsilon)}_{B'(x,\varepsilon)}} \qquad \begin{array}{c} \deg_x(A') \leq 1422 \\ \deg_x(B') \leq 1405 \end{array}$$

Task: remove common factors in A' and B'

Observation: $B'(x,\varepsilon)$ factors nicely!

$$B'(x,\varepsilon) =$$

$$\varepsilon^8(\varepsilon+1)^2(2\varepsilon-1)^3(2\varepsilon+1)^4(3\varepsilon-2)^2(3\varepsilon-1)(3\varepsilon+1)(4\varepsilon-3)(4\varepsilon-1)(4\varepsilon+1)^3(4\varepsilon+3)(5\varepsilon-2)$$

$$(5\varepsilon-1)(6\varepsilon-1)(6\varepsilon+1)^2(8\varepsilon+5)(x-2)^{24}(x-1)^{17}x^{28}(x+1)^2(2x-1)^2(x^2-22x+22)^9$$

$$(x^2-15x+15)^{10}(x^2-6x+6)^{10}(x^2-5x+5)^{10}(x^2-2x+2)^{20}(x^2-x+1)^{15}(x^2+x-1)^{10}$$

$$(x^2+2x-2)^9x^2+3x-3)^9(x^2+4x-4)^9(2x^2+3x-3)^9(3x^2-14x+14)^{11}(3x^2-10x+10)$$

$$(3x^2-8x+8)^{11}(3x^2-4x+4)^8(3x^2-2x+2)^9(3x^2+x-1)^8(3x^2+2x-2)^{10}(4x^2+3x-3)^9$$

$$(5x^2-18x+18)^9(5x^2-16x+16)^{10}(5x^2-2x+2)^9(5x^2+12x-12)^7(7x^2-6x+6)^9(11x^2+2x-2)^8$$

$$(27x^2+32x-32)^9(29x^2-2x+2)^8(99x^2-238x+238)^{10}(\varepsilon x^2-8\varepsilon x+2x+8\varepsilon-2)$$

$$(2\varepsilon x^2-x^2-16\varepsilon x+16\varepsilon)(3\varepsilon x^2-4\varepsilon x+x+4\varepsilon-1)(3\varepsilon x^2-x^2-4\varepsilon x+x+4\varepsilon-1)$$

$$(3\varepsilon x^2-x^2-2\varepsilon x+x+2\varepsilon-1)(7\varepsilon x^2+x^2-6\varepsilon x-x+6\varepsilon+1)(22\varepsilon x^2-5x^2-68\varepsilon x+18x+68\varepsilon-18)$$

$$(2x^2\varepsilon^2-4x\varepsilon^2+4\varepsilon^2+x^2\varepsilon+2x\varepsilon-2\varepsilon+x^2-x+1)(2x^2\varepsilon^2-4x\varepsilon^2+4\varepsilon^2+3x^2\varepsilon+x^2+3x-3)$$

$$(5x^2\varepsilon^2+15x\varepsilon^2-15\varepsilon^2-4x^2\varepsilon-14x\varepsilon+14\varepsilon+x^2+3x-3)$$

$$(6x^2\varepsilon^2-12x\varepsilon^2+12\varepsilon^2+5x^2\varepsilon+2x\varepsilon-2\varepsilon+x^2)$$

$$(28x^2\varepsilon^2-64x\varepsilon^2+64\varepsilon^2-25x^2\varepsilon+64x\varepsilon-64\varepsilon+5x^2-16x+16)$$

$$(30x^2\varepsilon^3-120x\varepsilon^3+120\varepsilon^3-77x^2\varepsilon^2+74x\varepsilon^2-74\varepsilon^2+57x^2\varepsilon-14x\varepsilon+14\varepsilon-11x^2-2x+2)$$

$$(-x^4-16\varepsilon^2x^3+8x^3+80\varepsilon^2x^2+8\varepsilon x^2-28x^2-128\varepsilon^2x-16\varepsilon x+40x+64\varepsilon^2+8\varepsilon-20)$$

$$(x^4-6x^3+18x^2-24x+12)^8(x^4-4x^3+5x^2-2x+1)^9(x^4+8x^3-32x^2+48x-24)^7$$

$$(3x^4+2x^3-6x^2+8x-4)^8(5x^4-152x^3+272x^2-240x+120)^{10}(5x^4-29x^3+27x^2+4x-2)^{10}$$

$$(5x^4+16x^3-40x^2+48x-24)^{11}(5x^4+184x^3-352x^2+336x-168)^9(7x^4-31x^3+25x^2+12x-6)^{10}$$

$$(9x^4-43x^3+37x^2+12x-6)^8(9x^4-11x^3+15x^2-8x+4)^8(9x^4+29x^3-15x^2-28x+14)^8\bullet\bullet\bullet$$

• • • $(9x^4 + 80x^3 - 12x^2 - 136x + 68)^9 (10x^4 + 9x^3 - 103x^2 + 188x - 94)^{10}$
$(12x^4 - 85x^3 + 115x^2 - 60x + 30)^{10} (13x^4 - 16x^3 + 40x^2 - 48x + 24)^{11}$
$(23x^4 + 16x^3 - 40x^2 + 48x - 24)^{11} (26x^4 - 83x^3 - 19x^2 + 204x - 102)^{10}$
$(60x^4 - 79x^3 - 215x^2 + 588x - 294)^8$
$(5\varepsilon x^4 - x^4 - 6\varepsilon x^3 + 2x^3 - 34\varepsilon x^2 + 6x^2 + 80\varepsilon x - 16x - 40\varepsilon + 8)$
$(11\varepsilon^2 x^4 - 15\varepsilon x^4 + 5x^4 + 60\varepsilon^2 x^3 - 62\varepsilon x^3 + 16x^3 - 156\varepsilon^2 x^2$
$+ 158\varepsilon x^2 - 40x^2 + 192\varepsilon^2 x - 192\varepsilon x + 48x - 96\varepsilon^2 + 96\varepsilon - 24)$
$(54\varepsilon^2 x^4 - 9\varepsilon x^4 - 3x^4 - 400\varepsilon^2 x^3 + 72\varepsilon x^3 + 20x^3 + 888\varepsilon^2 x^2$
$- 164\varepsilon x^2 - 48x^2 - 976\varepsilon^2 x + 184\varepsilon x + 56x + 488\varepsilon^2 - 92\varepsilon - 28)$
$(48x^4\varepsilon^6 - 9792x^3\varepsilon^6 + 19296x^2\varepsilon^6 - 19008x\varepsilon^6 + 9504\varepsilon^6 + 1006x^4\varepsilon^5 + 9264x^3\varepsilon^5$
$- 16896x^2\varepsilon^5 + 15264x\varepsilon^5 - 7632\varepsilon^5 - 686x^4\varepsilon^4 - 4136x^3\varepsilon^4 + 7352x^2\varepsilon^4$
$- 6432x\varepsilon^4 + 3216\varepsilon^4 + 95x^4\varepsilon^3 + 2416x^3\varepsilon^3 - 6664x^2\varepsilon^3 + 8496x\varepsilon^3$
$- 4248\varepsilon^3 - 32x^4\varepsilon^2 - 1270x^3\varepsilon^2 + 4438x^2\varepsilon^2 - 6336x\varepsilon^2 + 3168\varepsilon^2$
$+ 25x^4\varepsilon + 330x^3\varepsilon - 1266x^2\varepsilon + 1872x\varepsilon - 936\varepsilon - 4x^4 - 32x^3 + 128x^2 - 192x + 96)$
$(x^6 - 6x^5 + 11x^4 - 8x^3 - x^2 + 6x - 2)^9$

Example:

$$\frac{\overbrace{A_1(x,\varepsilon)B_2(x)+A_2(x,\varepsilon)B_1(x,\varepsilon)}^{A'(x,\varepsilon)=}}{\underbrace{B_1(x,\varepsilon)B_2(x,\varepsilon)}_{B'(x,\varepsilon)}} \qquad \begin{array}{l}\deg_x(A') \leq 1422\\ \deg_x(B') \leq 1405\end{array}$$

Task: remove common factors in A' and B'

Observation: $B'(x,\varepsilon)$ factors nicely!

$$B'(x,\varepsilon') = \mathsf{factor}_1(x,\varepsilon)\mathsf{factor}_2(x,\varepsilon)\ldots\mathsf{factor}_{71}(x,\varepsilon)$$

Example:

$$\frac{\overbrace{A_1(x,\varepsilon)B_2(x)+A_2(x,\varepsilon)B_1(x,\varepsilon)}^{A'(x,\varepsilon)=}}{\underbrace{B_1(x,\varepsilon)B_2(x,\varepsilon)}_{B'(x,\varepsilon)}} \qquad \begin{array}{l}\deg_x(A') \le 1422\\ \deg_x(B') \le 1405\end{array}$$

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Idea: Filter out factors in B' which do not cancel:

$$\mathsf{factor}(x,1234) \nmid A(x,1234) \quad\Longrightarrow\quad \mathsf{factor}(x,\varepsilon) \nmid A(x,\varepsilon)$$

Example:

$$\frac{\overbrace{A_1(x,\varepsilon)B_2(x) + A_2(x,\varepsilon)B_1(x,\varepsilon)}^{A'(x,\varepsilon)=}}{\underbrace{B_1(x,\varepsilon)B_2(x,\varepsilon)}_{B'(x,\varepsilon)}} \qquad \begin{array}{l}\deg_x(A') \le 1422\\ \deg_x(B') \le 1405\end{array}$$

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Idea: Filter out factors in B' which do not cancel:

$$\mathsf{factor}(x,1234) \nmid A(x,1234) \quad \Longrightarrow \quad \mathsf{factor}(x,\varepsilon) \nmid A(x,\varepsilon)$$

For the remaining candidates, carry out polynomial division:

$$A(x,\varepsilon) = q(x,\varepsilon)\mathsf{factor}(x,\varepsilon) + r(x,\varepsilon)$$

If $r = 0$: remove the factor from B and replace A by q.

If $r \neq 0$: do nothing (never happened so far)

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + \boxed{q_i(x)\hat{I}_i(x)} + \cdots + q_{2506}(x)\hat{I}_{2506}(x)$$

106 GB in MMA

$$\| $$

$$\underbrace{\frac{a_1(x)}{b_1(x)} + \cdots + \frac{a_{606}(x)}{b_{606}(x)}} + \underbrace{\frac{a_{607}(x)}{b_{607}(x)} + \cdots + \frac{a_{1213}(x)}{b_{1213}(x)}}$$

$\downarrow$ $\vdots$ $\downarrow$ MyTogether $\downarrow$ $\vdots$ $\downarrow$

Divide & Conquer

$$\frac{A_1(x)}{B_1(x)} + \frac{A_2(x)}{B_2(x)}$$

$$\|$$

$$\frac{A_1(x)B_2(x) + A_2(x)B_1(x)}{B_1(x)B_2(x)}$$

$\|$ MyCancel

$$\boxed{\frac{A(x)}{B(x)}} \qquad \gcd(A, B) = 1$$

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + q_i(x)\hat{I}_i(x) + q_{2506}(x)\hat{I}_{2506}(x)$$

106 GB
in MMA

$$\Big\downarrow \text{MyTogether}$$

$$\hat{P}(x) = \frac{A_1(x)}{B_1(x)}\hat{I}_1(x) + \cdots + \frac{A_i(x)}{B_i(x)}\hat{I}_i(x) + \cdots + \frac{A_{2506}(x)}{B_{2506}(x)}\hat{I}_{2506}(x)$$

4.7 GB
in MMA

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4.7 GB in MMA

$=$ ...expand each term in parallel...

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$=$ ...expand each term in parallel...

$$= \sum_{n=0}^{8000} P(n)x^n + O(x^{8001})$$

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106 GB in MMA

$$\Big\downarrow \text{MyTogether}$$

$$\hat{P}(x) = \frac{A_1(x)}{B_1(x)}\hat{I}_1(x) + \cdots + \boxed{\frac{A_i(x)}{B_i(x)}\hat{I}_i(x)} + \cdots + \frac{A_{2506}(x)}{B_{2506}(x)}\hat{I}_{2506}(x)$$

4.7 GB in MMA

$=$ ...expand each term in parallel...

$$= \sum_{n=0}^{8000} P(n)x^n + O(x^{8001})$$

Goal: Compute $c(n)$ for $n = 0, \ldots, 8000$ s.t.

$$\boxed{\frac{A(x)}{B(x)}\,\hat{I}(x)} = \sum_{n=0}^{8000} c(n) x^n + O(x^{8001})$$

Goal: Compute $c(n)$ for $n = 0, \ldots, 8000$ s.t.

$$\boxed{\frac{A(x)}{B(x)}\,\hat{I}(x)} = \sum_{n=0}^{8000} c(n) x^n + O(x^{8001})$$

- The large moment method gives $I(n)$ for $i = 0, \ldots, 8000$ s.t.

$$\hat{I}(x) = \sum_{n=0}^{8000} I(n) x^n + O(x^{8001})$$

Goal: Compute $c(n)$ for $n = 0, \ldots, 8000$ s.t.

$$\boxed{\frac{A(x)}{B(x)}\,\hat{I}(x)} = \sum_{n=0}^{8000} c(n) x^n + O(x^{8001})$$

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- Task: Compute $h(n)$ for $n = 0, \ldots, 8000$ s.t.

$$\frac{1}{B(x)} = \sum_{n=0}^{8000} h(n) x^n + O(x^{8001})$$

Goal: Compute $c(n)$ for $n = 0, \ldots, 8000$ s.t.

$$\boxed{\frac{A(x)}{B(x)}\,\hat{I}(x)} = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

- The large moment method gives $I(n)$ for $i = 0, \ldots, 8000$ s.t.

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- Task: Compute $h(n)$ for $n = 0, \ldots, 8000$ s.t.

$$\frac{1}{B(x)} = \sum_{n=0}^{8000} h(n)x^n + O(x^{8001})$$

Classical result (C-finite sequences): Let

$$\frac{1}{b_0 + q_1 x + \cdots + b_d x^d} = \sum_{n=0}^{\infty} h(n)x^n$$

Then

$$h(n+d) = \frac{-1}{b_0}\Big(b_1 \cdot h(n+d-1) + \cdots + b_d \cdot h(n)\Big)$$

Example.

$$B(x) = b_0 + b_1 x + \cdots + b_{1393} x^{1393}$$

Example.

$$B(x) = b_0+b_1x+\cdots+b_{1393}x^{1393} = \overbrace{(x-2)^{26}(x-1)^{19}x^{32}(x+1)^2(2x-1)^2}^{=F_{\mathsf{lin}}(n)} \times$$

$$\begin{aligned}\times\underbrace{\begin{aligned}&(x^2-22x+22)^9(x^2-15x+15)^{10}(x^2-6x+6)^9(x^2-5x+5)^{10}(x^2-2x+2)^{21}\\&(x^2-x+1)^{19}(x^2+x-1)^{10}(x^2+2x-2)^{10}(x^2+3x-3)^{11}(x^2+4x-4)^9\\&(2x^2+3x-3)^9(3x^2-14x+14)^{12}(3x^2-10x+10)(3x^2-8x+8)^{11}\\&(3x^2-4x+4)^8(3x^2-2x+2)^9(3x^2+x-1)^8(3x^2+2x-2)^{10}(4x^2+3x-3)^9\\&(5x^2-18x+18)^{10}(5x^2-16x+16)^{11}(5x^2-2x+2)^9(5x^2+12x-12)^7\\&(7x^2-6x+6)^9(11x^2+2x-2)^9(27x^2+32x-32)^9(29x^2-2x+2)^8\\&(99x^2-238x+238)^{10}(x^4-8x^3+28x^2-40x+20)(x^4-6x^3+18x^2-24x+12)^8\\&(x^4-4x^3+5x^2-2x+1)^9(x^4+8x^3-32x^2+48x-24)^8(3x^4+2x^3-6x^2+8x-4)^8\\&(5x^4-152x^3+272x^2-240x+120)^{10}(5x^4-29x^3+27x^2+4x-2)^9\\&(5x^4+16x^3-40x^2+48x-24)^{12}(5x^4+184x^3-352x^2+336x-168)^9\\&(7x^4-31x^3+25x^2+12x-6)^{10}(9x^4-43x^3+37x^2+12x-6)^8\\&(9x^4-11x^3+15x^2-8x+4)^8(9x^4+29x^3-15x^2-28x+14)^8\\&(9x^4+80x^3-12x^2-136x+68)^9(10x^4+9x^3-103x^2+188x-94)^9\\&(12x^4-85x^3+115x^2-60x+30)^{10}(13x^4-16x^3+40x^2-48x+24)^{11}\\&(23x^4+16x^3-40x^2+48x-24)^{11}(26x^4-83x^3-19x^2+204x-102)^{10}\\&(60x^4-79x^3-215x^2+588x-294)^8(x^6-6x^5+11x^4-8x^3-x^2+6x-2)^9\end{aligned}}_{=F_{\mathsf{nonlin}}(n)}\end{aligned}$$

Compute polynomials $s(n)$ and $t(n)$ such that

$$1 = s(n)F_{\mathsf{nonlin}}(n) + t(n)F_{\mathsf{lin}}(n)$$

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Method 1: Extended Euclidean algorithm

Compute polynomials $s(n)$ and $t(n)$ such that

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Method 2: Solve linear system
(compare the Sylvester matrix/ resultants)

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Method 1: Extended Euclidean algorithm
Method 2: Solve linear system
(compare the Sylvester matrix/ resultants)

$$\Big\downarrow \times \frac{1}{F_{\textsf{lin}}(n)\, F_{\textsf{nonlin}}(n)}$$

$$\frac{1}{B(x)} = \frac{1}{F_{\textsf{lin}}(n)\, F_{\textsf{nonlin}}(n)} = \frac{s(n)}{F_{\textsf{lin}}(n)} + \frac{t(n)}{F_{\textsf{nonlin}}(n)}$$

Compute polynomials $s(n)$ and $t(n)$ such that

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classical PFD

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Method 2: Solve linear system
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classical PFD

iterative application
of the above technique

$$\frac{1}{B(x)} = \frac{a_1(x)}{(x-2)^{26}} + \frac{p_2(x)}{(x-1)^{19}} + \cdots + \frac{p_{54}(x)}{(x^6 - 6x^5 + 11x^4 - 8x^3 - x^2 + 6x - 2)^9}$$

$$\frac{1}{B(x)} = \frac{a_1(x)}{(x-2)^{26}} + \frac{p_2(x)}{(x-1)^{19}} + \cdots + \frac{p_{54}(x)}{(x^6 - 6x^5 + 11x^4 - 8x^3 - x^2 + 6x - 2)^9}$$

$$\Big\downarrow \text{ expand in parallel (using recurrences)}$$

$$\frac{1}{B(x)} = \sum_{n=0}^{8000} h(n)x^n + O(x^{8001})$$

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$$\Big\downarrow \text{ expand in parallel (using recurrences)}$$

$$\frac{1}{B(x)} = \sum_{n=0}^{8000} h(n)x^n + O(x^{8001})$$

Thus we have

$$\frac{A(x)}{B(x)}\hat{I}(x) = \left(\sum_{n=0}^{8000} a(n)x^n\right)\left(\sum_{n=0}^{8000} h(n)x^n\right)\left(\sum_{n=0}^{8000} I(n)x^n\right) + O(x^{8001})$$

$$\frac{1}{B(x)} = \frac{a_1(x)}{(x-2)^{26}} + \frac{p_2(x)}{(x-1)^{19}} + \cdots + \frac{p_{54}(x)}{(x^6-6x^5+11x^4-8x^3-x^2+6x-2)^9}$$

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$$\frac{1}{B(x)} = \sum_{n=0}^{8000} h(n)x^n + O(x^{8001})$$

Thus we have

$$\frac{A(x)}{B(x)}\hat{I}(x) = \underbrace{\underbrace{\Big(\sum_{n=0}^{8000} p(n)x^n\Big)\Big(\sum_{n=0}^{8000} h(n)x^n\Big)}_{\text{Cauchy}}\Big(\sum_{n=0}^{8000} I(n)x^n\Big)}_{\text{Cauchy}} + O(x^{8001})$$

$$= \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Cauchy product:

$$c(n) = \sum_{k=0}^{n} a(k)b(n-k)$$

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Note: Mathematica's `Expand` is fast

– it parallelizes (even in user parallelized code)

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Note: Mathematica's `Expand` is fast

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Naive approach: execute

Expand[A*B]

and drop the terms $c(8001)x^{8001} + \cdots + c(16000)x^{16000}$

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Note: Mathematica's `Expand` is fast

– it parallelizes (even in user parallelized code)

Improved version:

$$\Big(\sum_{n=0}^{8000} a(n)x^n\Big)\Big(\sum_{n=0}^{8000} b(n)x^n\Big) =$$

$$\Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)$$

$$+ \Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big) + \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)$$

$$+ \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big)$$

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Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Note: Mathematica's `Expand` is fast

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Improved version:

$$\sum_{n=0}^{8000} c(n)x^n + O(x^{8001}) =$$

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$$+ \Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big) + \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)$$
$$+ \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big)$$

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

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$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

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Improved version:

$$\sum_{n=0}^{8000} c(n)x^n + O(x^{8001}) =$$

$$\texttt{Expand}\Big[\Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)\Big]$$

$$+ \Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big) + \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)$$

$$+ \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big)$$

Given $A(x) = \sum_{n=0}^{8000} a(n)x^n$ and $B(x) = \sum_{n=0}^{8000} b(n)x^n$

Find $c(n)$ with

$$A(x)B(x) = \sum_{n=0}^{8000} c(n)x^n + O(x^{8001})$$

Note: Mathematica's `Expand` is fast

– it parallelizes (even in user parallelized code)

Improved version:

$$\sum_{n=0}^{8000} c(n)x^n + O(x^{8001}) =$$

$$\texttt{Expand}\Big[\Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)\Big]$$

$$+\Big(\sum_{n=0}^{4000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big) + \Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=0}^{4000} b(n)x^n\Big)$$

$$+\xcancel{\Big(\sum_{n=4001}^{8000} a(n)x^n\Big)\Big(\sum_{n=4001}^{8000} b(n)x^n\Big)}$$

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Apply improved version recursively

$$\hat{P}(x) \overset{\text{DESY}}{=} q_1(x)\hat{I}_1(x) + \cdots + q_i(x)\hat{I}_i(x) + q_{2506}(x)\hat{I}_{2506}(x)$$

106 GB in MMA

$$\Big\downarrow \text{MyTogether}$$

$$\hat{P}(x) = \frac{A_1(x)}{B_1(x)}\hat{I}_1(x) + \cdots + \frac{A_i(x)}{B_i(x)}\hat{I}_i(x) + \cdots + \frac{A_{2506}(x)}{B_{2506}(x)}\hat{I}_{2506}(x)$$

4.7 GB in MMA

$=$ ...expand each term in parallel...

$$= \sum_{n=0}^{8000} P(n)x^n + O(x^{8001})$$

More flexible tactic: compute large moments (SolveCoupledSystem package)

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$

uncoupled DE system
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$$

extract coefficient
SumProduction package

holonomic closure prop.

linear recurrence
$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

Sigma's REC-solver

closed form solutions of $I_1(n)$
in the class of nested sums and products
– if thi

Given $r'(0), r'(1), \dots, r'(8000)$,
compute $I_1(0), I_1(1), \dots, I_1(8000)$

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   here various non-trivial challenges arise:

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   - series expansion of large rational functions
   - efficient multiplication of large expansions
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5. techniques will be useful for new challenges