Computing Mellin representations and asymptotics of nested binomial sums in a symbolic way: the RICA package

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Different problems in combinatorics, analysis of algorithms or even physics involve binomially weighted sums

- **Example 1 (combinatorics)**: In the paper [*Evaluation of Binomial Double Sums Involving Absolute Values* of C. Krattenthaler and C. Schneider], sums of the following form appear for the study of double sums with binomial coefficients:

$$-2^{2m+1}n\binom{2n}{n}\boxed{\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}}+2\binom{2m}{m}\binom{2n}{n}+2^{2m+2n}$$

  If we want the asymptotic expansion at $m\to+\infty$ for fixed m, this involves in particular computing the asymptotics of the boxed sum

- **Example 2 (physics)**: Particle physics computations are often done in Mellin space, and for example in the paper [*The $\mathcal{O}(\alpha_s^3 T_F^2)$ contributions to the gluonic operator matrix element* by J.Abligner, J. Blümlein, C. Schneider et al.], sums of the following form pop up:

$$\frac{1}{4^n}\binom{2n}{n}\left(\boxed{\sum_{i=1}^{n}\frac{4^i}{i^2\binom{2i}{i}}S_1(i-1)}-7\zeta_3\right),\quad S_1(i-1):=\sum_{k=1}^{i-1}\frac{1}{k},\quad \zeta_k=\sum_{n=1}^{\infty}\frac{1}{n^k}$$

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Sums can be nested, for example in [*Iterated Binomial Sums and their Associated Iterated Integrals* by J.Ablinger, J.Blümlein, C.G. Raab and C. Schneider], we also have sums such as:

$$\sum_{i=1}^{n}\binom{2i}{i}S_2(i), \quad \sum_{i=1}^{n}\frac{1}{i\binom{2i}{i}}\sum_{j=1}^{i}\binom{2j}{j}(-2)^j$$

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Aim: Being able to deal automatically with those kind of sums in all generality, in particular **Mellin inversion** and **asymptotic expansion**

- We define the binomially weighted sums as follows:

$$BS_{\{a_1,\dots,a_k\}}(n) := \sum_{i_1=1}^{n} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

with

$$a_j(p) = a_j(p; b, c, m) = \binom{2p}{p}^b \frac{c^p}{p^m}, \quad b \in \{-1, 0, 1\}, \, c \in \mathbb{R}^\star, \, m \in \mathbb{N}$$

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Example :

$$\sum_{i=1}^{n} \frac{1}{i\binom{2i}{i}} \sum_{j=1}^{i} \binom{2j}{j} \frac{(-1)^j}{j^3} \sum_{k=1}^{j} \frac{\left(\frac{1}{2}\right)^k}{k^2}$$

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- More generic summands can also be considered, such as:

$$\frac{c^n}{(2n+1)\binom{2n}{n}} \text{ or } \frac{p(i)}{q(i)}\binom{2i}{i},\ p,q \in \mathbb{C}[X],\ \deg p \leq \deg q,\ \text{roots}(q) \subset \mathbb{C}\backslash\mathbb{R}_+$$

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Example :

$$\sum_{i=1}^{n} \frac{1+i}{i^2+5i+6} \sum_{j=1}^{i} \frac{(-2)^j}{(2j+1)\binom{2j}{j}} \sum_{k=1}^{j} \frac{3^k}{k^2}$$

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We remind here some definitions and useful properties of Mellin transforms:

- Definition:

$$M\left[f(x)\right](n) := \int_0^1 \mathrm{d}x\, x^n f(x)$$

- Summation formula ($c \in \mathbb{C}$):

$$\sum_{i=1}^{n} c^i M\left[f(x)\right](i) = c^n M\left[\frac{x}{x-\frac{1}{c}} f(x)\right](n) - M\left[\frac{x}{x-\frac{1}{c}} f(x)\right](0) \qquad (1)$$

- Convolution:

$$f(x) * g(x) := \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, \delta(x - x_1 x_2) f(x_1) g(x_2) = \int_x^1 \mathrm{d}y\, \frac{f(y)}{y} g\left(\frac{x}{y}\right)$$

$$M\left[f(x) * g(x)\right](n) = M\left[f(x)\right](n) \cdot M\left[g(x)\right](n)$$

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- **Second method**: compute it recursively from the BS using fundamental **properties of Mellin transforms** and "rule-theorems" that allow us to compute in an automatic way Mellin convolutions [*Iterated Binomial Sums and their Associated Iterated Integrals* by J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider]

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  Note: RICA relies on C. Schneider's `Sigma` and J. Ablinger's `HarmonicSums`

Note: These Mellin representations will involve general polylogarithms

$$\mathrm{H}^*_{\emptyset}(x) := 1,\ \mathrm{H}^*_{\mathrm{b(t)},\vec{c}\,\mathrm{(t)}}(x) = \mathrm{H}^*_{\mathrm{b},\vec{c}}(x) := \int_x^1 \mathrm{d}t\, b(t)\mathrm{H}^*_{\vec{c}}(t)$$

Defined over a 37 letter alphabet $\{f_0, \ldots, f_{w_{32}}\}$ containing root singularities such that all iterated integrals are linearly independent over the algebraic functions, and obeying shuffle algebra

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Example:

$$\begin{aligned}\mathrm{H}^*_{\mathrm{f_{w_{11}}},\mathrm{f_2},\mathrm{f_{w_8}}} &= \int_x^1 \mathrm{d}t_1\ f_{w_{11}}(t_1) \int_{t_1}^1 \mathrm{d}t_2\ f_2(t_2) \int_{t_2}^1 \mathrm{d}t_3\ f_{w_8}(t_3) \\ &= \int_x^1 \mathrm{d}t_1\ \frac{1}{t_1\sqrt{1-t_1}\sqrt{2-t_1}} \int_{t_1}^1 \mathrm{d}t_2\ \frac{1}{2-t_2} \int_{t_2}^1 \mathrm{d}t_3\ \frac{1}{t_3\sqrt{t_3-\frac{1}{4}}}\end{aligned}$$

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Example: (linearization by shuffle relations)

$$\mathrm{H}^*_{\mathrm{f}_1}(x)\mathrm{H}^*_{\mathrm{f}_0,\mathrm{f}_{-1}}(x) = \mathrm{H}^*_{\mathrm{f}_1,\mathrm{f}_0,\mathrm{f}_{-1}}(x) + \mathrm{H}^*_{\mathrm{f}_0,\mathrm{f}_{-1},\mathrm{f}_1}(x) + \mathrm{H}^*_{\mathrm{f}_0,\mathrm{f}_1,\mathrm{f}_{-1}}(x)$$

where

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x}$$

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Example:

$$BS(n) = \sum_{k=1}^{n} \binom{2i}{i} S_2(i) = \sum_{k=1}^{n} \binom{2i}{i} \sum_{j=1}^{i} \frac{1}{i^2}$$

- First we compute the Mellin representation of $\frac{1}{i^2}$ by convolving $\frac{1}{i} = M\left[\frac{1}{x}\right](i)$ with itself. We get:

$$\frac{1}{i^2} = M\left[\frac{1}{x}\right](i) \cdot M\left[\frac{1}{x}\right](i) = M\left[\frac{1}{x} * \frac{1}{x}\right](i) = M\left[\frac{\mathrm{H}_0^*(x)}{x}\right](i)$$

where

$$\mathrm{H}_0^*(x) := \int_x^1 \mathrm{d}t\, f_0(t) = \int_x^1 \mathrm{d}t\, \frac{1}{t} = -\log x$$

- Using the summation formula (1), we can then obtain:

$$S_2(i) = \sum_{k=1}^{i} M\left[\frac{\mathrm{H}_0^*(x)}{x}\right](i) = \underbrace{\int_0^1 \mathrm{d}x\, x^i \frac{\mathrm{H}_0^*(x)}{x-1}}_{M\left[\frac{x}{x-1}\frac{\mathrm{H}_0^*(x)}{x}\right](i)} - \underbrace{\int_0^1 \mathrm{d}x\, \frac{\mathrm{H}_0^*(x)}{x-1}}_{M\left[\frac{x}{x-1}\frac{\mathrm{H}_0^*(x)}{x}\right](0)} = \int_0^1 \mathrm{d}x\, x^i \frac{\mathrm{H}_0^*(x)}{x-1} + \zeta_2$$

where

$$\zeta_2 = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

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- Now that the innermost sum has as integral representation, we shift to the next and last level. First, one can show (e.g. direct integration) that:

$$\binom{2i}{i} = \frac{4^i}{\pi} M\left[\frac{1}{\sqrt{x(1-x)}}\right](i)$$

So that

$$\sum_{i=1}^{k} \binom{2i}{i} S_2(i) = \frac{1}{\pi}\sum_{i=1}^{n} 4^i M\left[\frac{1}{\sqrt{x(1-x)}}\right](i) \cdot M\left[\frac{\mathrm{H}_0^*(x)}{x-1}\right](i) + \frac{\zeta_2}{\pi}\sum_{i=1}^{k} 4^i M\left[\frac{1}{\sqrt{x(1-x)}}\right](i)$$

- We apply again the summation formula to obtain first the second part:

$$\frac{\zeta_2}{\pi}\sum_{i=1}^{k} 4^i M\left[\frac{1}{\sqrt{x(1-x)}}\right](i) = \frac{\zeta_2}{\pi}\int_0^1 \mathrm{d}x\, \frac{(4x)^n-1}{x-\frac{1}{4}}\sqrt{\frac{x}{1-x}}$$

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- Then we switch to the first part of the binomial and convolve the functions:

$$M\left[\frac{1}{\sqrt{x(1-x)}}\right](i)\cdot M\left[\frac{\mathrm{H}_0^*(x)}{x-1}\right](i) = M\left[\int_x^1 \mathrm{d}y\,\frac{\mathrm{H}_0^*(y)}{(y-1)\sqrt{y-x}}\right](i)$$

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- A set of several "rule-theorems" have been proven in [*Iterated Binomial Sums and their Associated Iterated Integrals* by J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider] to simplify further such expressions. One of them allows us to get:

$$\int_x^1 \mathrm{d}y\,\frac{\mathrm{H}_0^*(y)}{(y-1)\sqrt{y-x}} = \frac{\mathrm{H}_{\mathrm{b},\mathrm{w}_1}^*(x)}{\sqrt{x-1}},\; f_b(x) = \frac{1}{\sqrt{x(x-1)}},\; f_{w_1}(x) = \frac{1}{\sqrt{x(1-x)}}$$

- Using the shuffle algebra, we can reduce the expression down to:

$$M\left[\frac{\mathrm{H}^*_{\mathrm{b},\mathrm{w}_1}(x)}{\sqrt{x(x-1)}}\right](i) = -M\left[\frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2\sqrt{x(1-x)}}\right](i)$$

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- Finally, using once again the summation formula we get:

$$\sum_{i=1}^{n} 4^i M\left[\frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2\sqrt{x(1-x)}}\right](i) = \int_0^1 \mathrm{d}x\, \frac{(4x)^n - 1}{x - \frac{1}{4}} \sqrt{\frac{x}{1-x}} \frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2}$$

and resumming everything, we get:

$$\sum_{i=1}^{n} \binom{2i}{i} S_2(i) = -\frac{1}{\pi} \int_0^1 \mathrm{d}x\, \frac{(4x)^n - 1}{x - \frac{1}{4}} \sqrt{\frac{x}{1-x}} \left(\frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2} - \zeta_2\right)$$

We now want to obtain an asymptotic expansion for $n \to +\infty$ up to order p of a general expression of the form:

$$\tilde{M}_a[f(x)](n) := \int_0^1 \mathrm{d}x \, \frac{(ax)^n - 1}{x - \frac{1}{a}} f(x) = \int_0^1 \mathrm{d}x \, [(ax)^n - 1] \, \tilde{f}(x) \qquad (2)$$

where

$$\tilde{f}(x) := \frac{f(x)}{x - \frac{1}{a}}$$

- There exist several method to compute this expansion, depending mostly on the regularity of f and whether the integral can be split [*Iterated Binomial Sums and their Associated Iterated Integrals* by J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider]
- We will present one of the methods, all of them are implemented in RICA

We suppose that f is regular on $[0;1]$ and $a < -1$, so that $\frac{1}{x-\frac{1}{a}}$ is regular on $[0;1]$ and we can simply split the integral in two, then use a **change of variables**

- Split the Mellin integral, factor out the a^n:

$$\int_0^1 \mathrm{d}x \, \frac{ax^n - 1}{x - \frac{1}{a}} f(x) = a^n M\left[\frac{f(x)}{x - \frac{1}{a}}\right](n) - \underbrace{M\left[\frac{f(x)}{x - \frac{1}{a}}\right](0)}_{=:C}$$

Note: When $|a| > 1$, the constant C is exponentially suppressed

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- In $M\left[\frac{f(x)}{x-\frac{1}{a}}\right](n)$, we make the following change of variables:

$$x = e^{-z}, \quad \mathrm{d}x = -e^{-z}\mathrm{d}z$$

and end up with:

$$M\left[\frac{f(x)}{x-\frac{1}{a}}\right](n) = a^n \int_0^{+\infty} \mathrm{d}z \, e^{-zn} \underbrace{\frac{e^{-z}}{e^{-z}-1} f(e^{-z})}_{=:g(z)}$$

- We expand $g(z)$ around $z = 0$ up to the order p:

$$g(z) \underset{z\to 0}{=} \sum_{\alpha \le p} g_\alpha z^\alpha + \mathcal{O}(z^{\alpha+1}), \quad \alpha \in \frac{1}{2}\mathbb{Z}_{\ge -1},\ g_\alpha \in \mathbb{R}$$

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- Finally we integrate $M[\frac{f(x)}{x-\frac{1}{a}}](n)$ using the expansion above, and adding the a^n coefficient back:

$$\tilde{M}_a[f(x)](n) \underset{n\to+\infty}{=} a^n \sum_{\alpha\leq p} \int_0^{+\infty} \mathrm{d}z \, e^{-zn} g_\alpha z^\alpha$$

$$= a^n \sum_{\alpha\leq p} \frac{h_\alpha}{n^\alpha}, \quad h_\alpha \in \mathbb{R}$$

First we preload Sigma and HarmonicSums:

In[1]:= << **Sigma.m;**

Sigma - A summation package by Carsten Schneider – © RISC

In[2]:= << **HarmonicSums.m;**

HarmonicSums by Jakob Ablinger – © RISC

And then our package:

In[3]:= << **RICA.m;**

Rule Induced Convolutions for Asymptotics (RICA) package by Nikolai Fadeev © RISC-JKU

First we preload Sigma and HarmonicSums:

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In[7]:= $<<$ **HarmonicSums.m;**

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And then our package:

In[8]:= $<<$ **RICA.m;**

> Rule Induced Convolutions for Asymptotics (RICA) package by Nikolai Fadeev © RISC-JKU

We define the sum that we want to study using HarmonicSums' GS function:

In[9]:= $\mathbf{sum1} = \mathbf{GS}\left[\left\{\text{Binomial}[\mathbf{2}\ \text{VarGL}, \text{VarGL}], \frac{\mathbf{1}}{\text{VarGL}^{\mathbf{2}}}\right\}, \mathbf{n}\right];$

First we preload Sigma and HarmonicSums:

In[11]:= $<<$ **Sigma.m;**

Sigma - A summation package by Carsten Schneider – © RISC

In[12]:= $<<$ **HarmonicSums.m;**

HarmonicSums by Jakob Ablinger – © RISC

And then our package:

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Rule Induced Convolutions for Asymptotics (RICA) package by Nikolai Fadeev © RISC-JKU

We define the sum that we want to study using HarmonicSums' GS function:

In[14]:= $\mathbf{sum1} = \mathbf{GS}\left[\left\{\text{Binomial}[\mathbf{2}\ \text{VarGL}, \text{VarGL}], \frac{\mathbf{1}}{\text{VarGL}^{\mathbf{2}}}\right\}, \mathbf{n}\right];$

We can now compute the Mellin representation:

In[15]:= $\mathbf{mel1} = \mathbf{SumToMellin}[\mathbf{sum1}, \mathbf{C}, \mathbf{x}]$

Out[15]= $$\left\{ -\frac{2\text{Mellin}\left(4^{n}x^{n}-1, \frac{\sqrt{x}\left(\text{Hwb}[\{\text{fw1}\},x]^{2}-2z2\right)}{\sqrt{1-x}(4x-1)}\right)}{\pi}, \{\} \right\}$$

We can now compute the asymptotics, either from the Mellin representation...

In[16]:= **asymp1 = AsymptoticsMellint[mel1[[1]], x, n, 4]**

Out[16]=

$$-\frac{2}{\pi}\left(-\frac{(\pi-12)(12+\pi)\sqrt{\pi}2^{2n-3}}{27n^{3/2}}-\frac{\left(288+59\pi^2\right)\sqrt{\pi}2^{2n-7}}{27n^{5/2}}-\frac{97\left(25\pi^2-432\right)\sqrt{\pi}2^{2n-10}}{81n^{7/2}}-\frac{17\left(33929\pi^2-440640\right)\sqrt{\pi}2^{2n-15}}{243n^{9/2}}-\frac{\pi^{5/2}2^{2n}}{9\sqrt{n}}\right)$$

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In[17]:= $\mathbf{asymp1 = AsymptoticsMellint[mel1[[1]], x, n, 4]}$

Out[17]=
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...or directly from the sum representation:

In[18]:= $\mathbf{asymp1P = AsymptoticsSum[sum1, n, x, 3]}$

Out[18]=
$$-\frac{2}{\pi}\left(-\frac{(\pi-12)(12+\pi)\sqrt{\pi}2^{2n-3}}{27n^{3/2}}-\frac{\left(288+59\pi^2\right)\sqrt{\pi}2^{2n-7}}{27n^{5/2}}-\frac{97\left(25\pi^2-432\right)\sqrt{\pi}2^{2n-10}}{81n^{7/2}}-\frac{17\left(33929\pi^2-440640\right)\sqrt{\pi}2^{2n-15}}{243n^{9/2}}-\frac{\pi^{5/2}2^{2n}}{9\sqrt{n}}\right)$$

We can now compute the asymptotics, either from the Mellin representation...

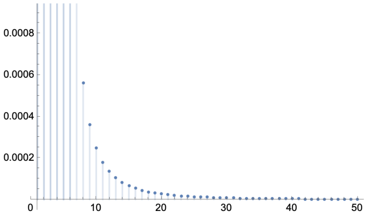
In[19]:= $\mathbf{asymp1 = AsymptoticsMellint[mel1[[1]], x, n, 4]}$

Out[19]=

$$-\frac{2}{\pi}\left(-\frac{(\pi-12)(12+\pi)\sqrt{\pi}2^{2n-3}}{27n^{3/2}}-\frac{\left(288+59\pi^2\right)\sqrt{\pi}2^{2n-7}}{27n^{5/2}}-\frac{97\left(25\pi^2-432\right)\sqrt{\pi}2^{2n-10}}{81n^{7/2}}-\frac{17\left(33929\pi^2-440640\right)\sqrt{\pi}2^{2n-15}}{243n^{9/2}}-\frac{\pi^{5/2}2^{2n}}{9\sqrt{n}}\right)$$

In[20]:= $\text{DiscretePlot}\left[\frac{|\text{sum1}-\text{asymp1}|}{|\text{sum1}|},\{n,1,50\}\right]$

Out[20]=



Here's another example:

$$\text{In[21]:=}\ \text{sum2} = \mathbf{GS}\left[\left\{(-\mathbf{2})^{\text{VarGL}}\text{Binomial}[\mathbf{2}\text{VarGL},\text{VarGL}], \frac{\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\text{VarGL}}}{\text{VarGL}}, \frac{\mathbf{1}}{\text{VarGL}^{\mathbf{2}}}\right\}, \mathbf{n}\right];$$

Here's another example:

In[22]:= $\text{sum2} = \mathbf{GS}\left[\left\{(-2)^{\text{VarGL}}\text{Binomial}[2\text{VarGL}, \text{VarGL}], \frac{\left(\frac{1}{2}\right)^{\text{VarGL}}}{\text{VarGL}}, \frac{1}{\text{VarGL}^2}\right\}, \mathbf{n}\right];$

In[23]:= ToHarmonicSumsSum[sum2]

Out[23]= $$\sum_{\tau_1=1}^{n}(-2)^{\tau_1}\text{Binomial}[2\tau_1, \tau_1]\left(\sum_{\tau_2=1}^{\tau_1}\frac{2^{-\tau_2}\left(\sum_{\tau_3=1}^{\tau_2}\frac{1}{\tau_3^2}\right)}{\tau_2}\right)$$

Here's another example:

In[24]:= $\text{sum2} = \mathbf{GS}\left[\left\{(-2)^{\text{VarGL}}\text{Binomial}[2\text{VarGL}, \text{VarGL}], \frac{\left(\frac{1}{2}\right)^{\text{VarGL}}}{\text{VarGL}}, \frac{1}{\text{VarGL}^2}\right\}, \mathbf{n}\right];$

In[25]:= $\mathbf{mel2} = \mathbf{SumToMellin}[\mathbf{sum2}, \mathbf{C}, \mathbf{x}, \text{ToGLbBasis} \to \text{False}]$

Out[25]= $$\left\{\frac{2\sqrt{2}\ \texttt{Mellin}\left((-4)^{n}x^{n}-1, \frac{\sqrt{x}(\text{Hwb}[\{\text{fw6},\text{fw1},\text{fw1}\},x]-\text{z2 Hwb}[\{\text{fw6}\},x])}{\sqrt{1-\frac{x}{2}(4x+1)}}\right)}{\pi} + \frac{5\ \text{z3}\ \texttt{Mellin}\left((-8)^{n}x^{n}-1, \frac{\sqrt{x}}{\sqrt{1-x}\left(x+\frac{1}{8}\right)}\right)}{8\pi}, \{\}\right\}$$

where

$$f_{w_1}(x) = \frac{1}{\sqrt{x(1-x)}}, \quad f_{w_6}(x) = \frac{1}{\sqrt{1-x}\sqrt{2-x}}$$

In[26]:= asymp2 = **AsymptoticsSum[sum2, n, x, 5]**

Out[26]=

$$\frac{5\left(-\frac{13\sqrt{\pi}(-8)^{n}}{81n^{3/2}}-\frac{13\sqrt{\pi}(-1)^{n}2^{3n-4}}{243n^{5/2}}+\frac{2195\sqrt{\pi}(-1)^{n}2^{3n-7}}{2187n^{7/2}}+\frac{806953\sqrt{\pi}(-1)^{n}8^{n-4}}{19683n^{9/2}}+\frac{\sqrt{\pi}(-1)^{n}8^{n+1}}{9\sqrt{n}}\right)z3}{8\pi}$$

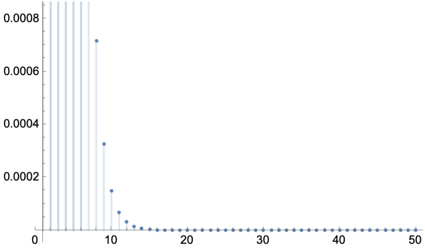
In[28]:= asymp2 = **AsymptoticsSum[sum2, n, x, 5]**

Out[28]=

$$\frac{5\left(-\frac{13\sqrt{\pi}(-8)^{n}}{81n^{3/2}}-\frac{13\sqrt{\pi}(-1)^{n}2^{3n-4}}{243n^{5/2}}+\frac{2195\sqrt{\pi}(-1)^{n}2^{3n-7}}{2187n^{7/2}}+\frac{806953\sqrt{\pi}(-1)^{n}8^{n-4}}{19683n^{9/2}}+\frac{\sqrt{\pi}(-1)^{n}8^{n+1}}{9\sqrt{n}}\right)z3}{8\pi}$$

In[29]:= DiscretePlot$\left[\frac{|\text{sum2}-\text{asymp2}|}{|\text{sum2}|},\{n,1,50\}\right]$

Out[29]=



Conclusion

- Both Mellin inversion and asymptotics computation presented [*Iterated Binomial Sums and their Associated Iterated Integrals* by J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider] implemented in the package
- We have extended the inversion method to make it work with some new classes of binomial nested sums (e.g. involving some classes of rational functions)
- Fully symbolic representation of constants
- Asymptotic expansion of sums with several possible schemes

Work in progress

- Explicit computation/simplification of constants is highly non-trivial, structure of binomial sums needs to be explored further (building a basis of binomial sums, unicity of root alphabet/relation between letters,...)
- Some classes of convolution involve difficult integrals that need to be tackled properly

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Thank you for listening!

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