THE LOCALIZATION METHOD APPLIED TO k-ELONGATED PLANE PARTITIONS AND DIVISIBILITY BY 5

KOUSTAV BANERJEE AND NICOLAS ALLEN SMOOT

Abstract. The enumeration $d_k(n)$ of k-elongated plane partition diamonds has emerged as a generalization of the classical integer partition function $p(n)$. We have discovered an infinite congruence family for $d_5(n)$ modulo powers of 5. Classical methods cannot be used to prove this family of congruences. Indeed, the proof employs the recently developed localization method, and utilizes a striking internal algebraic structure which has not yet been seen in the proof of any congruence family. We believe that this discovery poses important implications on future work in partition congruences.

1. Introduction

The theory of partition congruences has developed substantially over the last century, beginning with Ramanujan's revolutionary congruences for the integer partition function $p(n)$. Prior to the twentieth century, the sequence $(p(n))_{n\geq 0}$ was thought to be pseudorandom with respect to its divisibility properties. Ramanujan changed this perception almost overnight, with his discovery of three remarkable infinite congruence families for $p(n)$. We give these families here, with an appropriate adjustment on the powers of 7.

$$\text{If } n, \alpha \in \mathbb{Z}_{\geq 1} \text{ and } 24n \equiv 1 \pmod{\ell^{\alpha}}, \text{ then } p(n) \equiv 0 \pmod{\ell^{\beta}}, \tag{1.1}$$

with

$$\beta := \begin{cases} \alpha & \text{if } \ell \in \{5, 11\}, \\ \left\lfloor \frac{\alpha}{2} \right\rfloor + 1 & \text{if } \ell = 7. \end{cases} \tag{1.2}$$

Other congruence families have been found to exist for a variety of partition functions and related arithmetic sequences which at least superficially resemble those of (1.1). However, the difficulty of proving such congruence families can vary substantially, with some contemporary families still resisting proof today.

The congruence family discussed in this paper was discovered by the first author, and regards the enumeration $d_5(n)$ of 5-elongated plane partitions of n. The function is part of a large class of partition functions, $d_k(n)$, which generalize $p(n)$, and whose properties have been closely studied by partition theorists over the last few years.

The congruence family is interesting on its own; however, the proof method is the most important aspect of our work. The classical techniques used to prove Ramanujan's results (1.1) are insufficient to complete a proof. Instead, the recently developed localization method is utilized. The most critical part of the proof involves a surprising internal algebraic structure on the rational polynomials representing the individual cases of the family. It is this structure and possible analogues for other functions that we believe may lead to a deeper understanding of the theory of partition congruences, and possible resolution of standing conjectures.

1.1. k-Elongated Plane Partitions.

Over the years George Andrews and Peter Paule have produced an extremely influential series of papers developing an algorithmic methodology on MacMahon's Partition Analysis [13, Vol. II, Section VIII]. Among the many contributions of this series of papers has been the development of the k-elongated plane partition function $d_k(n)$, which is enumerated by the following generating function:

$$D_k(q) := \sum_{n=0}^{\infty} d_k(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^k}{(1-q^m)^{3k+1}}. \tag{1.3}$$

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This function counts the number of directed graphs with the form shown in the diagrams below, in which each $a_j \in \mathbb{Z}_{\geq 0}$, a directed edge $a_b \to a_c$ indicates that $a_b \geq a_c$, and $\sum_{j=1}^{(2k+1)m+1} a_j = n$.

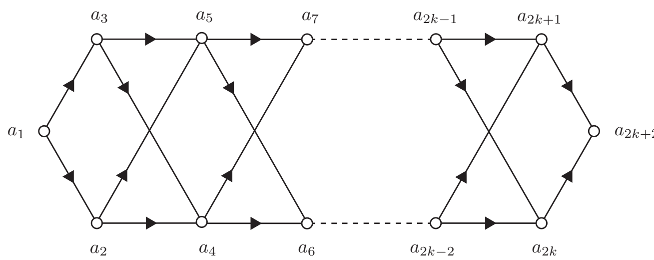

FIGURE 1. A typical length 1 k-elongated partition diamond.

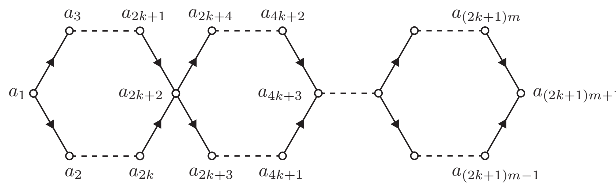

FIGURE 2. A typical length m k-elongated partition diamond.

Notice that $d_k(n)$ serves as a generalization of the unrestricted integer partition function $p(n) = d_0(n)$. As such, it may be expected that $d_k(n)$ may exhibit many arithmetic properties similar to those of $p(n)$. For example, a famous result for $p(n)$ due to Ramanujan [17] is

$$p(5n + 4) \equiv 0 \pmod 5 \tag{1.4}$$

for all $n \geq 0$. Recently da Silva, Hirschhorn, and Sellers [9] discovered that

$$d_5(5n + 4) \equiv 0 \pmod 5 \tag{1.5}$$

for all $n \geq 0$. It is certainly tempting to suspect that *infinite* families of congruence might exist for $d_k(n)$, in form similar to that of Ramanujan's family of congruences for $p(n)$ modulo powers of 5. These are generally much more difficult to prove than a congruence in a simple arithmetic progression, e.g., (1.4) or (1.5). Some have already been discovered and proved, e.g., Conjecture 3 of [2], which was proved in [19].

Our central result is another such family—a simple and elegant extension of (1.5) to an infinite family of congruences of $d_5(n)$:

Theorem 1.1. *Let* $n, \alpha \in \mathbb{Z}_{\geq 1}$ *such that* $4n \equiv 1 \pmod{5^\alpha}$. *Then* $d_5(n) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor + 1}}$.

It is this result that we will prove in this paper.

1.2. **History.** Much of our modern day understanding of the arithmetic properties of the integer partition function $p(n)$ derives from Ramanujan's groundbreaking work of (1.1) in 1919, albeit without the key modification (1.2).

This incredible result took decades to prove. Ramanujan himself appears to have understood the proof in the case that $\ell = 5$ [8]. The first published proof for $\ell = 5, 7$ appeared by Watson in 1938 [22]. The case for $\ell = 11$ was much more resistant, and a proof was not found before Atkin's work in 1967 [3].

This result, together with the techniques used to prove it, has stimulated a massive amount of work in partition theory. In particular, congruence families of a similar form have been found for a wide variety of more restrictive partition functions. Notably, there exists an extraordinary range of results in terms of the difficulty of proof. Some results are relatively unremarkable, and are proved regulary using the same methods that were used to prove (1.1) for $\ell = 5, 7$.

On the other hand, others are much more ambitious, and resistant to proof even today. For example, a congruence family for 2-colored Frobenius partitions modulo powers of 5 was proposed by Sellers in 1994 [18], but it was not proved until the work of Paule and Radu in 2012 [15].

Until recently, it was assumed that the genus of the underlying modular curve was responsible for the complications. For a given partition-like function $a(n)$, a typical infinite congruence family modulo powers of a given prime ℓ will have the form

$$a(n) \equiv 0 \pmod{\ell^\beta},$$

in which n is the inverse of some fixed positive integer modulo ℓ^α, and

$$\beta \to \infty \text{ as } \alpha \to \infty.$$

Such a family is generally associated with a sequence of modular functions $\mathcal{L} := (L_\alpha)_{\alpha\geq 1}$, in which each L_α effectively enumerates the congruence of $a(n)$ modulo ℓ^α. This sequence of modular functions is in turn associated with a given compact Riemann surface X, a classical modular curve. The topological properties of this curve have a profound impact on the difficulty in proving the associated congruence family.

In particular, it was found that for (1.1) with $\ell = 5, 7$, the associated modular curves have genus 0. On the other hand, in the case of (1.1) for $\ell = 11$, and for the Andrews–Sellers congruences proved in [15], the associated modular curves have genus 1. It was therefore assumed that our understanding of partition congruences when X has genus 0 was complete. The discovery that classical techniques are not sufficient to prove all genus 0 congruences is quite recent.

Recent work done in [19] and [20] has shown that a second important topological property of X must be taken into account—namely, the cusp count, i.e., the number of points required to properly compactify X.

When the cusp count is 2, the associated functions L_α generally have a pole at one cusp and a zero at the other. The space of functions on X with a pole only at a single point is isomorphic to a $\mathbb{C}[x]$ module with rank depending on the genus of X. As a result, we can usually express L_α as a polynomial of a finite number of basis functions which have a pole at a single cusp. Indeed, when the genus is 0, L_α is expressible in terms of a single function, i.e., a Hauptmodul. This is the classical method used to prove nearly all congruence families in the past.

However, when the cusp count is greater than 2, the functions L_α may have poles at multiple places. Thus, a polynomial expression of L_α in terms of the basis functions at a given cusp is generally unlikely. The solution to this dilemma is to express L_α as a *rational* polynomial in the basis functions, in which the denominator is governed by some multiplicative set $\mathcal{S}$.

For example, for $\alpha = 1$ we have

$$L_1 = \frac{(q^5;q^5)_\infty^{16}}{(q^{10};q^{10})_\infty^5}\sum_{n=0}^{\infty} d_5\left(5n+4\right)q^{n+2}. \tag{1.6}$$

For a more general definition of L_α, see Section 4. With the Hauptmodul

$$x = q\frac{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^3}{(q;q)_\infty^3(q^5;q^5)_\infty} \tag{1.7}$$

we have found that

$$\begin{aligned}
L_1 =&\frac{1}{(1+5x)^6}\cdot\Big(5705x^2 + 6840120x^3 + 2034152125x^4 + 280484938650x^5 + 22921365211325x^6 + 1260917405154520x^7 \\
&+ 50400843190048480x^8 + 1539115922208139200x^9 + 37183654303328448000x^{10} + 728924483359472640000x^{11} \\
&+ 11816089262411136000000x^{12} + 160681440628058880000000x^{13} + 1853291134193264640000000x^{14} \\
&+ 18284160727362809856000000x^{15} + 155286793010086625280000000x^{16} + 1140657222505472000000000000x^{17} \\
&+ 7269894420215070720000000000x^{18} + 40277647277404979200000000000x^{19} \\
&+ 194099187864646451200000000000x^{20} + 813054581193729638400000000000x^{21} \\
&+ 2954545150241538048000000000000x^{22} + 9282005730758492160000000000000x^{23} \\
&+ 25080951875200614400000000000000x^{24} + 57872525958316032000000000000000x^{25} \\
&+ 112916020309524480000000000000000x^{26} + 183812885074411520000000000000000x^{27} \\
&+ 245082228994867200000000000000000x^{28} + 260725452832768000000000000000000x^{29} \\
&+ 212837104353280000000000000000000x^{30} + 125198296678400000000000000000000x^{31} \\
&+ 47244640256000000000000000000000x^{32} + 8589934592000000000000000000000x^{33}\Big).
\end{aligned} \tag{1.8}$$

Inspection of the coefficients quickly reveals that L_1 is divisible by 5. Similar identities can be computed for any L_α, albeit with a rapidly increasing polynomial degree and mean coefficient size. We now give our main theorem, from which Theorem 1.1 immediately follows:

Theorem 1.2. *Let*

$$\psi := \psi(\alpha) := \left\lfloor \frac{5^{\alpha+1}}{4} \right\rfloor + 1 - \gcd(\alpha, 2),$$
$$\beta := \beta(\alpha) = \lfloor \alpha/2 \rfloor + 1.$$

Then for all $\alpha \geq 1$, we have

$$\frac{(1+5x)^\psi}{5^\beta} \cdot L_\alpha \in \mathbb{Z}[x]. \tag{1.9}$$

This theorem is proved by induction. What has surprised us is that in order to properly complete the induction, we had to take advantage of a striking internal structure attached to the coefficients of x^n in the numerator of the expressions for L_α. This structure is defined by the fact that each $L_\alpha/5^{\lfloor \alpha/2 \rfloor + 1}$ is effectively a member of the kernel of a homomorphism Ω onto a finite-dimensional $\mathbb{F}_5$-vector space (see Section 5). In retrospect, this was true in the previous applications of localization in [19] and [20]. Indeed, we recognize this property in all classical cases of partition congruences, including those of $p(n)$. However, in the cases of [19] and [20], the homomorphism was very simple, and it was only interpreted at the time as a minor idiosyncrasy on the coefficients of the corresponding L_α. In the classical cases, e.g., those of $p(n)$, the homomorphism is completely trivial.

We now hypothesize that this kernel structure plays a critical role in the theory of congruence families. Certainly, it allows a proof to be completed in a relatively straightforward manner, and we are extremely interested to know if this algebraic structure persists—or fails—in other examples.

1.3. **Additional Congruence Results.** Theorem 1.1 constitutes our central result, and we believe that our methods used to prove it will prove extremely useful in future work. However, in searching for Theorem 1.1 and any other congruence families of similar form, we have also discovered many other interesting congruence results by various powers of 5.

As such, we include an examination of divisibility properties of $d_k(n)$ by powers of 5 for various values of k and n. These results subsume a significant amount of work done in [9], [5]. We have since proved these results using Ramanujan's theta functions and their dissections, and we provide them here for some additional context on the behavior of $d_k(n)$ with respect to progressions with powers of 5.

Theorem 1.3. *For $k \geq 0$,*

$$d_{25k+1}(25n+23) \equiv 0 \pmod 5, \tag{1.10}$$

and

$$d_{75k+16}(25n+8) \equiv 0 \pmod 5. \tag{1.11}$$

For $k=0$ in (1.10), we have the following result.

Corollary 1.4. [5, Eqn. (6.1), Thm. 6.1]

$$d_1(25n+23) \equiv 0 \pmod 5.$$

The following theorem provides an elegant intricate connection between the two parameters; the index of the function k, and the initial of the progression ℓ considered in the context:

Theorem 1.5. *If $k \in \{1,3,5,8,10,13,15,16,18,20,23\}$, then for any non-negative integer n:*

$$d_k(25n+\ell) \equiv 0 \pmod 5; \tag{1.12}$$

and for $k \in \{5,8,10\}$,

$$d_k(25n+\ell) \equiv 0 \pmod{25}, \tag{1.13}$$

where $k+\ell=24$.

Finally, we give an interesting infinite family, in which k varies by a linear progression (with 5^3 as the base), and in which n varies modulo arbitrarily high odd powers of 5, but in which the congruence itself is fixed modulo 5.

Theorem 1.6. *For $n,k \geq 0$, $\alpha \geq 1$ and $j \in \{1,2\}$,*

$$d_{125k+2}\left(5^{2\alpha+1}n + 5^{2\alpha}j + \frac{23 \times 5^{2\alpha}+1}{8}\right) \equiv 0 \pmod 5. \tag{1.14}$$

Precisely,

$$d_{125k+2}\left(5^{2\alpha+1}n - \frac{1}{8}(5^{2\alpha}-1) + 4 \times 5^{2\alpha}\right) \equiv d_{125k+2}\left(5^{2\alpha+1}n - \frac{1}{8}(5^{2\alpha}-1) + 5 \times 5^{2\alpha}\right) \equiv 0 \pmod 5. \tag{1.15}$$

Plugging $k=0$ and $\alpha=1$ into (1.15), we get:

Corollary 1.7. [5, Eqn. (6.3), Thm. 6.1] *For $k=0$ and $\alpha=1$,*

$$d_2(125n+97) \equiv 0 \pmod 5,$$
$$d_2(125n+122) \equiv 0 \pmod 5.$$

1.4. **Outline.** The remainder of this paper is organized as follows: In Section 2 we review the properties of Ramanujan's theta functions, together with the theory of modular functions. In Section 3 we prove Theorems 1.3-1.6.

Thereafter, we commit to proving Theorem 1.1. In Section 4 we more precisely define the functions L_α, together with the operators $U^{(\alpha)}$ which map each L_α to $L_{\alpha+1}$. We also give some of the key properties of the Hauptmodul x, including the modular equations (4.25), (4.26) which will allow us to build useful recurrence relations for $U^{(\alpha)}\left(x^m/(1+5x)^n\right)$. In Section 5 we construct the function spaces $\mathcal{V}_n^{(\alpha)}$ in which our functions L_α live. In particular, we define the critical homomorphism Ω whose kernel is closely related to our function spaces. We then give Theorem 5.3, which demonstrate how the operator $U^{(\alpha)}$ affects elements of $\mathcal{V}_n^{(\alpha)}$. We also give some important congruence properties of an associated auxiliary function. In Section 6 we build the induction necessary to finish the proof of Theorem 1.2, and with it Theorem 1.1. In Section 7 we show how to finish the proof of Theorem 5.3, together with the proof of (4.25), (4.26), using the tools of the modular cusp analysis. In the Appendix we give the ten initial relations of $U^{(\alpha)}(x^m)$ needed to complete the proof of Theorem 5.3.

Throughout Sections 4-7 we devote many of the more tedious computations to a self-contained Mathematica supplement which can be found online at `https://www.risc.jku.at/people/nsmoot/d5congsuppA.nb`.

2. Basic Theory

We will recall a few preliminary definitions and lemmas in context of Ramanujan's theta fucntions necessary to prove Theorems 1.3-1.6. Thereafter, we will give some background in the theory of modular functions required to prove Theorem 1.2.

2.1. **Ramanujan's theta functions and its allied results.** For complex numbers a and q, we define

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ n \geq 1;$$
$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad |q|<1.$$

Ramanujan's two-variable general theta function is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{2.1}$$

Three special cases of (2.1) are defined by, in Ramanujan's notation

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$
$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$
$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Following Ramanujan's definition (2.1), Jacobi's famous triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = (-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty, \quad |q|<1 \text{ and } z\neq 0$$

takes the form

$$f(a,b) = (-a,ab)_\infty(-b,ab)_\infty(ab,ab)_\infty. \tag{2.2}$$

Recall the Jacobi's identity [6, Thm. 1.3.9, p. 14] that reads

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}. \tag{2.3}$$

From [6, Entry 31,eq. (31.1)], can express $f(a,b)$ as the n-linear combination of theta functions in the following form

$$f(a,b) = \sum_{r=0}^{n-1} a^{r(r+1)/2}b^{r(r-1)/2}f(a^{n(n+1)/2+nr}b^{n(n-1)/2+nr}, a^{n(n-1)/2-nr}b^{n(n+1)/2-nr}). \tag{2.4}$$

Lemma 2.1. [6, Entry 10(i), 10(v), and 10(iv), p. 262]

$$\psi(q^{1/5}) = q^{3/5}\psi(q^5) + f(q^2,q^3) + q^{1/5}f(q,q^4) \tag{2.5}$$
$$\psi^2(q) = \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty} + q\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^2} \tag{2.6}$$
$$\phi^2(q^5) = \phi^2(q) - 4qf(q,q^9)f(q^3,q^7). \tag{2.7}$$

Now we recall the Roger-Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1}\,_{+}\,\frac{q}{1}\,_{+}\,\frac{q^2}{1}\,_{+}\,\frac{q^3}{1}\,_{+\ldots} = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}, \ |q|<1. \tag{2.8}$$

Lemma 2.2. [7, p. 161 and p. 164] *If* $T(q) := \frac{q^{1/5}}{R(q)} = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}$,

$$T(q^5) - q - \frac{q^2}{T(q^5)} = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty}. \tag{2.9}$$

Lemma 2.3. [7, p. 165, Eqn. (7.4.14)]

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\Bigg(T^4(q^5) + qT^3(q^5) + 2q^2T^2(q^5) + 3q^3T(q^5) + 5q^4 - \frac{3q^5}{T(q^5)} + \frac{2q^6}{T^2(q^5)} - \frac{q^7}{T^3(q^5)} + \frac{q^8}{T^4(q^5)}\Bigg). \tag{2.10}$$

Lemma 2.4. [23, Lemma 2.1]

$$\sum_{n=0}^{\infty} a(5n+2)q^n \equiv -2(q;q)_\infty^3(q^2;q^2)_\infty^3 \ (mod\ 5), \tag{2.11}$$

where $a(n)$ is the cubic partition function and its generating function is

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}.$$

2.2. **Modular Functions and Riemann Surface Structure.** As with the unrestricted partition function $p(n)$, the generating function of $d_k(n)$ gives our first indication that modular functions may be useful in determining its properties.

Recall Dedekind's eta function [11, Chapter 3]:

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right),$$

with domain $\tau \in \mathbb{H}$. This function is a modular form of half-integral weight with an associated multiplier system [11, Chapter 3]. In particular, this means that

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \left(-i(c\tau+d)\right)^{1/2}\epsilon(a,b,c,d)\eta(\tau), \tag{2.12}$$

in which $a,b,c,d \in \mathbb{Z}$ such that $ad-bc=1$ and $c>0$, and $\epsilon(a,b,c,d)$ is a specific root of unity (the factor $-i$ is introduced to insure the principal branch of the half-integer power).

It can be quickly demonstrated that for any j with $0 \le j \le 4$,

$$\sum_{r=0}^{4}\exp\left(-2\pi ij\cdot\frac{\tau+r}{5}\right)D_k\left(\exp\left(\frac{\tau+r}{5}\right)\right) = 5\sum_{n=0}^{\infty} d_k(5n+j)e^{2\pi in\tau}.$$

Since $D_k(q)$ is in effect a quotient of the eta function, the function $\sum_{n=0}^{\infty} d_k(5n+j)q^n$ should hold similar properties to (2.12). Indeed, we would like to adjust this function so that it follows a cleaner functional equation, i.e., exact symmetry for $\tau \to (a\tau+b)(c\tau+d)^{-1}$, for a large range of matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$.

The subset of matrices we need are the congruence subgroups

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : N|c\right\},$$

for $N \in \mathbb{Z}_{\ge 1}$. Indeed, $\Gamma_0(N)$ acts on the extended complex plane

$$\hat{\mathbb{H}} := \mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$$

by

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\tau\right) \longrightarrow \frac{a\tau+b}{c\tau+d}.$$

We express this group action by

$$\gamma\tau := \frac{a\tau + b}{c\tau + d},$$

for any $\tau \in \hat{\mathbb{H}}$ and any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define the orbits of this group action as the sets

$$[\tau]_N := \{\gamma\tau : \gamma \in \Gamma_0(N)\}.$$

Definition 2.5. For any $N \in \mathbb{Z}_{\geq 1}$, the classical modular curve of level N is the set of all orbits of $\Gamma_0(N)$ applied to $\hat{\mathbb{H}}$:

$$\mathrm{X}_0(N) := \left\{[\tau]_N : \tau \in \hat{\mathbb{H}}\right\}.$$

Most importantly, one can construct a complex structure for the modular curves $\mathrm{X}_0(N)$, and in so doing show that the curves are *Riemann surfaces*. This allows us to study meromorphic functions on $\mathrm{X}_0(N)$ using most of the tools of classical complex analysis.

A description of the necessary complex structure is given in [10, Chapters 2,3]. The topology of $\mathrm{X}_0(N)$ includes two important nonnegative integers: the genus, denoted $\mathfrak{g}(\mathrm{X}_0(N))$, and the number of cusps, denoted $\epsilon_\infty(\mathrm{X}_0(N))$. These can be computed according to formulae in [10, Chapter 3, Sections 3.1, 3.8].

We note that the group action of $\Gamma_0(N)$ can be restricted to $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. Indeed, for $\tau \in \hat{\mathbb{Q}}$, we must have $[\tau]_N \subseteq \hat{\mathbb{Q}}$. Clearly such orbits partition $\hat{\mathbb{Q}}$ (in the set-theoretic sense). It can be proved that only a finite number of such orbits exist for any $\mathrm{X}_0(N)$ [10, Section 3.8].

Definition 2.6. For any $N \in \mathbb{Z}_{\geq 1}$, the cusps of $\mathrm{X}_0(N)$ are the orbits of $\Gamma_0(N)$ applied to $\hat{\mathbb{Q}}$.

To determine whether two elements of $\hat{\mathbb{Q}}$ are in the same cusp, we have the following [10, Proposition 3.8.3]:

Lemma 2.7. *Two rational elements $a_1/c_1, a_2/c_2$ are in the same cusp of $\mathrm{X}_0(N)$ if and only if there exist some $j, y \in \mathbb{Z}$ with $\gcd(y, N) = 1$ and*

$$y \cdot a_2 \equiv a_1 + j \cdot c_1 \pmod{N}, \tag{2.13}$$

$$c_2 \equiv y \cdot c_1 \pmod{N}. \tag{2.14}$$

The appeal of $\mathrm{X}_0(N)$ is that we can construct meromorphic functions which are constrained by the surface's manifold structure. With this in mind, we give the following definition of a modular function:

Definition 2.8. Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be holomorphic on $\mathbb{H}$. Then f is a modular function over $\Gamma_0(N)$ if the following properties are satisfied for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$:

1. If $\gamma \in \Gamma_0(N)$, we have $f(\gamma\tau) = f(\tau)$.
2. We have

$$f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} \alpha_\gamma(n) q^{n\gcd(c^2,N)/N},$$

with $n_\gamma \in \mathbb{Z}$, and $\alpha_\gamma(n_\gamma) \neq 0$. If $n_\gamma \geq 0$, then f is holomorphic at the cusp $[a/c]_N$. Otherwise, f has a pole of order n_γ, and principal part

$$\sum_{n=n_\gamma}^{-1} \alpha_\gamma(n) q^{n\gcd(c^2,N)/N}, \tag{2.15}$$

at the cusp $[a/c]_N$.

We refer to $\mathrm{ord}_{a/c}^{(N)}(f) := n_\gamma(f)$ as the order of f at the cusp $[a/c]_N$.

Definition 2.9. Let $\mathcal{M}(\Gamma_0(N))$ be the set of all modular functions over $\Gamma_0(N)$, and $\mathcal{M}^{a/c}(\Gamma_0(N)) \subset \mathcal{M}(\Gamma_0(N))$ to be those modular functions over $\Gamma_0(N)$ with a pole only at the cusp $[a/c]_N$. These are commutative algebras with 1, and standard addition and multiplication [16, Section 2.1].

A modular function $f \in \mathcal{M}\left(\Gamma_0(N)\right)$ induces a meromorphic function

$$\begin{aligned}\hat{f} &: \mathrm{X}_0(N) \longrightarrow \mathbb{C} \cup \{\infty\}\\ &: [\tau]_N \longrightarrow f(\tau).\end{aligned}$$

In fact, there is a one-to-one correspondence between the set of meromorphic functions on $\mathrm{X}_0(N)$ with poles only at the cusps and $\mathcal{M}\left(\Gamma_0(N)\right)$, with a matching correspondence between the function orders [12, Chapter VI, Theorem 4A]. Moreover, all possible poles for $\hat{f}$ exist as a subset of $[\tau]_N \subseteq \hat{\mathbb{Q}}$, and (2.15) represents the principal part of $\hat{f}$ in local coordinates near the cusp $[a/c]_N$.

Theorem 2.10. *Let* X *be a compact Riemann surface, and let* $\hat{f} : \mathrm{X} \longrightarrow \mathbb{C}$ *be analytic on the entirety of* X*. Then* $\hat{f}$ *is be a constant function.*

This is the most important underlying theorem in the subject, as the following corollary demonstrates:

Corollary 2.11. *For any* $N \in \mathbb{Z}_{\geq 1}$*, if* $f \in \mathcal{M}\left(\Gamma_0(N)\right)$ *has nonnegative order at every cusp of* $\Gamma_0(N)$*, then* f *is a constant.*

This is a useful tool for demonstrating equivalence between modular functions. If we consider $f, g \in \mathcal{M}^{\infty}\left(\Gamma_0(N)\right)$, in which f and g contain matching principal parts at $[\infty]_N$, then $f - g$, and therefore $\hat{f} - \hat{g}$, has no poles at all, making $\hat{f} - \hat{g}$ (and therefore $f - g$), a constant.

2.3. **Construction of Modular Functions.** We have defined the Riemann surfaces that we will work over, and whose topologies play a significant role in our work. We now wish to construct explicit modular functions, using η as a guide. The following theorem is due to Newman [14, Theorem 1]:

Theorem 2.12. *Let* $f = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$*, with* $\hat{r} = (r_\delta)_{\delta|N}$ *an integer-valued vector, for some* $N \in \mathbb{Z}_{\geq 1}$*. Then* $f \in \mathcal{M}\left(\Gamma_0(N)\right)$ *if and only if the following apply:*

(1) $\sum_{\delta|N} r_\delta = 0$;
(2) $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$;
(3) $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$;
(4) $\prod_{\delta|N} \delta^{|r_\delta|}$ *is a perfect square.*

An additional advantage in the use of η is the comparable ease in computing the order of the associated modular functions at each cusp. We can compute the orders using the following theorem [16, Theorem 23], generally attributed to Ligozat:

Theorem 2.13. *If* $f = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta} \in \mathcal{M}\left(\Gamma_0(N)\right)$*, then the order of* f *at the cusp* $[a/c]_N$ *is given by the following:*

$$\mathrm{ord}_{a/c}^{(N)}(f) = \frac{N}{24\gcd\left(c^2, N\right)} \sum_{\delta|N} r_\delta \frac{\gcd\left(c,\delta\right)^2}{\delta}.$$

Finally, we give a theorem from [16, Theorem 39] which we can use to compute the orders of modular functions with the coefficients taken in a given arithmetic progression.

Theorem 2.14. *Suppose that for some* $M \in \mathbb{Z}_{\geq 1}$ *and an index-valued vector* $(r_\delta)_{\delta|M}$ *we define the function* $a(n)$ *by*

$$\sum_{n=1}^{\infty} a(n)q^n := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta}$$

Moreover, suppose that $m, t, N \in \mathbb{Z}_{\geq 0}$ *such that* $0 \leq t < m$*, and that*

$$f = \prod_{\lambda|N} \eta(\lambda\tau)^{s_\delta} \sum_{n=1}^{\infty} a(mn+t)q^n \in \mathcal{M}\left(\Gamma_0(N)\right).$$

Then

$$\mathrm{ord}_{a/c}^{(N)}(f) \geq \frac{N}{\gcd(c^2,N)}\left(\min_{0\leq l\leq m-1}\frac{1}{24}\sum_{\delta|M} r_\delta \frac{\gcd(\delta(a+l\cdot c\cdot \gcd(m^2-1,24)),mc)^2}{\delta m}+\frac{1}{24}\sum_{\lambda|N}\frac{s_\lambda \gcd(\lambda,c)^2}{\lambda}\right).$$

2.4. **The Hecke Operator.** The central functions in our work are L_α, $\alpha \geq 1$. To relate L_α to $L_{\alpha+1}$, we need the Hecke U_ℓ operator, for a prime ℓ:

Definition 2.15. Let $f(q) = \sum_{m\geq M} a(m)q^m$. Then

$$U_\ell\left(f(q)\right) := \sum_{\ell m\geq M} a(\ell m)q^m. \tag{2.16}$$

We give some properties of U_ℓ. The proofs can be found in [1, Chapter 10] and [11, Chapter 8].

Lemma 2.16. *Given two functions*

$$f(q) = \sum_{m\geq M} a(m)q^m,\ \ g(q) = \sum_{m\geq N} b(m)q^m,$$

any $\alpha \in \mathbb{C}$, a primitive ℓ-th root of unity ζ, and the convention that $q^{1/\ell} := e^{2\pi i\tau/\ell}$, we have the following:

(1) $U_\ell\left(\alpha\cdot f+g\right) = \alpha\cdot U_\ell\left(f\right)+U_\ell\left(g\right)$;
(2) $U_\ell\left(f(q^\ell)g(q)\right) = f(q)U_\ell\left(g(q)\right)$;
(3) $\ell\cdot U_\ell\left(f\right) = \sum_{r=0}^{4} f\left(\zeta^r q^{1/\ell}\right)$.

We also give an important result from [4, Lemma 7], which we will use in Section 7:

Theorem 2.17. *For any $N\in\mathbb{Z}_{\geq 1}$ with $\ell|N$ and $f\in\mathcal{M}\left(\Gamma_0(N)\right)$,*

$$U_\ell\left(f\right)\in\mathcal{M}\left(\Gamma_0(N)\right).$$

Moreover, if $\ell^2|N$, then

$$U_\ell\left(f\right)\in\mathcal{M}\left(\Gamma_0(N/\ell)\right).$$

3. Proofs of Theorems 1.3-1.6

Throughout this section, for a formal power series, say $S(q) := \sum_{n=0}^{\infty} s(n)q^n$, we define

$$[q^{An+B}]S(q) := s(An+B).$$

Proof of Theorem 1.3. Recall that

$$\begin{aligned}\sum_{n=0}^{\infty} d_1(n)q^n &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^4}\\ &\equiv \frac{1}{(q^5;q^5)_\infty} f(-q)f(-q^2) \pmod 5.\end{aligned} \tag{3.1}$$

From (2.9) with $q\mapsto q^2$, we obtain

$$f(-q^2) = f(-q^{50})\left(T(q^{10})-q^2-\frac{q^4}{T(q^{10})}\right). \tag{3.2}$$

Putting (2.9) and (3.2) together, it follows that

$$[q^{5n+3}]f(-q)f(-q^2) = q^3 f(-q^{25})f(-q^{50}). \tag{3.3}$$

Substituting (3.3) into (3.1), we get

$$\sum_{n=0}^{\infty} d_1(5n+3)q^{5n+3} \equiv \frac{1}{(q^5;q^5)_\infty} q^3 f(-q^{25})f(-q^{50}) \pmod 5,$$

which implies that

$$\sum_{n=0}^{\infty} d_1(5n+3)q^{n} \equiv \frac{1}{(q;q)_\infty} f(-q^5)f(-q^{10}) \pmod 5. \tag{3.4}$$

Now, from (2.10), we see that

$$[q^{5n+4}]\frac{1}{(q;q)_\infty} = 5q^4\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}. \tag{3.5}$$

Therefore (3.4) finally reduces to

$$d_1(25n+23) \equiv 0 (\text{mod } 5). \tag{3.6}$$

Finally, (1.10) follows from (3.6) and the following observation: for $k \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} d_{25k+1}(n)q^n &= \frac{(q^2;q^2)_\infty^{25k+1}}{(q;q)_\infty^{75k+4}} \\ &\equiv \Big(\sum_{n=0}^{\infty} d_1(n)q^n\Big)\frac{(q^{10};q^{10})_\infty^{5k}}{(q^5;q^5)_\infty^{15k}} \pmod 5.\end{aligned}$$

This concludes the proof of (1.10).

Now, for all $k \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} d_{75k+16}(n)q^n &= \frac{(q^2;q^2)_\infty^{75k+16}}{(q;q)_\infty^{225k+49}} \\ &\equiv \frac{(q^{10};q^{10})_\infty^{15k+3}}{(q^5;q^5)_\infty^{45k+10}} f(-q)f(-q^2) \pmod 5. \end{aligned} \tag{3.7}$$

Using (3.3), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} d_{75k+16}(5n+3)q^n &\equiv \frac{(q^2;q^2)_\infty^{15k+3}}{(q;q)_\infty^{45k+10}} f(-q^5)f(-q^{10}) \pmod 5 \\ &\equiv \frac{(q^{10};q^{10})_\infty^{3k}}{(q^5;q^5)_\infty^{9k+1}} f(-q^5)f(-q^{10})(q^2;q^2)_\infty^3 \pmod 5. \end{aligned} \tag{3.8}$$

Applying the substitution $q \mapsto q^2$ into (2.3), it follows that the coefficients of q^{5n+1} in $(q^2;q^2)_\infty^3$ is divisible by 5 and therefore from (3.8), we conclude that

$$d_{75k+16}(25n+8) \equiv 0 \pmod 5.$$

□

Proof of Theorem 1.5. For $k \in \{3,5,8,10,13,15,18,20,23\}$, (1.12) is an immediate consequence of Corollary 14 and 15 of [2] and Corollary 4.5 of [9]. Equation (1.12) for $k \in \{1,16\}$ is a special instance of (1.10) and (1.11). For the proof of (1.13), one can use the Mathematica package RaduRK developed by the second author [21]. □

Proof of Theorem 1.6. From the definition of $d_k(n)$ for $k = 2$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} d_2(n)q^n &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2} \\ &\equiv \frac{1}{(q^5;q^5)_\infty}\frac{\psi^2(q)}{(q^2;q^2)_\infty^2} \pmod 5 \\ &\equiv \frac{1}{(q^5;q^5)_\infty}\frac{1}{(q^2;q^2)_\infty^2}\left[\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty} + q\frac{(q^{10};q^{10})_\infty}{(q^5;q^5)_\infty^2}\right] \pmod 5 \text{ (by (2.6))} \\ &\equiv \frac{(q^5;q^5)_\infty^2}{(q^{10};q^{10})_\infty}\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} + q\frac{(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty^3}(q^2;q^2)_\infty^3 \pmod 5. \end{aligned} \tag{3.9}$$

Applying (2.3) with the substitution $q \mapsto q^2$, we get

$$(q^2;q^2)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)},$$

and consequently, observe that the coefficients of q^{5n+1} in $(q^2;q^2)_\infty^3$ is divisible by 5; i.e.,

$$[q^{5n+1}](q^2;q^2)_\infty^3 \equiv 0 \pmod 5. \tag{3.10}$$

Using (3.10) into (3.9), it follows that

$$\sum_{n=0}^{\infty} d_2(5n+2)q^{5n+2} \equiv \frac{(q^5;q^5)_\infty^2}{(q^{10};q^{10})_\infty}\sum_{n=0}^{\infty} a(5n+2)q^{5n+2} \pmod 5,$$

and therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} d_2(5n+2)q^n &\equiv \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty} a(5n+2)q^n \pmod 5\\ &\equiv -2(q;q)_\infty^5(q^2;q^2)_\infty^3 \pmod 5 \text{ (by (2.11))}\\ &\equiv -2(q^5;q^5)_\infty\psi(q)f(-q) \pmod 5. \end{aligned} \tag{3.11}$$

Now, from (2.9), it follows that

$$[q^{5n+3}]f(-q) = 0 \text{ and } [q^{5n+1}]f(-q) = -q(q^{25};q^{25})_\infty, \tag{3.12}$$

whereas, applying the substitution $q \mapsto q^5$ into (2.5), we get

$$[q^{5n+3}]\psi(q) = q^3\psi(q^{25}). \tag{3.13}$$

Plugging (3.12) and (3.13) into (3.11), we have

$$\sum_{n=0}^{\infty} d_2(25n+22)q^{5n+4} \equiv 2q^4(q^5;q^5)_\infty\psi(q^{25})f(-q^{25}) \pmod 5,$$

and therefore,

$$\sum_{n=0}^{\infty} d_2(25n+22)q^n \equiv 2f(-q)\psi(q^5)f(-q^5) \pmod 5. \tag{3.14}$$

By induction, one can conclude that for all $\alpha \in \mathbb{Z}_{\geq 1}$,

$$\sum_{n=0}^{\infty} d_2\left(5^{2\alpha}n - \frac{1}{8}(5^{2\alpha}-1) + 5^{2\alpha}\right)q^n \equiv 2f(-q)\psi(q^5)f(-q^5) \pmod 5. \tag{3.15}$$

From Lemma 2.2, it follows that

$$[q^{5n+3}]f(-q) = [q^{5n+4}]f(-q) = 0. \tag{3.16}$$

Applying (3.16) into (3.15), we get

$$d_2\left(5^{2\alpha+1}n - \frac{1}{8}(5^{2\alpha}-1) + 4\times 5^{2\alpha}\right) \equiv d_2\left(5^{2\alpha+1}n - \frac{1}{8}(5^{2\alpha}-1) + 5\times 5^{2\alpha}\right) \equiv 0 \pmod 5. \tag{3.17}$$

Finally, we see that for all $k \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} d_{125k+2}(n)q^n &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^7}\frac{(q^2;q^2)_\infty^{125k}}{(q;q)_\infty^{375k}}\\ &\equiv \left(\sum_{n=0}^{\infty} d_2(n)q^n\right)\frac{(q^{10};q^{10})_\infty^{25k}}{(q^5;q^5)_\infty^{75k}} \pmod 5. \end{aligned} \tag{3.18}$$

Equations (3.17) and (3.18) together imply (1.15). □

4. Setup for Proof of Theorem 1.2

The remainder of our paper is dedicated to the proof of Theorem 1.1. The initial setup is standard to the theory. We define the function

$$\mathcal{A} := q^6\frac{D_5(q)}{D_5(q^{25})} = q^6\frac{(q^2;q^2)_\infty^5(q^{25};q^{25})_\infty^{16}}{(q;q)_\infty^{16}(q^{50};q^{50})_\infty^5}. \tag{4.1}$$

Using this function together with the U_5 operator, we can construct a sequence of linear operators:

$$U^{(1)}(f) := U_5(f), \tag{4.2}$$
$$U^{(0)}(f) := U_5(\mathcal{A} \cdot f), \tag{4.3}$$
$$U^{(\alpha)}(f) := U^{(i)}(f), \ \alpha \equiv i \pmod{2}. \tag{4.4}$$

We next define our function sequence $\mathcal{L} := (L_\alpha)_{\alpha \geq 1}$:

$$L_0 := 1, \tag{4.5}$$
$$L_{2\alpha-1}(\tau) = \frac{(q^5;q^5)_\infty^{16}}{(q^{10};q^{10})_\infty^5} \cdot \sum_{n=0}^{\infty} d_5(5^{2\alpha-1}n + \lambda_{2\alpha-1})q^{n+2}, \tag{4.6}$$
$$L_{2\alpha}(\tau) = \frac{(q;q)_\infty^{16}}{(q^2;q^2)_\infty^5} \cdot \sum_{n=0}^{\infty} d_5(5^{2\alpha}n + \lambda_{2\alpha})q^{n+1}. \tag{4.7}$$

We define λ_α as

$$\lambda_\alpha := \frac{1 + 5 \cdot 3^\alpha}{4}, . \tag{4.8}$$

and it can easily be shown that $y = \lambda_\alpha$ is the minimal positive integral solution to

$$4y \equiv 1 \pmod{5^\alpha}.$$

If we apply $U^{(0)}$ to the identity, we get

$$U^{(0)}(1) = U_5\left(q^6 \frac{D_5(q)}{D_2(q^{25})}\right) = \frac{1}{D_2(q^5)} \cdot U_5\left(q^6 D_5(q)\right) = \frac{1}{D_5(q^5)} \cdot U_5\left(\sum_{n \geq 6} d_5(n-6)q^n\right) \tag{4.9}$$
$$= \frac{1}{D_5(q^5)} \cdot \sum_{5n \geq 6} d_5(5n-6)q^n = \frac{1}{D_5(q^5)} \cdot \sum_{n=0}^{\infty} d_5(5n+4)q^{n+2} \tag{4.10}$$
$$= L_1. \tag{4.11}$$

More generally, we state the following:

Lemma 4.1. *For all $\alpha \geq 0$, we have*

$$L_{\alpha+1} = U^{(\alpha)}\left(L_\alpha\right). \tag{4.12}$$

Proof.

$$\begin{aligned}
U^{(2\alpha-1)}\left(L_{2\alpha-1}\right) &= U_5\left(L_{2\alpha-1}\right)\\
&= U_5\left(\frac{1}{D_5(q^5)}\sum_{n\geq 0} d_5\left(5^{2\alpha-1}n+\lambda_{2\alpha-1}\right)q^{n+2}\right)\\
&= \frac{1}{D_5(q)}\cdot U_5\left(\sum_{n\geq 2} d_5\left(5^{2\alpha-1}(n-2)+\lambda_{2\alpha-1}\right)q^{n}\right)\\
&= \frac{1}{D_5(q)}\cdot \sum_{5n\geq 2} d_5\left(5^{2\alpha-1}(5n-2)+\lambda_{2\alpha-1}\right)q^{n}\\
&= \frac{1}{D_5(q)}\cdot \sum_{n\geq 1} d_5\left(5^{2\alpha}n-2\cdot 5^{2\alpha-1}+\lambda_{2\alpha-1}\right)q^{n}\\
&= \frac{1}{D_5(q)}\cdot \sum_{n\geq 0} d_5\left(5^{2\alpha}n+5^{2\alpha}-2\cdot 5^{2\alpha-1}+\lambda_{2\alpha-1}\right)q^{n+1}\\
&= \frac{1}{D_5(q)}\cdot \sum_{n\geq 0} d_5\left(5^{2\alpha}n+\lambda_{2\alpha}\right)q^{n+1}.
\end{aligned}$$

$$\begin{aligned}
U^{(2\alpha)}\left(L_{2\alpha}\right) &= U_5\left(\mathcal{A}\cdot L_{2\alpha}\right)\\
&= U_5\left(q^6\frac{D_5(q)}{D_2(q^{25})}\frac{1}{D_5(q)}\sum_{n\geq 0} d_5\left(5^{2\alpha}n+\lambda_{2\alpha}\right)q^{n+1}\right)\\
&= \frac{1}{D_5(q^3)}\cdot U_5\left(\sum_{n\geq 7} d_5\left(5^{2\alpha}(n-7)+\lambda_{2\alpha}\right)q^{n+7}\right)\\
&= \frac{1}{D_5(q^3)}\cdot \sum_{5n\geq 7} d_5\left(5^{2\alpha}(5n-7)+\lambda_{2\alpha}\right)q^{n}\\
&= \frac{1}{D_5(q^3)}\cdot \sum_{n\geq 2} d_5\left(5^{2\alpha+1}n-7\cdot 5^{2\alpha}+\lambda_{2\alpha}\right)q^{n}\\
&= \frac{1}{D_5(q^3)}\cdot \sum_{n\geq 0} d_5\left(5^{2\alpha+1}(n+2)-7\cdot 5^{2\alpha}+\lambda_{2\alpha}\right)q^{n+2}\\
&= \frac{1}{D_5(q^3)}\cdot \sum_{n\geq 0} d_5\left(5^{2\alpha+1}n+\lambda_{2\alpha+1}\right)q^{n+2}.
\end{aligned}$$

□

4.1. **Our Hauptmodul.** We now have our key function sequence $\mathcal{L}$ defined, as well as the means of constructing $L_{\alpha+1}$ from L_α. We now need a reference function with which to properly represent each L_α. Using Theorem 2.14, we can compute

$$\mathrm{ord}_{\infty}^{(10)}(L_1) \geq 1, \tag{4.13}$$

$$\mathrm{ord}_{1/3}^{(10)}(L_1) \geq 5, \tag{4.14}$$

$$\mathrm{ord}_{1/2}^{(10)}(L_1) \geq -6, \tag{4.15}$$

$$\mathrm{ord}_{0}^{(10)}(L_1) \geq -27. \tag{4.16}$$

We need to compute a function, preferably a Hauptmodul at $[0]_{10}$, which has positive order at $[1/2]_{10}$. Only one eta quotient exists which has this property, which we denote

$$z = z(\tau) := \frac{(q^2;q^2)_\infty^5 (q^5;q^5)_\infty}{(q;q)_\infty^5 (q^{10};q^{10})_\infty}. \tag{4.17}$$

The orders of z at the cusps of $\mathrm{X}_0(10)$ can be computed by Theorem 2.13:

$$\mathrm{ord}_\infty^{(10)}(z) = 0, \tag{4.18}$$
$$\mathrm{ord}_{1/5}^{(10)}(z) = 0, \tag{4.19}$$
$$\mathrm{ord}_{1/2}^{(10)}(z) = 1, \tag{4.20}$$
$$\mathrm{ord}_0^{(10)}(z) = -1. \tag{4.21}$$

Therefore, we know that

$$z^6 L_1 \in \mathcal{M}^0\left(\Gamma_0(10)\right).$$

Because z is a Hauptmodul, we should be able to express $z^6 L_1$ as a polynomial in z. However, not only are the coefficients of such an expression are rational, but they possess denominators divisible by very large powers of 5. Indeed, the expression begins

$$z^6 L_1 = \frac{1}{5^{12}} + \frac{22}{5^{12}} z + \frac{198}{5^{12}} z^2 + ... \tag{4.22}$$

This is clearly not a useful representation.

However, we can repair this problem with a small adjustment. Notice that, because $(1-q)^5 \equiv 1 - q^5 \pmod 5$, we have

$$z \equiv 1 \pmod 5. \tag{4.23}$$

Indeed, if we take (4.22) and express $z = 1 + 5x$, then we recover (1.8).

Lemma 4.2. *If x is defined as in (1.7), then*

$$z = 1 + 5x \tag{4.24}$$

It is the function x which will be our most useful reference function. This lemma may be proved using the modular cusp analysis. However, we give an elementary proof here.

Proof. Notice that

$$z = \frac{f(-q^2)^5 f(-q^5)}{f(-q)^5 f(-q^{10})} \quad \textit{and} \quad x = q\frac{f(-q^2)f(-q^{10})^3}{f(-q)^3 f(-q^5)}.$$

From here we have

$$\begin{aligned}
z &= \frac{f(-q^2)f(-q^5)}{f(-q)^3 f(-q^{10})}\psi^2(q) \\
&= \frac{f(-q^2)f(-q^5)}{f(-q)^3 f(-q^{10})}\left(\frac{f(-q^2)f(-q^5)^3}{f(-q)f(-q^{10})} + q\frac{f(-q^{10})^4}{f(-q^5)}\right) \quad \text{(by (2.6))} \\
&= \frac{f(-q^2)^2}{f(-q)^4}\phi^2(-q^5) + x \\
&= \frac{f(-q^2)^2}{f(-q)^4}\left(\phi^2(-q) + 4qf(-q,-q^9)f(-q^3,-q^7)\right) + x \quad \text{(from (2.7) with } q \mapsto -q\text{)} \\
&= 1 + 4q\frac{f(-q^2)^2}{f(-q)^4}\frac{(q;q^2)_\infty f(-q^{10})^2}{(q^5;q^{10})_\infty} + x = 1 + 4x + x = 1 + 5x.
\end{aligned}$$

□

Both z and x will prove useful to us. As such, we will define two modular equations in each function which will become useful later.

Theorem 4.3. *Define*

$$\begin{aligned}
a_0(\tau) &= -(625x^5 + 500x^4 + 150x^3 + 20x^2 + x)\\
a_1(\tau) &= -(15x + 305x^2 + 2325x^3 + 7875x^4 + 10000x^5)\\
a_2(\tau) &= -(85x + 1750x^2 + 13125x^3 + 46500x^4 + 60000x^5)\\
a_3(\tau) &= -(215x + 4475x^2 + 35000x^3 + 122000x^4 + 160000x^5)\\
a_4(\tau) &= -(205x + 4300x^2 + 34000x^3 + 120000x^4 + 160000x^5)
\end{aligned}$$

$$x^5 + \sum_{j=0}^{4} a_j(5\tau)x^j = 0. \tag{4.25}$$

Proof. See the final section. □

Theorem 4.4. *Define*

$$\begin{aligned}
b_0(\tau) &= -z^5\\
b_1(\tau) &= 1 + 5z + 5z^2 + 5z^3 + 5z^4 - 16z^5\\
b_2(\tau) &= -4 - 15z + 10z^2 + 35z^3 + 60z^4 - 96z^5\\
b_3(\tau) &= 6 + 15z - 35z^2 + 40z^3 + 240z^4 - 256z^5\\
b_4(\tau) &= -4 - 5z + 20z^2 - 80z^3 + 320z^4 - 256z^5\\
b_5(\tau) &= 1.
\end{aligned}$$

$$z^3 + \sum_{k=0}^{4} b_k(5\tau)z^k = 0. \tag{4.26}$$

Proof. Take (4.25) and change variables by $z = (x-1)/5$, and simplify. □

5. Algebra Structure

We will now define the spaces of modular functions which our sequence $\mathcal{L}$ will live in, and how the operators $U^{(\alpha)}$ will affect them.

5.1. **Localized Ring.** We will express each member of $\mathcal{L}$ as a rational polynomial in which the denominator consists of powers of $1 + 5x$. With this in mind, we define the multiplicatively closed set

$$\mathcal{S} := \left\{(1+5x)^n : n \in \mathbb{Z}_{n\geq 0}\right\}. \tag{5.1}$$

We will be working with subspaces of the localized ring

$$\mathbb{Z}[x]_{\mathcal{S}} := \mathcal{S}^{-1}\mathbb{Z}[x]. \tag{5.2}$$

To define these subspaces, we begin by defining functions which will serve as a useful lower bound on the 5-adic valuation of the coefficients of powers of x for the associated functions.

$$\theta_1(m) := \begin{cases} 0, & 1 \leq m \leq 7,\\ \left\lfloor \frac{5m-2}{7} \right\rfloor - 5, & m \geq 8, \end{cases}$$

$$\theta_0(m) := \begin{cases} 0, & 1 \leq m \leq 4, \\ \left\lfloor \frac{5m-1}{7} \right\rfloor - 2, & m \geq 5, \end{cases}$$

We now define the critical mapping Ω which we discussed in Section 1.2. Let

$$\Omega : \bigoplus_{m=2}^{\infty} \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}^2, \tag{5.3}$$

$$: \mathbf{s} \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0... \\ 0 & 0 & 1 & 0 & 4 & 4 & 4 & 0 & 0 & 0... \end{pmatrix} \cdot \mathbf{s}, \tag{5.4}$$

with each component taken modulo 5. Of course, the index m used in the domain of Ω can begin with any integer we want. We begin with $m = 2$ because it is convenient when we build the following associated function spaces. We denote $s(m)$ as an arbitrary integer-valued function which is *discrete*, i.e., with finite support.

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m \geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot x^m \right\}, \tag{5.5}$$

$$\hat{\mathcal{V}}_n := \left\{ \frac{1}{(1+5x)^n} \sum_{m \geq 2} s(m) \cdot 5^{\theta_1(m)} \cdot x^m \right\}, \tag{5.6}$$

$$\mathcal{V}_n^{(1)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m \geq 2} s(m) \cdot 5^{\theta_1(m)} \cdot x^m : (s(m))_{m \geq 2} \in \ker(\Omega) \right\}. \tag{5.7}$$

By examining (1.8), we can verify that

$$\frac{1}{5} L_1 \in \mathcal{V}_6^{(1)}. \tag{5.8}$$

Indeed, we will prove that every odd-indexed member of $\mathcal{L}$ will live in $\mathcal{V}_n^{(1)}$ for some associated n, and similarly for the even-indexed members of $\mathcal{L}$ with respect to $\mathcal{V}_n^{(0)}$.

5.2. **Recurrence Relation.** We know from Lemma 4.12 how $U^{(\alpha)}$ affects members of $\mathcal{L}$. We now need to understand how $U^{(\alpha)}$ affects members of $\mathbb{Z}[x]_{\mathcal{S}}$.

Lemma 5.1. *For all $m, n \in \mathbb{Z}$, and $i \in \{0, 1\}$, we have*

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right) = -\frac{1}{(1+5x)^5} \sum_{j=0}^{4} \sum_{k=1}^{5} a_j(\tau) b_k(\tau) \cdot U^{(i)}\left(\frac{x^{m+j-5}}{(1+5x)^{n-k}}\right). \tag{5.9}$$

Proof. We take advantage of the modular equation (4.26). We isolate and divide by $b_0(5\tau)$, then multiply by z^{-n} for an arbitrary n:

$$b_0(5\tau) = -\sum_{k=1}^{5} b_k(5\tau) z^k,$$

$$1 = -\frac{1}{b_0(5\tau)} \sum_{k=1}^{5} b_k(5\tau) z^k,$$

$$z^{-n} = -\frac{1}{b_0(5\tau)} \sum_{k=1}^{5} b_k(5\tau) z^{-(n-k)}, \tag{5.10}$$

Now we change variables to express z in terms of x:

$$(1+5x)^{-n} = -\frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)(1+5x)^{-(n-k)}. \tag{5.11}$$

We multiply both sides by x^m for arbitrary integer m:

$$\begin{aligned}\frac{x^m}{(1+5x)^n} &= -\frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)\cdot\frac{x^m}{(1+5x)^{n-k}}\\ &= \frac{1}{(1+5x(5\tau))^5}\sum_{k=1}^{5} b_k(5\tau)\cdot\frac{x^m}{(1+5x)^{n-k}}.\end{aligned} \tag{5.12}$$

We can now take advantage of the modular equation (4.25):

$$\begin{aligned}\frac{x^m}{(1+5x)^n} &= \frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)\cdot\sum_{j=0}^{4} a_j(5\tau)\frac{x^{m+j-5}}{(1+5x)^{n-k}}\\ &= -\frac{1}{(1+5x(5\tau))^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(5\tau)b_k(5\tau)\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}.\end{aligned} \tag{5.13}$$

We multiply by $\mathcal{A}^{1-i}\cdot$, with $i=0,1$:

$$\begin{aligned}\mathcal{A}^{1-i}\cdot\frac{x^m}{(1+5x)^n} &= \frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)\cdot\sum_{j=0}^{4} a_j(5\tau)\cdot\mathcal{A}^{1-i}\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}\\ &= -\frac{1}{(1+5x(5\tau))^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(5\tau)b_k(5\tau)\cdot\mathcal{A}^{1-i}\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}.\end{aligned} \tag{5.14}$$

Remembering that

$$U_5(f(5\tau)\cdot g(\tau)) = f(\tau)\cdot U_5(g(\tau)),$$

we can take the U_5 operator of both sides and simplify:

$$\begin{aligned}&U_5\left(\mathcal{A}^{1-i}\cdot\frac{x^m}{(1+5x)^n}\right)\\ &= -\frac{1}{(1+5x)^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(\tau)b_k(\tau)\cdot U_5\left(\mathcal{A}^{1-i}\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}\right).\end{aligned} \tag{5.15}$$

□

5.3. **The Action of $U^{(i)}$ on $\mathbb{Z}[x]_{\mathcal{S}}$.** The previous lemma will be useful in proving a more direct representation of $U^{(i)}\left(x^m/(1+5x)^n\right)$. As in the cases of $\mathcal{V}_n^{(i)}$, we will define lower bounds on the 5-adic valuations of the numerator coefficients of x^r:

$$\begin{aligned}\pi_0(m,r) &:= \max\left(0, \left\lfloor\frac{5r-m+2}{7}\right\rfloor - 5\right),\\ \pi_1(m,r) &:= \left\lfloor\frac{5r-m}{7}\right\rfloor.\end{aligned}$$

We also define a useful auxiliary function:

$$\phi(l) := \left\lfloor\frac{5l+13}{7}\right\rfloor. \tag{5.16}$$

Definition 5.2. Let $n \geq 1$. A function $h : \mathbb{Z}^n \to \mathbb{Z}$ is a *discrete array* if for any fixed $(m_1, m_2, ..., m_{n-1}) \in \mathbb{Z}^{n-1}$, $h(m_1, m_2, ..., m_{n-1}, m)$ has finite support with respect to m.

Theorem 5.3. *There exist discrete arrays $h_1, h_0 : \mathbb{Z}^3 \to \mathbb{Z}$ such that*

$$U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n}} \sum_{r \geq \lceil m/5 \rceil} h_1(m,n,r) \cdot 5^{\pi_1(m,r)} \cdot x^r,$$
$$U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n+6}} \sum_{r \geq \lceil (m+1)/5 \rceil} h_0(m,n,r) \cdot 5^{\pi_0(m,r)} \cdot x^r.$$

Proof. We prove the the theorem by induction over m and n. Let us suppose that Theorem 5.3 is true for all $m \leq m_0 - 1$, $n \leq n_0 - 1$ for some fixed $m_0, n_0 \geq 5$. In that case we can write

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$$
$$= -\frac{1}{(1+5x)^5} \sum_{j=0}^{4} \sum_{k=1}^{5} a_j(\tau) b_k(\tau) \cdot U^{(i)}\left(\frac{x^{m_0+j-5}}{(1+5x)^{n_0-k}}\right) \tag{5.17}$$
$$= -\frac{1}{(1+5x)^5} \sum_{j=0}^{4} \sum_{k=1}^{5} \frac{a_j(\tau) b_k(\tau)}{(1+5x)^{5(n_0-k)+\kappa}}$$
$$\times \sum_{r \geq \lceil (m_0+j-5+\delta)/5 \rceil} h_i(m_0+j-5, n_0-k, r) \cdot 5^{\pi_i(m_0+j-5,r)} \cdot x^r \tag{5.18}$$
$$= \frac{1}{(1+5x)^{5n_0+\kappa}} \sum_{j=0}^{4} \sum_{k=1}^{5} w(j,k)$$
$$\times \sum_{r \geq \lceil (m_0+j-5+\delta)/5 \rceil} h_i(m_0+j-5, n_0-k, r) \cdot 5^{\pi_i(m_0+j-5,r)} \cdot x^r, \tag{5.19}$$

wherein we define

$$w(j,k) := -\, a_j(\tau) b_k(\tau)(1+5x)^{5(k-1)}$$
$$= \sum_{l=1}^{30} v(j,k,l) \cdot 5^{\left\lfloor \frac{5l+j+1}{7} \right\rfloor} \cdot x^l. \tag{5.20}$$

Expanding and recombining, we have

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$$
$$= \frac{1}{(1+5x)^{5n_0+\kappa}} \sum_{j=0}^{4} \sum_{k=1}^{5} \sum_{l=1}^{30} \sum_{r \geq \lceil (m_0+j-5+\delta)/5 \rceil}$$
$$v(j,k) \cdot h_i(m_0+j-5, n_0-k, r) \cdot 5^{\pi_i(m_0+j-5,r) + \left\lfloor \frac{5l+j+1}{7} \right\rfloor} \cdot x^{r+l}. \tag{5.21}$$

We need to verify that the coefficients of x^{r+l} are divisible by $5^{\pi_i(m_0, r+l)}$, and that $r+l$ is bounded below by $\lceil (m_0+\delta)/5 \rceil$. On the latter matter, we note that

$$r+l \geq \left\lceil \frac{m_0+j-5+\delta}{5} \right\rceil + l \tag{5.22}$$
$$\geq \left\lceil \frac{m_0+\delta}{5} - \frac{5-j}{5} \right\rceil + l \tag{5.23}$$
$$\geq \left\lceil \frac{m_0+\delta}{5} \right\rceil - 1 + l \tag{5.24}$$
$$\geq \left\lceil \frac{m_0+\delta}{5} \right\rceil. \tag{5.25}$$

Regarding the former matter, we first take $i=1$. Notice that

$$\left\lfloor \frac{M}{7} \right\rfloor + \left\lfloor \frac{N}{7} \right\rfloor \geq \left\lfloor \frac{M+N-6}{7} \right\rfloor.$$

From this, we have

$$\pi_1(m_0+j-5,r) + \left\lfloor \frac{5l+j+1}{7} \right\rfloor = \left\lfloor \frac{5r-m_0-j+5}{7} \right\rfloor + \left\lfloor \frac{5l+j+1}{7} \right\rfloor \tag{5.26}$$
$$\geq \left\lfloor \frac{5(r+l)-m_0}{7} \right\rfloor \tag{5.27}$$
$$= \pi_1(m_0,r). \tag{5.28}$$

For $i=0$, we have to more carefully expand

$$U^{(0)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$$
$$=\frac{1}{(1+5x)^{5n_0+6}}\sum_{j=0}^{4}\sum_{k=1}^{5} w(j,k)$$
$$\times \Bigg(\sum_{\lceil (m_0+j-5+1)/5\rceil \leq r \leq \lceil (m_0+j-5+33)/5\rceil -1} h_i(m_0+j-5,n_0-k,r)\cdot x^r \tag{5.29}$$
$$+ \sum_{r\geq \lceil (m_0+j-5+33)/5\rceil} h_0(m_0+j-5,n_0-k,r)\cdot 5^{\left\lfloor \frac{5r-(m_0+j-5)+2}{7}\right\rfloor -5}\cdot x^r\Bigg). \tag{5.30}$$

Expanding $w(j,k)$ again,

$$U^{(0)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$$
$$=\frac{1}{(1+5x)^{5n_0+6}}\sum_{j=0}^{4}\sum_{k=1}^{5}\sum_{l=1}^{30}\Bigg(\sum_{\lceil (m_0+j-5+1)/5\rceil \leq r \leq \lceil (m_0+j-5+33)/5\rceil -1}$$
$$v(j,k)\cdot h_0(m_0+j-5,n_0-k,r)\cdot 5^{\left\lfloor \frac{5l+j+1}{7}\right\rfloor}\cdot x^{r+l} \tag{5.31}$$
$$+ \sum_{r\geq \lceil (m_0+j-5+33)/5\rceil}$$
$$v(j,k)\cdot h_0(m_0+j-5,n_0-k,r)\cdot 5^{\left\lfloor \frac{5r-(m_0+j-5)+2}{7}\right\rfloor -5+\left\lfloor \frac{5l+j+1}{7}\right\rfloor}\cdot x^{r+l}\Bigg). \tag{5.32}$$

For $r \geq \lceil (m_0+j-5+33)/5\rceil$, we have $\pi_0(m_0+j-5,r) = \left\lfloor \frac{5r-(m_0+j-5)+2}{7}\right\rfloor - 5 \geq 0$. We find that the coefficient of x^{r+l} has a 5-adic valuation of at least

$$\left(\left\lfloor\frac{5r-(m_0+j-5)+2}{7}\right\rfloor-5\right)+\left\lfloor\frac{5l+j+1}{7}\right\rfloor\geq\left\lfloor\frac{5(r+l)-m_0+2}{7}\right\rfloor-5=\pi_0(m_0,r+l). \tag{5.33}$$

For $\lceil(m_0+j-5+1)/5\rceil \leq r \leq \lceil(m_0+j-5+33)/5\rceil-1$, we have $\pi_0(m_0+j-5,r)=0$. Let us suppose that $r+l\leq\lceil(m_0+j-5+33)/5\rceil-1$. In this case, we only need the power of 5 to be 0.

On the other hand, if $r+l\geq\lceil(m_0+j-5+33)/5\rceil$, then we have

$$\left\lfloor\frac{5l+j+1}{7}\right\rfloor-\left(\left\lfloor\frac{5(r+l)-m_0+2}{7}\right\rfloor-5\right) \tag{5.34}$$

$$=\left\lfloor\frac{5l+j+36}{7}\right\rfloor-\left\lfloor\frac{5(r+l)-m_0+2}{7}\right\rfloor \tag{5.35}$$

$$\geq\left\lfloor\frac{5(r+l)-m_0+2}{7}\right\rfloor+\left\lfloor\frac{m-5r+34}{7}\right\rfloor-\left\lfloor\frac{5(r+l)-m_0+2}{7}\right\rfloor \tag{5.36}$$

$$=\left\lfloor\frac{m_0-5r+34}{7}\right\rfloor. \tag{5.37}$$

We need to show that $m_0-5r+34\geq 0$. But of course this is equivalent to

$$r\leq\frac{m_0+34}{5}, \tag{5.38}$$

which must be true since

$$r\leq\left\lceil\frac{m_0+j-5+33}{5}\right\rceil-1\leq\left\lceil\frac{m_0+32}{5}\right\rceil. \tag{5.39}$$

Therefore in all cases for either value of i we have the coefficient of x^{r+l} divisible by $5^{\pi_i(m_0,r+l)}$.

$$\begin{aligned}&U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)\\ =&\frac{1}{(1+5x)^{5n_0+\kappa}}\sum_{\substack{0\leq j\leq 4,\\1\leq k\leq 5,\\1\leq l\leq 30}}\sum_{r\geq\lceil\frac{m_0+\delta}{5}\rceil}\\ &v(j,k)\cdot h_i(m_0+j-5,n_0-k,r-l)\cdot 5^{\pi_i(m_0+j-5,r)+\lfloor\frac{5l+j+1}{7}\rfloor}\cdot x^r.\end{aligned} \tag{5.40}$$

$$H_i(m,n,r):=\begin{cases}\sum_{\substack{0\leq j\leq 4,\\1\leq k\leq 5,\\1\leq l\leq 30}}\quad\sum_{r\geq\lceil\frac{m+\delta}{5}\rceil-1+l}\hat{H}(i,j,k,l,r),\ r\geq l\\ 0,\ r<l\end{cases} \tag{5.41}$$

$$\hat{H}(i,j,k,l,r):=v(j,k)\cdot h_i(m+j-5,n-k,r-l)\cdot 5^{\epsilon(i,j,l,m,r)},$$

$$\epsilon(i,j,l,m,r):=\pi_i(m+j-5,r-l)+\left\lfloor\frac{5l+j+1}{7}\right\rfloor-\pi_i(m,r)$$

$$\begin{aligned}U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)=&\frac{1}{(1+5x)^{5n_0+\kappa}}\\ &\times\sum_{r\geq\lceil\frac{m_0+\delta}{5}\rceil}H_i(m_0,n_0,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r.\end{aligned} \tag{5.42}$$

□

We will need a more detailed understanding of the arithmetic properties of the functions h_i. With this in mind, we give the following useful inequality:

Lemma 5.4. *For $l \geq 0$ and $i \in \{0, 1\}$, we have*

$$\pi_i(m, r-l) + \phi(l) - \pi_i(m, r) \geq 1. \tag{5.43}$$

Proof. Notice that for $l = 0$, the inequality reduces to

$$\phi(0) \geq 1, \tag{5.44}$$

which is true since $\phi(0) = 1$. We therefore consider the case in which $l > 0$.

i = 0

Consider first that $\pi_0(m, r-l), \pi_0(m, r) = 0$. This reduces to

$$\phi(l) \geq 1, \tag{5.45}$$

which follows from $\phi(l) \geq \phi(0) \geq 1$. If $\pi_0(m, r-l) = 0$, and $\pi_0(m, r) > 0$, then

$$r \geq \left\lceil \frac{m+33}{5} \right\rceil \text{ and } r - l \leq \left\lceil \frac{m+33}{5} \right\rceil - 1 = \left\lceil \frac{m+28}{5} \right\rceil, \tag{5.46}$$

$$l \geq r - \left\lceil \frac{m+28}{5} \right\rceil. \tag{5.47}$$

Knowing this, we can rewrite

$$\begin{aligned} \phi(l) = \left\lfloor \frac{5l+13}{7} \right\rfloor &\geq \left\lfloor \frac{5r-m-28+13}{7} \right\rfloor && (5.48)\\ &= \left\lfloor \frac{5r-m-15}{7} \right\rfloor && (5.49)\\ &\geq \left(\left\lfloor \frac{5r-m+2}{7} \right\rfloor - 5 \right) + 1 && (5.50)\\ &= \pi_0(m, r) + 1. && (5.51) \end{aligned}$$

If $\pi_0(m, r-l) > 0$, and $\pi_0(m, r) > 0$, then

$$\begin{aligned} \pi_0(m, r-l) + &\phi(l) && (5.52)\\ &= \left\lfloor \frac{5(r-l)-m+2}{7} \right\rfloor + \left\lfloor \frac{5l+13}{7} \right\rfloor && (5.53)\\ &\geq \left\lfloor \frac{5r-m+9}{7} \right\rfloor && (5.54)\\ &= \left\lfloor \frac{5r-m+2}{7} \right\rfloor + 1 && (5.55)\\ &= \pi_0(m, r) + 1. && (5.56) \end{aligned}$$

i = 1

We simply have

$$\begin{aligned}\pi_1(m, r-l)+&\phi(l) && (5.57)\\ &= \left\lfloor \frac{5(r+l)-m}{7} \right\rfloor + \left\lfloor \frac{5l+13}{7} \right\rfloor && (5.58)\\ &\geq \left\lfloor \frac{5r-m+7}{7} \right\rfloor && (5.59)\\ &= \left\lfloor \frac{5r-m}{7} \right\rfloor + 1 && (5.60)\\ &= \pi_1(m,r)+1. && (5.61)\end{aligned}$$

□

We now state a result which will prove extremely useful in the proof of our main theorem:

Theorem 5.5. *For all $m, n, r \in \mathbb{Z}_{\geq 1}$ and $i \in \{0,1\}$, we have*

$$h_i(m,n,r) \equiv h_i(m,n-5,r) \pmod 5. \tag{5.62}$$

Proof. In contrast to the proof of Theorem 5.3, let us examine $U^{(i)}\left(x^m/(1+5x)^n\right)$ while holding m fixed, and varying n through the modular equation of $1+5x$. In this case we have

$$\begin{aligned}U^{(i)}&\left(\frac{x^{m_0}}{(1+5x)^n}\right)\\ =&\frac{1}{(1+5x)^5}\sum_{k=1}^{5} b_k(\tau)\cdot U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n-k}}\right) && (5.63)\\ =&\frac{1}{(1+5x)^5}\sum_{k=1}^{5} \frac{b_k(\tau)}{(1+5x)^{5(n-k)-\kappa}}\sum_{r\geq 1} h_i(m_0,n-k,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r && (5.64)\\ =&\frac{1}{(1+5x)^{5n+\kappa}}\sum_{k=1}^{5} \hat{w}(k)\sum_{r\geq 1} h_i(m_0,n-k,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r, && (5.65)\end{aligned}$$

in which we can expand

$$\begin{aligned}\hat{w}(k) &:= b_k(\tau)(1+5x)^{5(k-1)}\\ &= \begin{cases} \displaystyle\sum_{l=0}^{20} \hat{v}(k,l)\cdot 5^{\phi(l)}\cdot x^l, \ k<5\\ \displaystyle 1+\sum_{l=1}^{20} \hat{v}(5,l)\cdot 5^{\phi(l)}\cdot x^l, \ k=5. \end{cases} && (5.66)\end{aligned}$$

We can now express

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n+\kappa}}$$

$$\times\Bigg(\sum_{\substack{1\le k\le 4,\\ 0\le l\le 20,\\ r\ge\lceil\frac{m_0+\delta}{5}\rceil}} \hat{v}(k,l)\cdot h_i(m_0,n-k,r)\cdot 5^{\pi_i(m_0,r)+\phi(l)}\cdot x^{r+l} \tag{5.67}$$

$$+\sum_{\substack{1\le l\le 20,\\ r\ge\lceil\frac{m_0+\delta}{5}\rceil}} \hat{v}(5,l)\cdot h_i(m_0,n-5,r)\cdot 5^{\pi_i(m_0,r)+\phi(l)}\cdot x^{r+l} \tag{5.68}$$

$$+\sum_{r\ge\lceil\frac{m_0+\delta}{5}\rceil} h_i(m_0,n-5,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^{r}\Bigg). \tag{5.69}$$

Relabeling our powers of x, we have

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n+\kappa}}\times\Bigg(\sum_{\substack{1\le k\le 4,\\ 0\le l\le 20,\\ r\ge l+\lceil\frac{m_0+\delta}{5}\rceil}} \hat{v}(k,l)\cdot h_i(m_0,n-k,r-l)\cdot 5^{\pi_i(m_0,r-l)+\phi(l)}\cdot x^{r} \tag{5.70}$$

$$+\sum_{\substack{1\le l\le 20,\\ r\ge l+\lceil\frac{m_0+\delta}{5}\rceil}} \hat{v}(5,l)\cdot h_i(m_0,n-5,r-l)\cdot 5^{\pi_i(m_0,r-l)+\phi(l)}\cdot x^{r} \tag{5.71}$$

$$+\sum_{r\ge\lceil\frac{m_0+\delta}{5}\rceil} h_i(m_0,n-5,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^{r}\Bigg) \tag{5.72}$$

$$= \frac{1}{(1+5x)^{5n+\kappa}}\sum_{r\ge\lceil\frac{m_0+\delta}{5}\rceil} h_i(m_0,n,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^{r}. \tag{5.73}$$

Notice that $\pi_i(m,r-l)+\phi(l)\ge\pi_i(m,r)+1$ by Lemma 5.4. Therefore, if we divide out $5^{\pi_i(m,r)}$, we must have

$$h_i(m,n,r)\equiv h_i(m,n-5,r)\pmod 5.$$

□

Corollary 5.6. *The following apply for all $n\ge 1$:*

$$h_1(m,n,1)\equiv 1 \pmod 5 \text{ for } m=2,3,5, \tag{5.74}$$

$$h_1(4,n,1)\equiv 2 \pmod 5, \tag{5.75}$$

$$h_1(4,n,2)\equiv 4 \pmod 5, \tag{5.76}$$

$$h_1(5,n,2)\equiv 0 \pmod 5, \tag{5.77}$$

$$h_1(m,n,2)\equiv 1 \pmod 5 \text{ for } m=6,7,8. \tag{5.78}$$

6. Main Theorem

We are now at the point that we may begin proving our main theorem. Since we know that

$$\frac{1}{5}L_1\in\mathcal{V}_6^{(1)},$$

we want to show that applying $U^{(1)}$ to a member of $\mathcal{V}_n^{(1)}$ will produce a member of $\mathcal{V}_{n'}^{(0)}$ in which all coefficients are divisible by an extra power of 5. Then applying $U^{(0)}$ to $\mathcal{V}_n^{(0)}$, one should achieve a member of $\mathcal{V}_{n'}^{(1)}$.

However, this strategy will need to be slightly modified. We start by proving the following slightly less ambitious theorem:

Theorem 6.1. *Let $f \in \mathcal{V}_n^{(0)}$. Then*

$$U^{(0)}(f) \in \hat{\mathcal{V}}_{5n+6}. \tag{6.1}$$

Proof. Let $f \in \mathcal{V}_n^{(0)}$. Then we can express f as

$$f = \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot x^m.$$

$$U^{(0)}(f) = \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) \tag{6.2}$$

$$= \frac{1}{(1+5x)^{5n+6}} \sum_{m\geq 1}\sum_{r\geq\lceil (m+1)/5\rceil} s(m)\cdot h_0(m,n,r)5^{\theta_0(m)+\pi_0(m,r)}\cdot x^r \tag{6.3}$$

$$= \frac{1}{(1+5x)^{5n+6}} \sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_0(m,n,r)5^{\theta_0(m)+\pi_0(m,r)}\cdot x^r. \tag{6.4}$$

We need to prove that for $m+1\leq 5r$,

$$\theta_0(m)+\pi_0(m,r)-\theta_1(r)\geq 0. \tag{6.5}$$

$\mathbf{\left\lceil \frac{m+1}{5}\right\rceil \leq r \leq \left\lceil \frac{m+33}{5}\right\rceil - 1}$

We have $\pi_0(m,r)=0$. Notice also that $m\geq 5r-28$.

For $1\leq m\leq 4$, we have $\theta_0(m)=0$. We also have $1\leq r\leq 7$, and $\theta_1(r)=0$. Thus (6.5) follows trivially.

For $m\geq 5$ we have $\theta_0(m)=\left\lfloor\frac{5m-1}{7}\right\rfloor-2$, and $r\geq 2$. If $2\leq r\leq 7$, then (6.5) again follows trivially. If $r\geq 8$, then

$$\begin{aligned}\theta_0(m)+\pi_0(m,r) &= \left(\left\lfloor\frac{5m-1}{7}\right\rfloor-2\right)+0-\left(\left\lfloor\frac{5r-2}{7}\right\rfloor-5\right)\\ &\geq \left\lfloor\frac{25m-29}{7}\right\rfloor-\left\lfloor\frac{5r-2}{7}\right\rfloor+3\\ &\geq \left\lfloor\frac{5r-2}{7}\right\rfloor+\left\lfloor\frac{20r-1}{7}\right\rfloor-\left\lfloor\frac{5r-2}{7}\right\rfloor-1\\ &\geq \left\lfloor\frac{20r-1}{7}\right\rfloor-1\geq 0.\end{aligned}$$

$\mathbf{r \geq \left\lceil \frac{m+33}{5}\right\rceil}$

We have $\pi_0(m,r)=\left\lfloor\frac{5r-m+2}{7}\right\rfloor-5$.

For $1\leq m\leq 4$, we have $\theta_0(m)=0$ and $r\geq 7$. For $r=7$, $\theta_1(r)=0$, and

$$\theta_0(m)+\pi_0(m,r)=\left\lfloor\frac{5r-m+2}{7}\right\rfloor-5\geq\theta_1(r).$$

For $r\geq 8$, we have

$$\begin{aligned}\theta_0(m)+\pi_0(m,r) &= \left\lfloor\frac{5r-m+2}{7}\right\rfloor-5\\ &\geq \left\lfloor\frac{5r-2}{7}\right\rfloor-5=\theta_1(r).\end{aligned}$$

For $m\geq 5$,

$$\begin{aligned}\theta_0(m)+\pi_0(m,r) &= \left\lfloor \frac{5m-1}{7} \right\rfloor - 2 + \left\lfloor \frac{5r-m+2}{7} \right\rfloor - 5\\ &\geq \left\lfloor \frac{5r+4m-5}{7} \right\rfloor - 7\\ &\geq \left(\left\lfloor \frac{5r-2}{7} \right\rfloor - 5 \right) + \left\lfloor \frac{4m-3}{7} \right\rfloor - 2\\ &\geq \theta_1(r).\end{aligned}$$

□

So far, this is what one would more or less expect from the classical proofs of congruence families. However, the matter becomes more complicated on application of $U^{(1)}$, as the following theorem shows:

Theorem 6.2. *Let $f \in \mathcal{V}_n^{(1)}$ with $n \equiv 1 \pmod 5$. Then*

$$\frac{1}{5} U^{(1)}(f) \in \mathcal{V}_{5n}^{(0)}. \tag{6.6}$$

Proof.

$$f = \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot x^m.$$

$$\begin{aligned}U^{(1)}(f) &= \frac{1}{(1+5x)^{5n}} \sum_{m\geq 2} \sum_{r\geq \lceil m/5 \rceil} s(m) \cdot h_1(m,n,r) \cdot 5^{\theta_1(m)+\pi_1(m,r)} \cdot x^r\\ &= \frac{1}{(1+5x)^{5n}} \sum_{r\geq 1} \sum_{m\geq 2} s(m) \cdot h_1(m,n,r) \cdot 5^{\theta_1(m)+\pi_1(m,r)} \cdot x^r.\end{aligned}$$

For $m \leq 5r$ we consider the relation

$$\theta_1(m) + \pi_1(m,r) - \theta_0(r) - 1 \geq 0. \tag{6.7}$$

We first verify that (6.7) is true for $r \geq 3$.
$\mathbf{r=3}$

Here $\theta_0(3) = 0$, and m ranges over $2 \leq m \leq 15$.
For $2 \leq m \leq 7$, $\theta_1(m) = 0$ and

$$\begin{aligned}\theta_1(m) + \pi_1(m,3) &= \left\lfloor \frac{15-m}{7} \right\rfloor\\ &\geq \left\lfloor \frac{15-7}{7} \right\rfloor\\ &\geq 1 = \theta_0(3)+1.\end{aligned}$$

For $m = 8$,

$$\theta_1(8) + \pi_1(8,3) = 2 + 1 \geq 1 = \theta_0(3) + 1.$$

For $m \geq 9$,

$$\begin{aligned}\theta_1(m)+\pi_1(m,3) &= \left(\left\lfloor\frac{5m-2}{7}\right\rfloor - 5\right) + \left\lfloor\frac{15-m}{7}\right\rfloor \\ &\geq \left\lfloor\frac{15+4m-8}{7}\right\rfloor - 5 \\ &\geq 1 = \theta_0(3)+1.\end{aligned}$$

r = 4

Here $\theta_0(4) = 0$, and m ranges over $2 \leq m \leq 20$.
For $2 \leq m \leq 7$, $\theta_1(m) = 0$ and

$$\begin{aligned}\theta_1(m)+\pi_1(m,4) &= \left\lfloor\frac{20-m}{7}\right\rfloor \\ &\geq \left\lfloor\frac{20-7}{7}\right\rfloor \\ &\geq 1 = \theta_0(4)+1.\end{aligned}$$

For $m \geq 8$,

$$\begin{aligned}\theta_1(m)+\pi_1(m,4) &= \left(\left\lfloor\frac{5m-2}{7}\right\rfloor - 5\right) + \left\lfloor\frac{20-m}{7}\right\rfloor \\ &\geq \left\lfloor\frac{20+4m-8}{7}\right\rfloor - 5 \\ &\geq 1 = \theta_0(4)+1.\end{aligned}$$

r ≥ 5

Here $\theta_0(r) = \left(\left\lfloor\frac{5r-1}{7}\right\rfloor - 2\right)$, and $m \geq 2$.
For $2 \leq m \leq 7$, $\theta_1(m) = 0$ and

$$\begin{aligned}\theta_1(m)+\pi_1(m,r) &= \left\lfloor\frac{5r-7}{7}\right\rfloor \\ &\geq \left\lfloor\frac{5r}{7}\right\rfloor - 1 \\ &\geq \left\lfloor\frac{5r-1}{7}\right\rfloor - 1 \\ &= \theta_0(r)+1.\end{aligned}$$

For $m = 8$,

$$\begin{aligned}\theta_1(8)+\pi_1(8,r) &= 0 + \left\lfloor\frac{5r-8}{7}\right\rfloor \\ &= \left\lfloor\frac{5r-1}{7}\right\rfloor - 1 \\ &= \theta_0(r)+1.\end{aligned}$$

For $m \geq 9$,

$$\begin{aligned}\theta_1(m)+\pi_1(m,r) &= \left(\left\lfloor\frac{5m-2}{7}\right\rfloor-5\right)+\left\lfloor\frac{5r-m}{7}\right\rfloor\\ &\geq \left\lfloor\frac{5r+4m-8}{7}\right\rfloor-5\\ &\geq \left\lfloor\frac{5r-1}{7}\right\rfloor+\left\lfloor\frac{4m}{7}\right\rfloor-6\\ &\geq \left\lfloor\frac{5r-1}{7}\right\rfloor-1\\ &= \theta_0(r)+1.\end{aligned}$$

$1 \leq r \leq 2$

Let us first consider the case for $r = 1$. We know that we will receive a contribution for $2 \leq m \leq 5$. In each of these cases, we see that

$$\begin{aligned}\theta_1(m)+\pi_1(m,1) &= 0+\left\lfloor\frac{5-m}{7}\right\rfloor = 0\\ &\neq \theta_0(1)+1=1.\end{aligned}$$

On the other hand, for $r = 2$, we have already shown that we get the correct contribution for $m \geq 9$. Moreover, for $2 \leq m \leq 3$,

$$\begin{aligned}\theta_1(m)+\pi_1(m,2) &= 0+\left\lfloor\frac{10-m}{7}\right\rfloor \geq 1\\ &= \theta_0(2)+1.\end{aligned}$$

However, for $4 \leq m \leq 7$,

$$\begin{aligned}\theta_1(m)+\pi_1(m,2) &= 0+\left\lfloor\frac{10-m}{7}\right\rfloor = 0\\ &\neq \theta_0(1)+1=1.\end{aligned}$$

The only possible resolution to this problem is that

$$\begin{aligned}\sum_{m=2}^{5} s(m)\cdot h_1(m,n,1) &\equiv 0 \pmod{5},\\ \sum_{m=4}^{8} s(m)\cdot h_1(m,n,2) &\equiv 0 \pmod{5}.\end{aligned}$$

By Corollary 5.6, we can reduce this to

$$\begin{aligned}s(2)+s(3)+2s(4)+s(5) &\equiv 0 \pmod{5},\\ 4s(4)+s(6)+s(7)+s(8) &\equiv 0 \pmod{5}.\end{aligned}$$

But notice that these are the components of the image of Ω applied to $(s(m))_{m\geq 2}$. Because $(s(m))_{m\geq 2} \in \ker(\Omega)$, these relations must both reduce to 0, and we have accounted for the necessary powers of 5 in all cases.

□

We now see the necessity of the kernel properties of $\mathcal{V}_n^{(1)}$. However, we have not done enough. We need to prove *stability* of the properties of $\mathcal{V}_n^{(1)}$. That is, we need to show the following:

Theorem 6.3. *Let* $f \in \mathcal{V}_n^{(1)}$, *with* $n \equiv 1 \pmod 5$. *Then*

$$\frac{1}{5}U^{(0)} \circ U^{(1)}(f) \in \mathcal{V}_{25n+6}^{(1)}. \tag{6.8}$$

It is this which will finally allow us to complete the proof of Theorem 1.2, and with it Theorem 1.1.

Proof. We suppose that f has the form

$$f = \frac{1}{(1+5x)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot x^m.$$

We proved in Theorem 6.2 above that

$$\frac{1}{5}U^{(1)}(f) \in \mathcal{V}_{5n}^{(0)}.$$

Moreover, we know from Theorem 6.1 that

$$\frac{1}{5}U^{(0)} \circ U^{(1)}(f) \in \hat{\mathcal{V}}_{25n+6}.$$

Because of this, we can expand

$$\frac{1}{5}U^{(0)} \circ U^{(1)}(f) = \frac{1}{(1+5x)^{25n+6}}\sum_{w\geq 2} t(w)\cdot 5^{\theta_1(w)}\cdot x^w,$$

with $t(w)$ defined as

$$t(w) := \sum_{r=1}^{5w-6}\sum_{m=1}^{5r} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n,w) \times 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-1}.$$

All that remains is to prove that $(t(w))_{w\geq 2} \in \ker(\Omega)$.

Notice that for any fixed w, the values m, r are bounded. The variable n is unbounded. However, we can now take advantage of Theorem 5.5, and the fact that $n \equiv 1 \pmod 5$ by hypothesis, and we define

$$\hat{t}(w) := \sum_{r=1}^{5w-6}\sum_{m=1}^{5r} s(m)\cdot h_1(m,1,r)\cdot h_0(r,5,w) \times 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-1},$$

Obviously, $\hat{t}(w) \equiv t(w) \pmod 5$. If we examine $\hat{t}(w)$ for $2 \leq w \leq 8$, we have

$$\hat{t}(2) =\frac{1}{5}\left(-624s(2) - 1664s(3) + 94204s(4) + 99616s(5) + 57078s(6) + 19008s(7) + 3708s(8)\right),$$
$$\hat{t}(3) =\frac{1}{5}\left(28224s(2) + 75264s(3) - 3621954s(4) - 3834516s(5) - 2197503s(6) - 731808s(7) - 142758s(8)\right),$$
$$\hat{t}(4) =715008s(2) + 1906688s(3) - 67390288s(4) - 71546272s(5) - 41020056s(6) - 13660416s(7) - 2664816s(8),$$
$$\hat{t}(5) =25337256s(2) + 67566016s(3) - 6656426s(4) - 33820304s(5) - 21775257s(6) - 7251552s(7) - 1414602s(8),$$
$$\hat{t}(6) =457837968s(2) + 1220901248s(3) + 140880948572s(4) + 147501656288s(5) + 84383715654s(6) + 28101294144s(7) + 5481881244s(8),$$
$$\hat{t}(7) =\frac{1}{5}(18893919144s(2) + 50383784384s(3) + 38559332136626s(4) + 40484108853704s(5) + 23170599952857s(6) + 7716226285152s(7) + 1505248688202s(8)),$$
$$\hat{t}(8) =\frac{1}{5}(-116748977604s(2) - 311330606944s(3) + 1303629422734184s(4) + 1369503027522686s(5) + 783890939008863s(6) + 261049773444768s(7) + 50924482319718s(8)).$$

It is clear that (so long as $s(m)$ is always integer-valued) $\hat{t}(4), \hat{t}(5), \hat{t}(6) \in \mathbb{Z}$. For the remainder, we define the ideal

$$I := \langle s(2) + s(3) + 2s(4) + s(5), 4s(4) + s(6) + 2s(7) + s(8)\rangle \leq \mathbb{Z}/5\mathbb{Z}[s(2), s(3), ..., s(8)].$$

By a simple polynomial reduction through a computer algebra system, it can be shown that

$$5\hat{t}(2), 5\hat{t}(3), 5\hat{t}(7), 5\hat{t}(8) \in I.$$

This ensures that

$$\hat{t}(2), \hat{t}(3), \hat{t}(7), \hat{t}(8) \in \mathbb{Z}.$$

We now need to verify that

$$\hat{t}(2) + \hat{t}(3) + 2\hat{t}(4) + \hat{t}(5) \equiv 0 \pmod 5, \tag{6.9}$$
$$4\hat{t}(4) + \hat{t}(6) + \hat{t}(7) + \hat{t}(8) \equiv 0 \pmod 5. \tag{6.10}$$

Expanding (6.9) and (6.10), we find that

$$\hat{t}(2) + \hat{t}(3) + 2\hat{t}(4) + \hat{t}(5) =26772792s(2) + 71394112s(3) - 142142552s(4) - 177659828s(5) - 104243454s(6) - 34714944s(7) - 6772044s(8), \tag{6.11}$$
$$4\hat{t}(4) + \hat{t}(6) + \hat{t}(7) + \hat{t}(8) = - 19110313692s(2) - 50960836512s(3) + 268578362361582s(4) + 282144642746478s(5) + 161496527427774s(6) + 53781246598464s(7) + 10491417423564s(8). \tag{6.12}$$

We reduce both of these polynomials modulo I to achieve 0.

Because $t(w) \equiv \hat{t}(w) \pmod 5$, we have $(t(w))_{w\geq 2} \in \ker(\Omega)$, and we have therefore proved (6.8).

□

Proof of Theorem 1.2. From (4.9)-(4.11), we know that

$$L_1 = U^{(0)}(1).$$

Moreover, after dividing each coefficient by 5, we can confirm by a simple computation that

$$\frac{1}{5} \cdot L_1 = f_1 = \frac{1}{(1 + 5x)^6}\left(1141x^2 + 1368024x^3 + 406830425x^4 + 56096987730x^5 + ...\right) \in \mathcal{V}_1^{(1)}.$$

In particular, $(1141, 1368024, 406830425, 56096987730, ...) \in \ker(\Omega)$.

Let us suppose that for some $\alpha \geq 1$, we have

$$\frac{1}{5^\alpha} \cdot L_{2\alpha-1} \in \mathcal{V}_n^{(1)}.$$

Then there exists some $f_{2\alpha-1} \in \mathcal{V}_n^{(1)}$ such that

$$L_{2\alpha-1} = 5^\alpha \cdot f_{2\alpha-1}. \tag{6.13}$$

On applying $U^{(1)}$, we have

$$L_{2\alpha} = U_5\left(L_{2\alpha-1}\right) = U_5\left(5^\alpha \cdot f_{2\alpha-1}\right) = 5^\alpha \cdot U^{(1)}\left(f_{2\alpha-1}\right). \tag{6.14}$$

But by Theorem 6.2, there exists some $f_{2\alpha} \in \mathcal{V}_{5n}^{(0)}$ such that

$$U^{(1)}\left(f_{2\alpha-1}\right) = 5 \cdot f_{2\alpha} \tag{6.15}$$

Therefore, we have

$$L_{2\alpha} = 5^{\alpha+1} \cdot f_{2\alpha}. \tag{6.16}$$

Moreover, on applying $U^{(0)}$ we have

$$L_{2\alpha+1} = U_5\left(\mathcal{A} \cdot L_{2\alpha}\right) = U_5\left(\mathcal{A} \cdot 5^{\alpha+1} \cdot f_{2\alpha}\right) = 5^{\alpha+1} \cdot U^{(0)}\left(f_{2\alpha}\right). \tag{6.17}$$

We recall from (6.15) that

$$U^{(0)}\left(f_{2\alpha}\right) = \frac{1}{5} U^{(0)} \circ U^{(1)}\left(f_{2\alpha-1}\right). \tag{6.18}$$

Because $f_{2\alpha-1} \in \mathcal{V}_n^{(1)}$, we know that by Theorem 6.3,

$$U^{(0)}\left(f_{2\alpha}\right) \in \mathcal{V}_{25n+6}^{(1)}. \tag{6.19}$$

Therefore, there exists some $f_{2\alpha+1} \in \mathcal{V}_{25n+6}^{(1)}$ such that

$$U^{(0)}\left(f_{2\alpha}\right) = f_{2\alpha+1}. \tag{6.20}$$

Inserting this into 6.17, we have

$$L_{2\alpha+1} = 5^{\alpha+1} \cdot f_{2\alpha+1}. \tag{6.21}$$

By induction, we have

$$\begin{aligned} \frac{L_{2\alpha-1}}{5^\alpha} &\in \mathcal{V}_{n_{2\alpha-1}}^{(1)}, \\ \frac{L_{2\alpha}}{5^{\alpha+1}} &\in \mathcal{V}_{n_{2\alpha}}^{(0)}. \end{aligned}$$

We need only verify that

$$\begin{aligned} n_{2\alpha-1} &= \psi(2\alpha-1), \\ n_{2\alpha} &= \psi(2\alpha) - 1. \end{aligned}$$

We note that for all $\alpha \geq 1$,

$$5^\alpha \equiv 1 \pmod 4.$$

We can then write

$$\psi(\alpha) = \left\lfloor \frac{5^{\alpha+1}}{4} \right\rfloor = \frac{5^{\alpha+1}}{4} - \frac{1}{4}.$$

With this in mind we can quickly verify that

$$\begin{aligned} 5\cdot\psi(\alpha) &= 5\cdot\left\lfloor \frac{5^{\alpha+1}}{4} \right\rfloor \\ &= 5\cdot\left(\frac{5^{\alpha+1}}{4} - \frac{1}{4}\right) \\ &= \frac{5^{\alpha+2}}{4} - \frac{1}{4} - 1 \\ &= \psi(\alpha+1) - 1. \end{aligned}$$

We then have

$$\begin{aligned} 5\cdot\psi(2\alpha-1) &= \psi(2\alpha)-1, \\ 5\cdot(\psi(2\alpha)-1)+6 &= 5\cdot\psi(2\alpha)-5+6 = \psi(2\alpha+1). \end{aligned}$$

Because applying $U^{(1)}$ to a rational polynomial with a denominator of $(1+5x)^n$ will result in a rational polynomial with denominator $(1+5x)^{5n}$, and similarly applying $U^{(0)}$ will result in a denominator $(1+5x)^{5n+6}$, we need only verify that the denominator of L_1 has power $\psi(1) = 6$. This is true from (1.8).

□

7. Initial Relations

We now need to finish the proofs of Theorem 5.3, Corollary 5.6, and the modular equation (4.25).

To finish the proof of Theorem 5.3, we need to verify the existence of $h_i(m,n,r)$ for $1 \le m \le 5$, $1 \le n \le 5$. This gives us 25 initial relations; for $i = 0, 1$, we have 50 such relations.

However, these relations are algebraically dependent on a much smaller number of relations. Notice that

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{5^m}\cdot U^{(i)}\left(\frac{(z-1)^m}{z^n}\right) \tag{7.1}$$

$$= \frac{1}{5^m}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\cdot U^{(i)}\left(z^{r-n}\right) \tag{7.2}$$

$$= \frac{1}{5^m}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\cdot U^{(i)}\left((1+5x)^{r-n}\right). \tag{7.3}$$

Therefore if we understand $U^{(i)}(z^n)$ for all n, then we can compute $U^{(i)}\left(x^m/(1+5x)^n\right)$. Of course,

$$U^{(i)}\left((1+5z)^n\right) = -\sum_{k=0}^{n} b_k(\tau)\cdot U^{(i)}\left((1+5x)^{k+n-5}\right). \tag{7.4}$$

If we consider $U^{(i)}\left((1+5z)^n\right)$ for positive n, then we have

$$U^{(i)}\left((1+5x)^n\right) = \sum_{k=0}^{n}\binom{n}{k}\cdot 5^k\cdot U^{(i)}\left(x^k\right). \tag{7.5}$$

So to completely construct $U^{(i)}\left(x^m/(1+5x)^n\right)$, we need to know how to construct $U^{(i)}(x^n)$ for all integers n. Using (4.25), we only need five initial relations, e.g., the relations for $0\leq n\leq 4$. Again accounting for two different values of i, this gives us a total of ten initial relations. We give those relations in the Appendix below.

From these relations, one can work algorithmically to the construction of $U^{(i)}\left(x^m/(1+5x)^n\right)$ for any $m,n\in\mathbb{Z}$. We provide such a construction in our Mathematica supplement, which can be found online at `https://www.risc.jku.at/people/nsmoot/d5congsuppA.nb`. We emphasize that our construction is by no means the most efficient. Rather, we have sacrificed efficiency for an easy and easily comprehensible construction process. Nevertheless, the 50 initial cases for Theorem 5.3 can still be constructed almost immediately.

Once these relations are constructed, we can quickly compute $h_i(m,n,r)$. Due to Theorem 5.5, we can verify Corollary 5.6 by checking the congruence properties of $h_i(m,n,r)$, with the appropriate m,r, for $1\leq n\leq 5$. This is also computed in our Mathematica supplement.

7.1. **Computing the Fundamental Relations.** We need to verify the relations (8.1)-(8.10) in the Appendix. Notice that these relations have the form

$$U^{(1)}\left(x^l\right)=p_{1,l}(x)\in\mathbb{Z}[x], \tag{7.6}$$

$$z^6\cdot U^{(0)}\left(x^l\right)=p_{0,l}(x)\in\mathbb{Z}[x], \tag{7.7}$$

for $0\leq l\leq 4$.

On the other hand, x,z have poles at the cusp $[0]_{10}$. If we divide both sides of each relation by the highest power of x appearing on the right-hand side, then we have

$$\frac{1}{x^m}\cdot U^{(1)}\left(x^l\right)\in\mathbb{Z}[x^{-1}]\subseteq\mathcal{M}^{\infty}(\Gamma_0(10)), \tag{7.8}$$

$$\frac{1}{x^m}\cdot z^6\cdot U^{(0)}\left(x^l\right)\in\mathbb{Z}[x^{-1}]\subseteq\mathcal{M}^{\infty}(\Gamma_0(10)). \tag{7.9}$$

This would more convenient to check, given that it is much easier to compute the principal part of a modular function with a pole only at $[\infty]_{10}$. Using Ligozat's theorem, we can compute the order of $\mathcal{A}(\tau)$, $x(\tau)$, $x(5\tau)$, $z(5\tau)$ as functions on the congruence subgroup $\Gamma_0(50)$.

Cusp Representative	$\mathcal{A}(\tau)$	$x(\tau)$	$x(5\tau)$	$z(5\tau)$
∞	6	1	5	0
1/25	27	0	0	0
1/10	0	1	0	1
1/5	0	0	-1	-1
3/10	0	1	0	1
2/5	0	0	-1	-1
1/2	-6	0	0	1
3/5	0	0	-1	-1
7/10	0	1	0	1
4/5	0	0	-1	-1
9/10	0	1	0	1
0	-27	-5	-1	-1

TABLE 1. Order at Cusps of $X_0(50)$

We can now pull the factors $1/x^m$ and z^6 inside the $U^{(i)}$ operators, so that we have

$$\frac{1}{x^m}\cdot U^{(1)}\left(x^l\right)=U_5\left(x(5\tau)^{-m}x(\tau)^l\right),$$

$$\frac{1}{x^m}\cdot z^6\cdot U^{(0)}\left(x^l\right)=U_5\left(\mathcal{A}(\tau)x(5\tau)^{-m}z(5\tau)^6x^l\right).$$

By Table 1, we can verify that

$$x(5\tau)^{-5}x(\tau)^l \in \mathcal{M}^\infty(\Gamma_0(50)),$$
$$\mathcal{A}(\tau)x(5\tau)^{-5}z(5\tau)^6x(\tau)^l \in \mathcal{M}^\infty\left(\Gamma_0(50)\right).$$

Lemma 7.1. *If $f \in \mathcal{M}^\infty(\Gamma_0(50))$, then $U_5\left(f\right) \in \mathcal{M}^\infty(\Gamma_0(10))$.*

Proof. We can express $U_5\left(f\right)$ as

$$5 \cdot U_5\left(f(\tau)\right) = \sum_{r=0}^{4} f\left(\frac{\tau+r}{5}\right).$$

As such, $U_5\left(f\right)$ will have a pole wherever $f\left(\frac{\tau+r}{5}\right)$ has a pole for $0 \leq r \leq 4$.

Let us suppose that τ approaches the rational point $h/k \in \mathbb{Q}$, with $\gcd(h,k) = 1$. Then

$$\frac{\tau+r}{5} \to \frac{h+kr}{5k}.$$

If f has a pole at $\frac{h+kr}{5k}$, then $\frac{h+kr}{5k} \in [\frac{1}{50}]_{50}$. Thus, by Lemma 2.7, there exist integers j, y such that

$$y \equiv h + kr + 5jk \pmod{50}, \tag{7.10}$$
$$50 \equiv 5ky \pmod{50}, \tag{7.11}$$
$$\gcd(y, 50) = 1. \tag{7.12}$$

By (7.11), (7.11), we must have $k \equiv 0 \pmod{10}$, and therefore $5k \equiv 0 \pmod{50}$. We therefore set $k = 10m$ for $m \in \mathbb{Z}$.

We want to show that $\frac{h+kr}{50m} \in [\frac{1}{10}]_{10}$. Again using Lemma 2.7, we need to demonstrate that there exist $y, j \in \mathbb{Z}$ such that

$$y \equiv h + kr + 50mj \pmod{10},$$
$$10 \equiv 50my \pmod{10},$$
$$\gcd(y, 10) = 1.$$

Simplifying, we have

$$y \equiv h + kr \pmod{10},$$
$$\gcd(y, 10) = 1.$$

Notice that $\gcd(h + kr, 10) = \gcd(h + 50m, 10) = \gcd(h, 10) = 1$, since $\gcd(h,k) = \gcd(h, 50m) = 1$. Therefore, we can simply set $y = h + kr$ and $j = 0$. Therefore, $\frac{h+kr}{50m} \in [\frac{1}{10}]_{10}$. In other words, if f has a pole only at the cusp $[\infty]_{50}$, then $U_5\left(f\right)$ has a pole only at $[\infty]_{10}$.

□

Therefore, we have

$$U_5\left(x(5\tau)^{-5}x(\tau)^l\right) \in \mathcal{M}^\infty(\Gamma_0(10)),$$
$$U_5\left(\mathcal{A}(\tau)x(5\tau)^{-5}z(5\tau)^6x(\tau)^l\right) \in \mathcal{M}^\infty(\Gamma_0(10)).$$

It is now only a matter of computing these principal parts and confirming that they match the right-hand sides of (8.1)-(8.10), adjusted by multiplying $z^{6\cdot(1-i)}/x^m$, where m is the degree of the right-hand side in x.

7.2. **Proof of the Modular Equation.** As a final application of the modular cusp analysis, we can confirm using Table 1 that

$$x(5\tau)^{-5} \cdot x(\tau) \in \mathcal{M}^{\infty}\left(\Gamma_0(50)\right).$$

Therefore, we can multiply $x(5\tau)^{-5}$ onto the left-hand side of (4.25) to show

$$x(5\tau)^{-5} \cdot \left(x^5 + \sum_{j=0}^{4} a_j(5\tau)x^j\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(50)\right). \tag{7.13}$$

As we show in our Mathematica supplement, it can be quickly computed that this modular function has no principal part, and constant term 0, confirming that it must be equal to 0. This verifies (4.25).

8. Appendix

$$U^{(1)}\left(1\right) = 1 \tag{8.1}$$

$$U^{(1)}\left(x\right) = 41x + 860x^2 + 6800x^3 + 24000x^4 + 32000x^5 \tag{8.2}$$

$$\begin{aligned} U^{(1)}\left(x^2\right) &= 86x + 10195x^2 + 366600x^3 + 6534800x^4 + 68384000x^5 + 450720000x^6 + 1907200000x^7 \\ &+ 5056000000x^8 + 7680000000x^9 + 5120000000x^{10} \end{aligned} \tag{8.3}$$

$$\begin{aligned} U^{(1)}\left(x^3\right) &= 51x + 27495x^2 + 2836265x^3 + 128688900x^4 + 3343692000x^5 + 56283680000x^6 \\ &+ 656205600000x^7 + 5502096000000x^8 + 33821312000000x^9 + 153192960000000x^{10} \\ &+ 506956800000000x^{11} + 1195008000000000x^{12} + 1904640000000000x^{13} + 1843200000000000x^{14} \\ &+ 819200000000000x^{15} \end{aligned} \tag{8.4}$$

$$\begin{aligned} U^{(1)}\left(x^4\right) &= 12x + 32674x^2 + 8579260x^3 + 831492275x^4 + 42958434000x^5 + 1396773180000x^6 \\ &+ 31314949600000x^7 + 511802288800000x^8 + 6319880448000000x^9 + 60349364480000000x^{10} \\ &+ 452174745600000000x^{11} + 2679038592000000000x^{12} + 12574269440000000000x^{13} \\ &+ 46561935360000000000x^{14} + 134544588800000000000x^{15} + 297365504000000000000x^{16} \\ &+ 485949440000000000000x^{17} + 553779200000000000000x^{18} + 393216000000000000000x^{19} \\ &+ 131072000000000000000x^{20} \end{aligned} \tag{8.5}$$

$$\begin{aligned} U^{(0)}\left(1\right) &= \frac{1}{(1+5x)^6}(5705x^2 + 6840120x^3 + 2034152125x^4 + 280484938650x^5 + 22921365211325x^6 \\ &+ 1260917405154520x^7 + 50400843190048480x^8 + 1539115922208139200x^9 \\ &+ 37183654303328448000x^{10} + 728924483359472640000x^{11} + 11816089262411136000000x^{12} \\ &+ 160681440628058880000000x^{13} + 1853291134193264640000000x^{14} \\ &+ 18284160727362809856000000x^{15} + 155286793010086625280000000x^{16} \\ &+ 1140657222505472000000000000x^{17} + 7269894420215070720000000000x^{18} \\ &+ 40277647277404979200000000000x^{19} + 194099187864646451200000000000x^{20} \\ &+ 813054581193729638400000000000x^{21} + 2954545150241538048000000000000x^{22} \\ &+ 9282005730758492160000000000000x^{23} + 25080951875200614400000000000000x^{24} \\ &+ 57872525958316032000000000000000x^{25} + 112916020309524480000000000000000x^{26} \\ &+ 183812885074411520000000000000000x^{27} + 245082228994867200000000000000000x^{28} \\ &+ 260725452832768000000000000000000x^{29} + 212837104353280000000000000000000x^{30} \\ &+ 125198296678400000000000000000000x^{31} + 47244640256000000000000000000000x^{32} \\ &+ 8589934592000000000000000000000x^{33}). \end{aligned} \tag{8.6}$$

$$U^{(0)}(x) = \frac{1}{(1+5x)^6}\big(1596x^2 + 5311629x^3 + 3020673965x^4 + 693946917880x^5 + 88012336687140x^6 \\
+ 7203973630079449x^7 + 417078090095103516x^8 + 18123681491802321200x^9 \\
+ 615843754566814808000x^{10} + 16857245810889643680000x^{11} + 380061235251469769600000x^{12} \\
+ 7179360858330987609600000x^{13} + 115153019537919900211200000x^{14} \\
+ 1584889574408762616832000000x^{15} + 18875287036126384148480000000x^{16} \\
+ 195815273618900539392000000000x^{17} + 1778815480050553692160000000000x^{18} \\
+ 14206927272980568637440000000000x^{19} + 100058873107538207703040000000000x^{20} \\
+ 622718357721614503116800000000000x^{21} + 3428656886761288105984000000000000x^{22} \\
+ 16707165479275661885440000000000000x^{23} + 72013094304097396326400000000000000x^{24} \\
+ 274190033219424878592000000000000000x^{25} + 920048921836076924928000000000000000x^{26} \\
+ 2711506126769477386240000000000000000x^{27} + 6985969318446812364800000000000000000x^{28} \\
+ 15637700398221885440000000000000000000x^{29} \\
+ 30166681246666588160000000000000000000x^{30} \\
+ 49621782059863244800000000000000000000x^{31} \\
+ 68625507039156633600000000000000000000x^{32} \\
+ 78284718944026624000000000000000000000x^{33} \\
+ 71718179952394240000000000000000000000x^{34} \\
+ 50720986785382400000000000000000000000x^{35} \\
+ 25993142075392000000000000000000000000x^{36} \\
+ 8589934592000000000000000000000000000x^{37} \\
+ 1374389534720000000000000000000000000x^{38}\big) \tag{8.7}$$

$$
\begin{aligned}
U^{(0)}\left(x^2\right) = \frac{1}{(1+5x)^6}\big(&268x^2 + 2847432x^3 + 3155658820x^4 + 1202043333790x^5 + 233251365647870x^6 \\
&+ 27857600543181592x^7 + 2282412359335489853x^8 + 137474599581860685500x^9 \\
&+ 6382595114107468178000x^{10} + 236333942045815117200000x^{11} \\
&+ 7159344666536530274720000x^{12} + 180943148092126406540160000x^{13} \\
&+ 3874442157232846507737600000x^{14} + 71154385951639684561408000000x^{15} \\
&+ 1131931871787660010234880000000x^{16} + 15724017231814160548864000000000x^{17} \\
&+ 191993970181152296671232000000000x^{18} + 2071679521214318766489600000000000x^{19} \\
&+ 19840725410941052260024320000000000x^{20} + 169240207863514163393331200000000000x^{21} \\
&+ 1289257956753574377619456000000000000x^{22} + 8789154839425847410032640000000000000x^{23} \\
&+ 53694599262146012197683200000000000000x^{24} \\
&+ 294194062113759948701696000000000000000x^{25} \\
&+ 1445922789195393074724864000000000000000x^{26} \\
&+ 6372283630042655710248960000000000000000x^{27} \\
&+ 25156363490877335666688000000000000000000x^{28} \\
&+ 88814063288837761662976000000000000000000x^{29} \\
&+ 279743502586000693002240000000000000000000x^{30} \\
&+ 783588083068217747046400000000000000000000x^{31} \\
&+ 1943797721585087728844800000000000000000000x^{32} \\
&+ 4247467234308952948736000000000000000000000x^{33} \\
&+ 8120384728408910725120000000000000000000000x^{34} \\
&+ 13465927624462381875200000000000000000000000x^{35} \\
&+ 19155038490681409536000000000000000000000000x^{36} \\
&+ 23036102962346721280000000000000000000000000x^{37} \\
&+ 22969782997939650560000000000000000000000000x^{38} \\
&+ 18482412505792512000000000000000000000000000x^{39} \\
&+ 11532502585835520000000000000000000000000000x^{40} \\
&+ 5236424127283200000000000000000000000000000x^{41} \\
&+ 1539316278886400000000000000000000000000000x^{42} \\
&+ 219902325555200000000000000000000000000000x^{43}\big)
\end{aligned}
\tag{8.8}
$$

$$
\begin{aligned}
U^{(0)}\left(x^3\right) = \frac{1}{(1+5x)^6}\big(&25x^2 + 1083750x^3 + 2419268600x^4 + 1533044850875x^5 + 451892277223875x^6 \\
&+ 77776397020017600x^7 + 8876969029993551625x^8 + 727751880723215938525x^9 \\
&+ 45237746915792486076500x^{10} + 2216202089061921720156000x^{11} \\
&+ 88063763926314004467152000x^{12} + 2901622367042821273526240000x^{13} \\
&+ 80660306943461135362236800000x^{14} + 1918093917299266390588800000000x^{15} \\
&+ 39459730054782721266716160000000x^{16} + 708795736244147980459443200000000x^{17} \\
&+ 11201811678733852133717299200000000x^{18} + 156752971920695775627182080000000000x^{19} \\
&+ 1952565980789217283863347200000000000x^{20} \\
&+ 21745861234519182941633740800000000000x^{21} \\
&+ 217330181296111203156344832000000000000x^{22} \\
&+ 1954986367634769250929868800000000000000x^{23} \\
&+ 15867347909658771160720998400000000000000x^{24} \\
&+ 116421209420849214282399744000000000000000x^{25} \\
&+ 773301464019936672352829440000000000000000x^{26} \\
&+ 4654578055099292525988413440000000000000000x^{27} \\
&+ 25401759242372849348693196800000000000000000x^{28} \\
&+ 125704489323417659460550656000000000000000000x^{29} \\
&+ 563904329114669957135728640000000000000000000x^{30} \\
&+ 2291384934896565019580825600000000000000000000x^{31} \\
&+ 8423518909348688068345856000000000000000000000x^{32} \\
&+ 27966210361515509645574144000000000000000000000x^{33} \\
&+ 83658769333926385994956800000000000000000000000x^{34} \\
&+ 224821089074620615622656000000000000000000000000x^{35} \\
&+ 540741258958646666067968000000000000000000000000x^{36} \\
&+ 1158650329533680577413120000000000000000000000000x^{37} \\
&+ 2198994699321618726912000000000000000000000000000x^{38} \\
&+ 3670226839781351882752000000000000000000000000000x^{39} \\
&+ 5338951094289689477120000000000000000000000000000x^{40} \\
&+ 6691954927703970283520000000000000000000000000000x^{41} \\
&+ 7121217229055422627840000000000000000000000000000x^{42} \\
&+ 6307992271770668236800000000000000000000000000000x^{43} \\
&+ 4525788871530643456000000000000000000000000000000x^{44} \\
&+ 2526853642489692160000000000000000000000000000000x^{45} \\
&+ 1030022492900556800000000000000000000000000000000x^{46} \\
&+ 272678883688448000000000000000000000000000000000x^{47} \\
&+ 35184372088832000000000000000000000000000000000x^{48}\big)
\end{aligned} \tag{8.9}
$$

$$
\begin{aligned}
U^{(0)}\left(x^4\right) = \frac{1}{(1+5x)^6}\big(&x^2 + 296324x^3 + 1400331440x^4 + 1491537289180x^5 + 666654357758190x^6 \\
&+ 164127484896644144x^7 + 25818034832153226421x^8 + 2843942677492333687050x^9 \\
&+ 233247182152327524438975x^{10} + 14876273050366789151536400x^{11} \\
&+ 761935276117868629545420000x^{12} + 32118457754484194871176800000x^{13} \\
&+ 1135956912571962510189130400000x^{14} + 34231269539693917027064800000000x^{15} \\
&+ 889959895359737292913435520000000x^{16} + 20168696740441852125087521280000000x^{17} \\
&+ 401853770227841145701015936000000000x^{18} + 7090293594621615513260024832000000000x^{19} \\
&+ 111455620826575249212238069760000000000x^{20} \\
&+ 1568967903706184088975184281600000000000x^{21} \\
&+ 19865297696614393785791188992000000000000x^{22} \\
&+ 227065713010913459721992798208000000000000x^{23} \\
&+ 2350395554799549839893043609600000000000000x^{24} \\
&+ 22090326137698781193612296192000000000000000x^{25} \\
&+ 188920535524484098164060192768000000000000000x^{26} \\
&+ 1472786798447713704939919769600000000000000000x^{27} \\
&+ 10480689728875682601093981798400000000000000000x^{28} \\
&+ 68152666405070998605987315712000000000000000000x^{29} \\
&+ 405254490887530431856613785600000000000000000000x^{30} \\
&+ 2204414239119153895179708006400000000000000000000x^{31} \\
&+ 10970036975349264582008478105600000000000000000000x^{32} \\
&+ 49929921460604339849107865600000000000000000000000x^{33} \\
&+ 207727909385809764269938442240000000000000000000000x^{34} \\
&+ 789224428145826945154469068800000000000000000000000x^{35} \\
&+ 2734666462414638550878257152000000000000000000000000x^{36} \\
&+ 8626866866620198090933534720000000000000000000000000x^{37} \\
&+ 24722506007012126262789406720000000000000000000000000x^{38} \\
&+ 64185847480498768484237312000000000000000000000000000x^{39} \\
&+ 150467914734947763855818752000000000000000000000000000x^{40} \\
&+ 317205685483135843705028608000000000000000000000000000x^{41} \\
&+ 598387657370971073308262400000000000000000000000000000x^{42} \\
&+ 1004032323013383412514816000000000000000000000000000000x^{43} \\
&+ 1487355820084245534081024000000000000000000000000000000x^{44} \\
&+ 1927442844590278705152000000000000000000000000000000000x^{45} \\
&+ 2159710523624808618393600000000000000000000000000000000x^{46} \\
&+ 2061307549285828316364800000000000000000000000000000000x^{47} \\
&+ 1642771482986634280960000000000000000000000000000000000x^{48} \\
&+ 1063549153298423480320000000000000000000000000000000000x^{49}
\end{aligned} \tag{8.10}
$$

$$+ 53732165679180677120000000000000000000000000000000000x^{50}$$
$$+ 198721333557723136000000000000000000000000000000000000x^{51}$$
$$+ 47850746040811520000000000000000000000000000000000000x^{52}$$
$$+ 5629499534213120000000000000000000000000000000000000x^{53})$$

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