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Refined telescoping algorithms in $R\Pi\Sigma$-extensions to reduce the degrees of the denominators

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FWF Der Wissenschaftsfonds.

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\operatorname{den}(f')$ is minimal

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Example:

$$f = \frac{2x^5 + 6x^4 + 8x^3 + 5x^2 + 6x + 4}{x(x+2)\left(x^2+1\right)\left(x^2+2x+2\right)}$$

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$$\frac{p(x+r)}{q(x+r)} = \frac{p(x)}{q(x)} + \gamma(x+1) - \gamma(x)$$

with

$$\gamma(x) = \sum_{i=0}^{r-1} \frac{p(x+i)}{q(x+i)}$$

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Example:

$$f = \overbrace{\frac{2}{x}}^{=f'} + g(x+1) - g(x) \quad \to \quad \sum_{k=1}^{n} \frac{2k^5+6k^4+8k^3+5k^2+6k+4}{k(k+2)(k^2+1)(k^2+2k+2)}$$

$$= -\frac{n(7+12n+8n^2+2n^3)}{(1+n)(2+n)(2+2n+n^2)} + 2\sum_{k=1}^{n} \frac{1}{k}$$

with $g(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x^2+1}$

Symbolic summation in a difference ring

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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Sk!=(k+1)k! $\quad \leftrightarrow \quad \sigma(y_1) = (x+1)y_1$

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		$\vdots$	
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$$\mathcal{S}(-1)^k = -(-1)^k \quad \leftrightarrow \quad \sigma(\mathbf{z}) = -\mathbf{z} \qquad \mathbf{z}^2 = \mathbf{1} \quad \leftrightarrow \quad ((-\mathbf{1})^{\mathbf{k}})^{\mathbf{2}} = \mathbf{1}$$

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$$\mathcal{S} \sum_{i=1}^{k} \frac{1}{i} = \sum_{i=1}^{k} \frac{1}{i} + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{x+1}$$

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(nested) sum $\leftrightarrow$

$$\sigma(s_1) = s_1 + f_1 \qquad f_1 \in \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][z]$$

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Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,Karr81,CS16,CS17,CS18)

$$\mathbb{A} := \mathbb{K}(x)[y_1, y_1^{-1}][y_2, y_2^{-1}]\dots[y_e, y_e^{-1}][z][s_1][s_2][s_3]\dots$$

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such that $\text{const}_\sigma\mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

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α is a primitive λth root of unity

GIVEN $f \in \mathbb{A}$;
FIND, in case of existence, a $g \in \mathbb{A}$ such that

$$\sigma(g) - g = f.$$

(nested) su... $y_e^{-1}][z]$

$$\sigma(s_2) = s_2 + f_2 \qquad f_2 \in \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][z][s_1]$$
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Refined telescoping:

Given $f \in \mathbb{A} = \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_u]$;
find $g, f' \in \mathbb{A}$ such that

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Suppose that $\mathbb{A} = \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_u]$ is "nice"

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Given $f \in \mathbb{A} = \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_u]$;
find, if possible, $g, f' \in \mathbb{A}$ such that

$$\sigma(g) - g + f' = f$$

and $\mathrm{den}_x(f')$ is "nice"

Fix $d \in \mathbb{N}$.

Suppose that $\mathbb{A} = \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_u]$ has x-degree$\leq d$:

- all multiplicands
$$a_i = \frac{\sigma(y_i)}{y_i} \in \mathbb{K}(x)^*$$
are built by irreducible factors of degree $\leq d$.
- all denominators of the summands
$$f_i = \sigma(s_i) - s_i \in \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_{i-1}]$$
have only irreducible factors w.r.t. x of degree $\leq d$.

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have only irreducible factors w.r.t. x of degree $\leq d$.

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Given $f \in \mathbb{A} = \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_u]$;
find, if possible, $g, f' \in \mathbb{A}$ such that

$$\sigma(g) - g + f' = f$$

and the irreducible factors w.r.t. x in $\operatorname{den}_x(f')$ have only degrees $\leq d$

Example:

$$\sum_{k=1}^{n}\left(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)}{10(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i}+\frac{(1-4k-2k^2)}{5(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i^3}\right)$$

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Take the $R\Pi\Sigma$-ring $\mathbb{A}=\mathbb{Q}(x)[s_1][s_2]$ with

$$\sigma(x)=x+1,\quad \sigma(s_1)=s_1+\frac{1}{x+1},\quad \sigma(s_2)=s_2+\frac{1}{(x+1)^3}$$

Example:

$$\sum_{k=1}^{n}\Big(\tfrac{-2+k}{10(1+k^2)}+\tfrac{(1-4k-2k^2)}{10(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i}+\tfrac{(1-4k-2k^2)}{5(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i^3}\Big)$$

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Note: $\mathbb{A}$ has x-degree $\leq 1 =: d$

Example:

$$\sum_{k=1}^{n}\Big(\tfrac{-2+k}{10(1+k^2)}+\tfrac{(1-4k-2k^2)}{10(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i}+\tfrac{(1-4k-2k^2)}{5(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i^3}\Big)$$

Take the $R\Pi\Sigma$-ring $\mathbb{A}=\mathbb{Q}(x)[s_1][s_2]$ with

$$\sigma(x)=x+1,\quad \sigma(s_1)=s_1+\frac{1}{x+1},\quad \sigma(s_2)=s_2+\frac{1}{(x+1)^3}$$

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Fix $d \in \mathbb{N}$.

Suppose that $\mathbb{A} = \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_u]$ has x-degree$\leq d$:

- all multiplicands
$$a_i = \frac{\sigma(y_i)}{y_i} \in \mathbb{K}(x)^*$$
are built by irreducible factors of degree $\leq d$.
- all denominators of the summands
$$f_i = \sigma(s_i) - s_i \in \mathbb{K}(x)[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}][s_1] \dots [s_{i-1}]$$
have only irreducible factors w.r.t. x of degree $\leq d$.

Crucial property (for the proof): For all $f \in \mathbb{A}$ and $k \in \mathbb{Z}$,

$\mathrm{den}_x(f)$ has only factors with degree $\leq d$

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Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{1}{x} + \frac{1}{x} - \frac{1}{x^2+1} + \frac{1}{(x+1)^2+1}$$
$$+ g(x+1) - g(x)$$

with $g(x) = \frac{1}{x} + \frac{1}{x+1}$

$$\frac{p(x+r)}{q(x+r)} = \frac{p(x)}{q(x)} + \gamma(x+1) - \gamma(x)$$

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Theorem. Let $\mathbb{A}$, f, f', g as above. Then:

$$\boxed{\exists h \in \mathbb{A} : \sigma(h) - h = f}$$

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1. $p_1 = p_2 = \cdots = p_r = 0$
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Not all $p_i = 0$: no solution in any extension with x-degree$\leq d$. **Otherwise:**

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Case 1: We find such a g'. Then we get

$$h = g + g' \in \mathbb{A}$$

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Case 2: There is no g'. Then take $\mathbb{E} = \mathbb{A}[s]$ with $\sigma(s) = s + \frac{p}{q}$ and we get

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In this case, $\text{const}_\sigma \mathbb{E} = \mathbb{K}$, i.e., we stay in an $R\Pi\Sigma$-ring (by our Σ-theory) which has x-degree$\leq d$

Back to our example

$$\sum_{k=1}^{n}\Big(\tfrac{-2+k}{10(1+k^2)}+\tfrac{(1-4k-2k^2)}{10(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i}+\tfrac{(1-4k-2k^2)}{5(1+k^2)(2+2k+k^2)}\sum_{i=1}^{k}\frac{1}{i^3}\Big)$$

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$$=\frac{(3+2n)}{10\big(2+2n+n^2\big)}\sum_{i=1}^{n}\frac{1}{i}+\frac{(3+2n)}{5\big(2+2n+n^2\big)}\sum_{i=1}^{n}\frac{1}{i^3}+\sum_{k=1}^{n}\frac{k^2-4k-2}{10k^3}$$

Take the $R\Pi\Sigma$-ring $\mathbb{E}=\mathbb{Q}(x)[s_1][s_2][s_3]$ with

$\sigma(x)=x+1,\ \sigma(s_1)=s_1+\frac{1}{x+1},\ \sigma(s_2)=s_2+\frac{1}{(x+1)^3},\ \sigma(s_3)=s_3+\frac{-2-4x+x^2}{10x^3},$

$$f=\frac{-2+x}{10(1+x^2)}+\frac{(1-4x-2x^2)}{10(1+x^2)(2+2x+x^2)}s_1+\frac{(1-4x-2x^2)}{5(1+x^2)(2+2x+x^2)}s_2$$

$$\downarrow$$

$$h=g+s_3=\frac{(1+2x)}{10(1+x^2)}s_1+\frac{h_3(1+2x)}{5(1+x^2)}s_2-\frac{(1+2x)(2+x^2)}{10x^3(1+x^2)}+s_3\in\mathbb{E}$$

with

$$\sigma(h)-h=f$$

Note: $\mathbb{E}$ is an $R\Pi\Sigma$-ring with x-degree≤ 1

Solving of large recurrences

In[1]:=
```
<< Sigma.m
```

Sigma – A summation package by Carsten Schneider – © RISC – V 2.895 (September 20, 2022) click for Help

In[2]:=
```
rec = -(2 419 791 630 618 001 724 349 719 825 822 515 200 n^3 +
    46 893 058 834 334 803 550 048 105 380 734 566 400 n^4 + 442 193 057 465 681 392 730 605 627 834 886 258 688 n^5 +
    2 705 163 824 287 918 413 734 939 189 834 632 593 408 n^6 + 12 077 169 282 974 589 920 658 629 516 473 115 295 744 n^7 +
    41 966 837 290 479 779 427 524 408 810 546 219 827 200 n^8 + 118 219 101 947 163 710 354 472 195 018 903 784 452 096 n^9 +
    277 636 548 018 756 319 924 282 582 437 965 801 250 816 n^10 + 554 818 694 316 590 989 606 680 673 486 591 181 169 664 n^11 +
    958 215 852 325 427 069 871 263 145 576 988 323 096 576 n^12 + 1 447 807 538 463 763 344 480 353 057 138 198 170 764 544 n^13 +
    1 932 634 574 589 957 327 485 740 054 131 528 807 341 312 n^14 + 2 297 546 075 527 773 923 214 701 010 279 556 037 859 200 n^15 +
    2 448 775 269 439 910 075 996 659 926 897 517 474 168 064 n^16 + 2 353 082 458 409 861 559 967 790 811 417 795 090 328 800 n^17 +
    2 048 299 114 135 230 176 735 374 555 025 680 366 259 616 n^18 + 1 621 741 955 236 083 712 069 918 941 593 026 341 937 328 n^19 +
    1 171 973 077 100 571 998 520 049 114 011 408 940 864 324 n^20 + 775 366 789 376 058 136 594 755 591 722 578 221 832 754 n^21 +
    470 845 228 207 878 464 545 078 102 494 550 968 145 626 n^22 + 263 029 261 282 480 708 488 833 600 406 560 031 424 266 n^23 +
    135 432 625 185 170 494 152 646 639 679 878 430 576 366 n^24 + 64 380 525 008 058 346 133 125 819 015 855 307 039 890 n^25 +
    28 294 941 869 896 718 238 659 823 138 529 447 643 409 n^26 + 11 510 639 826 509 868 241 332 665 511 273 019 883 245 n^27 +
    4 338 601 127 644 953 578 265 748 144 467 101 651 566 n^28 + 1 516 347 765 344 935 407 227 461 978 628 652 931 140 n^29 +
    491 707 203 871 221 783 491 014 552 667 264 654 348 n^30 + 147 998 315 309 037 544 218 073 839 785 461 595 524 n^31 +
    41 357 982 388 489 608 344 142 528 638 546 419 812 n^32 + 10 731 279 186 864 834 895 938 990 612 954 501 362 n^33 +
    2 585 221 832 059 644 562 908 138 564 950 975 983 n^34 + 578 081 901 021 769 208 062 617 955 849 607 255 n^35 +
    119 933 836 608 195 636 314 895 593 780 004 412 n^36 + 23 072 349 655 406 047 775 407 594 541 682 536 n^37 +
    4 112 383 801 613 288 221 238 341 717 431 870 n^38 + 678 441 475 758 950 513 418 671 792 499 790 n^39 +
    103 470 631 338 396 383 215 642 368 754 950 n^40 + 14 567 035 475 277 568 652 763 736 374 038 n^41 +
    1 889 820 701 704 575 278 082 350 304 095 n^42 + 225 464 981 620 744 825 055 854 151 899 n^43 +
    24 677 977 218 000 736 361 042 595 318 n^44 + 2 471 158 501 904 322 420 447 817 484 n^45 +
    225 650 752 266 000 765 364 410 056 n^46 + 18 718 039 227 261 648 047 794 832 n^47 + 1 404 175 763 444 631 451 277 224 n^48 +
    94 757 256 722 843 508 771 238 n^49 + 5 715 830 429 727 778 230 433 n^50 + 305 843 846 362 020 680 945 n^51 +
```

14 381 720 253 879 875 868 n^{52} + 587 432 077 055 797 654 n^{53} + 20 535 798 664 117 844 n^{54} + 602 646 805 448 372 n^{55} +
14 460 796 607 552 n^{56} + 273 265 680 480 n^{57} + 3 838 554 144 n^{58} + 36 289 696 n^{59} + 187 328 n^{60} + 256 n^{61}) f[n] +
(2 184 637 011 646 089 260 832 263 446 242 263 040 + 60 213 944 484 149 858 917 493 971 997 278 863 360 n +
765 667 472 926 191 423 796 966 419 188 114 522 112 n^{2} + 6 105 365 080 510 451 396 982 635 521 026 428 239 872 n^{3} +
34 708 860 405 036 002 296 270 093 362 827 805 720 576 n^{4} + 151 012 015 780 667 833 836 461 483 157 418 876 403 712 n^{5} +
525 927 125 380 365 588 824 032 094 032 506 978 336 768 n^{6} + 1 512 196 534 433 079 466 364 883 142 044 247 431 979 008 n^{7} +
3 671 470 718 711 877 962 933 422 750 534 015 631 462 400 n^{8} + 7 656 732 454 200 652 158 809 870 797 561 463 793 930 240 n^{9} +
13 900 418 857 159 591 484 188 788 012 697 295 475 566 592 n^{10} + 22 205 000 857 672 892 951 155 779 495 290 643 697 085 440 n^{11} +
31 486 034 126 294 839 958 942 742 570 718 414 414 535 680 n^{12} + 39 919 692 863 067 272 123 801 901 519 465 590 396 631 552 n^{13} +
45 531 432 069 748 835 165 613 722 710 992 028 682 930 304 n^{14} + 46 961 436 588 813 572 025 777 511 789 038 901 263 777 888 n^{15} +
43 994 959 907 297 109 868 687 362 235 336 122 063 739 280 n^{16} + 37 579 715 685 313 851 159 102 021 797 411 799 861 122 656 n^{17} +
29 364 773 248 686 683 915 239 812 679 461 616 136 356 200 n^{18} + 21 050 691 034 785 332 915 492 476 057 834 318 260 647 992 n^{19} +
13 878 863 958 177 278 906 363 320 901 764 654 180 674 068 n^{20} + 8 433 920 328 583 831 248 209 974 031 887 597 144 856 166 n^{21} +
4 732 703 540 813 596 136 660 683 547 266 937 537 829 654 n^{22} + 2 456 402 281 573 059 079 053 298 938 442 251 846 697 230 n^{23} +
1 180 884 360 800 419 306 100 550 729 721 645 208 294 214 n^{24} + 526 441 908 393 454 455 371 612 300 421 050 477 868 182 n^{25} +
217 853 112 783 218 022 232 146 626 128 806 211 983 463 n^{26} + 83 754 020 847 842 024 722 911 201 861 265 143 892 515 n^{27} +
29 933 835 570 003 312 829 692 764 177 995 129 463 678 n^{28} + 9 950 649 553 145 001 634 744 062 228 609 687 611 636 n^{29} +
3 077 685 237 402 338 181 588 673 390 623 771 087 768 n^{30} + 885 864 683 981 666 005 570 651 087 823 908 853 468 n^{31} +
237 303 300 491 689 480 050 502 026 965 768 922 040 n^{32} + 59 155 090 124 648 479 353 044 037 293 149 067 126 n^{33} +
13 719 117 466 936 551 766 792 191 652 359 249 405 n^{34} + 2 958 917 780 333 050 813 266 891 451 512 650 209 n^{35} +
593 157 044 864 289 496 291 653 275 756 884 512 n^{36} + 110 438 535 500 745 724 875 815 791 275 373 824 n^{37} +
19 080 579 219 381 871 689 558 731 310 265 194 n^{38} + 3 055 650 762 530 616 827 066 946 087 618 154 n^{39} +
452 987 258 893 586 741 202 300 418 715 142 n^{40} + 62 067 608 935 535 379 147 520 211 092 898 n^{41} +
7 846 086 266 834 729 051 216 715 205 681 n^{42} + 913 130 773 095 847 812 486 509 019 205 n^{43} +
97 597 613 016 813 524 555 747 415 182 n^{44} + 9 552 877 857 235 409 775 526 229 148 n^{45} +
853 450 995 298 842 653 501 332 876 n^{46} + 69 325 405 419 294 490 888 090 944 n^{47} + 5 096 893 257 290 771 080 869 644 n^{48} +
337 358 454 813 758 704 464 466 n^{49} + 19 974 662 258 188 401 941 987 n^{50} + 1 049 859 496 149 284 305 279 n^{51} +
48 525 482 667 944 885 632 n^{52} + 1 949 521 858 753 528 714 n^{53} + 67 075 940 936 545 308 n^{54} + 1 938 510 016 023 020 n^{55} +
45 836 200 086 880 n^{56} + 854 037 611 552 n^{57} + 11 836 382 688 n^{58} + 110 495 776 n^{59} + 564 032 n^{60} + 768 n^{61}) f[1 + n] –
(– 22 022 314 605 592 690 531 751 005 935 108 096 000 – 289 135 277 001 687 902 353 264 231 969 613 414 400 n –
1 488 037 313 005 848 582 765 208 942 457 703 628 800 n^{2} – 1 818 726 834 376 404 926 186 570 730 184 369 766 400 n^{3} +

$23\,382\,204\,546\,977\,767\,384\,985\,015\,673\,369\,488\,916\,480\,n^4 + 188\,082\,685\,608\,246\,892\,321\,802\,873\,635\,659\,960\,877\,056\,n^5 +$
$825\,801\,477\,256\,291\,595\,460\,486\,042\,061\,384\,946\,712\,576\,n^6 + 2\,628\,541\,026\,330\,989\,080\,112\,440\,997\,321\,895\,434\,330\,112\,n^7 +$
$6\,629\,655\,427\,580\,241\,497\,225\,184\,251\,016\,863\,653\,908\,480\,n^8 + 13\,848\,216\,502\,549\,635\,881\,680\,294\,337\,223\,876\,185\,911\,296\,n^9 +$
$24\,611\,563\,389\,512\,504\,073\,584\,247\,446\,699\,535\,922\,575\,360\,n^{10} + 37\,909\,511\,298\,937\,488\,708\,854\,223\,877\,932\,503\,102\,176\,256\,n^{11} +$
$51\,299\,918\,554\,927\,297\,426\,860\,529\,427\,157\,650\,519\,514\,112\,n^{12} + 61\,629\,468\,272\,462\,855\,574\,906\,800\,764\,536\,339\,189\,128\,192\,n^{13} +$
$66\,280\,124\,475\,672\,744\,973\,400\,889\,288\,785\,644\,773\,739\,136\,n^{14} + 64\,247\,409\,978\,580\,992\,116\,427\,545\,385\,119\,399\,679\,833\,312\,n^{15} +$
$56\,449\,472\,002\,389\,683\,597\,707\,337\,063\,406\,362\,546\,728\,272\,n^{16} + 45\,171\,322\,891\,361\,752\,871\,580\,719\,683\,909\,268\,264\,681\,440\,n^{17} +$
$33\,053\,824\,818\,470\,328\,765\,890\,536\,318\,636\,510\,140\,913\,496\,n^{18} + 22\,194\,286\,968\,981\,437\,345\,392\,564\,520\,988\,385\,907\,626\,416\,n^{19} +$
$13\,715\,606\,341\,403\,352\,934\,357\,761\,159\,783\,608\,231\,286\,212\,n^{20} + 7\,820\,919\,263\,011\,960\,290\,532\,633\,923\,574\,013\,757\,937\,322\,n^{21} +$
$4\,124\,101\,824\,382\,320\,133\,598\,766\,513\,489\,500\,768\,514\,642\,n^{22} + 2\,014\,900\,658\,274\,015\,221\,853\,754\,922\,074\,596\,765\,809\,754\,n^{23} +$
$913\,552\,367\,745\,160\,482\,379\,609\,090\,421\,491\,101\,869\,914\,n^{24} + 384\,912\,898\,173\,545\,720\,900\,056\,698\,586\,842\,503\,122\,918\,n^{25} +$
$150\,881\,012\,199\,256\,958\,147\,419\,290\,021\,749\,103\,219\,985\,n^{26} + 55\,074\,624\,810\,472\,925\,390\,185\,481\,773\,716\,375\,577\,417\,n^{27} +$
$18\,733\,933\,409\,151\,309\,041\,626\,750\,964\,983\,474\,454\,358\,n^{28} + 5\,941\,570\,334\,258\,463\,328\,802\,562\,879\,066\,834\,157\,404\,n^{29} +$
$1\,757\,621\,107\,214\,989\,234\,449\,316\,717\,174\,814\,410\,560\,n^{30} + 485\,046\,390\,298\,804\,579\,275\,932\,040\,902\,973\,332\,692\,n^{31} +$
$124\,877\,689\,424\,959\,601\,610\,199\,276\,249\,591\,868\,356\,n^{32} + 29\,989\,412\,247\,749\,884\,342\,281\,895\,499\,180\,335\,754\,n^{33} +$
$6\,715\,821\,939\,958\,386\,382\,569\,103\,248\,806\,082\,447\,n^{34} + 1\,401\,758\,065\,674\,213\,210\,274\,260\,438\,674\,133\,863\,n^{35} +$
$272\,527\,587\,406\,079\,568\,538\,323\,090\,035\,115\,684\,n^{36} + 49\,312\,358\,430\,897\,053\,286\,589\,310\,725\,757\,704\,n^{37} +$
$8\,296\,078\,956\,189\,808\,509\,997\,550\,437\,883\,390\,n^{38} + 1\,296\,103\,778\,071\,525\,102\,182\,189\,470\,635\,774\,n^{39} +$
$187\,776\,581\,998\,229\,163\,368\,113\,485\,479\,490\,n^{40} + 25\,186\,081\,309\,681\,650\,194\,834\,084\,101\,162\,n^{41} +$
$3\,121\,520\,966\,967\,059\,012\,579\,790\,235\,663\,n^{42} + 356\,696\,551\,136\,881\,314\,117\,558\,024\,007\,n^{43} +$
$37\,484\,536\,898\,709\,537\,869\,638\,087\,550\,n^{44} + 3\,612\,012\,600\,567\,270\,037\,248\,636\,788\,n^{45} +$
$318\,063\,209\,276\,420\,217\,186\,951\,276\,n^{46} + 25\,493\,505\,002\,286\,210\,193\,173\,232\,n^{47} + 1\,851\,374\,597\,173\,806\,436\,884\,944\,n^{48} +$
$121\,157\,534\,952\,438\,954\,351\,366\,n^{49} + 7\,099\,015\,219\,827\,843\,837\,177\,n^{50} + 369\,549\,841\,897\,016\,581\,817\,n^{51} +$
$16\,930\,667\,101\,318\,371\,492\,n^{52} + 674\,702\,033\,398\,352\,670\,n^{53} + 23\,042\,430\,467\,695\,396\,n^{54} + 661\,436\,747\,087\,492\,n^{55} +$
$15\,543\,736\,473\,056\,n^{56} + 288\,011\,538\,912\,n^{57} + 3\,971\,907\,936\,n^{58} + 36\,920\,096\,n^{59} + 187\,840\,n^{60} + 256\,n^{61})\ f[2 + n] -$
$(27\,276\,878\,942\,915\,969\,107\,316\,087\,331\,005\,399\,040 + 705\,128\,506\,368\,571\,028\,133\,010\,447\,867\,324\,661\,760\,n +$
$8\,795\,984\,422\,030\,468\,128\,177\,510\,606\,635\,859\,443\,712\,n^2 + 69\,923\,155\,581\,931\,005\,683\,616\,191\,106\,190\,820\,769\,792\,n^3 +$
$397\,777\,067\,188\,645\,895\,439\,894\,469\,915\,116\,729\,532\,416\,n^4 + 1\,729\,204\,532\,204\,996\,997\,152\,583\,407\,397\,237\,453\,225\,984\,n^5 +$
$5\,995\,525\,164\,632\,894\,191\,922\,695\,828\,861\,346\,535\,505\,920\,n^6 + 17\,088\,735\,819\,395\,764\,913\,747\,809\,298\,043\,883\,399\,888\,896\,n^7 +$
$40\,950\,242\,468\,668\,902\,568\,149\,188\,809\,126\,378\,671\,144\,960\,n^8 + 83\,944\,922\,396\,996\,847\,271\,825\,479\,812\,337\,521\,047\,842\,816\,n^9 +$
$149\,241\,076\,173\,129\,905\,329\,165\,504\,863\,175\,575\,091\,332\,096\,n^{10} +$

$232 683 501 679 364 482 345 805 030 869 544 077 599 268 352 n^{11} + 321 069 920 926 096 778 719 573 512 107 074 163 678 806 528$
$n^{12} + 395 099 424 821 926 843 732 271 053 703 633 605 285 841 664 n^{13} +$
$436 399 473 903 028 869 076 842 967 038 797 890 370 004 032 n^{14} + 435 026 967 213 159 405 048 515 684 405 715 110 378 338 720$
$n^{15} + 393 230 111 147 319 510 033 909 063 747 616 966 700 500 160 n^{16} +$
$323 623 592 515 063 057 301 937 486 575 186 179 314 548 656 n^{17} + 243 346 679 569 443 698 265 032 683 268 250 796 082 355 872$
$n^{18} + 167 699 480 512 372 076 874 814 970 574 805 341 647 426 008 n^{19} +$
$106 198 031 145 430 172 090 232 700 406 650 650 176 362 252 n^{20} + 61 942 356 363 090 378 267 431 321 236 277 379 702 842 498 n^{21} +$
$33 344 350 577 247 386 451 113 806 267 465 360 943 413 242 n^{22} + 16 595 113 813 714 428 964 861 691 068 549 243 862 900 514 n^{23} +$
$7 647 457 411 737 976 747 992 069 047 332 333 397 916 774 n^{24} + 3 267 331 144 899 381 542 768 858 934 068 187 529 943 662 n^{25} +$
$1 295 632 087 778 176 811 117 167 430 324 892 967 262 289 n^{26} + 477 281 953 036 655 577 658 172 066 157 254 622 874 249 n^{27} +$
$163 451 731 801 877 305 191 319 996 120 051 969 522 930 n^{28} + 52 067 844 190 687 573 163 358 660 941 762 172 198 124 n^{29} +$
$15 434 420 976 233 983 819 168 350 126 595 595 202 140 n^{30} + 4 258 528 038 477 820 285 037 384 293 734 708 124 148 n^{31} +$
$1 093 751 039 753 805 052 074 448 772 782 500 867 240 n^{32} + 261 482 348 137 970 277 384 591 678 834 281 844 450 n^{33} +$
$58 175 398 225 026 023 823 129 376 451 604 957 051 n^{34} + 12 040 624 671 354 180 854 266 520 205 844 399 991 n^{35} +$
$2 317 073 223 658 459 084 340 945 199 597 363 112 n^{36} + 414 290 751 988 803 620 905 542 484 937 609 472 n^{37} +$
$68 763 751 517 799 771 571 460 168 482 707 342 n^{38} + 10 583 544 268 756 001 126 111 015 357 786 438 n^{39} +$
$1 508 533 267 822 499 512 906 498 412 337 430 n^{40} + 198 821 183 372 782 469 752 192 989 049 026 n^{41} +$
$24 186 388 725 302 176 057 341 026 698 695 n^{42} + 2 709 982 082 198 798 782 794 681 162 855 n^{43} +$
$278 988 479 854 541 096 081 379 081 954 n^{44} + 26 314 522 078 796 930 198 189 306 612 n^{45} +$
$2 266 504 237 219 506 659 730 716 168 n^{46} + 177 578 797 780 695 304 231 916 832 n^{47} + 12 598 833 333 382 387 109 147 196 n^{48} +$
$805 098 410 198 134 438 811 902 n^{49} + 46 044 417 764 504 993 310 349 n^{50} + 2 338 717 637 745 860 215 073 n^{51} +$
$104 514 427 357 536 583 000 n^{52} + 4 061 699 225 871 964 542 n^{53} + 135 250 177 600 636 980 n^{54} + 3 784 921 061 092 388 n^{55} +$
$86 707 444 117 568 n^{56} + 1 566 227 411 552 n^{57} + 21 060 262 112 n^{58} + 190 971 744 n^{59} + 949 440 n^{60} + 1280 n^{61}) f[3 + n] +$
$(-167 707 728 739 563 162 662 793 640 641 822 720 000 - 1 884 621 099 579 099 264 283 288 794 700 972 032 000 n -$
$5 041 377 848 806 910 695 366 548 334 073 767 526 400 n^{2} + 46 775 477 320 896 085 249 578 292 645 052 798 730 240 n^{3} +$
$556 885 835 525 484 915 713 810 095 094 705 435 443 200 n^{4} + 3 159 242 452 468 551 228 562 153 543 444 819 393 904 640 n^{5} +$
$12 415 541 071 618 540 660 307 393 273 559 673 826 377 728 n^{6} + 37 637 925 296 940 019 781 324 185 478 731 922 295 963 648 n^{7} +$
$92 737 714 460 401 157 604 501 008 974 332 095 412 830 208 n^{8} + 191 667 000 496 318 490 779 014 701 563 244 967 355 920 384 n^{9} +$
$339 452 263 097 369 421 820 859 975 846 282 755 107 529 728 n^{10} + 523 248 300 281 830 339 242 284 804 346 885 285 914 676 736$
$n^{11} + 710 386 898 168 009 760 958 599 932 526 536 214 775 457 280 n^{12} +$
$857 462 533 335 657 188 694 551 242 676 693 238 363 818 752 n^{13} + 927 185 892 315 490 057 264 223 661 958 052 089 087 716 416$

$n^{14} + 903\,794\,264\,491\,665\,943\,887\,994\,150\,618\,261\,070\,781\,888\,544\, n^{15} +$
$798\,358\,468\,088\,992\,068\,216\,784\,561\,678\,832\,305\,544\,914\,688\, n^{16} + 641\,910\,993\,104\,210\,400\,936\,977\,369\,822\,714\,460\,830\,564\,208$
$n^{17} + 471\,562\,530\,047\,346\,319\,754\,202\,040\,007\,262\,652\,757\,218\,624\, n^{18} +$
$317\,538\,797\,128\,843\,618\,565\,893\,715\,748\,339\,861\,357\,902\,640\, n^{19} + 196\,542\,800\,425\,272\,936\,030\,894\,969\,376\,573\,161\,178\,781\,716$
$n^{20} + 112\,089\,205\,036\,988\,299\,699\,706\,179\,845\,387\,148\,374\,717\,002\, n^{21} +$
$59\,022\,381\,396\,427\,738\,240\,092\,380\,898\,572\,083\,968\,461\,562\, n^{22} + 28\,746\,865\,359\,106\,416\,256\,416\,247\,321\,136\,578\,512\,534\,282\, n^{23} +$
$12\,970\,254\,953\,319\,120\,480\,983\,509\,703\,101\,810\,054\,182\,782\, n^{24} + 5\,428\,183\,614\,772\,190\,906\,005\,960\,866\,663\,410\,280\,479\,562\, n^{25} +$
$2\,109\,510\,758\,277\,426\,773\,606\,416\,875\,119\,872\,314\,811\,877\, n^{26} + 761\,938\,745\,869\,645\,201\,813\,190\,530\,563\,834\,432\,295\,465\, n^{27} +$
$255\,965\,695\,136\,511\,737\,849\,398\,726\,940\,189\,624\,590\,934\, n^{28} + 80\,021\,356\,179\,793\,507\,587\,079\,440\,531\,706\,016\,662\,668\, n^{29} +$
$23\,289\,571\,211\,370\,481\,847\,378\,953\,662\,633\,197\,929\,068\, n^{30} + 6\,311\,748\,066\,033\,135\,567\,176\,717\,489\,354\,950\,656\,844\, n^{31} +$
$1\,592\,960\,015\,056\,501\,397\,895\,148\,828\,890\,424\,971\,668\, n^{32} + 374\,365\,028\,306\,562\,451\,548\,967\,569\,889\,349\,299\,994\, n^{33} +$
$81\,907\,187\,008\,712\,905\,581\,886\,627\,733\,322\,657\,947\, n^{34} + 16\,676\,994\,045\,570\,734\,948\,021\,794\,434\,982\,584\,627\, n^{35} +$
$3\,158\,233\,696\,398\,401\,126\,430\,392\,498\,360\,079\,036\, n^{36} + 555\,889\,720\,171\,821\,324\,468\,177\,471\,226\,088\,216\, n^{37} +$
$90\,857\,083\,052\,539\,834\,834\,050\,188\,828\,261\,198\, n^{38} + 13\,774\,542\,091\,612\,157\,361\,488\,698\,113\,634\,638\, n^{39} +$
$1\,934\,519\,485\,240\,215\,372\,599\,275\,594\,201\,750\, n^{40} + 251\,288\,787\,124\,077\,923\,597\,876\,350\,839\,934\, n^{41} +$
$30\,136\,222\,767\,569\,363\,668\,692\,805\,964\,139\, n^{42} + 3\,329\,676\,213\,071\,448\,715\,046\,083\,817\,455\, n^{43} +$
$338\,100\,507\,320\,636\,683\,862\,629\,994\,494\, n^{44} + 31\,461\,631\,160\,467\,405\,736\,547\,997\,940\, n^{45} +$
$2\,674\,035\,177\,086\,832\,347\,621\,329\,096\, n^{46} + 206\,786\,377\,638\,960\,225\,950\,374\,136\, n^{47} + 14\,483\,513\,059\,358\,524\,937\,574\,360\, n^{48} +$
$913\,893\,080\,887\,697\,869\,410\,574\, n^{49} + 51\,619\,555\,838\,689\,353\,242\,549\, n^{50} + 2\,589\,956\,035\,399\,262\,629\,893\, n^{51} +$
$114\,354\,608\,201\,432\,199\,836\, n^{52} + 4\,391\,706\,562\,159\,684\,062\, n^{53} + 144\,543\,224\,194\,970\,180\, n^{54} + 3\,998\,894\,202\,305\,188\, n^{55} +$
$90\,585\,200\,571\,648\, n^{56} + 1\,618\,378\,146\,784\, n^{57} + 21\,530\,125\,088\, n^{58} + 193\,247\,264\, n^{59} + 952\,000\, n^{60} + 1280\, n^{61}) f[4 + n] +$
$(146\,973\,883\,013\,564\,258\,890\,458\,767\,567\,093\,760\,000 + 2\,229\,259\,984\,516\,059\,314\,164\,243\,577\,955\,680\,256\,000\, n +$
$16\,595\,245\,522\,055\,651\,232\,438\,964\,274\,595\,574\,579\,200\, n^{2} + 82\,110\,213\,857\,892\,331\,021\,373\,929\,579\,308\,684\,410\,880\, n^{3} +$
$310\,740\,084\,394\,832\,008\,017\,420\,625\,315\,940\,985\,077\,760\, n^{4} + 983\,450\,262\,875\,422\,499\,074\,965\,195\,835\,230\,498\,914\,304\, n^{5} +$
$2\,752\,795\,738\,160\,705\,474\,897\,634\,293\,607\,251\,897\,352\,192\, n^{6} + 6\,979\,426\,901\,876\,915\,027\,726\,954\,745\,414\,540\,323\,635\,200\, n^{7} +$
$16\,022\,042\,987\,367\,137\,971\,152\,393\,882\,056\,321\,496\,649\,728\, n^{8} + 32\,976\,239\,563\,629\,576\,537\,093\,753\,171\,267\,079\,363\,952\,640\, n^{9} +$
$60\,348\,223\,112\,048\,685\,326\,140\,378\,332\,733\,677\,958\,551\,552\, n^{10} + 97\,870\,234\,277\,454\,774\,181\,867\,539\,360\,398\,981\,127\,882\,752\, n^{11} +$
$140\,769\,737\,681\,664\,961\,820\,732\,259\,956\,248\,222\,002\,156\,544\, n^{12} + 180\,159\,949\,407\,962\,743\,950\,515\,756\,408\,355\,287\,251\,632\,128$
$n^{13} + 206\,063\,708\,436\,712\,977\,188\,583\,276\,398\,745\,589\,001\,139\,968\, n^{14} +$
$211\,629\,972\,252\,724\,984\,409\,247\,015\,325\,612\,755\,050\,864\,384\, n^{15} + 196\,052\,177\,074\,481\,560\,658\,330\,639\,762\,342\,332\,344\,444\,048$
$n^{16} + 164\,525\,447\,978\,078\,562\,567\,838\,918\,443\,591\,164\,707\,502\,416\, n^{17} +$

125 556 179 875 444 843 058 702 170 521 611 753 698 519 256 n^{18} + 87 435 155 865 587 777 892 765 113 376 420 118 121 960 328 n^{19} +
55 732 185 112 640 160 274 945 213 210 191 691 809 530 492 n^{20} + 32 603 848 226 154 254 694 224 926 180 497 667 742 692 898 n^{21} +
17 546 915 408 521 547 104 784 732 751 147 391 512 437 658 n^{22} + 8 705 564 957 790 628 615 250 340 784 589 164 051 237 410 n^{23} +
3 988 719 282 955 627 082 164 451 161 346 964 412 614 922 n^{24} + 1 690 355 246 982 168 411 514 553 093 925 250 510 197 586 n^{25} +
663 435 741 557 456 861 450 332 956 793 381 788 403 741 n^{26} + 241 419 754 068 235 849 516 846 929 157 324 919 604 649 n^{27} +
81 525 053 083 740 624 437 032 057 330 999 824 995 610 n^{28} + 25 566 108 133 626 930 799 108 150 870 844 970 750 372 n^{29} +
7 449 510 928 708 389 104 237 268 058 331 083 573 272 n^{30} + 2 017 628 218 981 886 286 590 169 884 480 431 538 092 n^{31} +
508 034 801 230 647 487 942 634 941 743 803 660 008 n^{32} + 118 932 982 740 371 640 583 748 836 229 570 720 114 n^{33} +
25 882 966 091 402 668 443 156 859 423 413 630 159 n^{34} + 5 234 831 252 327 150 258 495 816 833 335 987 083 n^{35} +
983 485 163 689 520 917 457 934 207 647 204 592 n^{36} + 171 526 535 385 991 276 074 256 199 958 715 648 n^{37} +
27 748 080 934 509 521 703 257 731 932 900 950 n^{38} + 4 159 340 594 584 083 287 370 449 291 142 406 n^{39} +
576 981 874 549 508 051 423 551 767 867 114 n^{40} + 73 960 066 614 630 153 761 116 279 570 486 n^{41} +
8 745 090 271 861 061 856 731 036 132 491 n^{42} + 951 844 347 105 863 253 364 908 309 823 n^{43} +
95 138 142 685 999 346 905 079 417 738 n^{44} + 8 707 815 519 389 802 473 929 517 372 n^{45} +
727 452 758 910 755 330 250 630 756 n^{46} + 55 255 418 993 869 721 987 243 656 n^{47} + 3 798 932 121 017 318 727 211 124 n^{48} +
235 152 612 408 584 314 355 478 n^{49} + 13 022 051 437 733 827 813 521 n^{50} + 640 211 109 261 103 389 509 n^{51} +
27 683 156 067 422 637 168 n^{52} + 1 040 643 976 182 868 542 n^{53} + 33 508 905 375 528 340 n^{54} + 906 557 733 231 620 n^{55} +
20 073 465 051 104 n^{56} + 350 427 896 160 n^{57} + 4 554 253 344 n^{58} + 39 936 608 n^{59} + 192 448 n^{60} + 256 n^{61}) f[5 + n] −
(− 229 133 653 589 293 103 588 089 989 773 131 776 000 − 2 707 218 686 086 927 844 219 339 121 202 456 166 400 n −
8 435 158 839 130 133 492 119 907 321 120 544 522 240 n^{2} + 57 312 106 282 972 020 092 113 166 164 705 144 995 840 n^{3} +
761 311 556 185 420 846 905 694 442 317 899 022 860 288 n^{4} + 4 470 070 472 792 421 402 760 998 791 768 772 851 073 024 n^{5} +
17 933 275 241 733 080 753 971 051 500 884 064 151 535 616 n^{6} + 55 163 890 293 619 386 176 182 666 327 824 835 758 047 232 n^{7} +
137 396 641 381 867 437 272 033 940 971 117 581 338 464 256 n^{8} + 286 236 922 551 698 791 964 818 718 459 230 151 232 184 320 n^{9} +
509 799 264 887 681 665 018 259 066 709 529 133 495 617 536 n^{10} + 788 651 555 674 921 555 121 999 082 128 867 307 071 072 256
n^{11} + 1 072 588 143 491 400 565 323 822 921 025 679 581 406 329 856 n^{12} +
1 294 744 188 580 567 907 082 105 180 269 849 560 142 171 648 n^{13} + 1 397 941 057 071 523 729 674 966 499 885 688 914 786 767 104
n^{14} + 1 358 660 215 905 239 775 278 697 600 361 668 905 916 384 384 n^{15} +
1 194 983 627 524 612 608 767 092 393 372 811 905 774 530 512 n^{16} + 955 428 594 075 553 278 209 673 752 244 502 528 652 871 632
n^{17} + 697 089 659 895 187 705 783 120 568 296 945 401 635 556 904 n^{18} +
465 656 730 189 159 752 034 253 308 339 764 890 945 988 496 n^{19} + 285 604 431 687 495 229 880 881 251 511 616 073 109 650 220
n^{20} + 161 232 532 865 686 268 214 919 319 101 899 922 033 636 862 n^{21} +

$83 955 845 991 827 386 122 821 327 404 717 592 026 870 110 n^{22} + 40 397 529 485 287 635 174 511 174 996 855 520 331 201 558 n^{23} +$
$17 990 657 209 554 183 341 396 289 732 046 387 408 087 174 n^{24} + 7 425 224 905 641 533 823 522 969 250 408 974 792 884 434 n^{25} +$
$2 843 367 741 942 140 922 093 698 660 861 113 923 900 963 n^{26} + 1 011 168 561 548 543 796 801 655 543 741 138 811 011 507 n^{27} +$
$334 201 980 134 186 841 985 596 581 694 146 281 914 418 n^{28} + 102 717 210 317 860 109 532 509 242 812 022 564 194 860 n^{29} +$
$29 370 354 351 546 973 498 662 057 262 843 294 604 096 n^{30} + 7 814 863 828 387 728 382 126 093 354 840 716 158 868 n^{31} +$
$1 935 205 500 844 458 362 351 907 362 574 701 252 076 n^{32} + 445 970 814 156 252 687 541 717 555 256 393 051 710 n^{33} +$
$95 625 156 036 928 688 663 360 232 186 306 822 077 n^{34} + 19 070 756 523 821 114 762 082 917 501 786 328 629 n^{35} +$
$3 535 622 715 616 424 883 553 216 938 987 081 532 n^{36} + 608 924 288 966 351 008 440 912 039 102 146 104 n^{37} +$
$97 336 892 660 236 397 648 749 897 843 495 650 n^{38} + 14 425 803 764 696 881 833 933 919 196 754 354 n^{39} +$
$1 979 652 359 568 410 734 625 780 917 990 814 n^{40} + 251 164 938 768 336 941 770 845 524 474 110 n^{41} +$
$29 408 294 486 395 333 803 864 974 921 629 n^{42} + 3 171 106 066 171 398 274 360 702 357 901 n^{43} +$
$314 139 274 560 185 545 478 142 702 954 n^{44} + 28 508 357 961 055 344 546 198 016 916 n^{45} +$
$2 362 252 408 109 032 902 335 223 380 n^{46} + 178 036 706 377 314 851 710 827 640 n^{47} + 12 149 450 550 976 441 759 660 448 n^{48} +$
$746 700 160 166 119 020 800 866 n^{49} + 41 068 868 731 260 336 676 891 n^{50} + 2 005 971 167 965 587 366 107 n^{51} +$
$86 201 183 906 778 839 132 n^{52} + 3 221 230 995 223 510 058 n^{53} + 103 140 014 798 676 940 n^{54} + 2 775 465 779 235 692 n^{55} +$
$61 145 713 649 888 n^{56} + 1 062 388 816 288 n^{57} + 13 746 710 944 n^{58} + 120 074 336 n^{59} + 576 832 n^{60} + 768 n^{61}) f[6 + n] +$
$(-205 329 880 537 617 010 501 022 298 748 747 776 000 - 2 696 078 286 141 154 627 238 041 660 556 732 006 400 n -$
$13 446 746 876 280 445 907 576 988 016 652 310 282 240 n^2 - 9 572 101 437 141 628 816 667 721 484 850 492 866 560 n^3 +$
$280 182 940 654 525 918 506 623 162 238 291 749 634 048 n^4 + 2 094 095 152 538 489 412 154 535 455 992 036 382 212 096 n^5 +$
$9 063 043 112 286 942 735 738 908 649 797 197 564 280 832 n^6 + 28 763 695 575 379 100 484 547 551 886 516 377 655 984 128 n^7 +$
$72 587 277 948 926 887 578 249 012 833 148 937 342 402 560 n^8 + 151 848 308 039 043 719 353 760 639 271 736 941 736 587 264 n^9 +$
$270 228 635 931 765 132 109 743 959 410 128 481 769 127 936 n^{10} +$
$416 475 851 930 582 110 790 956 894 347 255 568 214 041 600 n^{11} + 563 273 528 848 780 505 801 853 290 651 985 532 735 597 568$
$n^{12} + 675 378 713 606 734 164 964 894 487 190 643 586 775 823 616 n^{13} +$
$723 772 257 202 863 431 434 275 435 199 614 271 966 064 896 n^{14} + 697 846 465 839 303 446 043 836 853 429 870 244 915 884 800$
$n^{15} + 608 706 380 872 990 973 730 902 890 434 221 992 542 828 992 n^{16} +$
$482 560 435 172 785 135 279 241 439 251 298 345 923 241 248 n^{17} + 349 054 980 225 759 116 262 635 139 230 522 365 632 991 808$
$n^{18} + 231 147 595 596 532 620 592 379 457 194 557 347 102 467 288 n^{19} +$
$140 536 192 755 338 951 691 788 258 703 532 233 289 176 732 n^{20} + 78 644 201 689 438 279 267 336 949 795 701 480 453 983 370 n^{21} +$
$40 593 143 275 902 089 690 541 181 091 180 591 688 214 746 n^{22} + 19 361 740 620 088 456 118 995 567 543 717 926 580 867 170 n^{23} +$
$8 547 241 017 147 324 196 672 102 915 309 670 663 358 310 n^{24} + 3 496 868 703 574 705 660 432 884 441 070 148 864 445 574 n^{25} +$

1 327 376 171 171 369 752 308 048 746 304 225 126 827 565 n^{26} + 467 923 258 436 459 092 038 397 095 600 958 333 553 789 n^{27} +
153 301 662 682 604 734 680 685 528 474 464 248 493 050 n^{28} + 46 704 858 741 585 641 410 966 274 980 319 548 456 852 n^{29} +
13 237 336 261 385 959 202 565 843 547 692 916 281 036 n^{30} + 3 491 195 603 476 702 931 319 945 067 604 568 162 828 n^{31} +
856 894 095 632 874 258 492 412 142 645 342 853 768 n^{32} + 195 720 752 557 255 112 678 769 676 790 740 646 042 n^{33} +
41 592 327 466 920 617 640 131 171 642 573 300 047 n^{34} + 8 220 478 083 640 342 014 901 813 382 323 247 819 n^{35} +
1 510 281 458 336 947 632 567 400 840 635 396 600 n^{36} + 257 744 957 982 899 404 990 201 966 847 018 128 n^{37} +
40 823 363 729 408 037 724 579 306 410 090 910 n^{38} + 5 994 359 838 556 501 099 719 702 325 281 158 n^{39} +
814 944 222 044 225 628 890 907 403 778 678 n^{40} + 102 422 776 251 835 173 864 123 492 495 018 n^{41} +
11 878 625 574 103 989 575 227 584 857 771 n^{42} + 1 268 598 347 854 663 415 431 294 550 179 n^{43} +
124 454 005 507 987 360 466 745 640 362 n^{44} + 11 183 726 605 231 249 216 080 033 340 n^{45} +
917 531 675 574 530 213 244 008 152 n^{46} + 68 459 846 335 433 575 358 003 800 n^{47} + 4 624 495 807 310 652 011 314 684 n^{48} +
281 309 213 929 495 034 504 854 n^{49} + 15 311 862 610 709 369 056 793 n^{50} + 740 055 853 160 834 519 261 n^{51} +
31 464 751 682 713 946 856 n^{52} + 1 163 191 934 453 543 606 n^{53} + 36 840 351 024 083 332 n^{54} + 980 508 441 304 116 n^{55} +
21 362 825 138 432 n^{56} + 367 049 792 736 n^{57} + 4 696 674 016 n^{58} + 40 576 224 n^{59} + 192 960 n^{60} + 256 n^{61}) f[7 + n] == 0;

In[3]:= **recSol = SolveRecurrence[rec, f[n], IndefiniteSummation → False, SigmaPrint → True]**

Out[3]=
$$\left\{\{0,1\},\left\{0,-\left(\sum_{\iota_1=4}^{n}\frac{6+2\iota_1-3\iota_1^2+6\iota_1^3+\iota_1^4}{(-1+\iota_1)^3\,\iota_1\,(1+\iota_1)^3}\right)\right\},\right.$$

$$\left\{0,\sum_{\iota_1=4}^{n}\frac{-68-23\iota_1+37\iota_1^2-66\iota_1^3-13\iota_1^4+\iota_1^5}{(-1+\iota_1)^3\,\iota_1\,(1+\iota_1)^3}\right\},\left\{0,-\left(\sum_{\iota_1=4}^{n}\frac{424+146\iota_1-253\iota_1^2+396\iota_1^3+92\iota_1^4-14\iota_1^5+\iota_1^6}{(-1+\iota_1)^3\,\iota_1\,(1+\iota_1)^3}\right)\right\},$$

$$\left\{0,-2\left(\sum_{\iota_1=12}^{n}\frac{1}{(-1+\iota_1)^3\,\iota_1\,(1+\iota_1)^3}(6+2\iota_1-3\iota_1^2+6\iota_1^3+\iota_1^4)\left(\sum_{\iota_2=10}^{\iota_1}\left((8-64\iota_2+92\iota_2^2+39\iota_2^3-128\iota_2^4+125\iota_2^5-57\iota_2^6+8\iota_2^7+\iota_2^8)\right.\right.\right.\right.$$

$$\left(\sum_{\iota_3=8}^{\iota_2}\left((-1+\iota_3)^2(-4+22\iota_3-15\iota_3^2+2\iota_3^3+\iota_3^4)(144-1392\iota_3+2884\iota_3^2-568\iota_3^3-3625\iota_3^4+4604\iota_3^5-2797\iota_3^6+980\iota_3^7-175\iota_3^8+8\iota_3^9+\iota_3^{10})\right.\right.$$

$$\left(\sum_{\iota_4=6}^{\iota_3}((-192+1396\iota_4-3038\iota_4^2+3165\iota_4^3-1818\iota_4^4+579\iota_4^5-85\iota_4^6+\iota_4^8)(-1\,036\,800+50\,108\,544\iota_4-378\,482\,688\iota_4^2+\right.$$

$$1\,203\,879\,360\iota_4^3-1\,866\,851\,440\iota_4^4+901\,685\,472\iota_4^5+1\,958\,467\,456\iota_4^6-4\,787\,603\,188\iota_4^7+5\,528\,964\,798\iota_4^8-$$

$$4\,152\,500\,203\iota_4^9+2\,141\,149\,820\iota_4^{10}-715\,311\,261\iota_4^{11}+98\,930\,622\iota_4^{12}+42\,122\,506\iota_4^{13}-32\,513\,904\iota_4^{14}+$$

$$11\,176\,602\iota_4^{15}-2\,396\,078\iota_4^{16}+327\,953\iota_4^{17}-26\,348\iota_4^{18}+935\iota_4^{19}+2\iota_4^{20}))\Big/((-3+\iota_4)^2(-2+\iota_4)^3(-1+\iota_4)^3\iota_4^2(1+\iota_4)$$

$$(-7200+53\,760\iota_4-130\,880\iota_4^2+160\,564\iota_4^3-115\,948\iota_4^4+52\,522\iota_4^5-15\,019\iota_4^6+2548\iota_4^7-202\iota_4^8-2\iota_4^9+\iota_4^{10})$$

$$\left.\left.(144-1392\iota_4+2884\iota_4^2-568\iota_4^3-3625\iota_4^4+4604\iota_4^5-2797\iota_4^6+980\iota_4^7-175\iota_4^8+8\iota_4^9+\iota_4^{10}))\right)\right)\Big/$$

$$((-192+1396\iota_3-3038\iota_3^2+3165\iota_3^3-1818\iota_3^4+579\iota_3^5-85\iota_3^6+\iota_3^8)(8-64\iota_3+92\iota_3^2+39\iota_3^3-128\iota_3^4+125\iota_3^5-$$

$$\left.\left.\left.\left.57\iota_3^6+8\iota_3^7+\iota_3^8))\right)\right)\Big/((-4+22\iota_2-15\iota_2^2+2\iota_2^3+\iota_2^4)(6+2\iota_2-3\iota_2^2+6\iota_2^3+\iota_2^4))\right)\right)\right\},$$

$$\left\{0,-2\left(\sum_{\iota_1=12}^{n}\frac{1}{(-1+\iota_1)^3\,\iota_1\,(1+\iota_1)^3}(6+2\iota_1-3\iota_1^2+6\iota_1^3+\iota_1^4)\left(\sum_{\iota_2=10}^{\iota_1}\left((8-64\iota_2+92\iota_2^2+39\iota_2^3-128\iota_2^4+125\iota_2^5-57\iota_2^6+8\iota_2^7+\iota_2^8)\right.\right.\right.\right.$$

$$\left(\sum_{\iota_3=8}^{\iota_2}\left((-1+\iota_3)^2(-4+22\iota_3-15\iota_3^2+2\iota_3^3+\iota_3^4)(144-1392\iota_3+2884\iota_3^2-568\iota_3^3-3625\iota_3^4+4604\iota_3^5-2797\iota_3^6+980\iota_3^7-175\iota_3^8+8\iota_3^9+\iota_3^{10})\right.\right.$$

$$\Bigg(\sum_{\imath_4=6}^{\imath_3} \Bigg(\big(-192 + 1396\,\imath_4 - 3038\,\imath_4^2 + 3165\,\imath_4^3 - 1818\,\imath_4^4 + 579\,\imath_4^5 - 85\,\imath_4^6 + \imath_4^8\big)\big(-1\,036\,800 + 50\,108\,544\,\imath_4 - 378\,482\,688\,\imath_4^2 +$$

$$1\,203\,879\,360\,\imath_4^3 - 1\,866\,851\,440\,\imath_4^4 + 901\,685\,472\,\imath_4^5 + 1\,958\,467\,456\,\imath_4^6 - 4\,787\,603\,188\,\imath_4^7 + 5\,528\,964\,798\,\imath_4^8 -$$

$$4\,152\,500\,203\,\imath_4^9 + 2\,141\,149\,820\,\imath_4^{10} - 715\,311\,261\,\imath_4^{11} + 98\,930\,622\,\imath_4^{12} + 42\,122\,506\,\imath_4^{13} - 32\,513\,904\,\imath_4^{14} + 11\,176\,602$$

$$\imath_4^{15} - 2\,396\,078\,\imath_4^{16} + 327\,953\,\imath_4^{17} - 26\,348\,\imath_4^{18} + 935\,\imath_4^{19} + 2\,\imath_4^{20}\big)\Bigg(\sum_{\imath_5=3}^{\imath_4} \big(\big(-7200 + 53\,760\,\imath_5 - 130\,880\,\imath_5^2 + 160\,564\,\imath_5^3 -$$

$$115\,948\,\imath_5^4 + 52\,522\,\imath_5^5 - 15\,019\,\imath_5^6 + 2548\,\imath_5^7 - 202\,\imath_5^8 - 2\,\imath_5^9 + \imath_5^{10}\big)\big(-268\,061\,736\,960 + 2\,824\,243\,780\,608\,\imath_5 +$$

$$4\,318\,123\,156\,992\,\imath_5^2 - 201\,123\,626\,821\,632\,\imath_5^3 + 1\,391\,666\,715\,407\,232\,\imath_5^4 - 5\,277\,178\,048\,925\,952\,\imath_5^5 +$$

$$12\,962\,808\,461\,111\,200\,\imath_5^6 - 21\,305\,292\,915\,332\,800\,\imath_5^7 + 21\,166\,111\,959\,705\,696\,\imath_5^8 - 2\,890\,677\,091\,472\,172$$

$$\imath_5^9 - 33\,448\,944\,651\,882\,246\,\imath_5^{10} + 73\,993\,322\,390\,651\,126\,\imath_5^{11} - 99\,697\,525\,001\,566\,438\,\imath_5^{12} +$$

$$100\,147\,841\,927\,725\,059\,\imath_5^{13} - 79\,546\,250\,838\,626\,709\,\imath_5^{14} + 50\,955\,754\,267\,106\,775\,\imath_5^{15} -$$

$$26\,260\,396\,787\,425\,091\,\imath_5^{16} + 10\,563\,977\,846\,009\,515\,\imath_5^{17} - 2\,989\,102\,604\,911\,489\,\imath_5^{18} + 315\,718\,564\,141\,377$$

$$\imath_5^{19} + 242\,070\,039\,538\,587\,\imath_5^{20} - 192\,965\,061\,962\,541\,\imath_5^{21} + 85\,146\,456\,468\,263\,\imath_5^{22} - 27\,671\,025\,884\,951\,\imath_5^{23} +$$

$$7\,076\,120\,002\,347\,\imath_5^{24} - 1\,452\,474\,125\,119\,\imath_5^{25} + 239\,656\,119\,375\,\imath_5^{26} - 31\,463\,978\,263\,\imath_5^{27} +$$

$$3\,217\,398\,027\,\imath_5^{28} - 247\,073\,926\,\imath_5^{29} + 13\,382\,758\,\imath_5^{30} - 452\,800\,\imath_5^{31} + 6936\,\imath_5^{32} + 16\,\imath_5^{33}\big)(-1)^{\imath_5}\big) \big/ \big(\big(14\,892\,318\,720 -$$

$$139\,725\,950\,976\,\imath_5 + 544\,478\,712\,960\,\imath_5^2 - 1\,213\,653\,772\,800\,\imath_5^3 + 1\,752\,329\,541\,504\,\imath_5^4 - 1\,727\,022\,580\,784\,\imath_5^5 +$$

$$1\,160\,998\,559\,536\,\imath_5^6 - 479\,387\,084\,584\,\imath_5^7 + 42\,444\,457\,372\,\imath_5^8 + 101\,229\,213\,030\,\imath_5^9 - 88\,422\,557\,953\,\imath_5^{10} +$$

$$43\,907\,846\,563\,\imath_5^{11} - 15\,419\,001\,366\,\imath_5^{12} + 4\,044\,155\,236\,\imath_5^{13} - 802\,119\,874\,\imath_5^{14} + 119\,208\,478\,\imath_5^{15} -$$

$$12\,898\,848\,\imath_5^{16} + 959\,822\,\imath_5^{17} - 43\,733\,\imath_5^{18} + 895\,\imath_5^{19} + 2\,\imath_5^{20}\big)\big(-1\,036\,800 + 50\,108\,544\,\imath_5 - 378\,482\,688\,\imath_5^2 +$$

$$1\,203\,879\,360\,\imath_5^3 - 1\,866\,851\,440\,\imath_5^4 + 901\,685\,472\,\imath_5^5 + 1\,958\,467\,456\,\imath_5^6 - 4\,787\,603\,188\,\imath_5^7 + 5\,528\,964\,798\,\imath_5^8 -$$

$$4\,152\,500\,203\,\imath_5^9 + 2\,141\,149\,820\,\imath_5^{10} - 715\,311\,261\,\imath_5^{11} + 98\,930\,622\,\imath_5^{12} + 42\,122\,506\,\imath_5^{13} - 32\,513\,904\,\imath_5^{14} +$$

$$11\,176\,602\,\imath_5^{15} - 2\,396\,078\,\imath_5^{16} + 327\,953\,\imath_5^{17} - 26\,348\,\imath_5^{18} + 935\,\imath_5^{19} + 2\,\imath_5^{20}\big)\big)\Bigg)\Bigg) \Big/ \big((-3 + \imath_4)^2 (-2 + \imath_4)^3 (-1 + \imath_4)^3$$

$$\imath_4^2 (1 + \imath_4)\big(-7200 + 53\,760\,\imath_4 - 130\,880\,\imath_4^2 + 160\,564\,\imath_4^3 - 115\,948\,\imath_4^4 + 52\,522\,\imath_4^5 - 15\,019\,\imath_4^6 + 2548\,\imath_4^7 -$$

$$202\,\imath_4^8 - 2\,\imath_4^9 + \imath_4^{10}\big)\big(144 - 1392\,\imath_4 + 2884\,\imath_4^2 - 568\,\imath_4^3 - 3625\,\imath_4^4 + 4604\,\imath_4^5 - 2797\,\imath_4^6 + 980\,\imath_4^7 - 175\,\imath_4^8 + 8\,\imath_4^9 + \imath_4^{10}\big)\big)\Bigg)\Bigg) \Big/$$

$$\big(\big(-192 + 1396\,\imath_3 - 3038\,\imath_3^2 + 3165\,\imath_3^3 - 1818\,\imath_3^4 + 579\,\imath_3^5 - 85\,\imath_3^6 + \imath_3^8\big)\big(8 - 64\,\imath_3 + 92\,\imath_3^2 + 39\,\imath_3^3 - 128\,\imath_3^4 + 125\,\imath_3^5 - 57\,\imath_3^6 + 8\,\imath_3^7 + \imath_3^8\big)\big)\Bigg)\Bigg) \Big/$$

$$\left(\left(-4+22\, i_2-15\, i_2^2+2\, i_2^3+i_2^4\right)\left(6+2\, i_2-3\, i_2^2+6\, i_2^3+i_2^4\right)\right)\Bigg)\Bigg)\Bigg\}, \Bigg\{0,\ 2\left(\sum_{i_1=18}^{n} \frac{1}{(-1+i_1)^3\, i_1\,(1+i_1)^3}(6+2\right.$$

$$i_1-3$$

$$i_1^2+6$$

$$i_1^3+i_1^4)$$

$$\left(\sum_{i_2=16}^{i_1}\left(\left(8-64\, i_2+92\, i_2^2+39\, i_2^3-128\, i_2^4+125\, i_2^5-57\, i_2^6+8\, i_2^7+i_2^8\right)\right.\right.$$

$$\left(\sum_{i_3=14}^{i_2}\left((-1+i_3)^2\left(-4+22\, i_3-15\, i_3^2+2\, i_3^3+i_3^4\right)\left(144-1392\, i_3+2884\, i_3^2-568\, i_3^3-3625\, i_3^4+4604\, i_3^5-2797\, i_3^6+980\, i_3^7-175\, i_3^8+8\, i_3^9+i_3^{10}\right)\right.\right.$$

$$\left(\sum_{i_4=12}^{i_3}\left(\left(-192+1396\, i_4-3038\, i_4^2+3165\, i_4^3-1818\, i_4^4+579\, i_4^5-85\, i_4^6+i_4^8\right)\left(-1\,036\,800+50\,108\,544\, i_4-378\,482\,688\, i_4^2+\right.\right.\right.$$

$$1\,203\,879\,360\, i_4^3-1\,866\,851\,440\, i_4^4+901\,685\,472\, i_4^5+1\,958\,467\,456\, i_4^6-4\,787\,603\,188\, i_4^7+5\,528\,964\,798\, i_4^8-$$

$$4\,152\,500\,203\, i_4^9+2\,141\,149\,820\, i_4^{10}-715\,311\,261\, i_4^{11}+98\,930\,622\, i_4^{12}+42\,122\,506\, i_4^{13}-32\,513\,904\, i_4^{14}+11\,176\,602$$

$$\left.i_4^{15}-2\,396\,078\, i_4^{16}+327\,953\, i_4^{17}-26\,348\, i_4^{18}+935\, i_4^{19}+2\, i_4^{20}\right)\left(\sum_{i_5=10}^{i_4}\left(\left(-7200+53\,760\, i_5-130\,880\, i_5^2+160\,564\, i_5^3-\right.\right.\right.$$

$$\left.115\,948\, i_5^4+52\,522\, i_5^5-15\,019\, i_5^6+2548\, i_5^7-202\, i_5^8-2\, i_5^9+i_5^{10}\right)\left(-268\,061\,736\,960+2\,824\,243\,780\,608\, i_5+\right.$$

$$4\,318\,123\,156\,992\, i_5^2-201\,123\,626\,821\,632\, i_5^3+1\,391\,666\,715\,407\,232\, i_5^4-5\,277\,178\,048\,925\,952\, i_5^5+$$

$$12\,962\,808\,461\,111\,200\, i_5^6-21\,305\,292\,915\,332\,800\, i_5^7+21\,166\,111\,959\,705\,696\, i_5^8-2\,890\,677\,091\,472\,172$$

$$i_5^9-33\,448\,944\,651\,882\,246\, i_5^{10}+73\,993\,322\,390\,651\,126\, i_5^{11}-99\,697\,525\,001\,566\,438\, i_5^{12}+$$

$$100\,147\,841\,927\,725\,059\, i_5^{13}-79\,546\,250\,838\,626\,709\, i_5^{14}+50\,955\,754\,267\,106\,775\, i_5^{15}-$$

$$26\,260\,396\,787\,425\,091\, i_5^{16}+10\,563\,977\,846\,009\,515\, i_5^{17}-2\,989\,102\,604\,911\,489\, i_5^{18}+315\,718\,564\,141\,377$$

$$i_5^{19}+242\,070\,039\,538\,587\, i_5^{20}-192\,965\,061\,962\,541\, i_5^{21}+85\,146\,456\,468\,263\, i_5^{22}-27\,671\,025\,884\,951\, i_5^{23}+$$

$$7\,076\,120\,002\,347\, i_5^{24}-1\,452\,474\,125\,119\, i_5^{25}+239\,656\,119\,375\, i_5^{26}-31\,463\,978\,263\, i_5^{27}+$$

$$\left.3\,217\,398\,027\, i_5^{28}-247\,073\,926\, i_5^{29}+13\,382\,758\, i_5^{30}-452\,800\, i_5^{31}+6936\, i_5^{32}+16\, i_5^{33}\right)(-1)^{i_5}$$

$$\left(\sum_{i_6=8}^{i_5}\left(\left(14\,892\,318\,720-139\,725\,950\,976\, i_6+544\,478\,712\,960\, i_6^2-1\,213\,653\,772\,800\, i_6^3+1\,752\,329\,541\,504\, i_6^4-\right.\right.\right.$$

$$1\,727\,022\,580\,784\, i_6^5+1\,160\,998\,559\,536\, i_6^6-479\,387\,084\,584\, i_6^7+42\,444\,457\,372\, i_6^8+$$

$$101\,229\,213\,030\, i_6^9-88\,422\,557\,953\, i_6^{10}+43\,907\,846\,563\, i_6^{11}-15\,419\,001\,366\, i_6^{12}+4\,044\,155\,236\, i_6^{13}-$$

$802\,119\,874\,\iota_6^{14} + 119\,208\,478\,\iota_6^{15} - 12\,898\,848\,\iota_6^{16} + 959\,822\,\iota_6^{17} - 43\,733\,\iota_6^{18} + 895\,\iota_6^{19} + 2\,\iota_6^{20})$

$(593\,519\,176\,032\,819\,609\,600 - 10\,446\,329\,177\,297\,398\,333\,440\,\iota_6 + 82\,630\,965\,219\,898\,476\,699\,648$

$\iota_6^2 - 357\,887\,593\,339\,887\,818\,373\,120\,\iota_6^3 + 690\,983\,909\,070\,546\,888\,723\,456\,\iota_6^4 +$

$1\,496\,932\,061\,461\,512\,796\,454\,400\,\iota_6^5 - 16\,372\,134\,529\,175\,981\,041\,662\,720\,\iota_6^6 +$

$66\,157\,720\,357\,287\,991\,244\,617\,728\,\iota_6^7 - 176\,234\,670\,958\,207\,456\,090\,009\,472\,\iota_6^8 +$

$340\,927\,853\,078\,980\,448\,736\,433\,216\,\iota_6^9 - 478\,069\,445\,198\,356\,978\,374\,270\,464\,\iota_6^{10} +$

$422\,259\,594\,550\,509\,437\,294\,594\,320\,\iota_6^{11} - 1\,696\,454\,047\,051\,999\,626\,760\,968\,\iota_6^{12} -$

$827\,295\,585\,729\,170\,477\,467\,194\,152\,\iota_6^{13} + 1\,891\,616\,818\,293\,149\,574\,583\,751\,698\,\iota_6^{14} -$

$2\,852\,358\,842\,192\,182\,537\,015\,567\,718\,\iota_6^{15} + 3\,379\,568\,978\,845\,931\,753\,930\,388\,712\,\iota_6^{16} -$

$3\,327\,813\,430\,252\,425\,974\,324\,904\,640\,\iota_6^{17} + 2\,790\,700\,377\,405\,510\,701\,834\,118\,192\,\iota_6^{18} -$

$2\,014\,914\,026\,478\,703\,290\,951\,734\,807\,\iota_6^{19} + 1\,255\,359\,759\,286\,899\,924\,457\,407\,965\,\iota_6^{20} -$

$670\,558\,775\,247\,631\,998\,805\,980\,073\,\iota_6^{21} + 300\,843\,773\,637\,614\,981\,124\,313\,231\,\iota_6^{22} -$

$107\,316\,931\,132\,693\,335\,986\,435\,430\,\iota_6^{23} + 25\,083\,773\,398\,052\,438\,537\,473\,950\,\iota_6^{24} +$

$1\,155\,372\,404\,682\,645\,330\,065\,542\,\iota_6^{25} - 5\,437\,314\,367\,180\,455\,039\,158\,408\,\iota_6^{26} +$

$3\,807\,094\,879\,666\,162\,288\,177\,828\,\iota_6^{27} - 1\,865\,177\,834\,427\,973\,405\,930\,258\,\iota_6^{28} +$

$741\,345\,294\,491\,308\,012\,170\,232\,\iota_6^{29} - 250\,745\,206\,073\,700\,514\,078\,036\,\iota_6^{30} +$

$73\,677\,243\,614\,668\,746\,237\,912\,\iota_6^{31} - 18\,995\,086\,772\,703\,864\,765\,888\,\iota_6^{32} +$

$4\,316\,734\,897\,083\,386\,501\,414\,\iota_6^{33} - 865\,953\,430\,370\,743\,330\,500\,\iota_6^{34} + 153\,218\,767\,943\,719\,612\,259$

$\iota_6^{35} - 23\,849\,092\,909\,382\,625\,867\,\iota_6^{36} + 3\,251\,585\,587\,952\,045\,721\,\iota_6^{37} - 385\,933\,937\,310\,432\,229\,\iota_6^{38} +$

$39\,548\,656\,479\,377\,428\,\iota_6^{39} - 3\,460\,902\,362\,671\,038\,\iota_6^{40} + 254\,885\,646\,365\,044\,\iota_6^{41} -$

$15\,487\,684\,236\,972\,\iota_6^{42} + 755\,062\,632\,096\,\iota_6^{43} - 28\,330\,156\,288\,\iota_6^{44} + 764\,221\,216\,\iota_6^{45} - 13\,001\,120\,\iota_6^{46} +$

$96\,704\,\iota_6^{47} + 256\,\iota_6^{48})) \big/ ((-5 + \iota_6)\,(-4 + \iota_6)\,\iota_6\,(-412\,166\,094\,467\,235\,840 + 5\,717\,811\,264\,849\,119\,232\,\iota_6 -$

$36\,138\,772\,924\,632\,064\,512\,\iota_6^2 + 140\,349\,855\,651\,877\,949\,184\,\iota_6^3 - 378\,777\,898\,422\,568\,498\,944\,\iota_6^4 +$

$758\,763\,972\,872\,561\,806\,848\,\iota_6^5 - 1\,172\,910\,092\,281\,534\,607\,296\,\iota_6^6 + 1\,430\,909\,630\,074\,881\,248\,480$

$\iota_6^7 - 1\,390\,341\,094\,820\,656\,259\,632\,\iota_6^8 + 1\,068\,467\,451\,890\,543\,337\,664\,\iota_6^9 -$

$624\,704\,974\,810\,303\,880\,820\,\iota_6^{10} + 239\,348\,483\,335\,483\,302\,230\,\iota_6^{11} - 7\,040\,707\,094\,697\,754\,888\,\iota_6^{12} -$

$79\,932\,743\,247\,661\,774\,058\,\iota_6^{13} + 81\,112\,200\,716\,630\,610\,092\,\iota_6^{14} - 53\,282\,439\,119\,215\,791\,359\,\iota_6^{15} +$

$27\,238\,737\,541\,464\,309\,451\,\iota_6^{16} - 11\,465\,983\,444\,871\,365\,568\,\iota_6^{17} + 4\,072\,118\,966\,331\,964\,204\,\iota_6^{18} -$

$1\,234\,664\,080\,649\,927\,971\,\iota_6^{19} + 321\,362\,780\,236\,805\,789\,\iota_6^{20} - 71\,923\,499\,057\,928\,948\,\iota_6^{21} +$

$13\,823\,877\,306\,122\,378\,\iota_6^{22} - 2\,273\,119\,505\,706\,269\,\iota_6^{23} + 317\,735\,193\,578\,077\,\iota_6^{24} -$

$37\,399\,473\,131\,128\,\iota_6^{25} + 3\,658\,075\,291\,020\,\iota_6^{26} - 291\,823\,509\,303\,\iota_6^{27} + 18\,484\,998\,795\,\iota_6^{28} -$

$892\,856\,506\, \iota_6^{29} + 30\,772\,518\, \iota_6^{30} - 666\,304\, \iota_6^{31} + 6408\, \iota_6^{32} + 16\, \iota_6^{33}) (-268\,061\,736\,960 +$
$2\,824\,243\,780\,608\, \iota_6 + 4\,318\,123\,156\,992\, \iota_6^{2} - 201\,123\,626\,821\,632\, \iota_6^{3} + 1\,391\,666\,715\,407\,232\, \iota_6^{4} -$
$5\,277\,178\,048\,925\,952\, \iota_6^{5} + 12\,962\,808\,461\,111\,200\, \iota_6^{6} - 21\,305\,292\,915\,332\,800\, \iota_6^{7} +$
$21\,166\,111\,959\,705\,696\, \iota_6^{8} - 2\,890\,677\,091\,472\,172\, \iota_6^{9} - 33\,448\,944\,651\,882\,246\, \iota_6^{10} +$
$73\,993\,322\,390\,651\,126\, \iota_6^{11} - 99\,697\,525\,001\,566\,438\, \iota_6^{12} + 100\,147\,841\,927\,725\,059\, \iota_6^{13} -$
$79\,546\,250\,838\,626\,709\, \iota_6^{14} + 50\,955\,754\,267\,106\,775\, \iota_6^{15} - 26\,260\,396\,787\,425\,091\, \iota_6^{16} +$
$10\,563\,977\,846\,009\,515\, \iota_6^{17} - 2\,989\,102\,604\,911\,489\, \iota_6^{18} + 315\,718\,564\,141\,377\, \iota_6^{19} +$
$242\,070\,039\,538\,587\, \iota_6^{20} - 192\,965\,061\,962\,541\, \iota_6^{21} + 85\,146\,456\,468\,263\, \iota_6^{22} - 27\,671\,025\,884\,951\, \iota_6^{23} +$
$7\,076\,120\,002\,347\, \iota_6^{24} - 1\,452\,474\,125\,119\, \iota_6^{25} + 239\,656\,119\,375\, \iota_6^{26} - 31\,463\,978\,263\, \iota_6^{27} +$
$3\,217\,398\,027\, \iota_6^{28} - 247\,073\,926\, \iota_6^{29} + 13\,382\,758\, \iota_6^{30} - 452\,800\, \iota_6^{31} + 6936\, \iota_6^{32} + 16\, \iota_6^{33}))))\ /$

$((14\,892\,318\,720 - 139\,725\,950\,976\, \iota_5 + 544\,478\,712\,960\, \iota_5^{2} - 1\,213\,653\,772\,800\, \iota_5^{3} +$
$1\,752\,329\,541\,504\, \iota_5^{4} - 1\,727\,022\,580\,784\, \iota_5^{5} + 1\,160\,998\,559\,536\, \iota_5^{6} - 479\,387\,084\,584\, \iota_5^{7} + 42\,444\,457\,372\, \iota_5^{8} +$
$101\,229\,213\,030\, \iota_5^{9} - 88\,422\,557\,953\, \iota_5^{10} + 43\,907\,846\,563\, \iota_5^{11} - 15\,419\,001\,366\, \iota_5^{12} + 4\,044\,155\,236\, \iota_5^{13} -$
$802\,119\,874\, \iota_5^{14} + 119\,208\,478\, \iota_5^{15} - 12\,898\,848\, \iota_5^{16} + 959\,822\, \iota_5^{17} - 43\,733\, \iota_5^{18} + 895\, \iota_5^{19} + 2\, \iota_5^{20}) (-1\,036\,800 +$
$50\,108\,544\, \iota_5 - 378\,482\,688\, \iota_5^{2} + 1\,203\,879\,360\, \iota_5^{3} - 1\,866\,851\,440\, \iota_5^{4} + 901\,685\,472\, \iota_5^{5} + 1\,958\,467\,456\, \iota_5^{6} -$
$4\,787\,603\,188\, \iota_5^{7} + 5\,528\,964\,798\, \iota_5^{8} - 4\,152\,500\,203\, \iota_5^{9} + 2\,141\,149\,820\, \iota_5^{10} - 715\,311\,261\, \iota_5^{11} + 98\,930\,622\, \iota_5^{12} +$
$42\,122\,506\, \iota_5^{13} - 32\,513\,904\, \iota_5^{14} + 11\,176\,602\, \iota_5^{15} - 2\,396\,078\, \iota_5^{16} + 327\,953\, \iota_5^{17} - 26\,348\, \iota_5^{18} + 935\, \iota_5^{19} + 2\, \iota_5^{20}))))\ /$

$((-3 + \iota_4)^2 (-2 + \iota_4)^3 (-1 + \iota_4)^3 \iota_4^2 (1 + \iota_4) (-7200 + 53\,760\, \iota_4 - 130\,880\, \iota_4^{2} + 160\,564\, \iota_4^{3} - 115\,948\, \iota_4^{4} +$
$52\,522\, \iota_4^{5} - 15\,019\, \iota_4^{6} + 2548\, \iota_4^{7} - 202\, \iota_4^{8} - 2\, \iota_4^{9} + \iota_4^{10})$
$(144 - 1392\, \iota_4 + 2884\, \iota_4^{2} - 568\, \iota_4^{3} - 3625\, \iota_4^{4} + 4604\, \iota_4^{5} - 2797\, \iota_4^{6} + 980\, \iota_4^{7} - 175\, \iota_4^{8} + 8\, \iota_4^{9} + \iota_4^{10}))))\ /$

$((-192 + 1396\, \iota_3 - 3038\, \iota_3^{2} + 3165\, \iota_3^{3} - 1818\, \iota_3^{4} + 579\, \iota_3^{5} - 85\, \iota_3^{6} + \iota_3^{8}) (8 - 64\, \iota_3 + 92\, \iota_3^{2} + 39\, \iota_3^{3} - 128\, \iota_3^{4} + 125\, \iota_3^{5} - 57\, \iota_3^{6} + 8\, \iota_3^{7} + \iota_3^{8}))))\ /$

$((-4 + 22\, \iota_2 - 15\, \iota_2^{2} + 2\, \iota_2^{3} + \iota_2^{4}) (6 + 2\, \iota_2 - 3\, \iota_2^{2} + 6\, \iota_2^{3} + \iota_2^{4})))\}, \{1, 0\}\}$

In[4]:= **recSol = SigmaReduce[recSol, n, ExtLowerBound → 1]**

Out[4]=
$$\Big\{\{0,\ 1\},\ \Big\{0,\ \frac{5184 + 10\,368\,n + 10\,368\,n^2 + 10\,745\,n^3 + 11\,499\,n^4 + 11\,499\,n^5 + 3833\,n^6}{3456\,n^3\,(1+n)^3} - \sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\Big\},$$

$$\Big\{0,\ \frac{57\,024 + 112\,320\,n + 112\,320\,n^2 + 116\,743\,n^3 + 115\,221\,n^4 + 111\,765\,n^5 + 37\,255\,n^6}{3456\,n^3\,(1+n)^3} - 11\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\right) + \sum_{\iota_1=1}^{n} \frac{1}{\iota_1^2}\Big\},$$

$$\Big\{0,\ \frac{6336 + 12\,224\,n + 12\,256\,n^2 + 12\,935\,n^3 + 11\,509\,n^4 + 10\,581\,n^5 + 3527\,n^6}{64\,n^3\,(1+n)^3} - 66\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\right) + 14\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^2}\right) - \sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\Big\},$$

$$\Big\{0,\ (177\,675\,989\,502\,318\,507\,987\,769\,006\,794\,000 - 1\,410\,712\,406\,669\,670\,719\,031\,723\,275\,496\,450\,n -$$
$$3\,439\,201\,029\,228\,791\,304\,796\,863\,424\,895\,400\,n^2 - 278\,184\,072\,616\,843\,994\,045\,760\,502\,422\,331\,n^3 -$$
$$883\,030\,032\,531\,882\,415\,215\,360\,348\,386\,824\,n^4 - 7\,513\,660\,184\,500\,149\,503\,676\,220\,207\,781\,286\,n^5 -$$
$$8\,002\,726\,448\,083\,248\,268\,192\,726\,344\,549\,124\,n^6 - 2\,000\,681\,612\,020\,812\,067\,048\,181\,586\,137\,281\,n^7)\,/$$
$$(118\,005\,405\,566\,765\,899\,551\,911\,942\,400\,000\,n^3\,(1+n)^4) + 10\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^4}\right) +$$
$$\left((-24\,175\,238\,997\,442\,478\,400 - 101\,208\,257\,838\,363\,114\,137\,n - 101\,208\,257\,838\,363\,114\,137\,n^2)\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\right)\right)\Big/$$
$$(8\,058\,412\,999\,147\,492\,800\,n\,(1+n)) + \frac{15\,030\,357\,843\,328\,637\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^2}\right)}{636\,612\,255\,552\,000} + \frac{1}{150\,508\,800\,n^3\,(1+n)^3}$$
$$(451\,526\,400 + 903\,052\,800\,n + 903\,052\,800\,n^2 - 735\,336\,127\,n^3 - 4\,012\,113\,981\,n^4 - 4\,012\,113\,981\,n^5 - 1\,337\,371\,327\,n^6)\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right) +$$
$$6\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\right)\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right) - 8\left(\sum_{\iota_1=1}^{n} \frac{\sum_{\iota_2=1}^{\iota_1} \frac{1}{\iota_2}}{\iota_1^3}\right)\Big\},$$

$$\Big\{0,\ (700\,234\,146\,825\,853\,061\,652\,231\,697\,542\,000 + 612\,491\,909\,087\,943\,228\,112\,655\,518\,897\,650\,n -$$
$$72\,997\,696\,738\,788\,093\,630\,806\,513\,650\,200\,n^2 + 542\,489\,232\,120\,728\,701\,146\,753\,995\,140\,367\,n^3 -$$

$$7\,155\,233\,842\,060\,606\,096\,274\,876\,982\,689\,032\, n^4 - 18\,523\,535\,178\,258\,533\,535\,196\,755\,578\,365\,698\, n^5 -$$

$$14\,357\,250\,578\,611\,080\,924\,459\,008\,196\,173\,932\, n^6 - 3\,589\,312\,644\,652\,770\,231\,114\,752\,049\,043\,483\, n^7\Big) \Big/$$

$$\left(145\,237\,422\,236\,019\,568\,679\,276\,236\,800\,000\, n^3 (1+n)^4\right) - \frac{(-1)^n}{2(1+n)^3} - \frac{5}{8}\left(\sum_{i_1=1}^{n} \frac{1}{i_1^4}\right) +$$

$$\left(\left(1\,859\,633\,769\,034\,036\,800 - 24\,715\,595\,986\,460\,348\,591\, n - 24\,715\,595\,986\,460\,348\,591\, n^2\right)\left(\sum_{i_1=1}^{n} \frac{1}{i_1^3}\right)\right) \Big/ \left(9\,918\,046\,768\,181\,529\,600\, n (1+n)\right) +$$

$$\frac{49\,542\,503\,717\,067\,073 \left(\sum_{i_1=1}^{n} \frac{1}{i_1^2}\right)}{2\,350\,568\,328\,192\,000} + \frac{1}{185\,241\,600\, n^3 (1+n)^3}$$

$$\left(-34\,732\,800 - 69\,465\,600\, n - 69\,465\,600\, n^2 - 153\,373\,849\, n^3 - 321\,190\,347\, n^4 - 321\,190\,347\, n^5 - 107\,063\,449\, n^6\right)\left(\sum_{i_1=1}^{n} \frac{1}{i_1}\right) -$$

$$\frac{3}{8}\left(\sum_{i_1=1}^{n} \frac{1}{i_1^3}\right)\left(\sum_{i_1=1}^{n} \frac{1}{i_1}\right) - \frac{1}{2}\left(\sum_{i_1=1}^{n} \frac{(-1)^{i_1}}{i_1^3}\right) - \frac{\sum_{i_1=1}^{n} \frac{(-1)^{i_1}}{i_1^2}}{2 n (1+n)} + \frac{1}{2}\left(\sum_{i_1=1}^{n} \frac{\sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1^3}\right) + \sum_{i_1=1}^{n} \frac{\sum_{i_2=1}^{i_1} \frac{(-1)^{i_2}}{i_2^2}}{i_1}\Bigg\},$$

$$\Big\{0,\ \Big(3\,172\,234\,341\,475\,250\,626\,930\,297\,481\,134\,513\,379\,369\,199\,146\,461\,714\,282\,409\,058\,443\,039\,193\,908\,156\,723\,878\,176\,000 +$$

$$13\,703\,554\,571\,889\,142\,208\,717\,900\,900\,024\,029\,018\,128\,403\,636\,121\,567\,746\,819\,142\,105\,887\,786\,951\,419\,281\,904\,288\,000\, n +$$

$$22\,633\,460\,555\,646\,280\,808\,897\,179\,070\,878\,879\,223\,803\,843\,391\,705\,379\,527\,299\,752\,322\,376\,282\,333\,308\,668\,827\,005\,500\, n^2 +$$

$$11\,304\,382\,367\,741\,888\,586\,734\,738\,749\,361\,753\,693\,950\,936\,232\,393\,349\,361\,895\,916\,271\,599\,894\,361\,356\,697\,645\,342\,117\, n^3 -$$

$$6\,986\,663\,480\,101\,931\,984\,122\,364\,874\,065\,777\,293\,195\,554\,539\,438\,227\,327\,641\,301\,200\,182\,615\,673\,505\,870\,929\,777\,032\, n^4 -$$

$$10\,205\,984\,114\,230\,730\,235\,014\,507\,892\,854\,746\,932\,757\,844\,192\,575\,565\,025\,725\,063\,670\,227\,880\,054\,214\,095\,527\,712\,798\, n^5 +$$

$$343\,656\,527\,009\,530\,370\,271\,509\,803\,386\,560\,965\,045\,735\,488\,494\,769\,691\,241\,444\,315\,621\,011\,511\,608\,295\,603\,350\,468\, n^6 +$$

$$85\,914\,131\,752\,382\,592\,567\,877\,450\,846\,640\,241\,261\,433\,872\,123\,692\,422\,810\,361\,078\,905\,252\,877\,902\,073\,900\,837\,617\, n^7\Big) \Big/$$

$$\left(31\,214\,471\,465\,127\,939\,229\,269\,120\,764\,957\,202\,733\,964\,008\,235\,142\,413\,082\,065\,770\,090\,529\,658\,371\,284\,992\,000\,000\, n^3 (1+n)^4\right) +$$

$$\frac{(-8\,506\,487\,402\,381\,487\,334\,439 - 36\,196\,399\,693\,653\,913\,720\,259\, n)(-1)^n}{13\,844\,956\,145\,636\,213\,192\,910 (1+n)^4} -$$

$$\frac{18\,616\,749\,581\,355\,305\,364\,061\,736\,932\,808\,331\,715 \left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^4}\right)}{108\,750\,658\,867\,754\,514\,577\,382\,646\,332\,440\,704} +$$

$$\Bigg(\big(20\,917\,701\,268\,652\,295\,809\,267\,579\,547\,713\,733\,159\,384\,876\,727\,482\,392\,030\,723\,365\,633\,500 +$$
$$52\,976\,355\,124\,310\,484\,968\,046\,618\,448\,080\,657\,301\,114\,394\,202\,075\,763\,794\,888\,248\,787\,991\, n +$$
$$52\,976\,355\,124\,310\,484\,968\,046\,618\,448\,080\,657\,301\,114\,394\,202\,075\,763\,794\,888\,248\,787\,991\, n^2\big) \left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\right)\Bigg) \Big/$$
$$(407\,305\,937\,953\,678\,391\,471\,694\,038\,954\,327\,303\,661\,676\,500\,524\,617\,715\,296\,209\,792\,000\, n\,(1+n)) -$$

$$\frac{1\,761\,581\,757\,523\,484\,659\,980\,748\,098\,142\,540\,570\,068\,606\,001\,349\,955\,558\,639\,878\,771 \left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^2}\right)}{16\,237\,493\,967\,058\,196\,596\,058\,193\,932\,913\,555\,018\,120\,254\,591\,220\,023\,959\,641\,600} +$$

$$\Bigg(\big(-1\,015\,141\,472\,348\,427\,762\,764\,626\,949\,709\,308\,784\,032\,307\,985\,302\,343\,392\,000 -$$
$$2\,030\,282\,944\,696\,855\,525\,529\,253\,899\,418\,617\,568\,064\,615\,970\,604\,686\,784\,000\, n -$$
$$2\,030\,282\,944\,696\,855\,525\,529\,253\,899\,418\,617\,568\,064\,615\,970\,604\,686\,784\,000\, n^2 +$$
$$1\,168\,764\,315\,806\,409\,784\,751\,529\,127\,984\,284\,444\,145\,041\,190\,597\,524\,494\,099\, n^3 +$$
$$7\,566\,858\,836\,812\,940\,405\,313\,095\,182\,790\,088\,468\,564\,355\,513\,001\,947\,050\,297\, n^4 +$$
$$7\,566\,858\,836\,812\,940\,405\,313\,095\,182\,790\,088\,468\,564\,355\,513\,001\,947\,050\,297\, n^5 +$$
$$2\,522\,286\,278\,937\,646\,801\,771\,031\,727\,596\,696\,156\,188\,118\,504\,333\,982\,350\,099\, n^6\big) \left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right)\Bigg) \Big/$$
$$\left(19\,766\,662\,896\,663\,217\,395\,576\,847\,477\,554\,475\,059\,862\,034\,507\,587\,584\,000\, n^3\,(1+n)^3\right) + \frac{(-1)^n \left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right)}{(1+n)^3} -$$

$$\frac{3\,723\,349\,916\,271\,061\,072\,812\,347\,386\,561\,666\,343 \left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^3}\right)\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right)}{36\,250\,219\,622\,584\,838\,192\,460\,882\,110\,813\,568} - 2\left(\sum_{\iota_1=1}^{n} \frac{(-1)^{\iota_1}}{\iota_1^4}\right) +$$

$$\left((20\,767\,434\,218\,454\,319\,789\,365 - 22\,351\,443\,548\,017\,700\,527\,349\, n - 22\,351\,443\,548\,017\,700\,527\,349\, n^2)\left(\sum_{\iota_1=1}^{n} \frac{(-1)^{\iota_1}}{\iota_1^3}\right)\right) \Big/$$

$$(13\,844\,956\,145\,636\,213\,192\,910\, n\,(1+n)) - 3\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right)\left(\sum_{\iota_1=1}^{n} \frac{(-1)^{\iota_1}}{\iota_1^3}\right) +$$

$$\left((6\,922\,478\,072\,818\,106\,596\,455 - 22\,351\,443\,548\,017\,700\,527\,349\, n - 36\,196\,399\,693\,653\,913\,720\,259\, n^2)\left(\sum_{\iota_1=1}^{n} \frac{(-1)^{\iota_1}}{\iota_1^2}\right)\right) \Big/$$

$$(13\,844\,956\,145\,636\,213\,192\,910\, n^2\,(1+n)^2) + \frac{\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right)\left(\sum_{\iota_1=1}^{n} \frac{(-1)^{\iota_1}}{\iota_1^2}\right)}{n\,(1+n)} +$$

$$\frac{3\,723\,349\,916\,271\,061\,072\,812\,347\,386\,561\,666\,343\left(\sum_{\iota_1=1}^{n} \frac{\sum_{\iota_2=1}^{\iota_1} \frac{1}{\iota_2}}{\iota_1^3}\right)}{27\,187\,664\,716\,938\,628\,644\,345\,661\,583\,110\,176} + 4\left(\sum_{\iota_1=1}^{n} \frac{(-1)^{\iota_1}\left(\sum_{\iota_2=1}^{\iota_1} \frac{1}{\iota_2}\right)}{\iota_1^3}\right) + \sum_{\iota_1=1}^{n} \frac{\sum_{\iota_2=1}^{\iota_1} \frac{(-1)^{\iota_2}}{\iota_2^2}}{\iota_1^2} +$$

$$\left((-6\,922\,478\,072\,818\,106\,596\,455 + 22\,351\,443\,548\,017\,700\,527\,349\, n + 22\,351\,443\,548\,017\,700\,527\,349\, n^2)\left(\sum_{\iota_1=1}^{n} \frac{\sum_{\iota_2=1}^{\iota_1} \frac{(-1)^{\iota_2}}{\iota_2^2}}{\iota_1}\right)\right) \Big/$$

$$(6\,922\,478\,072\,818\,106\,596\,455\, n\,(1+n)) + 2\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1}\right)\left(\sum_{\iota_1=1}^{n} \frac{\sum_{\iota_2=1}^{\iota_1} \frac{(-1)^{\iota_2}}{\iota_2^2}}{\iota_1}\right) - 4\left(\sum_{\iota_1=1}^{n} \frac{\left(\sum_{\iota_2=1}^{\iota_1} \frac{1}{\iota_2}\right)\left(\sum_{\iota_2=1}^{\iota_1} \frac{(-1)^{\iota_2}}{\iota_2^2}\right)}{\iota_1}\right)\Big\},\ \{1,\ 0\}\Big\}$$

In[5]:=
```
sol = FindLinearCombination[recSol,
  {3, {-63653/3888, -8802581/486000, -5234029/270000, -13978373/686000, -15827913259/746807040, -1578915745223/72013536000, -95194737209/4219543125}}, n, 7]
```

Out[5]=

$$-\frac{2\left(-132+106\,n+507\,n^2-571\,n^3-979\,n^4+383\,n^5+1080\,n^6+270\,n^7\right)}{27\,n^3\,(1+n)^4}+\frac{64\,(1+4\,n)\,(-1)^n}{9\,(1+n)^4}+\frac{80}{3}\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1^4}\right)-\frac{8\left(3+14\,n+14\,n^2\right)\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1^3}\right)}{3\,n\,(1+n)}+$$

$$\frac{1336}{27}\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1^2}\right)-\frac{4\left(-54-108\,n-108\,n^2+137\,n^3+627\,n^4+627\,n^5+209\,n^6\right)\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1}\right)}{27\,n^3\,(1+n)^3}-\frac{32\,(-1)^n\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1}\right)}{3\,(1+n)^3}+16\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1^3}\right)\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1}\right)+\frac{64}{3}\left(\sum_{\iota_1=1}^{n}\frac{(-1)^{\iota_1}}{\iota_1^4}\right)+$$

$$\frac{16\left(-9+10\,n+10\,n^2\right)\left(\sum_{\iota_1=1}^{n}\frac{(-1)^{\iota_1}}{\iota_1^3}\right)}{9\,n\,(1+n)}+32\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1}\right)\left(\sum_{\iota_1=1}^{n}\frac{(-1)^{\iota_1}}{\iota_1^3}\right)+\frac{16\left(-3+10\,n+16\,n^2\right)\left(\sum_{\iota_1=1}^{n}\frac{(-1)^{\iota_1}}{\iota_1^2}\right)}{9\,n^2\,(1+n)^2}-\frac{32\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1}\right)\left(\sum_{\iota_1=1}^{n}\frac{(-1)^{\iota_1}}{\iota_1^2}\right)}{3\,n\,(1+n)}-\frac{64}{3}\left(\sum_{\iota_1=1}^{n}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{1}{\iota_2}}{\iota_1^3}\right)-$$

$$\frac{128}{3}\left(\sum_{\iota_1=1}^{n}\frac{(-1)^{\iota_1}\left(\sum_{\iota_2=1}^{\iota_1}\frac{1}{\iota_2}\right)}{\iota_1^3}\right)-\frac{32}{3}\left(\sum_{\iota_1=1}^{n}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{(-1)^{\iota_2}}{\iota_2^2}}{\iota_1^2}\right)-\frac{32\left(-3+10\,n+10\,n^2\right)\left(\sum_{\iota_1=1}^{n}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{(-1)^{\iota_2}}{\iota_2^2}}{\iota_1}\right)}{9\,n\,(1+n)}-\frac{64}{3}\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1}\right)\left(\sum_{\iota_1=1}^{n}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{(-1)^{\iota_2}}{\iota_2^2}}{\iota_1}\right)+\frac{128}{3}\left(\sum_{\iota_1=1}^{n}\frac{\left(\sum_{\iota_2=1}^{\iota_1}\frac{1}{\iota_2}\right)\left(\sum_{\iota_2=1}^{\iota_1}\frac{(-1)^{\iota_2}}{\iota_2^2}\right)}{\iota_1}\right)$$

Further results in the article

- Extension 1: hypergeometric products $\rightarrow$ nested products

$$\mathbb{A} = \mathbb{K}(x)\underbrace{[y_1, y_1^{-1}]\dots[y_e, y_e^{-1}]}_{\text{nested products}}[s_1]\dots[s_u]$$

(including also nested products of roots of unity)

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(including also nested products of roots of unity)

Example:

$$\sum_{k=1}^{n} \frac{k(1+k^2)\left(2+2k+k^2+(k!)^2\prod_{i=1}^{k} i!\right) - \left(2+2k+k^2\right)k!\prod_{i=1}^{k} i!}{k(1+k^2)\left(2+2k+k^2\right)k!}$$

$$= -\frac{1}{2} + \frac{n!\prod_{i=1}^{n} i!}{2+2n+n^2} + \sum_{k=1}^{n} \frac{1}{k!}$$

Further results in the article

- Extension 1: hypergeometric products → nested products
- Extension 2: $\mathbb{K}(x) \to \mathbb{F}(x)$ where $\mathbb{F}(x)$ is a $\Pi\Sigma$-field (Karr81), i.e.,
  1. $\mathbb{F}(x) = \mathbb{K}(t_1)\dots(t_\lambda)$ is a rational function field,
  2. for $1 \leq i \leq \lambda$: one of the two cases hold:
     - $\sigma(t_i) = \alpha_i\, t_i$ with $\alpha_i \in \mathbb{K}(t_1)\dots(t_{i-1})$
     - $\sigma(t_i) = t_i + \beta_i$ with $\beta_i \in \mathbb{K}(t_1)\dots(t_{i-1})$
  3. the constants remain unchanged: $\text{const}_\sigma\mathbb{F}(x) = \mathbb{K}$.

  Special cases: q-hypergeometric products and mixed versions

Further results in the article

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  1. $\mathbb{F}(x) = \mathbb{K}(t_1)\dots(t_\lambda)$ is a rational function field,
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  3. the constants remain unchanged: $\mathrm{const}_\sigma \mathbb{F}(x) = \mathbb{K}$.

  Special cases: q-hypergeometric products and mixed versions

  Example: $x = k!$

$$\sum_{k=1}^{n} \frac{1+k+\Big(3+k(4+k(3+k))+(2+k)\Big(1+k+k^2-k(1+k)k!\sum_{i=1}^{k}\frac{1}{i!}\Big)k!^2\Big)k!^2}{(1+k)\big(1+(1+k)^2k!^2\big)k!^3\big(1+k!^2\big)}$$

$$= -\frac{-2+(1+n)n!+(1+n)^3(n!)^3}{2(1+n)n!+2(1+n)^3(n!)^3} + \frac{1}{1+(n!)^2+2n(n!)^2+n^2(n!)^2}\sum_{i=1}^{n}\frac{1}{i!} + \sum_{i=1}^{n}\frac{1}{(i!)^3}$$

Further results in the article

- Extension 1: hypergeometric products $\rightarrow$ nested products
- Extension 2: $\mathbb{K}(x) \rightarrow \mathbb{F}(x)$ where $\mathbb{F}(x)$ is a $\Pi\Sigma$-field (Karr81)
- Extension 3: Covering certain sums with "ugly" denominators

$$\mathbb{A} = \mathbb{F}(x) \underbrace{[y_1, y_1^{-1}] \dots [y_e, y_e^{-1}]}_{\text{nested products}} \quad \underbrace{[s_1] \dots [s_u]}_{\text{sums with nice den.}} \qquad \underbrace{[\tau_1] \dots [\tau_\mu]}_{\text{certain sums with ugly den.}}$$

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Example: Suppose we have adjoined in $\mathbb{A}$ the "ugly" sum:

$$T_1(n) = \sum_{j=1}^{n} \frac{(1-(-1)^j)j}{(3-3j+j^2)(j!)^2} \prod_{i=1}^{j} i! = \tau_1$$

Further results in the article

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Example: Suppose we have adjoined in $\mathbb{A}$ the "ugly" sum:

$$T_1(n) = \sum_{j=1}^{n} \frac{(1-(-1)^j)j}{(3-3j+j^2)(j!)^2} \prod_{i=1}^{j} i! = \tau_1$$

Then:

$$T_2(n) = \sum_{k=2}^{n} \Big((-1)^{k-1}(3+(-3+k)k)(1+(-1+k)k)k!^2 + (k-1)\Big(\prod_{i=1}^{k} i!\Big)\times \Big((-1+(-1)^k)k(1+n)(1+(-1+k)k) + (1+(-1)^k)(3+(-3+k)k)k!\Big)\Big/ \Big((-1+k)(3-3k+k^2)(1-k+k^2)k!^2\Big)$$

$\downarrow$simplify

$$T_2(n) = 2n - n\,T_1(n) + \frac{1+(-1)^n}{(1-n+n^2)n!}\prod_{i=1}^{n} i! + \sum_{i=2}^{n} \frac{(-1)^{i-1}}{i-1}$$

Further results in the article

- Extension 1: hypergeometric products $\to$ nested products
- Extension 2: $\mathbb{K}(x) \to \mathbb{F}(x)$ where $\mathbb{F}(x)$ is a $\Pi\Sigma$-field (Karr81)
- Extension 3: Covering certain sums with "ugly" denominators
- Extension 4: telescoping $\to$ parameterized telescoping
  (covering as special case Zeilberger's creative telescoping paradigm)

Further results in the article

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Possible future extensions/refinements

- Reduce w.r.t. refined statistics. E.g., not w.r.t. factors with certain degrees but w.r.t. certain factors which are equivalent
- the multivariate case: reduce factors not in $\mathbb{F}[x]$ but in $\mathbb{F}[x_1, \dots, x_l]$ (covering also the multibasic and mixed case)
- deal with more "ugly" $R\Pi\Sigma$-extensions
- global statements for nested sums