INEQUALITIES FOR THE PARTITION FUNCTION ARISING FROM TRUNCATED THETA SERIES

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Abstract. Positivity questions related to the partition function arising from classical theta identities have been studied in the combinatorial and q-series framework. Two such identities that emerge from truncation of Euler's pentagonal number theorem and an identity due to Gauss are the predominant ones among others. In this paper, we prove the asymptotic growth of coefficients of truncation of theta series directly from inequalities for the shifted partition function rather than taking a detour to Wright's circle method. Recently, Andrews and Merca conjectured that for n odd or k even,

$$M_k(n) \geq (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p(n-j(2j+1))-p(n-(j+1)(2j+1))\Big),$$

where $M_k(n)=(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\Big)$. We confirm the conjecture for all $n \geq N(k)$ with explicit information about $N(k)$ by determining the asymptotic growth of the difference between the alternating sums presented in the above inequality. This in turn shows that the conjecture of Andrews and Merca is even true for the excluded case; i.e., n even and k odd with $n > N(k)$. Moreover we modify the error bound in the asymptotic expansion of $M_k(n)$, obtained by Chern. We also present an unified structure to obtain asymptotic growths up to any order as we please for such alternating sums involving the partition function.

1. Introduction

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\pi_1, \pi_2, \dots, \pi_r$ such that $\sum_{i=1}^r \pi_i = n$. The partition $(\pi_1, \pi_2, \dots, \pi_r)$ will be denoted by π, and we shall write $\pi \vdash n$ to denote that π is a partition of n. The partition function $p(n)$ is the number of partitions of n. Due to Euler, the generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout the rest of this section, we follow the standard notation for the q-shifted factorial

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

One of the more well known result in the theory of partitions is Euler's pentagonal number theorem [1, Equation (1.3.1)] which states that

$$(q;q)_\infty = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}. \tag{1.1}$$

Applying the principle of mathematical induction and q-binomial theorem, Andrews and Merca [2] showed that the truncation of (1.1) has nonnegative coefficients.

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Theorem 1.1. *[2, Theorem 1.1] For $n > 0$, $k \geq 1$,*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\Big)=M_k(n), \tag{1.2}$$

where $M_k(n)$ is the number of partitions of n in which k is the least integer that is not a part and there are more parts $> k$ than there are $< k$.

As a corollary of Theorem 1.1, they proved that $M_k(n) \geq 0$ with strict inequality for $n \geq k(3k+1)/2$, see [2, Corollary 1.3]. Yee [19] gave a combinatorial proof of Theorem 1.1. Burnette and Kolitsch [12, 13] gave combinatorial interpretation for $M_k(n)$ using partition pairs. In [18], Wang explained $M_k(n)$ as the difference between size of two sets of partitions based on its rank enumeration. An asymptotic estimation for $M_k(n)$ was given by Chern [8] using Wright's circle method.

Theorem 1.2. *[8, Theorem 1.1] Let $\epsilon > 0$ be arbitrarily small. Then as $n \to \infty$, we have, for $k \ll n^{1/8-\epsilon}$,*

$$M_k(n)=\frac{\pi}{12\sqrt{2}}kn^{-3/2}e^{2\pi\sqrt{n}/\sqrt{6}}+O\Big(k^3n^{-7/4}e^{2\pi\sqrt{n}/\sqrt{6}}\Big). \tag{1.3}$$

Applying an extended version of Bailey's transform, Bachraoui [10, Corollary 1 and 2] obtained the following two inequalities for the partition function in the spirit of Andrews and Merca.

Apart from Euler's pentagonal number theorem, the following is another classical theta identity [1, Equation (2.2.13)] due to Gauss (or sometimes Jacobi):

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}=\sum_{j=0}^{\infty}(-q)^{j(j+1)/2}. \tag{1.4}$$

Starting from Rogers-Fine identity, Andrews and Merca [3] retrieved Theorem 1.1 and studying the truncated version of (1.4), obtained the following result.

Theorem 1.3. *[3, Theorem 1.9] For $n, k \geq 1$,*

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2}=1-(-1)^k\frac{(-q;q^2)_k}{(q^2;q^2)_k}\sum_{j=0}^{\infty}\frac{q^{k(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty}. \tag{1.5}$$

Consequently, they proved the following infinite family of inequalities for the partition function.

Corollary 1.4. *[3, Corollary 11] If at least one of n and k is odd,*

$$\widetilde{M}_k(n):=(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p(n-j(2j+1))-p(n-(j+1)(2j+1))\Big)\geq 0. \tag{1.6}$$

Ballantine, Merca, Passary, and Yee [4, Theorem 3] gave a combinatorial interpretation for $\widetilde{M}_k(n)$ in term of overpartitions. Andrews and Merca proposed the following conjecture with regards to $M_k(n)$ and $\widetilde{M}_k(n)$.

Conjecture 1.5. (*Andrews-Merca*)*[3] For n odd or k even,*

$$M_k(n)\geq \widetilde{M}_k(n). \tag{1.7}$$

In [15, Theorem 1.2], Merca and Katriel studied a family of non-trivial homogeneous partition inequalities from the framework of Prouhet-Tarry-Escott problem [9, Chapter XXIV] that arises in Diophantine equations. Using this set up, they proved that Conjecture 1.5 is true for k odd and for sufficiently large n.

The main motivation of this paper is to derive asymptotic growth of the aforementioned alternating sums involving the partition function. We construct an unified framework by employing the infinite family of inequalities obtained by the first author [5, Theorem 4.5] so as to get the desired asymptotic growth. Of course, the inequalities presented before are much stronger in the sense that it predicts the exact threshold,say $N(k)$ for n from which the inequality holds. For example, in context of Theorem 1.1, we already know that $M_k(n) > 0$ for all $n \geq k(3k+1)/2$ but here our goal is to get to the asymptotic growth. Nonetheless, we also derive an explicit threshold for n which is higher than the optimal one. Studies on truncated theta series identities already unfolded the combinatorial facets through the jargon of partitions, whereas in this paper, we unearth the other facet of such problems by studying asymptotic analysis for the partition function.

Asymptotic analysis for the partition function had begun with the work of Hardy and Ramanujan [11] in 1918 that reads:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{1.8}$$

Rademacher [16] improved the work of Hardy and Ramanujan by providing a convergent series for $p(n)$ and Lehmer [14] estimated the remainder term of the convergent series for $p(n)$. The Hardy-Ramanujan-Rademacher formula states that

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k}\right] + R_2(n,N), \tag{1.9}$$

where

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \quad A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor \frac{\mu}{k} \right\rfloor - \frac{1}{2}\right)\left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2}\right),$$

and

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)}\right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2\right]. \tag{1.10}$$

After Rademacher's work on the partition function, numerous research papers have been written on inequalities for the partition function. Recently Paule, Radu, Schneider and the first author [6] obtained a full asymptotic expansion of $p(n)$ along with estimations of error bounds. Based on their work, an infinite family of inequalities for shifted partition function $p(n-\ell)$ for $\ell \geq 0$ is given in [5, Theorem 4.5] which is the key machinery in proving all of the theorems stated below.

Theorem 1.6. *Define for all $k \geq 1$,*

$$\mathcal{M}_k^1(n) := \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-36\pi^2 + \frac{23\pi^2 - 216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}} \times \left(1296\pi^4 + \frac{31104\pi^2 - 2760\pi^4}{k^2} + \frac{31104 - 19872\pi^2 + 1681\pi^4}{k^4}\right).$$

Then for all $n > 121k^4$,

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\mathcal{M}_k^1(n) + \frac{\mathcal{E}_L^1(k)}{n^2}\right) < M_k(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\mathcal{M}_k^1(n) + \frac{\mathcal{E}_U^1(k)}{n^2}\right). \tag{1.11}$$

Explicit expressions for $\mathcal{E}_L^1(k)$ and $\mathcal{E}_L^1(k)$ are given in (3.11) and (3.14) respectively for k odd and even.

Corollary 1.7. *For $k \geq 1$ and $n > 121k^4$, as $n \to \infty$,*

$$M_k(n) \sim \frac{\pi e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}k + \frac{e^{\pi\sqrt{2n/3}}}{576\sqrt{3}n^2}k^3\Big(\frac{23\pi^2 - 216}{k^2} - 36\pi^2\Big) + O\Big(\frac{e^{\pi\sqrt{2n/3}}}{n^{5/2}}k^5\Big). \tag{1.12}$$

Remark 1.8. *Rewriting the asymptotic expansion* (1.12) *of $M_k(n)$ in the following way:*

$$M_k(n) \sim \frac{\pi e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}k + O\Big(\frac{e^{\pi\sqrt{2n/3}}}{n^2}k^3\Big) \quad \text{as} \quad n \to \infty,$$

we observe that the growth of error bound is in indeed the optimal one in comparison with Theorem 1.2.

Remark 1.9. *From the lower bound in* (1.11)*, one can retrieve positivity of $M_k(n)$ for $n > f_1(k)$ with minimal $f_1(k)$ such that $\mathcal{M}_k^1(n) + \frac{\mathcal{E}_L^1(k)}{n^2} > 0$ holds for all $n > f_1(k)$.*

Theorem 1.10. *Define for all $k \geq 1$,*

$$\mathcal{M}_k^2(n) := \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-48\pi^2 + \frac{35\pi^2 - 216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}} \times \left(2304\pi^4 + \frac{41472\pi^2 - 5472\pi^4}{k^2} + \frac{31104 - 30240\pi^2 + 3385\pi^4}{k^4}\right).$$

Then for all $n > 169k^4$,

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^2(n) + \frac{\mathcal{E}_L^2(k)}{n^2}\Big) < \widetilde{M}_k(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^2(n) + \frac{\mathcal{E}_U^2(k)}{n^2}\Big). \tag{1.13}$$

Explicit expressions for $\mathcal{E}_L^2(k)$ and $\mathcal{E}_L^2(k)$ are given in (3.27) and (3.30) respectively for k odd and even.

Corollary 1.11. *For $k \geq 1$ and $n > 169k^4$, as $n \to \infty$,*

$$\widetilde{M}_k(n) \sim \frac{\pi e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}k + \frac{e^{\pi\sqrt{2n/3}}}{576\sqrt{3}n^2}k^3\Big(\frac{35\pi^2 - 216}{k^2} - 48\pi^2\Big) + O\Big(\frac{e^{\pi\sqrt{2n/3}}}{n^{5/2}}k^5\Big). \tag{1.14}$$

Remark 1.12. *Similar to Remark 1.16, from the lower bound in* (1.13)*, one can prove positivity of $\widetilde{M}_k(n)$ for $n > f_2(k)$ such that $\mathcal{M}_k^2(n) + \frac{\mathcal{E}_L^2(k)}{n^2} > 0$ holds for all $n > f_2(k)$.*

Theorem 1.13. *Define for all $k \geq 1$,*

$$\mathcal{M}_k^3(n) := \mathcal{M}_k^1(n) - \mathcal{M}_k^2(n) = \frac{k^3 - k}{12n} - \frac{k^5}{6912\sqrt{6}\pi n^{3/2}}\left(1008\pi^4 + \frac{10368\pi^2 - 2712\pi^4}{k^2} + \frac{-10368\pi^2 + 1704\pi^4}{k^4}\right).$$

Then for all $n > 169k^4$,

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^3(n) + \frac{\mathcal{E}_L^1(k) - \mathcal{E}_U^2(k)}{n^2}\Big) < M_k(n) - \widetilde{M}_k(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^3(n) + \frac{\mathcal{E}_U^1(k) - \mathcal{E}_L^2(k)}{n^2}\Big). \tag{1.15}$$

Proof. Theorems 1.6 and 1.7 immediately imply (1.15). ☐

Corollary 1.14. *For $k \geq 1$ and $n > 169k^4$, as $n \to \infty$,*

$$M_k(n) - \widetilde{M}_k(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{48\sqrt{3}n^2}(k^3 - k) + O\Big(\frac{e^{\pi\sqrt{2n/3}}}{n^{5/2}}k^5\Big). \tag{1.16}$$

Remark 1.15. *Proving $M_k(n) > \widetilde{M}_k(n)$ for $n \geq N(k)$, it is enough to show that $\mathcal{M}_k^3(n) + \dfrac{\mathcal{E}_L^1(k) - \mathcal{E}_U^2(k)}{n^2} > 0$ holds for all $n \geq N(k)$.*

Remark 1.16. *Note that for $k = 1$, $M_k(n) - \widetilde{M}_k(n) = 0$ because $M_k(n) = \widetilde{M}_k(n) = p(n) - p(n-1)$, whereas for all $k \geq 2$, (1.15) suggests that $M_k(n) - \widetilde{M}_k(n)$ is positive for $n \geq N(k)$. This observation helps us to relax the condition given in Conjecture 1.5; i.e., instead of restricting to either n odd or k even, we can assume for all n and k with $n \geq N(k)$ that subsumes the excluded case k odd and n even. Still it is worthwhile to point out that whenever we consider n odd or k even, (1.7) is true for all $n \geq 1$ and $k \geq 1$. But when we assume the case k odd and n even, (1.5) doesn't hold for all $n, k \geq 1$, in other words, it remains to determine the optimal $N(k)$.*

By numerical verification with Mathematica, we listed down the values of $(N(k))_{1\leq k\leq 20}$ such that $M_{2k+1}(2n) > \widetilde{M}_{2k+1}(2n)$ for all $n \geq N(k)$.

k	1	2	3	4	5	6	7	8	9	10
$N(k)$	11	28	54	88	129	179	237	303	376	458

k	11	12	13	14	15	16	17	18	19	20
$N(k)$	548	646	752	866	988	1118	1256	1402	1558	1719

Based on the above data, a rough estimation predicts that as k become larger,

$$N(k) \approx \Big\lfloor 4k^2 + 7k - \sqrt{k}\log k \Big\rfloor - \Big\lfloor \frac{k}{3} \Big\rfloor := N_c(k).$$

Table of $N_c(k)$ is as follows:

k	1	2	3	4	5	6	7	8	9	10
$N_c(k)$	11	29	54	88	130	179	237	304	377	459

k	11	12	13	14	15	16	17	18	19	20
$N_c(k)$	550	647	753	868	989	1119	1258	1403	1558	1720

Extending the assumption of Conjecture 1.5, we propose the following question:

Problem 1.17. *For all $k \geq 1$ and $n \geq N_c(k)$, does the following inequality*

$$M_{2k+1}(2n) > \widetilde{M}_{2k+1}(2n) \tag{1.17}$$

hold?

The rest of the paper is organized as follows. In Section 2, we give all the necessary definitions and inequalities for $p(n - \ell)$ for all $\ell \geq 0$ (see Theorem 2.5 below) so as to ease to follow the later section. Section 3 presents the proofs of Theorems 1.6 and 1.7.

2. Preliminaries

First, we shall recall a few definitions from [5] which will be useful in the estimations worked out in Section 3.

Definition 2.1. *Following [5, Theorem 3.2], for $k \in \mathbb{Z}_{\geq 2}$, we define*

$$\widehat{g}(k) := \frac{1}{24}\left(\frac{36}{\pi^2}\cdot \nu(k)^2 + 1\right), \tag{2.1}$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$.

Definition 2.2. *[5, Definition 3.42] For all $k \geq 1$ and $\ell \geq 0$, define*

$$n_0(k,\ell) = \max_{k\geq 1, \ell \geq 0}\left\{\frac{(24\ell+1)^2}{16}, \frac{(k+3)(24\ell+1)}{24}\right\}.$$

Definition 2.3. *[5, Equation (3.45)] For all $\ell \geq 0$ and $t \geq 0$,*

$$g(t,\ell) = \frac{(1+24\ell)^t}{(-4\sqrt{6})^t}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}\frac{1}{(1+24\ell)^k}. \tag{2.2}$$

Definition 2.4. *[5, Definition 4.4] Let $g(t,\ell)$ be as in (2.1). If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$, define*

$$\mathcal{L}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\left(\frac{1}{\sqrt{n}}\right)^t + \frac{L(w,\ell)}{\sqrt{n}^w} \quad \text{and} \quad \mathcal{U}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\left(\frac{1}{\sqrt{n}}\right)^t + \frac{U(w,\ell)}{\sqrt{n}^w}.$$

The explicit expressions for $L(w,\ell)$ and $U(w,\ell)$ are given in [5, Definition 4.1].

Theorem 2.5. *[5, Theorem 4.5] For $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$ and $n > \max\{\widehat{g}(w)+\ell, n_0(w,\ell)\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell). \tag{2.3}$$

3. Proof of Theorems 1.6-1.13

Proof of Theorem 1.6: Let $k \geq 1$ be an odd integer. Following (1.2), we write

$$M_{2k+1}(n) = M^e_{2k+1}(n) - M^o_{2k+1}(n), \tag{3.1}$$

where

$$M^e_{2k+1}(n) = \sum_{j=0}^{k}\Big(p(n-j(6j+1)) - p(n-j(6j+5)-1)\Big)$$

and

$$M^o_{2k+1}(n) = \sum_{j=0}^{k-1}\Big(p(n-(2j+1)(3j+2)) - p(n-(2j+1)(3j+4)-1)\Big).$$

Applying Theorem 2.5 with $w=4$, we obtain

$$M^e_{2k+1}(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k}\sum_{t=0}^{3}\Big(g(t,j(6j+1)) - g(t,j(6j+5)+1)\Big)\frac{1}{\sqrt{n}^t} + \frac{U^e_1(2k+1)}{n^2}\right) \tag{3.2}$$

and

$$M^e_{2k+1}(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k}\sum_{t=0}^{3}\Big(g(t,j(6j+1)) - g(t,j(6j+5)+1)\Big)\frac{1}{\sqrt{n}^t} + \frac{L^e_1(2k+1)}{n^2}\right), \tag{3.3}$$

with

$$L^e_1(2k+1) = \sum_{j=0}^{k} L\big(4, j(6j+1)\big) - U\big(4, j(6j+5)+1\big) \tag{3.4}$$

and

$$U_1^e(2k+1)=\sum_{j=0}^{k}U\big(4,j(6j+1)\big)-L\big(4,j(6j+5)+1\big) \tag{3.5}$$

Analogously, for $M_{2k+1}^o(n)$, we get

$$M_{2k+1}^o(n)<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k-1}\sum_{t=0}^{3}\Big(g(t,(2j+1)(3j+2))-g(t,(2j+1)(3j+4)+1)\Big)\frac{1}{\sqrt{n}^t}+\frac{U_1^o(2k+1)}{n^2}\right) \tag{3.6}$$

and

$$M_{2k+1}^o(n)>\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k-1}\sum_{t=0}^{3}\Big(g(t,(2j+1)(3j+2))-g(t,(2j+1)(3j+4)+1)\Big)\frac{1}{\sqrt{n}^t}+\frac{L_1^o(2k+1)}{n^2}\right), \tag{3.7}$$

with

$$L_1^o(2k+1)=\sum_{j=0}^{k-1}L\big(4,(2j+1)(3j+2)\big)-U\big(4,(2j+1)(3j+4)+1\big) \tag{3.8}$$

and

$$U_1^o(2k+1)=\sum_{j=0}^{k-1}U\big(4,(2j+1)(3j+2)\big)-L\big(4,(2j+1)(3j+4)+1\big). \tag{3.9}$$

Combining (3.2)-(3.9) and applying to (3.1), if follows that

$$\frac{\mathcal{E}_L^1(2k+1)}{n^2}<\frac{M_{2k+1}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)}-\sum_{j=0}^{2k}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(3j+1)/2)-g(t,j(3j+5)/2+1)}{\sqrt{n}^t}<\frac{\mathcal{E}_U^1(2k+1)}{n^2}, \tag{3.10}$$

with

$$\mathcal{E}_L^1(2k+1)=L_1^e(2k+1)-U_1^o(2k+1)\ \text{ and }\ \mathcal{E}_U^1(2k+1)=U_1^e(2k+1)-L_1^o(2k+1). \tag{3.11}$$

Next assume $k\geq 1$ is even. We split $M_{2k}(n)$ as follows:

$$M_{2k}(n)=-M_{2k}^e(n)+M_{2k}^o(n), \tag{3.12}$$

with

$$M_{2k}^e(n)=\sum_{j=0}^{k-1}\Big(p(n-j(6j+1))-p(n-j(6j+5)-1)\Big)$$

and

$$M_{2k}^o(n)=\sum_{j=0}^{k-1}\Big(p(n-(2j+1)(3j+2))-p(n-(2j+1)(3j+4)-1)\Big).$$

Applying (2.3) separately to $M_{2k}^e(n)$ and $M_{2k}^o(n)$, we get

$$\frac{\mathcal{E}_L^1(2k)}{n^2}<\frac{M_{2k}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)}+\sum_{j=0}^{2k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(3j+1)/2)-g(t,j(3j+5)/2+1)}{\sqrt{n}^t}<\frac{\mathcal{E}_U^1(2k)}{n^2}, \tag{3.13}$$

where

$$\mathcal{E}_L^1(2k)=L_1^o(2k)-U_1^e(2k)\ \text{ and }\ \mathcal{E}_U^1(2k)=U_1^o(2k)-L_1^e(2k), \tag{3.14}$$

with

$$\begin{aligned} L_1^e(2k) &= \sum_{j=0}^{k-1} L(4, j(6j+1)) - U(4, j(6j+5)+1) \\ U_1^e(2k) &= \sum_{j=0}^{k-1} U(4, j(6j+1)) - L(4, j(6j+5)+1) \\ L_1^o(2k) &= \sum_{j=0}^{k-1} L(4, (2j+1)(3j+2)) - U(4, (2j+1)(3j+4)+1) \\ U_1^o(2k) &= \sum_{j=0}^{k-1} U(4, (2j+1)(3j+2)) - L(4, (2j+1)(3j+4)+1). \end{aligned}$$

Define $n_1(k) := \max\Big\{\widehat{g}(4) + (k-1)(3k+5)/2 + 1, n_0\Big(4, (k-1)(3k+5)/2+1\Big)\Big\}$. Putting (3.10) and (3.13) together, for all $n > n_1(k)$, it follows that

$$\frac{\mathcal{E}_L^1(k)}{n^2} < \frac{M_k(n)}{\Big(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\Big)} - (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j \sum_{t=0}^{3} \frac{g(t, j(3j+1)/2) - g(t, j(3j+5)/2+1)}{\sqrt{n}^t} < \frac{\mathcal{E}_U^1(k)}{n^2}. \tag{3.15}$$

Following (2.2), we get

$$\begin{aligned} &(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j \sum_{t=0}^{3} \frac{g(t, j(3j+1)/2) - g(t, j(3j+5)/2+1)}{\sqrt{n}^t} \\ &= \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-36\pi^2 + \frac{23\pi^2 - 216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}} \times \qquad (3.16) \\ &\qquad \left(1296\pi^4 + \frac{31104\pi^2 - 2760\pi^4}{k^2} + \frac{31104 - 19872\pi^2 + 1681\pi^4}{k^4}\right) \\ &= \mathcal{M}_k^1(n). \end{aligned}$$

Finally, it is easy to verify that for all $k \geq 1$,

$$n_1(k) \leq 121k^4.$$

This finishes the proof of Theorem 1.6. $\square$

Proof of Theorem 1.7: Assume $k \geq 1$ is odd. Following (1.6), rewrite

$$\widetilde{M}_{2k+1}(n) = \widetilde{M}^e_{2k+1}(n) - \widetilde{M}^o_{2k+1}(n), \tag{3.17}$$

where

$$\widetilde{M}^e_{2k+1}(n) = \sum_{j=0}^{k}\Big(p(n - 2j(4j+1)) - p(n-(2j+1)(4j+1))\Big)$$

and

$$\widetilde{M}^o_{2k+1}(n) = \sum_{j=0}^{k-1}\Big(p(n-(2j+1)(4j+3)) - p(n-(2j+2)(4j+3))\Big).$$

Applying Theorem 2.5 with $w = 4$, it follows that

$$\widetilde{M}^e_{2k+1}(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k}\sum_{t=0}^{3}\Big(g(t, 2j(4j+1)) - g(t, (2j+1)(4j+1))\Big)\frac{1}{\sqrt{n}^t} + \frac{U_2^e(2k+1)}{n^2}\right) \tag{3.18}$$

and

$$\widetilde{M}^e_{2k+1}(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k}\sum_{t=0}^{3}\Big(g(t,2j(4j+1))-g(t,(2j+1)(4j+1))\Big)\frac{1}{\sqrt{n}^t}+\frac{L^e_2(2k+1)}{n^2}\right), \quad (3.19)$$

with

$$L^e_2(2k+1) = \sum_{j=0}^{k} L\big(4, 2j(4j+1)\big) - U\big(4,(2j+1)(4j+1)\big) \quad (3.20)$$

and

$$U^e_2(2k+1) = \sum_{j=0}^{k} U\big(4, 2j(4j+1)\big) - L\big(4,(2j+1)(4j+1)\big) \quad (3.21)$$

Similarly for $\widetilde{M}^o_{2k+1}(n)$, we obtain

$$\widetilde{M}^o_{2k+1}(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k-1}\sum_{t=0}^{3}\Big(g(t,(2j+1)(4j+3))-g(t,(2j+2)(4j+3))\Big)\frac{1}{\sqrt{n}^t}+\frac{U^o_2(2k+1)}{n^2}\right) \quad (3.22)$$

and

$$\widetilde{M}^o_{2k+1}(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k-1}\sum_{t=0}^{3}\Big(g(t,(2j+1)(4j+3))-g(t,(2j+2)(4j+3))\Big)\frac{1}{\sqrt{n}^t}+\frac{L^o_2(2k+1)}{n^2}\right), \quad (3.23)$$

with

$$L^o_2(2k+1) = \sum_{j=0}^{k-1} L\big(4,(2j+1)(4j+3)\big) - U\big(4,(2j+2)(4j+3)\big) \quad (3.24)$$

and

$$U^o_2(2k+1) = \sum_{j=0}^{k-1} U\big(4,(2j+1)(4j+3)\big) - L\big(4,(2j+2)(4j+3)\big). \quad (3.25)$$

Applying (3.18)-(3.25) to (3.17), if follows that

$$\frac{\mathcal{E}^2_L(2k+1)}{n^2} < \frac{\widetilde{M}_{2k+1}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} - \sum_{j=0}^{2k}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1))-g(t,(j+1)(2j+1))}{\sqrt{n}^t} < \frac{\mathcal{E}^2_U(2k+1)}{n^2}, \quad (3.26)$$

with

$$\mathcal{E}^2_L(2k+1) = L^e_2(2k+1) - U^o_2(2k+1) \quad \text{and} \quad \mathcal{E}^2_U(2k+1) = U^e_2(2k+1) - L^o_2(2k+1). \quad (3.27)$$

Now assume $k \geq 1$ is even. Split $\widetilde{M}_{2k}(n)$ as follows:

$$\widetilde{M}_{2k}(n) = -\widetilde{M}^e_{2k}(n) + \widetilde{M}^o_{2k}(n), \quad (3.28)$$

with

$$\widetilde{M}^e_{2k}(n) = \sum_{j=0}^{k-1}\Big(p(n-2j(4j+1)) - p(n-(2j+1)(4j+1))\Big)$$

and

$$\widetilde{M}^o_{2k}(n) = \sum_{j=0}^{k-1}\Big(p(n-(2j+1)(4j+3)) - p(n-(2j+2)(4j+3))\Big).$$

Applying (2.3) to $\widetilde{M}^e_{2k}(n)$ and $\widetilde{M}^o_{2k}(n)$, it follows that

$$\frac{\mathcal{E}^2_L(2k)}{n^2} < \frac{\widetilde{M}_{2k}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} + \sum_{j=0}^{2k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1))-g(t,(j+1)(2j+1))}{\sqrt{n}^t} < \frac{\mathcal{E}^2_U(2k)}{n^2}, \tag{3.29}$$

where

$$\mathcal{E}^2_L(2k) = L^o_2(2k) - U^e_2(2k) \quad \text{and} \quad \mathcal{E}^2_U(2k) = U^o_2(2k) - L^e_2(2k), \tag{3.30}$$

with

$$\begin{aligned}
L^e_2(2k) &= \sum_{j=0}^{k-1} L(4,2j(4j+1)) - U(4,(2j+1)(4j+1))\\
U^e_2(2k) &= \sum_{j=0}^{k-1} U(4,2j(4j+1)) - L(4,(2j+1)(4j+1))\\
L^o_2(2k) &= \sum_{j=0}^{k-1} L(4,(2j+1)(4j+3)) - U(4,(2j+2)(4j+3))\\
U^o_2(2k) &= \sum_{j=0}^{k-1} U(4,(2j+1)(4j+3)) - L(4,(2j+2)(4j+3)).
\end{aligned}$$

Define $n_2(k) := \max\left\{\widehat{g}(4)+k(2k-1), n_0\left(4,k(2k-1)\right)\right\}$. Combining (3.26) and (3.29), for all $n > n_2(k)$, it follows that

$$\frac{\mathcal{E}^2_L(k)}{n^2} < \frac{\widetilde{M}_k(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} - (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1))-g(t,(j+1)(2j+1))}{\sqrt{n}^t} < \frac{\mathcal{E}^2_U(k)}{n^2}. \tag{3.31}$$

Following (2.2), we have

$$\begin{aligned}
&(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1))-g(t,(j+1)(2j+1))}{\sqrt{n}^t}\\
&= \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-48\pi^2 + \frac{35\pi^2-216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}}\times\\
&\qquad\left(2304\pi^4 + \frac{41472\pi^2-5472\pi^4}{k^2} + \frac{31104-30240\pi^2+3385\pi^4}{k^4}\right)\\
&= \mathcal{M}^2_k(n).
\end{aligned} \tag{3.32}$$

We conclude the proof of Theorem 1.7 by verifying that for all $k \geq 1$, $n_2(k) \leq 169k^4$. □

4. Conclusion

We conclude this paper by noting down a few possible follow ups.

(1) Extending the inequality (3.15) (resp. (3.31)) by letting $w \to \infty$, we obtain the full asymptotic expansion of $M_k(n)$ (resp. of $\widetilde{M}_k(n)$).

(2) We observe that all of the aforementioned inequalities with regard to the alternating sums for the partition function can be considered under the following framework:

$$\sum_{i=1}^{T} p(n+s_i) \geq \sum_{i=1}^{T} p(n+r_i),$$

where s_i, r_i are non-positive integers for all $1 \leq i \leq T$. In order to prove such inequalities, it is enough to choose the appropriate w in Theorem 2.5 and carry out similar work as done in Section 3. For the choice of w, it suffices to take the minimal $w_0 \geq 1$ such that $\sum_{i=1}^{T} g(w_0, s_i) - g(w_0, r_i) \neq 0$, where $g(t, \ell)$ as in (2.2).

(3) Wang and Yee [17, Theorem 1.2] considered the sum representation of $(q;q)_\infty^2$ due to Hecke and showed positivity of the following alternating sum in the 2-colored partition function (denoted by $pp(n)$):

$$(-1)^m \sum_{n=0}^{m} \sum_{j=-n}^{n} (-1)^j \Big(pp(N_j - n(2n+1)) - pp(N_j - (n+1)(2n+1)) \Big), \tag{4.1}$$

where $N_j = N + j(3j+1)/2$. Recently Bringmann et. al. [7] studied the asymptotic expansion of k-colored partition function. Setting $k = 2$, one has the asymptotic expansion for $pp(n)$ and working out to derive the infinite family of inequalities for $pp(n-\ell)$ as in Theorem 2.5 which in turn finally show the asymptotic growth of (4.1). Whereas for $k = 3$, similar synthesis for the 3-colored partitions can be done to derive the asymptotic growth of

$$J_k(n) = (-1)^k \sum_{j=0}^{k} (-1)^j (2j+1) t\Big(n - j(j+1)/2\Big),$$

given in [3].

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