

Mathematical Structures in Feynman Integrals
Emmy Noether Campus, Siegen, February 15, 2023

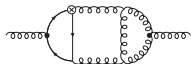
Computer Algebra Methods for Feynman Integrals

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

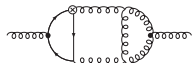


Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals



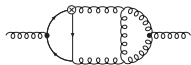
behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



behavior of particles



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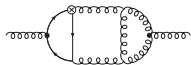
Feynman integrals

DESY

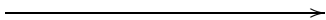
$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



behavior of particles



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Feynman integrals

DESY



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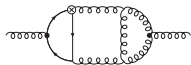
complicated
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expression in
special functions

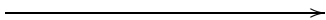


RISC
(Sigma-package)

Evaluation of Feynman Integrals



behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals



LHC at CERN

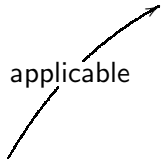
DESY



$$\sum f(N, \epsilon, k)$$

complicated multi-sums

applicable



expression in special functions



RISC
(Sigma-package)

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

||?

$$F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^0 + \dots$$

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon} k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon} k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon} k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta \cdot k_3)^N}{k_2^4 ((k_1 - k_3)^2 - m^2) (k_1 - k_2)^2 ((k_3 - p)^2 - m^2)}$$

||

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) \times \\ \times B\left(2 + k, \frac{\varepsilon}{2}\right) B(-\varepsilon + k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

where

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon} k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon} k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon} k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta \cdot k_3)^N}{k_2^4 ((k_1 - k_3)^2 - m^2) (k_1 - k_2)^2 ((k_3 - p)^2 - m^2)}$$

$$\parallel$$

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) \times$$

$$\underbrace{\times B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}}_{= f_{-3}(N, k)\varepsilon^{-3} + f_{-2}(N, k)\varepsilon^{-2} + f_{-1}(N, k)\varepsilon^{-1} + \dots}$$

for general expansion methods see

J. Blümlein, CS, M. Saragnese, 2021. arXiv:2111.15501 [math-ph]

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon} k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon} k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon} k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta \cdot k_3)^N}{k_2^4 ((k_1 - k_3)^2 - m^2) (k_1 - k_2)^2 ((k_3 - p)^2 - m^2)}$$

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$$\parallel$$

$$\left(\sum_{k=1}^N f_{-3}(N, k)\right)\varepsilon^{-3} + \left(\sum_{k=1}^N f_{-2}(N, k)\right)\varepsilon^{-2} + \left(\sum_{k=1}^N f_{-1}(N, k)\right)\varepsilon^{-1} + \dots$$

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

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Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left(\frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

where

$$S_a(N) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^a} \quad \text{and} \quad \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left(\frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

↓ (summation package Sigma.m)

$$\begin{aligned} & (16N^3 + 144N^2 + 413N + 384)(N+1)^2 F_{-1}(N) \\ & - (N+2)(2N+5)(16N^3 + 112N^2 + 221N + 113) F_{-1}(N+1) \\ & + (N+3)^2(16N^3 + 96N^2 + 173N + 99) F_{-1}(N+2) \\ & = \frac{1}{2}(4N^2 + 21N + 29)\zeta_2 + \frac{-64N^5 - 500N^4 - 1133N^3 + 203N^2 + 3516N + 3090}{3(N+2)(N+3)} \end{aligned}$$

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↓ (summation package Sigma.m)

$$\begin{aligned} & \left\{ c_1 \frac{1-4N}{N+1} + c_2 \frac{-14N-13}{(N+1)^2} \right. \\ & + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} \\ & \left. + \frac{175N^2 + 334N + 155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \mid c_1, c_2 \in \mathbb{Q} \right\} \end{aligned}$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left(\frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$



$$\left\{ c_1 \frac{1-4N}{N+1} + c_2 \frac{-14N-13}{(N+1)^2} + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} + \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \mid c_1, c_2 \in \mathbb{Q} \right\}$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left(\frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

|| (recurrence finding and solving)

$$\begin{aligned} & \left(\frac{1}{12} - \frac{1}{8}\zeta_2 \right) \frac{1-4N}{N+1} + 1 \frac{-14N-13}{(N+1)^2} \\ & + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} \\ & + \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \end{aligned}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^N f(N, k);$$

$f(N, k)$: indefinite nested product-sum in k ;
 N : extra parameter

FIND a recurrence for $F(N)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
 indefinite nested product-sum expressions.

$$a_0(N)F(N) + \dots + a_d(N)F(N + d) = h(N);$$

FIND all solutions expressible by **indefinite nested products/sums**

(Abramov/Bronstein/Petkovšek/CS, 2021)

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Special cases:

$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j} \quad (\text{harmonic sums})$$

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Special cases:

$$\sum_{k=1}^n \frac{2^k}{k} \sum_{i=1}^k \frac{2^{-i}}{i} \sum_{j=1}^i \frac{S_1(j)}{j}$$

(generalized harmonic sums)

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(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{k=1}^n \frac{1}{(1+2k)^2} \sum_{j=1}^k \frac{1}{j^2} \sum_{i=1}^j \frac{1}{1+2i} \quad (\text{cyclotomic harmonic sums})$$

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FIND all solutions expressible by indefinite nested products/sums

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Special cases:

$$\sum_{j=1}^n \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \quad (\text{binomial sums})$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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FIND all solutions expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{h=1}^n 2^{-2h} (1 - \eta)^h \binom{2h}{h} \sum_{k=1}^h \frac{2^{2k}}{k^2 \binom{2k}{k}} \quad (\text{generalized binomial sums})$$

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(Abramov/Bronstein/Petkovšek/CS, 2021)

A more general example:

$$\sum_{k=1}^n \left(\prod_{i=1}^k \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^k \frac{1}{j \binom{4j}{3j}^2}$$

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FIND all solutions expressible by **indefinite nested products/sums**

(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a “closed form”

$F(N)$ =combined solutions in terms of **indefinite nested sums**.

Sigma.m is based on difference ring/field theory

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In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[4]:= **mySum** =

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\epsilon\gamma}{2}} \left(-2 - \frac{3\epsilon}{2}\right)! B\left[2+k, \frac{\epsilon}{2}\right] B[-\epsilon+k, -\epsilon] B\left(1 - \frac{\epsilon}{2} + k, 1 + \frac{\epsilon}{2}\right) \binom{N}{k};$$

In[5]:= **EvaluateMultiSum**[mySum, {}, {N}, {1}, **ExpandIn** → {ε, -3, -3}]

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **mySum** =

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\epsilon\gamma}{2}} \left(-2 - \frac{3\epsilon}{2}\right)! B\left[2+k, \frac{\epsilon}{2}\right] B[-\epsilon+k, -\epsilon] B\left(1 - \frac{\epsilon}{2} + k, 1 + \frac{\epsilon}{2}\right) \binom{N}{k};$$

In[5]:= **EvaluateMultiSum**[**mySum**, {}, {N}, {1}, **ExpandIn** → { ϵ , -3, -3}]

$$\text{Out[5]} = \left\{ \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)} \right\}$$

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[5]:= **EvaluateMultiSum**[**mySum**, {}, {N}, {1}, **ExpandIn** → { ϵ , -3, -2}]

$$\text{Out[5]} = \left\{ \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)}, \right. \\ \left. - \frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right\}$$

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$$\sum_{k=1}^N (-1)^k e^{-\frac{3\epsilon\gamma}{2}} \left(-2 - \frac{3\epsilon}{2}\right)! B\left[2+k, \frac{\epsilon}{2}\right] B[-\epsilon+k, -\epsilon] B\left(1 - \frac{\epsilon}{2} + k, 1 + \frac{\epsilon}{2}\right) \binom{N}{k};$$

In[5]:= **EvaluateMultiSum**[**mySum**, {}, {N}, {1}, **ExpandIn** → { ϵ , -3, -1}]

$$\begin{aligned} \text{Out[5]} = & \left\{ \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)}, \right. \\ & - \frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1}, \\ & \left. \left(\frac{1}{12} - \frac{1}{8}\zeta(2) \right) \frac{1-4N}{N+1} + \frac{-14N-13}{(N+1)^2} + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \right. \\ & \left. \frac{(14N+13)S_1(N)}{3(N+1)^2} + \frac{175N^2 + 334N + 155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta(2)}{8(N+1)} \right\} \end{aligned}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

||

$$\boxed{\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

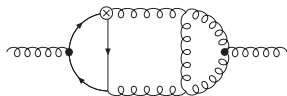
||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

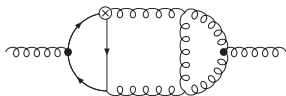
||

$$\frac{-n^2 - n - 1}{n^2 (n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2 (n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^n \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

||

$$\sum_{j=0}^{N-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+N-3} \sum_{s=1}^{-l+N-q-3} \sum_{r=0}^{-l+N-q-s-3} (-1)^{-j+k-l+N-q-3} \times$$

$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{N-1}{j+2} \binom{-j+N-3}{q} \binom{-l+N-q-3}{s} \binom{-l+N-q-s-3}{r} r! (-l+N-q-r-s-3)! (s-1)!}{(-l+N-q-2)! (-j+N-1) (N-q-r-s-2) (q+s+1)}$$

$$\left[4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k) \right.$$

$$\left. - (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)) \right.$$

$$\left. + 2S_1(s-1) - 2S_1(r+s) \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\boxed{F_0(N)} =$$

$$\begin{aligned} & \frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(-\frac{4(13N+5)}{N^2(N+1)^2} + \left(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \right. \\ & + \left(2 + 2(-1)^N \right) S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} S_1(N) + \left(\frac{3}{4} + (-1)^N \right) S_2(N)^2 \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N+1} \right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)} \right. \\ & + \left. \frac{4(3N-1)}{N(N+1)} \right) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left(\frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6 + 5(-1)^N) S_{-4}(N) \\ & + \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

$$\boxed{F_0(N)} =$$

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$$F_0(N) =$$

$$\begin{aligned}
 & \frac{7}{12} S_1(N)^3 + \frac{(17N+5)S_1(N)^3}{2N^2(N+1)^2} + \left(\frac{35N^2 - 2N - 5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\
 & + \left(-\frac{1}{N(N+1)} \right) S_1(N) = \sum_{i=1}^N \frac{1}{i} (-1)^N (2N+1) - \frac{13}{N} S_2(N) + \left(\frac{29}{2} - (-1)^N \right) S_3(N) \\
 & + (2 + 2(-1)^N) S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} S_2(N) = \sum_{i=1}^N \frac{1}{i^2} (-1)^N S_2(N)^2 \\
 & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) \frac{(-1)^N}{N+1} \right) \\
 & + \left(\frac{(-1)^N}{2N^2} \right)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)} \right) \\
 & + \frac{4(3N-5)}{N(N+1)} (-1)^N S_2(N) - \frac{16}{N(N+1)} \\
 & + \left(\frac{(-1)^N}{N} \right) S_{-2,1,1}(N) = \sum_{i=1}^N \frac{(-1)^i \sum_{j=1}^i \frac{1}{k}}{i^2} S_{-2,1,1}(N) + (-6 + 5(-1)^N) S_{-4}(N) \\
 & + \left(-\frac{2(-1)^N}{N(N+1)} \right) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\
 & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\
 & + 32S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)
 \end{aligned}$$

The general tactic

Feynman integrals

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms
of special functions

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

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compact expression in terms
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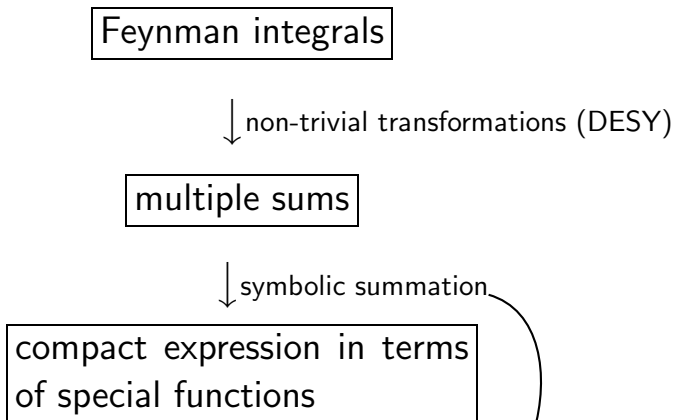
Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]

Blümlein, Hasselhuhn, CS, RADCOR'10, arXiv:1202.4303 [math-ph]

CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

The general tactic



Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - \left(2\zeta_2 - \frac{68}{3}\right)\varepsilon^0 + \dots$$

$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - (2\zeta_2 - \frac{68}{3})\varepsilon^0 + \dots$$

$$F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left(\frac{\zeta_2}{4} + \frac{79}{24}\right)\varepsilon^{-1} + \dots$$

$$F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \left(\frac{\zeta_2}{3} + \frac{1415}{324}\right)\varepsilon^{-1} + \dots$$

↓

$$F(N) = F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + \dots$$

$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - (2\zeta_2 - \frac{68}{3})\varepsilon^0 + \dots$$

$$F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left(\frac{\zeta_2}{4} + \frac{79}{24}\right)\varepsilon^{-1} + \dots$$

$$F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \left(\frac{\zeta_2}{3} + \frac{1415}{324}\right)\varepsilon^{-1} + \dots$$

↓ (summation package Sigma.m)

$$F(N) = \frac{4N}{3(N+1)}\varepsilon^{-3} - \left(\frac{2(2N+1)}{3(N+1)}S_1(N) + \frac{2N(2N+3)}{3(N+1)^2}\right)\varepsilon^{-2}$$

$$\left(\frac{(1-4N)}{6(N+1)}S_1(N)^2 - \frac{N(N^2-2)}{3(N+1)^3} + \frac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N) + \frac{(1-4N)}{6(N+1)}S_2(N) + \frac{N\zeta_2}{2(N+1)}\right)\varepsilon^{-1} + \dots$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1}$$

$$\underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

The integrand is

- hyperexponential in x, y, z :

$$\frac{D_x f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

The integrand is

- hyperexponential in x, y, z :

$$\frac{D_y f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

The integrand is

- hyperexponential in x, y, z :

$$\frac{D_z f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1}$$

$$\underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

The integrand is

- ▶ hyperexponential in x, y, z :
- ▶ hypergeometric in n :

$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1}$$

$$\underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

$$\begin{aligned}a_0(N, \varepsilon) = & (N + 1)(N + 2)(8\varepsilon^{10} + 104\varepsilon^9(N + 3) + 4\varepsilon^8(96N^2 + 601N + 887) \\ & + 4\varepsilon^7(12N^3 + 414N^2 + 1583N + 1393) \\ & - 8\varepsilon^6(264N^4 + 2436N^3 + 8643N^2 + 14518N + 9947) \\ & - 16\varepsilon^5(156N^5 + 1690N^4 + 6847N^3 + 12661N^2 + 9537N + 717) \\ & + 32\varepsilon^4(68N^6 + 1158N^5 + 8155N^4 + 30114N^3 + 61712N^2 + 67616N + 31693) \\ & + 64\varepsilon^3(40N^7 + 560N^6 + 2755N^5 + 3729N^4 - 14194N^3 - 61920N^2 - 89140N - 46600) \\ & - 128\varepsilon^2(N + 2)(12N^7 + 254N^6 + 2249N^5 + 10758N^4 + 30173N^3 + 50610N^2 \\ & + 49122N + 22706) \\ & + 256\varepsilon(N + 2)^2(N + 3)(N + 4)(44N^4 + 501N^3 + 2044N^2 + 3455N + 1976) \\ & - 512(N + 1)(N + 2)^3(N + 3)^2(N + 4)(6N^2 + 47N + 95)),\end{aligned}$$

$$\begin{aligned}
a_1(N, \varepsilon) = & (N + 2)(-22\varepsilon^{11} - 2\varepsilon^{10}(157N + 435) - \varepsilon^9(1500N^2 + 8611N + 11745) \\
& - \varepsilon^8(2548N^3 + 22936N^2 + 63597N + 54229) \\
& + 4\varepsilon^7(266N^4 + 1857N^3 + 6065N^2 + 14351N + 15987) \\
& + 8\varepsilon^6(994N^5 + 12961N^4 + 67246N^3 + 174692N^2 + 226821N + 116092) \\
& + 16\varepsilon^5(336N^6 + 5348N^5 + 33569N^4 + 104918N^3 + 165290N^2 + 108259N + 6100) \\
& - 16\varepsilon^4(404N^7 + 7578N^6 + 61778N^5 + 284762N^4 + 802660N^3 + 1382074N^2 \\
& + 1340455N + 560287) \\
& - 64\varepsilon^3(94N^8 + 1823N^7 + 14305N^6 + 55870N^5 + 96299N^4 - 37256N^3 \\
& - 447044N^2 - 704959N - 379338) \\
& + 128\varepsilon^2(N + 3)(30N^8 + 715N^7 + 7667N^6 + 48253N^5 + 194086N^4 + 507439N^3 \\
& + 835393N^2 + 785327N + 320382) \\
& - 256\varepsilon(N + 2)(N + 3)^2(107N^6 + 2070N^5 + 16342N^4 + 67226N^3 + 151557N^2 \\
& + 176932N + 83196) \\
& + 256(N + 2)^3(N + 3)^3(N + 4)(30N^3 + 331N^2 + 1193N + 1386)),
\end{aligned}$$

$$\begin{aligned}
a_2(N, \varepsilon) = & (12\varepsilon^{12} + 12\varepsilon^{11}(17N + 45) + 2\varepsilon^{10}(620N^2 + 3553N + 4795) \\
& + 2\varepsilon^9(1504N^3 + 14190N^2 + 41901N + 38907) \\
& + 4\varepsilon^8(172N^4 + 4983N^3 + 30942N^2 + 69119N + 50850) \\
& - 4\varepsilon^7(1996N^5 + 24056N^4 + 113313N^3 + 269119N^2 + 337198N + 185290) \\
& - 16\varepsilon^6(450N^6 + 8210N^5 + 59749N^4 + 227386N^3 + 486841N^2 + 563176N + 275664) \\
& + 16\varepsilon^5(340N^7 + 4314N^6 + 19137N^5 + 25532N^4 - 55105N^3 - 206516N^2 - 191528N \\
& - 23458) \\
& + 32\varepsilon^4(140N^8 + 2940N^7 + 26550N^6 + 139926N^5 + 493839N^4 + 1240186N^3 \\
& + 2161699N^2 + 2304248N + 1100084) \\
& + 64\varepsilon^3(4N^9 + 506N^8 + 8651N^7 + 63510N^6 + 236215N^5 + 395334N^4 - 105413N^3 \\
& - 1551017N^2 - 2362944N - 1217770) \\
& - 128\varepsilon^2(N + 3)(12N^9 + 314N^8 + 3782N^7 + 29105N^6 + 160727N^5 + 640273N^4 \\
& + 1750874N^3 + 3052505N^2 + 3017094N + 1276604) \\
& + 256\varepsilon(N + 2)(N + 3)^2(N + 4)(26N^6 + 825N^5 + 8967N^4 + 46529N^3 + 125411N^2 \\
& + 168628N + 88652) \\
& - 512(N + 1)(N + 2)^2(N + 3)^3(N + 4)^2(6N^3 + 98N^2 + 459N + 655)),
\end{aligned}$$

$$\begin{aligned}
a_3(N, \varepsilon) = & (-64\varepsilon^{12} - 8\varepsilon^{11}(113N + 298) - 8\varepsilon^{10}(519N^2 + 2948N + 3896) \\
& - 4\varepsilon^9(1444N^3 + 13839N^2 + 39746N + 34305) \\
& + 4\varepsilon^8(1948N^4 + 17868N^3 + 63837N^2 + 112966N + 84655) \\
& + 16\varepsilon^7(1456N^5 + 20460N^4 + 112365N^3 + 304963N^2 + 412258N + 221769) \\
& - 8\varepsilon^6(320N^6 + 2050N^5 + 4192N^4 + 27408N^3 + 174901N^2 + 411759N + 324872) \\
& - 16\varepsilon^5(1756N^7 + 33154N^6 + 265889N^5 + 1186719N^4 + 3218059N^3 + 5349388N^2 \\
& + 5071913N + 2113696) \\
& + 32\varepsilon^4(188N^8 + 4802N^7 + 59527N^6 + 439922N^5 + 2025336N^4 + 5813984N^3 \\
& + 10076450N^2 + 9621283N + 3878602) \\
& + 64\varepsilon^3(140N^9 + 2768N^8 + 22500N^7 + 99545N^6 + 287700N^5 + 723136N^4 \\
& + 1854572N^3 + 3714620N^2 + 4272517N + 2031600) \\
& - 128\varepsilon^2(24N^{10} + 830N^9 + 14362N^8 + 152630N^7 + 1053620N^6 + 4834279N^5 \\
& + 14824351N^4 + 29964399N^3 + 38244797N^2 + 27875896N + 8824032) \\
& + 256\varepsilon(N + 2)(N + 3)(N + 4)(118N^7 + 2639N^6 + 24247N^5 + 118311N^4 + 329565N^3 \\
& + 520306N^2 + 426076N + 136854) \\
& - 512(N + 1)(N + 2)^2(N + 3)^2(N + 4)^2(N + 5)(12N^3 + 97N^2 + 230N + 144)),
\end{aligned}$$

$$\begin{aligned}
a_4(N, \varepsilon) = & (64\varepsilon^{12} + 192\varepsilon^{11}(5N + 14) + 16\varepsilon^{10}(297N^2 + 1769N + 2451) \\
& + 16\varepsilon^9(453N^3 + 4462N^2 + 13094N + 11244) \\
& - 8\varepsilon^8(1084N^4 + 11117N^3 + 47258N^2 + 103981N + 94650) \\
& - 8\varepsilon^7(3304N^5 + 51138N^4 + 311957N^3 + 948722N^2 + 1440105N + 858544) \\
& + 16\varepsilon^6(420N^6 + 5507N^5 + 36275N^4 + 169650N^3 + 536911N^2 + 952507N + 694370) \\
& + 16\varepsilon^5(1828N^7 + 38868N^6 + 353301N^5 + 1801014N^4 + 5604391N^3 + 10664390N^2 \\
& + 11433064N + 5260048) \\
& - 32\varepsilon^4(316N^8 + 8356N^7 + 105800N^6 + 802421N^5 + 3836854N^4 + 11588223N^3 \\
& + 21401558N^2 + 22066744N + 9745752) \\
& - 64\varepsilon^3(116N^9 + 2424N^8 + 19923N^7 + 82966N^6 + 208191N^5 + 530980N^4 + 1847484N^3 \\
& + 4687014N^2 + 6120858N + 3111104) \\
& + 128\varepsilon^2(24N^{10} + 826N^9 + 14897N^8 + 172000N^7 + 1314686N^6 + 6710299N^5 \\
& + 22873183N^4 + 51298261N^3 + 72551278N^2 + 58573022N + 20544948) \\
& - 256\varepsilon(N + 2)(N + 3)(106N^8 + 3278N^7 + 42903N^6 + 310942N^5 + 1366350N^4 \\
& + 3729418N^3 + 6173159N^2 + 5657732N + 2191212) \\
& + 512(N + 1)(N + 2)^2(N + 3)^2(N + 4)(N + 5)(N + 6)(12N^3 + 121N^2 + 396N + 431)),
\end{aligned}$$

$$\begin{aligned}
a_5(N, \varepsilon) = & (N + 5)(-128\varepsilon^{11} - 128\varepsilon^{10}(11N + 26) - 32\varepsilon^9(115N^2 + 592N + 647) \\
& + 32\varepsilon^8(63N^3 + 430N^2 + 1665N + 2384) \\
& + 16\varepsilon^7(714N^4 + 7881N^3 + 33802N^2 + 66225N + 47654) \\
& - 16\varepsilon^6(234N^5 + 2444N^4 + 13989N^3 + 50862N^2 + 104083N + 87848) \\
& - 16\varepsilon^5(580N^6 + 10181N^5 + 76586N^4 + 319207N^3 + 772120N^2 + 1012046N + 547832) \\
& + 16\varepsilon^4(244N^7 + 5456N^6 + 61605N^5 + 401216N^4 + 1536277N^3 + 3408574N^2 \\
& + 4066436N + 2026928) \\
& + 64\varepsilon^3(26N^8 + 357N^7 + 583N^6 - 11139N^5 - 65193N^4 - 120264N^3 + 11864N^2 \\
& + 272830N + 222624) \\
& - 64\varepsilon^2(N + 3)(12N^8 + 298N^7 + 4684N^6 + 49024N^5 + 306907N^4 + 1122441N^3 \\
& + 2350650N^2 + 2607576N + 1185072) \\
& + 256\varepsilon(N + 2)(N + 3)(25N^7 + 743N^6 + 8856N^5 + 55358N^4 + 197497N^3 + 404131N^2 \\
& + 439902N + 196128) \\
& - 256(N + 1)(N + 2)^2(N + 3)^2(N + 4)(N + 6)(N + 7)(6N^2 + 35N + 54).
\end{aligned}$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1}$$

$$\underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

Example: A master integral from Ladder and V -topologies

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Ablinger's
MultiIntegrate.m \downarrow (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

Sigma.m \downarrow (2 hours)

$$F(\varepsilon, n) = F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \dots + F_4(n)\varepsilon^4 + O(\varepsilon^5)$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

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$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n(3n^3+18n^2+31n+18)}{3(n+1)^3(n+2)^2} - \frac{4(6n^3+32n^2+51n+26)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n(3n^3+18n^2+31n+18)}{3(n+1)^3(n+2)^2} - \frac{4(6n^3+32n^2+51n+26)}{3(n+1)^3(n+2)^2}$$

$$\begin{aligned} F_{-1}(n) &= (-1)^n \left(\frac{2(9n^5 + 81n^4 + 295n^3 + 533n^2 + 500n + 204)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)} \right) \\ &+ \frac{2(18n^5 + 150n^4 + 490n^3 + 755n^2 + 536n + 132)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)} \\ &+ \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)} \right) S_2(n) \\ &+ \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)} \right) S_{-2}(n) \end{aligned}$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

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$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum
Package (F. Stan)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

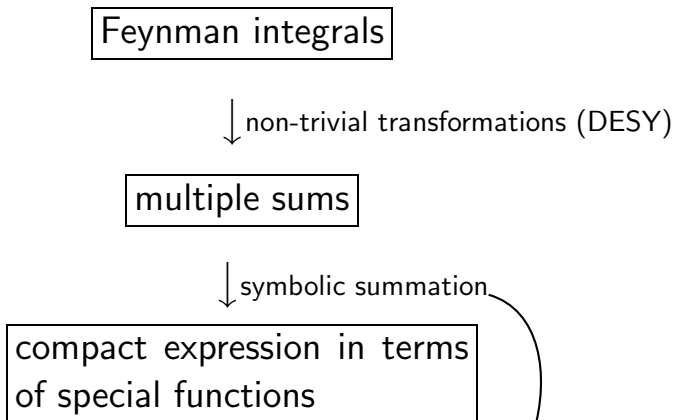
$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum
Package (F. Stan)

Holonomic/difference field
approach (M. Round)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

The general tactic



Tactic 2: Expand a recurrence in ϵ

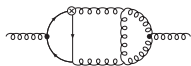
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Evaluation of Feynman Integrals



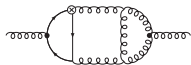
Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

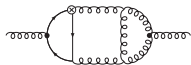


$\int \Phi(N, \epsilon, x) dx$
Feynman integrals

DESY

$Dy = Ay$
coupled systems of
linear DEs

Evaluation of Feynman Integrals



Behavior of particles



$\int \Phi(N, \epsilon, x) dx$
Feynman integrals

DESY



$Dy = Ay$
coupled systems of
linear DEs

RISC



(new coupled system solver)

expression in
special functions

Tactic 3: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)

Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
 Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

\downarrow
 uncoupling algorithms
 (Zürcher, Abramov/Zima, Gauss, ...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinComb}(\dots, D^i\hat{R}_i(x), \dots)$$

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DE-solver

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

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DECIDE constructively if $f(x)$ can be expressed in terms of **iterated integrals** defined over **hyperexponential functions**.

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Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \quad (\text{harmonic polylogarithms})$$

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

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Special cases of iterated integrals over hyperexponential functions:

$$H_{2,-2}(x) = \int_0^x \frac{1}{2 - \tau_1} \int_0^{\tau_1} \frac{1}{2 + \tau_2} d\tau_2 d\tau_1 \quad (\text{generalized polylogarithms})$$

S. Moch, P. Uwer and S. Weinzierl, *J. Math. Phys.* **43** (2002) 3363 [hep-ph/0110083];

J. Ablinger, J. Blümlein and CS, *J. Math. Phys.* **54** (2013) 082301 [arXiv:1302.0378].

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J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2} dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11 + 20x)f'(x) + (1 + x)(35 + 134x)f''(x) + 3(1 + x)^2(4 + 37x)f^{(3)}(x) + 18x(1 + x)^3f^{(4)}(x) = 0$$

↓

$$\left\{ c_1 + c_2 \int_0^x \frac{1}{1 + \tau_1} d\tau_1 + c_3 \int_0^x \frac{1}{1 + \tau_1} \int_0^{\tau_1} \frac{\sqrt[3]{1 + \sqrt{1 + \tau_2}}}{1 + \tau_2} d\tau_2 d\tau_1 + c_4 \int_0^x \frac{1}{1 + \tau_1} \int_0^{\tau_1} \frac{\sqrt[3]{1 - \sqrt{1 + \tau_2}}}{1 + \tau_2} d\tau_2 d\tau_1 \mid c_1, c_2, c_3, c_4 \in \mathbb{K} \right\}$$

Connection: DE \longleftrightarrow REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

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\Updownarrow algorithmic

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Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\begin{aligned} & - (x^4 - 64x^3) f^{(4)}(x) - 2(5x^3 - 144x^2) f^{(3)}(x) \\ & - (25x^2 - 208x) f''(x) - (15x - 8) f'(x) - f(x) = 0 \end{aligned}$$

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$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

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↓ Sigma.m

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Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2 \left[\begin{matrix} 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2} \end{matrix}; \frac{x}{16} \right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

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DE-solver

REC-solver

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$$

Tactic 3: the DE-REC approach

$$\begin{array}{c} \text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \end{array}$$

OreSys package (S. Gerhold)
uncoupling algorithm

$$\begin{array}{c} \text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1 \end{array}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

Tactic 3: the DE-REC approach

$$\text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)
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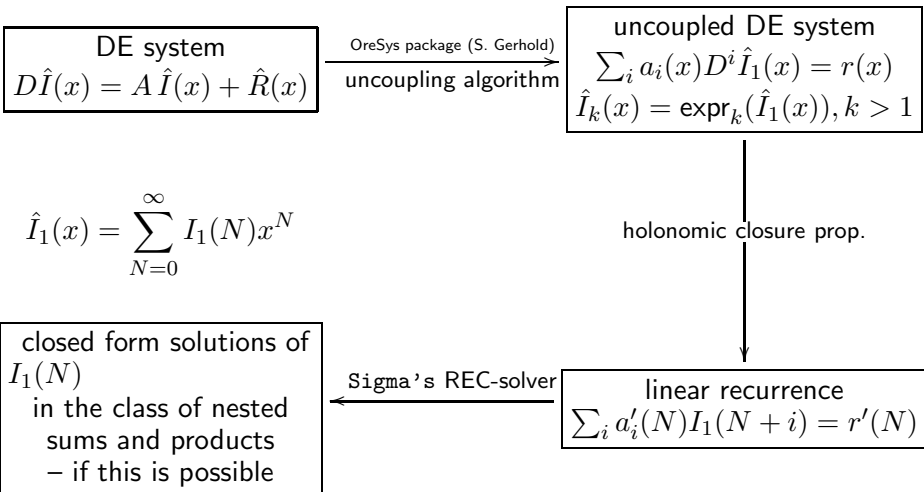
$$\text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$$

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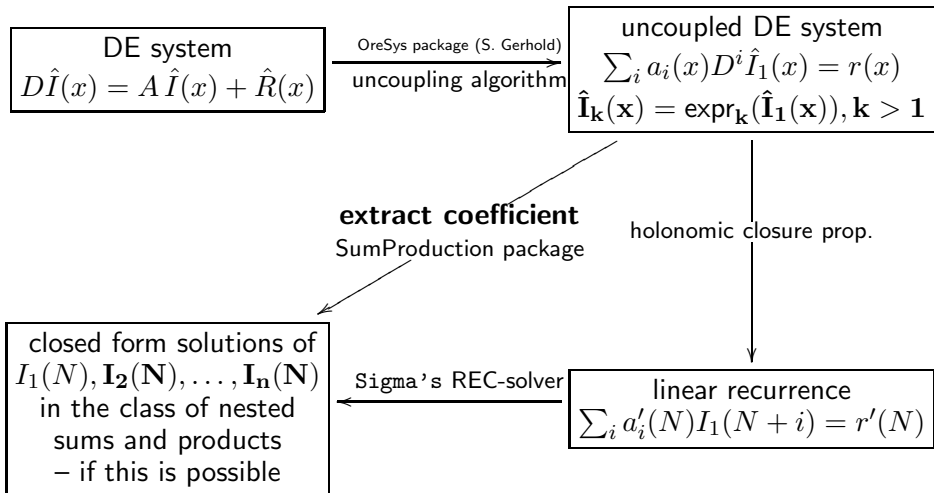
holonomic closure prop.

$$\text{linear recurrence} \\ \sum_i a'_i(N) I_1(N+i) = r'(N)$$

Tactic 3: the DE-REC approach



Tactic 3: the DE-REC approach (SolveCoupledSystem package)



General strategy: physical problem $\hat{P}(x)$

↓ IBP methods

- ▶ Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- ▶ $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

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physical problem $\hat{P}(x)$

$$\downarrow \text{IBP methods}$$

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$$\downarrow \text{solver for } \hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)x^N$$

$$I_i(N) = \varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \varepsilon^0F_0(N) + \dots$$

General strategy: physical problem $\hat{P}(x)$

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↓ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)x^N$

$$I_i(N) = \varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \varepsilon^0F_0(N) + \dots$$

↓ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N)x^N$

$$P(N) = \varepsilon^{-3}P_{-3}(N) + \varepsilon^{-2}P_{-2}(N) + \varepsilon^{-1}P_{-1}(N) + \varepsilon^0P_0(N) + \dots$$

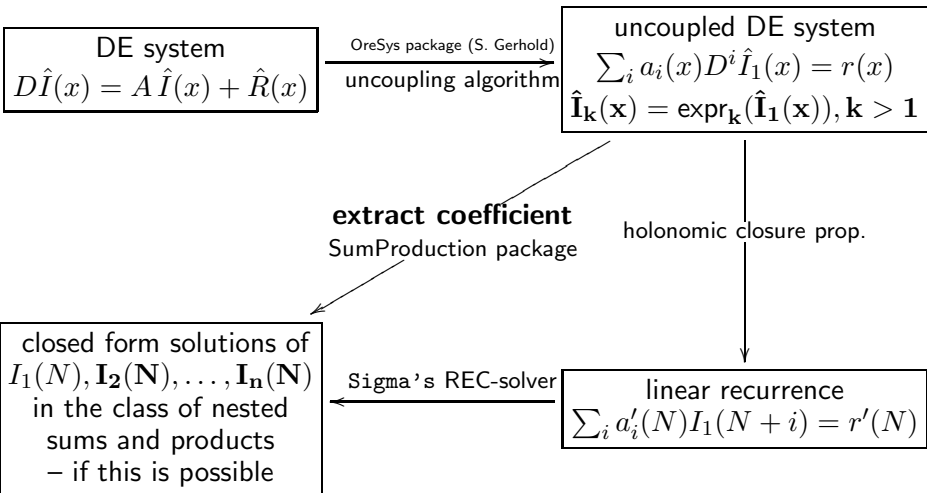
Calculations based on Tactic 3:

- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
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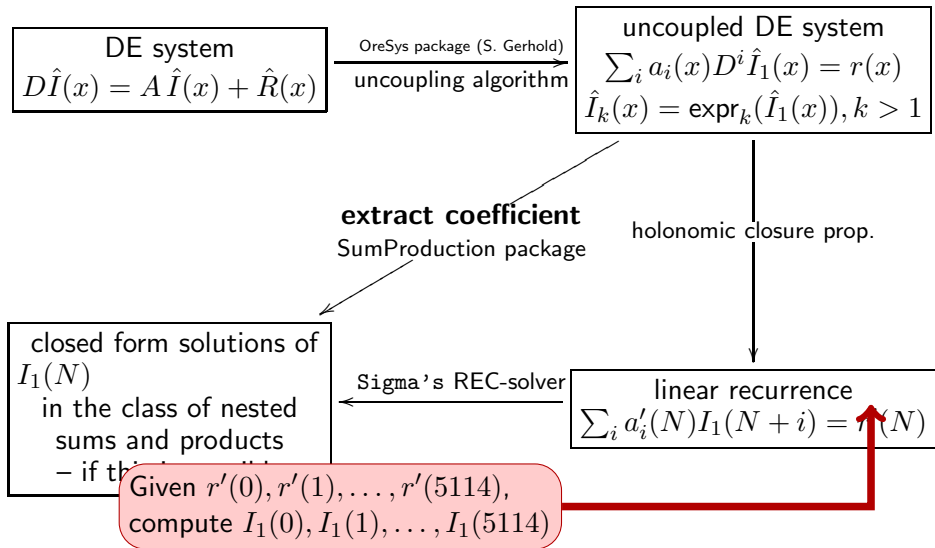
Tactic 4: Compute large moments and guessing recurrences

[coming, e.g., from IBP methods]

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



Tactic 4: compute large moments (SolveCoupledSystem package)



General strategy:

physical problem $\hat{P}(x)$

↓ IBP methods

- ▶ Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- ▶ $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

 ↓ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)x^N$

$$I_i(N) = \underbrace{\varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \varepsilon^0F_0(N) + \dots}_{\text{only numbers}}$$

 $N = 0, 1, \dots, 8000$

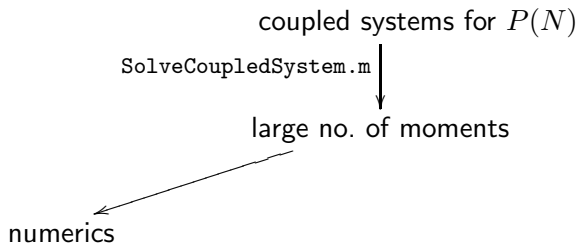
only numbers

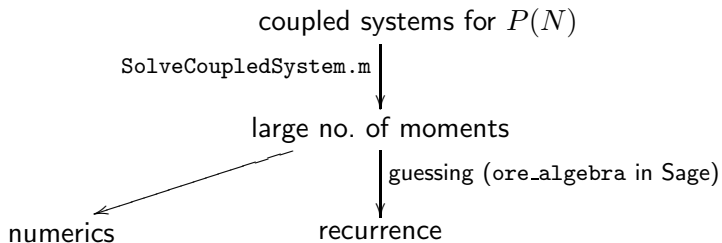
 ↓ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N)x^N$

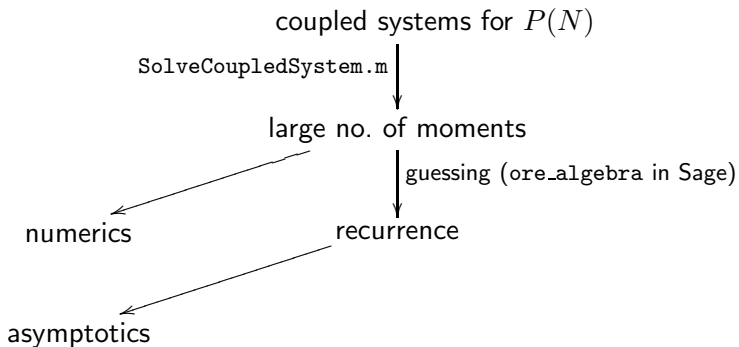
$$P(N) = \underbrace{\varepsilon^{-3}P_{-3}(N) + \varepsilon^{-2}P_{-2}(N) + \varepsilon^{-1}P_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^0P_0(N)}_{\text{numbers}} + \dots$$

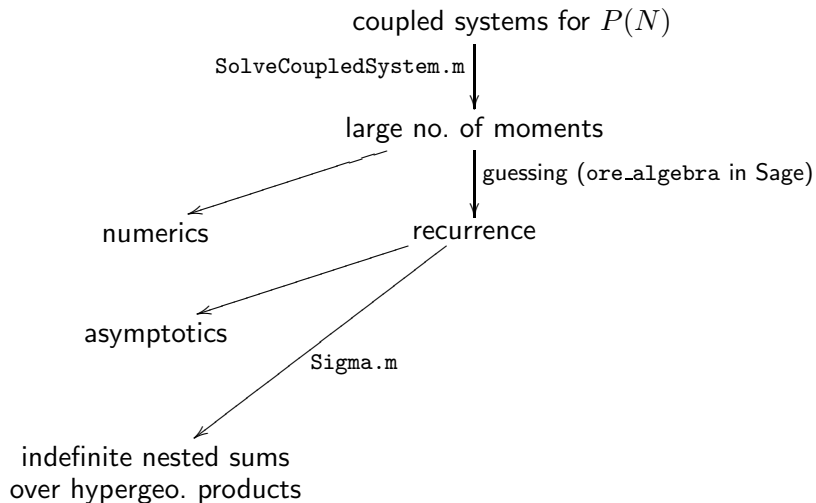
 $N = 0, 1, \dots, 8000$

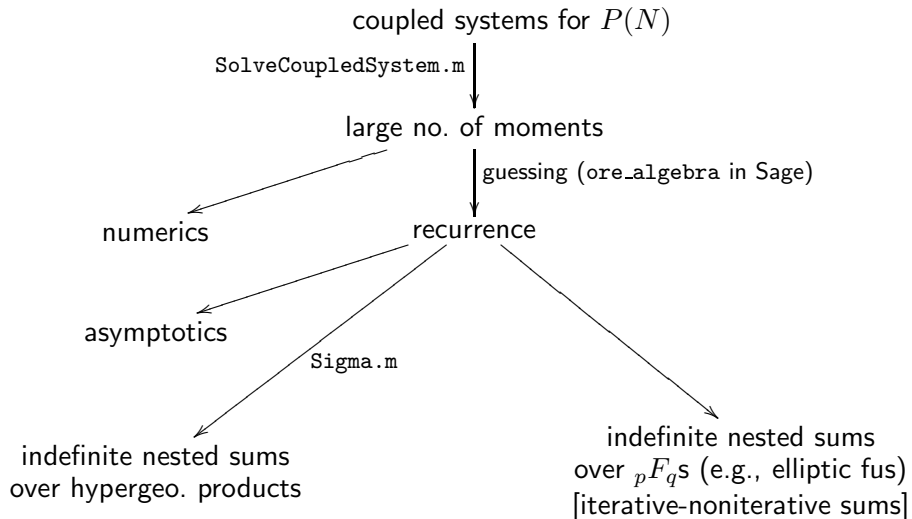
coupled systems for $P(N)$
`SolveCoupledSystem.m` ↓
large no. of moments

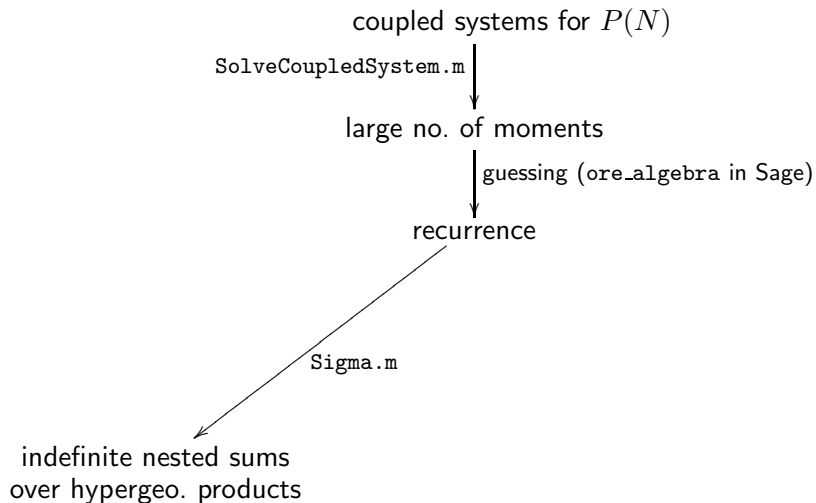












General strategy:

physical problem $\hat{P}(x)$

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- ▶ Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- ▶ $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

 ↓ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)x^N$

$$I_i(N) = \underbrace{\varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \varepsilon^0F_0(N) + \dots}_{\text{only numbers}}$$

 $N = 0, 1, \dots, 8000$

only numbers

 ↓ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N)x^N$

$$P(N) = \underbrace{\varepsilon^{-3}P_{-3}(N) + \varepsilon^{-2}P_{-2}(N) + \varepsilon^{-1}P_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^0P_0(N)}_{\text{numbers}} + \dots$$

 $N = 0, 1, \dots, 8000$

General strategy:

physical problem $\hat{P}(x)$


↓ IBP methods

- ▶ Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- ▶ $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

 ↓ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)x^N$

$$I_i(N) = \underbrace{\varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \varepsilon^0F_0(N) + \dots}_{\text{only numbers}}$$

only numbers

 ↓ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N)x^N$


$$P(N) = \underbrace{\varepsilon^{-3}P_{-3}(N) + \varepsilon^{-2}P_{-2}(N) + \varepsilon^{-1}P_{-1}(N)}_{\text{nice}} + \underbrace{\varepsilon^0P_0(N)}_{\text{partially nice}} + \dots$$

nice

partially nice

all N solution

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

```
In[8]:= << Sigma.m
```

Sigma - A summation package by Carsten Schneider © RISC-Linz

```
In[9]:= initial = << iFile16
```


In[10]:= **rec** ==<< **rFile16**

Out[10]= $(n + 1)^4(n + 2)^2(2n + 3)(2n + 5)(2n + 7)(2n + 9)(2n + 11) \left(309237645312n^{32} + 38256884318208n^{31} + 2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + 873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + 1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + 575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + 69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + 901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + 7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + 32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + 86710462147941775492301231896818n^9 + 112917328975807075881545543668548n^8 + 124873767581470867343743078943772n^7 + 115624836314544572769501784072647n^6 + 87938536330971046886456627610048n^5 + 53481897815980319933589323279298n^4 + 25000430622737750756669804052204n^3 + 8430930497463933665464836129855n^2 + 1825177817831282261293155379650n + 190428196025667395685609855000 \right) (2n + 1)^4 P[n]$

$$\begin{aligned}
& -(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11) \left(12369505812480n^{38} + 1613151061671936n^{37} + \right. \\
& 101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
& 2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
& 9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
& 7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
& 356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
& 9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
& 170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
& 1892149418896523531194676203153920n^{19} + 532117380629233448534132495165440n^{18} + \\
& 13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
& 59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
& 167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
& 284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
& 279679164311116651162116055961513301n^9 + 221781415386984655607595031093415136n^8 + \\
& 149214365004640710156345950062395186n^7 + 83882523964213110328265187672574356n^6 + \\
& 38609679702395410742361774562392789n^5 + 14149471988638475521561721269939086n^4 + \\
& 3963748138857399502678254252169734n^3 + 795659668131014454843348852372480n^2 + \\
& 101701393436276172443717692853400n + 6204709909986751913151675960000) P[n+1]
\end{aligned}$$

$$\begin{aligned}
& +2(n+3)^2(2n+5)^3(2n+9)(2n+11) \left(24739011624960n^{40} + 3317836466356224n^{39} + 215508170284466176n^{38} + 9032884062187945984n^{37} + \right. \\
& 274636134389959884800n^{36} + 6455501959255126179840n^{35} + 122094572934385260036096n^{34} + 1909387225793663151898624n^{33} + \\
& 25180108291969215434326016n^{32} + 284171960705270647479074816n^{31} + 2775794400720227034854326272n^{30} + \\
& 23677622163992853854566219776n^{29} + 177624312783583749157935120384n^{28} + 1178515602115604757944201871360n^{27} + \\
& 6947091965313419323781358354432n^{26} + 36515023100308314818702129258496n^{25} + 171621148571344894953594594017280n^{24} + \\
& 722837793013976317556258102507520n^{23} + 2732534027077907914497042720534528n^{22} + 9281028665970648470895368668485120n^{21} + \\
& 28337819215557708948254385336117248n^{20} + 77786125749274632150536464583130752n^{19} + 191877161455672780973502244537632256n^{18} + \\
& 424953221702140663089937921965135648n^{17} + 843818276409975584824720931649555264n^{16} + \\
& 1499359936674956711935311062995422344n^{15} + 2378007025570977662661938772843220240n^{14} + \\
& 3355671771434535852147325502571953770n^{13} + 4196375762867184563407432891655585484n^{12} + \\
& 4627675779563752366067861596232781096n^{11} + 4473175960511956000526499430851993603n^{10} + \\
& 3761696365025837909581516781307249585n^9 + 2726553473467254373993685951699145492n^8 + \\
& 1683383212304999468664293798012773485n^7 + 871926653651504419744271839781064837n^6 + \\
& 371307437598003570058538796122994147n^5 + 126427972742886389602285855482966072n^4 + 33048762330145623969058704448697313n^3 + \\
& 6217924746857741077419160100404560n^2 + 748298077423337427195946099994100n + 43181089548034246077698611794000) P[n+2]
\end{aligned}$$

$$\begin{aligned}
& -2(n+4)^2(2n+5)(2n+7)^3(2n+11) \left(24739011624960n^{40} + 3322784268681216n^{39} + 216160919414112256n^{38} + 9074528155284275200n^{37} + \right. \\
& 276348048819456311296n^{36} + 6506479077331107315712n^{35} + 123266585640616142569472n^{34} + 1931040885785102661976064n^{33} + \\
& 25510503383281445462081536n^{32} + 288418124175428279391485952n^{31} + 28224427990336303081019326464n^{30} + \\
& 24120717233320712351821332480n^{29} + 181295944719289040999116701696n^{28} + 1205246297785423925076555694080n^{27} + \\
& 7119049557560114436136213413888n^{26} + 37496933571993839665392189775872n^{25} + 176616172467048982234270428880896n^{24} + \\
& 745539218875020737621728364206080n^{23} + 2824909633156578132652259733712896n^{22} + 9618101958268071244680677589035520n^{21} + \\
& 29441860528446423517613263360742912n^{20} + 81033563306363873505877563416477312n^{19} + 200454769103641040142838133702338304n^{18} + \\
& 445286624972461749049425309485328992n^{17} + 887028447418790661018847407251573152n^{16} + \\
& 1581538101499869694224895701784875304n^{15} + 2517550244392724509968791166585362672n^{14} + \\
& 3566593026520465155504695877897282630n^{13} + 4479066125207404898722179511912639638n^{12} + \\
& 4962006990874351800791769650243464872n^{11} + 4819992643914265990647887896664485209n^{10} + \\
& 407489538669418224094153822230233221n^9 + 2970477229398746689186622534784613554n^8 + \\
& 1845274131994015990683957902602775337n^7 + 962091291302144537393228847830431614n^6 + \\
& 412595107814836563208757757032740146n^5 + 141540723940232563767779647013785485n^4 + 37292931812630561528276365992452010n^3 + \\
& 7074865777225416725452872895397100n^2 + 858794112392644074221312049837000n + 4999738673826011260361510478000) P[n+3]
\end{aligned}$$

$$\begin{aligned}
& + (n+5)^3(2n+5)(2n+7)(2n+9)^4 \left(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} + \right. \\
& 3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} + \\
& 38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} + \\
& 69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} + \\
& 32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} + \\
& 1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} + \\
& 20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} + \\
& 260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} + \\
& 2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} + \\
& 10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} + \\
& 34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} + \\
& 69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} + \\
& 82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^9 + \\
& 55935262205790259307904762197107653n^8 + 36322355479155199114489624391144238n^7 + \\
& 19756597118002557191991191826327042n^6 + 8822212911433711339358062994077203n^5 + \\
& 3145597282374650512689680780380605n^4 + 859907105684964990690798899478888n^3 + \\
& 168963309995629650025632011492580n^2 + 21205680751316222158938757272000n + \\
& 1274120732351744651125603886400) P[n+4]
\end{aligned}$$

$$\begin{aligned}
& -(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4 \left(309237645312n^{32} + 28361279668224n^{31} + \right. \\
& 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + \\
& 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + \\
& 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + \\
& 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + \\
& 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + \\
& 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + \\
& 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + \\
& 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^9 + 25770328899499991754425455738n^8 + \\
& 18817114309842270306167785140n^7 + 11424980760825630752861027739n^6 + 5656051955667821083952617134n^5 + \\
& 2221448212382554437709999491n^4 + 664859653803075491350122060n^3 + 142190920852333874895041748n^2 + \\
& 19313175036907229252501700n + 1248723341516324359641600) P[n+5] = 0
\end{aligned}$$


```
In[11]:= recSol = SolveRecurrence[rec, P[n]]
```

In[11]:= `recSol = SolveRecurrence[rec, P[n]]`

$$\begin{aligned} \text{Out[11]} = & \left\{ \left\{ 0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2} \right\} \right. \\ & \left. \left\{ 0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2} \right\} \right. \\ & \left. \left\{ 0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n) \sum_{i=1}^n \frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} \right\} \right. \\ & \left. \left\{ 0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n) \sum_{i=1}^n \frac{1}{i^2}}{(1+n)(1+2n)} + \left(-\frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} \right) \sum_{i=1}^n \frac{1}{i} \right. \right. \\ & \left. \left. + \frac{12(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^2}{(1+n)(1+2n)} + \frac{56(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)(3+44n+48n^2) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2}{(1+n)(1+2n)} \right\} \right. \end{aligned}$$

$$\begin{aligned}
& \{0, \frac{1}{16(1+n)^4(1+2n)^4} (72519 + 572343n + 1814716n^2 + 2918100n^3 + 2442240n^4 + 912896n^5 + 24576n^6 - \\
& 49152n^7) + \frac{16(3+2n) \sum_{i=1}^n \frac{1}{i^3}}{3(1+n)(1+2n)} + (-\frac{(3+2n)(29+307n+322n^2)}{4(1+n)^2(1+2n)^2} + \frac{44(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{i^2} + \\
& (\frac{(3+2n)(91+259n+974n^2+1784n^3+1024n^4)}{4(1+n)^3(1+2n)^3} + \frac{22(3+2n) \sum_{i=1}^n \frac{1}{i^2}}{(1+n)(1+2n)} + \frac{24(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \\
& \frac{4(3+2n)(-13-4n+16n^2) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{16(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^2}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{i} + (- \\
& \frac{(3+2n)(19+92n+80n^2)}{(1+n)^2(1+2n)^2} + \frac{40(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} (\sum_{i=1}^n \frac{1}{i})^2 + \frac{20(3+2n)(\sum_{i=1}^n \frac{1}{i})^3}{3(1+n)(1+2n)} + \\
& \frac{64(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)} - \frac{3(3+2n)(63+209n+150n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2} + \\
& (\frac{(3+2n)(347+1795n+4302n^2+4856n^3+2048n^4)}{2(1+n)^3(1+2n)^3} + \frac{48(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{-1+2i} - \\
& \frac{4(3+2n)(19+92n+80n^2)(\sum_{i=1}^n \frac{1}{-1+2i})^2}{(1+n)^2(1+2n)^2} + \frac{32(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^3}{3(1+n)(1+2n)} - \\
& \frac{8(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{i}}{(1+n)(1+2n)} - \frac{16(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{-1+2i}}{(1+n)(1+2n)} \\
& - \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{i}}{(1+n)(1+2n)} + \frac{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)} + \\
& \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{i}}{(1+n)(1+2n)} + \frac{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{-1+2i}}{(1+n)(1+2n)} \}, \{1, 0\} \}
\end{aligned}$$

```
In[12]:= sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]
```

In[12]:= sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]

$$\begin{aligned}
 \text{Out}[12]= & \frac{1}{3(1+n)^4(1+2n)^4} (111 + 1920n + 11765n^2 + 32545n^3 + 46476n^4 + 35376n^5 + 13440n^6 + 1968n^7) + \frac{32(3+2n) \sum_{i=1}^n \frac{1}{i^3}}{9(1+n)(1+2n)} - \\
 & \frac{(3+2n)(-3+101n+126n^2) \sum_{i=1}^n \frac{1}{i^2}}{(3+2n)(115+921n+1967n^2+1524n^3+340n^4) \sum_{i=1}^n \frac{1}{i}} + \\
 & \frac{3(1+n)^2(1+2n)^2}{44(3+2n) \left(\sum_{i=1}^n \frac{1}{i^2} \right) \sum_{i=1}^n \frac{1}{i}} - \frac{3(1+n)^3(1+2n)^3}{(3+2n)(23+139n+130n^2) \left(\sum_{i=1}^n \frac{1}{i} \right)^2} + \frac{40(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^3}{4(3+2n)(77+261n+190n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \\
 & \frac{3(1+n)(1+2n)}{128(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^3}} - \frac{3(1+n)^2(1+2n)^2}{4(3+2n)(77+261n+190n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \frac{9(1+n)(1+2n)}{16(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \\
 & \frac{9(1+n)(1+2n)}{2(3+2n)(13-153n-303n^2+12n^3+172n^4) \sum_{i=1}^n \frac{1}{-1+2i}} + \frac{3(1+n)^2(1+2n)^2}{88(3+2n) \left(\sum_{i=1}^n \frac{1}{i^2} \right) \sum_{i=1}^n \frac{1}{-1+2i}} - \\
 & \frac{3(1+n)^3(1+2n)^3}{4(3+2n)(-41-53n+2n^2) \left(\sum_{i=1}^n \frac{1}{i} \right) \sum_{i=1}^n \frac{1}{-1+2i}} + \frac{3(1+n)(1+2n)}{80(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^2 \sum_{i=1}^n \frac{1}{-1+2i}} + \\
 & \frac{3(1+n)^2(1+2n)^2}{32(3+2n) \left(\sum_{i=1}^n \frac{1}{(-1+2i)^2} \right) \sum_{i=1}^n \frac{1}{-1+2i}} - \frac{3(1+n)(1+2n)}{4(3+2n)(23+139n+130n^2) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2} + \\
 & \frac{(1+n)(1+2n)}{32(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2} + \frac{64(3+2n) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^3}{16(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{i}} - \\
 & \frac{3(1+n)(1+2n)}{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{-1+2i}} - \frac{9(1+n)(1+2n)}{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{i}} + \\
 & \frac{3(1+n)(1+2n)}{128(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{-1+2i}} - \frac{3(1+n)(1+2n)}{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{i}} + \\
 & \frac{3(1+n)(1+2n)}{128(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{-1+2i}} + \\
 & \frac{3(1+n)(1+2n)}{3(1+n)(1+2n)}
 \end{aligned}$$

```
In[13]:= << HarmonicSums.m
```

```
HarmonicSums by Jakob Ablinger © RISC-Linz
```

```
In[14]:= sol = TransformToSSums[sol];
```

```
In[15]:= sol = ReduceToBasis[MultipleSumLimit[sol,  
n, 2]//ToStandardForm, n]//CollectProdSum;
```

```
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```

```
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```
In[14]:= sol = TransformToSSums[sol];
```

```
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n, 2]//ToStandardForm, n]//CollectProdSum;
```

$$\begin{aligned} \text{Out[15]} = & \frac{1}{3(1+n)^4(1+2n)^4} (111 + 1920n + 11765n^2 + 32545n^3 + 46476n^4 + 35376n^5 + 13440n^6 + \\ & 1968n^7) + \frac{64(3+2n)^2 S[1, n]}{3(1+n)(1+2n)^2} + \frac{64(3+2n)(2+3n) S[1, n]^2}{3(1+n)(1+2n)^2} + (- \\ & \frac{2(3+2n)(147 + 985n + 1871n^2 + 1268n^3 + 212n^4)}{3(1+n)^3(1+2n)^3} + \frac{224(3+2n) S[2, 2n]}{3(1+n)(1+2n)} + \\ & \frac{128(3+2n) S[-2, 2n]}{3(1+n)(1+2n)}) S[1, 2n] - \frac{4(3+2n)(23 + 123n + 114n^2) S[1, 2n]^2}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[1, 2n]^3}{3(1+n)(1+2n)} + \frac{64(3+2n) S[2, n]}{3(1+n)(1+2n)} - \frac{4(3+2n)(53 + 229n + 190n^2) S[2, 2n]}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[3, 2n]}{3(1+n)(1+2n)} + (- \frac{64(3+2n)^2}{3(1+n)(1+2n)^2} - \frac{128(3+2n)(2+3n) S[1, 2n]}{3(1+n)(1+2n)^2}) S[-1, 2n] - \\ & \frac{64(3+2n)(2+3n) S[-1, 2n]^2}{3(1+n)(1+2n)^2} - \frac{32(3+2n)(1+8n+8n^2) S[-2, 2n]}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[-3, 2n]}{3(1+n)(1+2n)} - \frac{128(3+2n) S[-2, 1, 2n]}{3(1+n)(1+2n)} \end{aligned}$$

```
In[13]:= << HarmonicSums.m
```

```
HarmonicSums by Jakob Ablinger © RISC-Linz
```

```
In[14]:= sol = TransformToSSums[sol];
```

```
In[15]:= sol = ReduceToBasis[MultipleSumLimit[sol,
n, 2]//ToStandardForm, n]//CollectProdSum;
```

```
In[16]:= SExpansion[sol, n, 2]
```

$$\begin{aligned} \text{Out[16]} = & \ln^2 \left(\frac{64\text{LG}[n]}{n} + \frac{160}{3n^2} - \frac{44}{n} \right) + \\ & \ln 2 \left(\left(\frac{320}{3n^2} - \frac{88}{n} \right) \text{LG}[n] + \frac{64\text{LG}[n]^2}{n} - \frac{430}{3n^2} + \frac{160\zeta_2}{3n} - \frac{14}{n} \right) + \\ & \zeta_2 \left(\frac{160\text{LG}[n]}{3n} + \frac{40}{n^2} - \frac{84}{n} \right) + \left(\frac{160}{3n^2} - \frac{44}{n} \right) \text{LG}[n]^2 + \left(-\frac{430}{3n^2} - \frac{14}{n} \right) \text{LG}[n] + \frac{64\text{LG}[n]^3}{3n} + \\ & \frac{64\ln 2^3}{3n} + \frac{145}{2n^2} + \frac{32\zeta_3}{n} + \frac{41}{n} \end{aligned}$$

Calculations based on Tactic 4:

- ▶ J. Blümlein, CS. The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory. *Physics Letters B* 771, pp. 31-36. 2017.
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The Three-Loop Splitting Functions $P_{gg}^{(2)}$ and $P_{gg}^{(2, \text{NF})}$. *Nucl. Phys. B* 922, pp. 1-40. 2017.
- ▶ J. Blümlein, P. Marquard, N. Rana, CS. The Heavy Fermion Contributions to the Massive Three Loop Form Factors. *Nuclear Physics B* 949(114751), pp. 1-97. 2019.
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, CS, K. Schönwald. The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements. *Nuclear Physics B* 948(114753), pp. 1-41. 2019.
- ▶ J. Blümlein, A. Maier, P. Marquard, G. Schäfer, CS. From Momentum Expansions to Post-Minkowskian Hamiltonians by Computer Algebra Algorithms. *Physics Letters B* 801(135157), pp. 1-8. 2020.
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS, K. Schönwald. The three-loop single mass polarized pure singlet operator matrix element. *Nuclear Physics B* 953(114945), pp. 1-25. 2020.
- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Polarized Operator Matrix Element $A_{gg,Q}^{(3)}$. *Nuclear Physics B* 955, pp. 1-70. 2020.
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, K. Schönwald, CS. The Polarized Transition Matrix Element $A_{g,q}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. *Nuclear Physics B* 964, pp. 115331-115356, 2021.
- ▶ J. Blümlein, A. De Freitas, M. Saragnese, K. Schönwald, CS. The Logarithmic Contributions to the Polarized $O(\alpha_s^3)$ Asymptotic Massive Wilson Coefficients and Operator Matrix Elements in Deeply Inelastic Scattering. *Physical Review D* 104(3), pp. 1-73. 2021.
- ▶ J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements. *Nucl. Phys. B* 971, pp. 1-44. 2021.
- ▶ J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements. *Journal of High Energy Physics* 2022(193), pp. 0-32. 2022.
- ▶ J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The Two-Loop Massless Off-Shell QCD Operator Matrix Elements to Finite Terms. *Nuclear Physics B* 980(115794), pp. 1-131. 2022.
- ▶ J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1 . *Journal of High Energy Physics*. 1-83. 2022.

Symbolic tools for special functions

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Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remiddi, Blümlein, . . .)

$$\sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

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Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx,$$

$$\zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n) - \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O\left(\frac{\ln(n)}{n^6}\right).$$

limit computations

numerical evaluation

► Generalized algorithms for generalized harmonic sums

$$\begin{aligned}
 \sum_{k=1}^n \frac{2^k \sum_{i=1}^k \frac{2^{-i} \sum_{j=1}^i \frac{H_j}{j}}{i}}{k} &= -\frac{21\zeta(2)^2}{20n} + \frac{1}{8n^2} + \frac{295}{216n^3} - \frac{1115}{96n^4} + O(n^{-5}) \\
 &+ \left(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\right)\zeta(2) \\
 &+ 2^n \left(\frac{3}{2n} + \frac{3}{2n^2} + \frac{9}{2n^3} + \frac{39}{2n^4} + O(n^{-5})\right)\zeta(3) \\
 &+ \left(\frac{1}{n} + \frac{3}{4n^2} - \frac{157}{36n^3} + \frac{19}{n^4} + O(n^{-5})\right)(\log(n) + \gamma) \\
 &+ \left(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\right)(\log(n) + \gamma)^2
 \end{aligned}$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for cyclotomic harmonic sums

$$\begin{aligned}
 \sum_{k=1}^n \frac{\sum_{j=1}^k \frac{\sum_{i=1}^j \frac{1}{1+2i}}{j^2}}{(1+2k)^2} &= \left(-3 + \frac{35\zeta(3)}{16}\right)\zeta(2) - \frac{31\zeta(5)}{8} \\
 &+ \frac{1}{n} - \frac{33}{32n^2} + \frac{17}{16n^3} - \frac{4795}{4608n^4} + O(n^{-5}) \\
 &+ \log(2)\left(6\zeta(2) - \frac{1}{n} + \frac{9}{8n^2} - \frac{7}{6n^3} + \frac{209}{192n^4} + O(n^{-5})\right) \\
 &+ \left(-\frac{7}{4} - \frac{7}{16n} + \frac{7}{16n^2} - \frac{77}{192n^3} + \frac{21}{64n^4} + O(n^{-5})\right)\zeta(3) \\
 &+ \left(\frac{1}{16n^2} - \frac{1}{8n^3} + \frac{65}{384n^4} + O(n^{-5})\right)(\log(n) + \gamma)
 \end{aligned}$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for nested binomial sums

$$\sum_{j=1}^n \frac{4^j H_{j-1}}{\binom{2j}{j} j^2} = 7\zeta(3) + \sqrt{\pi}\sqrt{n} \left\{ \left[-\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} \right. \right. \\ \left. \left. + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} \right. \right. \\ \left. \left. + O(n^{-10}) \right] \ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} \right. \\ \left. - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} \right. \\ \left. + O(n^{-10}) \right\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence and DE solver
3. coupled systems solver
4. the large moment method

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6. stable and efficient software packages

Main CA-packages

In[17]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[18]:= << **MultiIntegrate.m**

MultIntegrate by Jakob Ablinger © RISC-Linz

In[19]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[20]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[21]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[22]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

In[23]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

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Within the **RISC-DESY** cooperation we expect that we will discover and explore many

new algorithms in CA and **results in QFT!**