RAMANUJAN AND COMPUTER ALGEBRA

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Abstract. This note describes developments in computer algebra which have been inspired by the mathematics of Ramanujan.

1. Introduction

The homage [3], written by George Andrews at the occasion of the 123th anniversary of Ramanujan's birth, contains a section "Ramanujan and computation." It begins with a relevant quote by Hardy [30, p. xxxv]:

"His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal". If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper [which lists the partition numbers up to $p(200)$ having 13 digits]. This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurate of the two."

Now, one hundred years later, methods of computation have changed drastically. The current world record in computing partition numbers is at $p(10^{20})$ which has slightly more than 11 billion digits [19]. To this end, Fredrik Johansson used his highly efficient implementation of the Hardy-Ramanujan-Rademacher formula [18]. Other aspects of the impact of Ramanujan's work on nowadays computing technology, e.g., can be found in [23].

One of the developments described in [23] is the Ramanujan Machine [29] that creates mathematical conjectures using AI and computer automation. Davide Castelvecchi [12] attempts to discuss this development in a broader context; e.g., he quotes from an interview with George Andrews:

"The fact that they have improved the irrationality exponent for the Catalan constant from 0.554 to 0.567 reveals that they are able to make contributions to really hard problems, [. . .] But the contributions made so far are not of the calibre that using Ramanujan's name would suggest. Calling this the Ramanujan Machine is over the top," says Andrews.

Castelvecchi [12] also quotes from his interview with Doron Zeilberger:

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"Eventually, humans will be obsolete," says Zeilberger, who adds: "And as the complexity of AI-generated mathematics grows, mathematicians will lose track of what computers are doing and will be able to understand the calculations only in broad outline."

In this note we discuss neither high precision- nor AI-related aspects of computer technology. Rather we restrict to one particular aspect of computation, namely to computer algebra based on algorithmic methods and related software developments.

2. Computer Algebra inspired by Ramanujan

In his monograph on q-series [2, p. 87], Andrews was led to speculate about Ramanujan and the age of computer algebra:

"Sometimes when studying his work I have wondered how much Ramanujan could have done if he had had MACSYMA or SCRATCHPAD or some other symbolic algebra package. More often I get the feeling that he was such a brilliant, clever, and intuitive computer himself that he really did not need them."

In the section "What If Ramanujan Had Mathematica?" of [35], Steven Wolfram speculates too:

"It's fun to imagine what Ramanujan would have done with these modern tools. I rather think he would have been quite an adventurer—going out into the mathematical universe and finding all sorts of strange and wonderful things, then using his intuition and aesthetic sense to see what fits together and what to study further.

Ramanujan unquestionably had remarkable skills. But I think the first step to following in his footsteps is just to be adventurous: not to stay in the comfort of well-established mathematical theories, but instead to go out into the wider mathematical universe and start finding—experimentally—what's true."

In the rest of this note we give a brief sketch in which ways Ramanujan's work has inspired developments in computer algebra. Needless to say, that we will restrict only to a small sample of such developments, owing to page limit but also to incompleteness of our knowledge.

A natural start is made by stressing George Andrews's role as pioneer also with regard to "Ramanujan and Computer Algebra". In his book [2] a whole chapter is devoted to computer algebra; it presents an account of how symbolic computation can be utilized in research of q-series. Much of this material is related to Ramanujan. For example, Frank Garvan, another pioneer in this topical area, developed the Maple package `BAILEY` for computing Bailey pairs. The usage of `BAILEY` requires another Garvan package, `QSERIES`, whose main features are: (i) conversion of q-series to infinite products of different types including eta-products and theta products; (ii) factorization of a given rational function into a finite q-product if one exists; (iii) generating probable algebraic relations (if they exist) among given q-series, and much more.

On Garvan's web page `https://qseries.org/fgarvan/qmaple/qseries/index.html` one finds a variety of other Maple packages related to or inspired by the mathematics of Ramanujan. For example, there is the `ETA` package for doing calculations with Dedekind eta functions and for proving eta-product identities using the valence formula for modular forms [16]. For proving generalized eta-product identities there is the package `THETAIDS` and the `RAMAROBINSIDS` package for finding and proving identities for generalizations of Ramanujan's functions $G(q)$ and $H(q)$ and Robins's extensions; see [15].

Andrews contributed substantially to the development of computer algebra algorithms and respective software. As described in [3], a generalization of Engel expansion [4], a certain kind of series representation, can be used to computationally "explain" how to go from the product sides of the Rogers-Ramanujan identities to their sum sides. Subsequently, this idea has been applied to other identities of Rogers-Ramanujan type, [5] and [6], and the Mathematica package `Engel`, written by Burkhard Zimmermann, has been developed [7]. For example, from the input $\prod_{n\geq 0}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}$ the first terms of the sum side are obtained with the procedure call

In[1]:= **Engel[1/qPochhammer[q, q^5]/qPochhammer[q^4, q^5], q, 20]**

Out[1]= $1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+O[q]^{16}$

Another major algorithmic contribution is George Andrews's revitalization of MacMahon's method of Partition Analysis. It was Andrews who, when he spent part of his sabbatical at RISC in spring 1998, initiated the project of studying the algorithmic potential of Partition Analysis—with the goal to produce corresponding software. Over the years a series of articles has been produced, [9] being the most recent one. The Mathematica package `Omega`, written by Axel Riese, has served as a fundumental tool in this project. Besides its important role in experimental mathematics, it has been of great help in deepening the understanding of MacMahon's method and various features of his Omega operator. We restrict to present only one result from [9, Thm. 5]:

The generating function for partitions with n copies of n with m parts in which the weighted difference between parts $\geq r$ with $r\geq -2$ is given by

$$\frac{x_1^{(m-1)r+(2m-1)}x_2^{(m-2)r+(2m-3)}\dots x_m^{0\cdot r+1}}{\prod_{i=1}^{m}(1-x_1x_2\dots x_i)\cdot(1-x_1)\prod_{i=2}^{m}(1-x_1^2x_2^2\dots x_{i-1}^2x_i)},$$

where the exponent of x_i accounts for the ith part of the partition in question.

Remarkably, the case $r=-2$, after setting the $x_i=q$ and summing over all $m\geq 0$, in a natural way rewrites in terms of Ramanujan's fifth order mock theta function $\phi_0(q)=1+\sum_{m\geq 1}q^{m^2}(1+q)(1+q^3)\dots(1+q^{2m-1})$.

3. "First guess, then prove" strategies

The usage of computers in mathematical research often comes in two flavors. There is the aspect of doing experiments (computing data for further inspection, correctness checks, etc.) with the goal to arrive at solid hypotheses or at properly specified problems to solve. The second aspect is to use symbolic computation systems for proving and for solving. The computing technology applied in these two processes, in short: guessing and proving, can be based on manifold methods (data bases, AI-methods, computational logic, numerical solvers, etc.). As already mentioned, in this note we restrict to computer algebra algorithms.

Guessing and proving, in practice, usually interact in various ways. For example, in many applications of MacMahon's Partition Analysis, the `Omega` package is used for producing expressions depending on a parameter n. Inspection of the first cases, for $n = 1, 2$, etc., then provides the basis for setting up a general hypothesis to prove. Remarkably, MacMahon's method is taylored in such a way, that the computational steps made by `Omega` in the concrete cases for $n = 1, 2$, etc., often suggest very concrete patterns for proving the general hypothesis by mathematical induction.

The next example should illustrate another different kind of "first guess, then prove" interplay: the task is to prove

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x_3(q)\right) = a(q), \quad \text{where } a(q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{3.1} $$

and

$$ x_3(q) = \frac{c(q)^3}{a(q)^3}, \quad \text{with} \quad c(q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}. $$

Here we use Shaun Cooper's notation; a classical proof of (3.1) can be found in his beautiful book [13, Thm. 4.4]. There one also finds how these functions relate to original work by Ramanujan and by the Borwein brothers; see also Cooper's contribution [14] to this volume.

Using Mallinger's Mathematica package `GeneratingFunctions`, which is freely available at `https://combinatorics.risc.jku.at/software`, one can discover (3.1) as follows. Open a Mathematica session, load the package,

In[2]:= **<< RISC\`GeneratingFunctions\`**

Version 0.8 written by Christian Mallinger © RISC-JKU

and input the q-series expansions of $a(q)$ and $x_3(q)$, for instance, as follows:

In[3]:= $\mathbf{a[N\_] := Series[Sum[q^{m^2+mn+n^2}, \{m, -N, N\}, \{n, -N, N\}], \{q, 0, N\}]}$

In[4]:= $\mathbf{c[N\_] := Series[Sum[q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}, \{m, -N, N\}, \{n, -N, N\}], \{q, 0, N\}]}$

In[5]:= $\mathbf{x3[N\_] := c[N]^3/a[N]^3}$

In[6]:= **a[7]**

Out[6]= $1 + 6q + 6q^3 + 6q^4 + 12q^7 + O[q]^8$

In[7]:= **x3[7]**

Out[7]= $27q - 405q^2 + 4617q^3 - 45333q^4 + 406458q^5 - 3428487q^6 + 27673704q^7 + O[q]^8$

Next, determine an initial string $R = (r(0), \dots, r(8))$ of coefficients $r(n)$ in the expansion

$$ a(q) = \sum_{n=0}^{\infty} r(n) x_3(q)^n, $$

which is done by using the compositorial inverse $X_3(q)$ such that $x_3(X_3(q)) = q$:

In[8]:= **ComposeSeries[a[22], InverseSeries[x3[22]]]; R = CoefficientList[Normal[%], q]**

Out[8]= $\left\{1, \frac{2}{9}, \frac{10}{81}, \frac{560}{6561}, \frac{3850}{59049}, \frac{28028}{531441}, \frac{1905904}{43046721}, \frac{14780480}{387420489}, \frac{116858170}{3486784401}\right\}$

Next, calling the `GeneratingFunctions` procedure `GuessRE`, automatically *guesses* a recurrence `Rec` satisfied by the coefficients $r(n)$:

In[9]:= **Rec = GuessRE[R, r[n]][[1]]**

Out[9]= $\{9(\mathtt{n}+1)^2\mathtt{r}(\mathtt{n}+1)-(3\mathtt{n}+1)(3\mathtt{n}+2)\mathtt{r}(\mathtt{n})=0, \mathtt{r}(0)=1\}$

In other words, we guessed that

$$\frac{r(n+1)}{r(n)} = \frac{(n+\frac{1}{3})(n+\frac{2}{3})}{(n+1)^2},\ n \geq 1, \text{ and } r(0)=1,$$

which is equivalent to guessing the ${}_2F_1$-series expansion in (3.1). From a general point of view, the recurrence `Rec` in `Out[9]` is a linear recurrence with polynomial coefficients. It is a well-known fact that sequences $r(n)$, which satisfy such kind of recurrences, have generating functions $y(x) := \sum_{n=0} r(n)x^n$ satisfying linear differential equations with polynomial coefficients. Algorithmic versions of such facts (conversions, etc.), for instance, are described in [21].

For example, the conversion of the recurrence `Rec` in `Out[9]` into the corresponding differential equation for $y(x) := \sum_{n=0} r(n)x^n$ is done by the procedure `RE2DE` of Mallinger's package:

In[10]:= **DE = RE2DE[Rec, r[n], y[x]]**

Out[10]= $\{9\mathtt{x}(1-\mathtt{x})\mathtt{y}''[\mathtt{x}]+9(1-2\mathtt{x})\mathtt{y}'[\mathtt{x}]-2\mathtt{y}[\mathtt{x}]=0, \mathtt{y}[0]=1, \mathtt{y}'[0]=2/9\}$

Consequently, to complete our task, i.e., to prove the correctness of the ${}_2F_1$-series expansion of $a(q)$ in (3.1), which so far was only guessed, we need to prove the validity of the equivalent differential equation `DE`. Algorithmically, this will be done using the equivalent form

$$9x_3(q)(1-x_3(q))y''(x_3(q)) + 9(1-2x_3(q))y'(x_3(q)) - 2y(x_3(q)) = 0, \tag{3.2}$$

with $q = e^{2\pi i\tau}$ and where τ is from the upper half of the complex plane such that its imaginary part is sufficiently large.

How the correctness of such differential equations as (3.2) can be proven algorithmically is explained in [27] and [28]. The key idea is to reduce the problem to a problem of zero recognition of modular functions; the problem transformation is done by using a conversion involving basis elements consisting of functions introduced by Yifan Yang [36].

The general framework for this "first guess, then prove" strategy is based on the classical fact [37] that modular forms g of weight k can be locally expanded, $g = y(h)$, in terms of modular functions h (i.e., the weight of h is 0) and where the y satisfy linear differential equations

$$p_d(h)y^{(d)}(h) + p_{d-1}(h)y^{(d-1)}(h) + \cdots + p_0(h)y(h) = 0$$

with $p_j(h)$ being polynomials in h. In general, the order d will be greater than $k+1$. In case the underlying modular curve has genus 0 and h is a Hauptmodul one has $d = k+1$; see [37] and [27].

Concretely, to prove (3.1) one has $g := a(q)$ which as a function in τ is a modular form of weight 1 for the congruence subgroup $\Gamma_0(3)$; see [11, Sec. 3]. Moreover, $h := x_3(q)$ is a modular function and a Hauptmodul for $\Gamma_0(3)$; this, e.g., follows from [13, (4.36), p. 258].

All the steps of this "first guess, then prove" method are explained in full detail [27, Ex. 4.2 and Sec. 6.3] in a classical case, namely, to derive and prove

$$\theta_3(\tau)^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{1}; 1; \lambda(\tau)\right),$$

where $\theta_3(\tau) = 1+2\sum_{n\geq 1} q^{n^2/2}$, $q = e^{2\pi i\tau}$, is the Jacobi theta function and $\lambda = \theta_2(\tau)^4/\theta_3(\tau)^4$ is the modular lambda function .

4. Conclusion

The "first guess, then prove" method described in the previous section can be applied to a variety of identities in Ramanujan's work. As pointed out in [27], a particular application domain concerns Ramanujan's approximating series for $1/\pi$; see [10] for history and a fine survey.

Cristian-Silviu Radu [31] has developed an algorithm which "in one run" discovers and proves identities such as,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6},$$

which was selected by Hardy [30, p. XXXV] as Ramanujan's most beautiful formula next to the Rogers-Ramanujan identities. A short description of Radu's algorithm can be found in this volume [32]; a Mathematica implementation was done by Nicolas Smoot [34], an implementation in FriCAS (a freely available descendent of Axiom) by Ralf Hemmecke [17].

Concerning (q-)hypergeometric sums and series arising in Ramanujan's work, the major computer algebra systems contain procedures which implement versions of Doron Zeilberger's univariate creative telescoping algorithm [38]. For q-hypergeometric summation, for example, "finite sum" versions of the Rogers-Ramanujan identitities as discussed in [26], the package `qZeil` can still be useful; it also includes tools for Bailey chain computations [33, Ch. 3] and other features.

When dealing with multivariate identities, i.e., multiple (q-)hypergeometric sums and multiple integrals (also mixed), fitting into Zeilberger's holonomic systems approach [39], there is Christoph Koutschan's powerful Mathematica package `HolonomicSystems` [22].

We conclude with other recent computer algebra developments.

The computer searches (in PARI/GP, Maple, and Mathematica) by James McLaughlin, Drew Sills, and Peter Zimmer [24] led to the discovery of a number of identities of Rogers-Ramanujan type and identities of false theta functions.

The Maple package `IdentityFinder` was designed by Shashank Kanade and Matthew Russell [20] to generate conjectures of identities of Rogers-Ramanujan type. Some of the identities found this way were known, some have remained unproven until today.

A particular experimental feature of the Mathematica package `qFunctions` developed by Jakob Ablinger and Ali Uncu [1] is that it provides various kinds of guesses, for instance, for q-difference equations. To uncouple a coupled system of recurrences they invoke Koutschan's `HolonomicSystems` package [1, Sec. 4].

This concludes our survey of computer algebra related to Ramanujan. No doubt, his work will continue to spark creativity, and the near future will see many more exciting computer algebra developments— inspired by the mathematics of Ramanujan.

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