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Linear functionals and Δ- coherent pairs of the second kind

Diego Dominici [*†]   Francisco Marcellán [‡]

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Abstract

We classify all the Δ-coherent pairs of measures of the second kind on the real line. We obtain 5 cases, corresponding to all the families of discrete semiclassical orthogonal polynomials of class $s \leq 1$.

Keywords: Discrete orthogonal polynomials, discrete semiclassical functionals, discrete Sobolev inner products, coherent pairs of discrete measures, coherent pairs of second kind for discrete measures.

2020 Mathematical Subject Classification: 42C05 (primary), 33C45, 46E39 (secondary).

1 Introduction

The aim of this contribution is to provide a characterization of all pairs of discrete measures $\{\rho_0, \rho_1\}$ supported on the real line such that the corresponding sequences of *monic orthogonal polynomials* (MOPs for short) $\{P_n(\rho_0; x)\}_{n\geq 0}$ and $\{P_n(\rho_1; x)\}_{n\geq 0}$ satisfy

$$P_n(\rho_1; x) - \tau_n P_{n-1}(\rho_1; x) = \frac{1}{n+1} \Delta P_{n+1}(\rho_0; x), \quad n \geq 1, \tag{1}$$

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where $\tau_n \neq 0$ for $n \geq 1$. We will solve this problem by dealing with a more general problem concerning the characterization of pairs of quasi-definite linear functionals (with complex moments) such that the corresponding sequences of MOPs satisfy (1).

As we show in this contribution, we get special pairs of linear functionals (in particular, positive definite linear functionals associated with positive measures supported on infinite subsets of the real line) and the corresponding associated sequences of orthogonal polynomials have interesting properties. These pairs are *discrete semiclassical linear functionals* of class at most $s = 1$ which have been studied in [8] and [9] in the framework of a classification problem based on a hierarchy structure.

On the other hand, these sequences of orthogonal polynomials are related to some problems in approximation theory. More precisely, the analysis of Fourier expansions in terms of sequences of polynomials orthogonal with respect to the Sobolev inner product

$$\langle f, g \rangle_{\mathfrak{S}} = \sum_{x=0}^{\infty} f(x)g(x)\rho_0(x) + \lambda \sum_{x=0}^{\infty} \Delta f(x)\Delta g(x)\rho_1(x), \tag{2}$$

defined by the pair of discrete measures $\{\rho_0, \rho_1\}$ of class $s \geq 1$. It turns out that the monic Sobolev orthogonal polynomials $S_n(\rho_0, \rho_1; x)$ with respect to the inner product $\langle\, , \,\rangle_{\mathfrak{S}}$ satisfy the connection formulas

$$\begin{aligned} &S_{n+1}(\rho_0, \rho_1; x) - \gamma_n S_n(\rho_0, \rho_1; x) = P_{n+1}(\rho_0; x), \\ &\Delta S_{n+1}(\rho_0, \rho_1; x) - \gamma_n \Delta S_n(\rho_0, \rho_1; x) = (n+1)\left[P_n(\rho_1; x) - \tau_n P_{n-1}(\rho_1; x)\right], \end{aligned} \quad n \geq 1, \tag{3}$$

with $S_1(\rho_0, \rho_1; x) = P_1(\rho_0; x)$. The above connection formulas between $\{S_n(\rho_0, \rho_1; x)\}_{n\geq 0}$ and the sequences of MOPs $\{P_n(\rho_0; x)\}_{n\geq 0}$ and $\{P_n(\rho_1; x)\}_{n\geq 0}$ yield a nice approach to the study of algebraic and analytic properties of the polynomials $S_n(\rho_0, \rho_1; x)$.

Based on historical information given below we will refer to pairs of measures $\{\rho_0, \rho_1\}$ satisfying property (1) as Δ-*coherent pairs of measures of the second kind on the real line.* The concept of coherence between a pair of probability measures $\{\nu_0, \nu_1\}$ supported on the real line was introduced in [13]. Indeed, a pair of probability measures supported on the real line is said to be a coherent pair if the corresponding sequences of MOPs $\{P_n(\nu_0;x)\}_{n\geq 0}$ and $\{P_n(\nu_1;x)\}_{n\geq 0}$, satisfy

$$P_n(\nu_1;x) = \frac{1}{n+1}\left[P'_{n+1}(\nu_0;x) - \rho_n P'_n(\nu_0;x)\right], \quad \rho_n \neq 0, \quad n \geq 1. \tag{4}$$

It was shown in this case that the sequence of monic orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_S = \int f(x)g(x)d\nu_0 + \lambda \int f'(x)g'(x)d\nu_1, \tag{5}$$

satisfies the connection formulas

$$\begin{aligned} S_{n+1}(\nu_0,\nu_1;x) - \gamma_n S_n(\nu_0,\nu_1;x) &= P_{n+1}(\nu_0;x) - \rho_n P_n(\nu_0;x), \\ S'_{n+1}(\nu_0,\nu_1;x) - \gamma_n S'_n(\nu_0,\nu_1;x) &= (n+1)P_n(\nu_1;x), \end{aligned} \quad n \geq 1. \tag{6}$$

These formulas are very useful in the study of analytic properties of the corresponding Sobolev orthogonal polynomials. For more information about polynomials orthogonal with respect to Sobolev inner products see the updated survey [19].

The motivation in [13] for introducing such pairs of measures was their applications in the framework of Fourier expansions of functions with respect to the Sobolev inner product $\langle\, , \,\rangle_{\mathfrak{S}}$. A particular case of such Fourier series expansions based on Legendre-Sobolev orthogonal polynomials had already been considered in [14], where some numerical tests comparing these Legendre-Sobolev Fourier series expansions and the ordinary Legendre Fourier series expansions are presented.

The pairs of probability measures supported on the real line with the property that the corresponding sequences of MOPs satisfy (4) were completely determined in 1997 by H. G. Meijer [20]. He showed that if (ν_0, ν_1) is a coherent pair of measures on the real line, then one of the measures must be classical (either Jacobi or Laguerre) and the other one a rational perturbation of it. Note that the condition for classical linear functionals to be part of a coherent pair was deduced in [17].

Thus, what was proved in [20] is more general than what is stated above. The starting point of the classification in [20] are certain functional relations established in [16] with respect to pairs of quasi-definite linear functionals such that the corresponding sequences of MOPs satisfy a relation like (4). Note that if the coherence property (4) holds, then you get the connection formulas in (6). According to [20], (6) holds when one of the measures in $\{\nu_0, \nu_1\}$ is a classical one.

An extension of the concept of coherence, known in the literature as $(1,1)$-coherence, is defined in terms of the corresponding MOPs as follows (see [7])

$$P_n(\nu_1;x) - \tau_n P_{n-1}(\nu_1;x) = \frac{1}{n+1}\left[P'_{n+1}(\nu_0;x) - \xi_n P'_n(\nu_0;x)\right], \quad \xi_n, \tau_n \neq 0, \quad n \geq 1, \tag{7}$$

In this case, the Sobolev orthogonal polynomials associated with the inner product $\langle\, , \,\rangle_{\mathfrak{S}}$ in (5) satisfy the connection formulas

$$\begin{aligned} S_{n+1}(\nu_0,\nu_1;x) - \gamma_n S_n(\nu_0,\nu_1;x) &= P_{n+1}(\nu_0;x) - \xi_n P_n(\nu_0;x), \\ S'_{n+1}(\nu_0,\nu_1;x) - \gamma_n S'_n(\nu_0,\nu_1;x) &= (n+1)\left[P_n(\nu_1;x) - \tau_n P_{n-1}(\nu_1;x)\right], \end{aligned} \quad n \geq 1. \tag{8}$$

We would like to emphasize that the characterization of probability measures satisfying the coherence property (7) is given in [7] assuming that $\xi_n, \tau_n \neq 0$ for $n \geq 1$. Therein you have semiclassical linear functionals of class at most 1. Its role in the study of the sequences of orthogonal polynomials with respect to the Sobolev inner product defined by a $(1,1)$ coherent pairs of measures has been emphasized in [12]. Notice that when $\tau_n = 0$ and $\xi_n \neq 0$ for $n \geq 1$, we get the results on coherent pairs presented in [20]. Thus, a natural question is to analyze the case $\xi_n = 0$,and $\tau_n \neq 0$ for $n \geq 0$.

In [24], the concept of *coherent pair of second kind* is introduced. A pair of probability measures $\{\nu_0, \nu_1\}$ supported on the real line is said to be a coherent pair of second kind if the corresponding sequences of MOPs $\{P_n(\nu_0;x)\}_{n\geq 0}$ and $\{P_n(\nu_1;x)\}_{n\geq 0}$ satisfy

$$\frac{1}{n+1}P'_{n+1}(\nu_0;x) = P_n(\nu_1;x) - \tau_n P_{n-1}(\nu_1;x) \quad n \geq 1, \tag{9}$$

where $\tau_n \neq 0$ for $n \geq 1$. The characterization of all pairs of probability measures as well as the pairs of quasi-definite linear functionals which are coherent pairs of second kind has been given in [24]. Some illustrative examples were shown therein.

The concept of Δ-coherent pair of linear functionals was introduced in [1] as well as in [3]. Indeed, a pair of linear functionals $\{L_0, L_1\}$ it is said to be a Δ-coherent pair if the corresponding sequences of MOPs $\{P_n^{(0)}(x)\}_{n\geq 0}$ and $\{P_n^{(1)}(x)\}_{n\geq 0}$ satisfy a discrete version of (4)

$$P_n^{(1)}(x) = \frac{1}{n+1}\left[\Delta P_{n+1}^{(0)}(x) - \rho_n \Delta P_n^{(0)}(x)\right], \quad \rho_n \neq 0, \quad n \geq 1. \tag{10}$$

It was shown in this case that the sequence of monic polynomials $S_n(x)$ orthogonal with respect to the inner product

$$\langle f, g\rangle_{\mathfrak{S}} = \langle L_0, fg\rangle + \langle L_1, \Delta f \Delta g\rangle$$

satisfies the connection formulas

$$\begin{aligned} S_{n+1}(x) - \gamma_n S_n(x) &= P_{n+1}^{(0)}(x) - \rho_n P_n^{(0)}(x), \\ \Delta S_{n+1}(x) - \gamma_n \Delta S_n(x) &= (n+1) P_n^{(1)}(x), \end{aligned} \qquad n \geq 1. \tag{11}$$

In [3], it was proved that one of them must be a Δ-classical linear functional. The corresponding companion linear functionals were described therein. Notice that this inner product is a discretization of a Sobolev inner product (5), i.e., ν_0 and ν_1 are discrete measures.

The paper is organized as follows. Section 2 summarizes the basic concepts about linear functionals and orthogonal polynomials to be used in the sequel. We emphasize a special family of linear functionals, the so called Δ-*semiclassical*, which will play a central role along the manuscript. In Section 3 we introduce the concept of Δ-coherent pair of the second kind for linear functionals and we give a characterization of them. We must point out that both functionals in the pair turn out to be discrete semiclassical functionals of class at most 1. In Section 4 we classify all the Δ- coherent pairs of the second kind. We show that the companions of the discrete classical functionals (Charlier, Kravchuk, Meixner, and Hahn) are discrete semiclassical of class $s = 1$.

2 Preliminary material

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{N}_0$ be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

We will denote by $\delta_{k,n}$ the *Kronecker delta*, defined by

$$\delta_{k,n} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}, \quad k, n \in \mathbb{N}_0,$$

and say that $\{\Lambda_n(x)\}_{n\geq 0} \subset \mathbb{P}$ is a *basis* of $\mathbb{P}$ if $\deg(\Lambda_n) = n$.

Suppose that $L : \mathbb{P} \to \mathbb{C}$ is a *linear functional*, $\{\Lambda_n(x)\}_{n\geq 0}$ is a basis of $\mathbb{P}$, and we choose a **nonzero** sequence $\{h_n\}_{n\geq 0} \subset \mathbb{C}$. If the system of linear equations

$$\sum_{i=0}^{n} L[\Lambda_k \Lambda_i] c_{n,i} = h_n \delta_{k,n}, \quad 0 \leq k \leq n, \tag{12}$$

has a **unique solution** $\{c_{n,i}\}_{0\leq i\leq n}$, then we can define a polynomial $P_n(x)$ by

$$P_n(x) = \sum_{i=0}^{n} c_{n,i} \Lambda_i(x), \quad n \in \mathbb{N}_0.$$

We say that $\{P_n(x)\}_{n\geq 0}$ is an *orthogonal polynomial sequence* with respect to the functional L. The system (12) can be written as

$$L[\Lambda_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,$$

and using linearity we see that the sequence $\{P_n(x)\}_{n\geq 0}$ satisfies the *orthogonality conditions*

$$L[P_k P_n] = h_n \delta_{k,n}, \quad k, n \in \mathbb{N}_0. \tag{13}$$

The linear functional L is said to be *quasi-definite* [6].

We define the multiplication of a linear functional by a polynomial $q(x)$ to be the linear functional qL such that

$$(qL)[p] = L[qp], \quad p \in \mathbb{P}, \tag{14}$$

and the adjoint operator Δ^* of a linear functional by

$$(\Delta^* L)[p] = -L[\Delta p], \quad p \in \mathbb{P}. \tag{15}$$

We say that L is a *semiclassical functional* with respect to the operator Δ, ($\Delta-$semiclassical, in short) [18] if there exist polynomials (ϕ, ψ) such that L satisfies the *Pearson equation*

$$\Delta^*(\phi L) + \psi L = 0, \tag{16}$$

where ϕ is monic and $\deg(\psi) \geq 1$. Note that using (14) and (15) we can rewrite the Pearson equation as

$$L[\phi \Delta p] = L[\psi p], \quad p \in \mathbb{P}. \tag{17}$$

Lemma 1 *If $f, g : \mathbb{Z} \to \mathbb{C}$, then we have the* summation by parts *formula*

$$\sum_{x=a}^{b} f(x) \Delta g(x) = [f(x-1) g(x)]_a^{b+1} - \sum_{x=a}^{b} g(x) \nabla f(x), \quad a, b \in \mathbb{Z}. \tag{18}$$

Proof. The formula follows from the telescoping sum

$$\begin{aligned} \sum_{x=a}^{b} [f(x) \Delta g(x) + g(x) \nabla f(x)] &= \sum_{x=a}^{b} f(x) g(x+1) - g(x) f(x-1) \\ &= f(b) g(b+1) - g(a) f(a-1). \end{aligned}$$

■

Using summation by parts, we can write an alternative form of the Pearson equation (16).

Proposition 2 *Let $L \in P^*$ be defined by*

$$L[p] = \sum_{x=0}^{\infty} p(x) \rho(x), \quad p \in \mathbb{P},$$

where $\rho(-1) = 0$ and

$$\lim_{x \to \infty} p(x) \rho(x) = 0, \quad p \in \mathbb{P}.$$

Then, L satisfies the Pearson equation (16) if and only if

$$\nabla(\phi \rho) + \psi \rho = 0. \tag{19}$$

Proof. Since (16) is equivalent to (17), we can use (18) and conclude that L satisfies (16) if and only if

$$\sum_{x=0}^{\infty} \psi(x) p(x) \rho(x) = -\sum_{x=0}^{\infty} p(x) \nabla(\phi \rho)(x), \quad p \in \mathbb{P}.$$

If the last equation holds for all $p \in \mathbb{P}$, then we must have

$$\psi\rho = -\nabla (\phi\rho),$$

and the result follows. ■

The *class* of a discrete semiclassical linear functional L is defined by

$$s = \min\{\max\{\deg(\phi) - 2,\ \deg(\psi)\} - 1\}, \tag{20}$$

where the minimum is taken among all pairs of polynomials ϕ, ψ such that the Pearson equation (16) holds for the linear functional L. Functionals of class $s = 0$ are called Δ-classical (see [11], [15], and [22] among others). Δ-semiclassical linear functionals of class $s = 1$ have been described in [8] and [9].

Remark 3 *Let* $\deg(\phi) = d_1$, $\deg(\psi) = d_2$. *If* $d_1 = d_2 + 1 = s + 2$, *we will always assume the admissibility condition*

$$\psi_{d_2} \neq n - s, \quad n \in \mathbb{N}_0, \tag{21}$$

where ψ_{d_2} *is the leading coefficient of* $\psi(x)$.

We will denote by $\{\varphi_n(x)\}_{n\geq 0}$ the basis of *falling factorial polynomials* defined by $\varphi_0(x) = 1$ and

$$\varphi_n(x) = \prod_{k=0}^{n-1} (x - k), \quad n \in \mathbb{N}. \tag{22}$$

The polynomials $\varphi_n(x)$ satisfy the basic identities

$$x\varphi_n(x) = \varphi_{n+1}(x) + n\varphi_n(x), \quad n \in \mathbb{N}_0, \tag{23}$$

and

$$\Delta\varphi_n = n\varphi_{n-1}, \quad n \in \mathbb{N}_0. \tag{24}$$

Lemma 4 *We have the representation*

$$\varphi_n(x-1) = \sum_{k=0}^{n} (-1)^k \varphi_k(n)\, \varphi_{n-k}(x), \quad n \in \mathbb{N}_0. \tag{25}$$

Proof. We use induction. The identity is true for $n = 0$ since $\varphi_0(x) = 1$. Assuming (25) to be true for all $0 \leq k \leq n$ and using the recurrence (23), we have

$$\varphi_{n+1}(x-1) = (x-1-n)\,\varphi_n(x-1) = \sum_{k=0}^{n} (-1)^k \varphi_k(n)\,(x-1-n)\,\varphi_{n-k}(x)$$
$$= \sum_{k=0}^{n} (-1)^k \varphi_k(n)\left[\varphi_{n+1-k}(x) - (k+1)\,\varphi_{n-k}(x)\right].$$

Since $\varphi_{n+1}(n) = 0$, we can rewrite the sum above as

$$\varphi_{n+1}(x-1) = \sum_{k=0}^{n+1} (-1)^k \varphi_k(n)\,\varphi_{n+1-k}(x) + \sum_{k=0}^{n} (-1)^{k+1}(k+1)\,\varphi_k(n)\,\varphi_{n-k}(x),$$

or shifting the index in the second sum

$$\varphi_{n+1}(x-1) = \sum_{k=0}^{n+1} (-1)^k \varphi_k(n)\,\varphi_{n+1-k}(x) + \sum_{k=1}^{n+1} (-1)^k k\varphi_{k-1}(n)\,\varphi_{n+1-k}(x).$$

But from (24) we see that

$$\varphi_k(n+1) = \varphi_k(n) + k\varphi_{k-1}(n),$$

and we conclude that

$$\varphi_{n+1}(x-1) = \sum_{k=0}^{n+1} (-1)^k \varphi_k(n+1)\,\varphi_{n+1-k}(x).$$

■

Proposition 5 *Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of polynomials orthogonal with respect to a $\Delta-$semiclassical functional L of class s. Then,*

$$L[\phi\Delta q\Delta p_n] = 0, \quad q \in \mathbb{P}, \quad \deg(q) < n - s. \tag{26}$$

Proof. The equation (26) is obviously true if $q = 1$, so we can assume that $\deg(q) \geq 1$. Using (25) with $k \geq 1$, we have

$$\Delta\varphi_k = -\sum_{j=1}^{k} (-1)^j \varphi_j(k)\,\varphi_{k-j}(x+1),$$

and multiplying by Δp_n we get

$$\Delta\varphi_k \Delta p_n = \sum_{j=1}^{k} (-1)^j \varphi_j(k) \left[\Delta\varphi_{k-j} p_n - \Delta(\varphi_{k-j} p_n)\right],$$

where we have used the identity

$$\Delta(fg) = g(x+1)\Delta f + f\Delta g = g\Delta f + f\Delta g + \Delta f \Delta g. \tag{27}$$

Using the Pearson equation (16) and (24), we obtain

$$\begin{aligned} L[\phi\Delta\varphi_k\Delta p_n] &= \sum_{j=1}^{k} (-1)^j \varphi_j(k) \left(L[\phi\Delta\varphi_{k-j}p_n] - L[\psi\varphi_{k-j}p_n]\right) \\ &= \sum_{j=1}^{k} (-1)^j \varphi_j(k) \left\{(k-j) L[\phi\varphi_{k-j-1}p_n] - L[\psi\varphi_{k-j}p_n]\right\}, \end{aligned}$$

and since the sequence of polynomials $\{p_n(x)\}_{n\geq 0}$ is orthogonal with respect to L we conclude that

$$L[\phi\Delta\varphi_k\Delta p_n] = 0, \quad k < \min\{n+2-\deg(\phi),\ n+1-\deg(\psi)\} = n-s.$$

Using the fact that $\{\varphi_k(x)\}_{k\geq 0}$ is a polynomial basis, the result follows. ■

Using Proposition 5, we can prove the following characterization of discrete semiclassical polynomials.

Theorem 6 *Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials orthogonal with respect to a functional L. The following statements are equivalent:*

Proposition 7 *(i) L is $\Delta-$semiclassical of class s.*

(ii) There exists a monic polynomial $\phi(x)$ such that the polynomials p_n satisfy the structure equation

$$\phi(x)\Delta p_{n+1} = \sum_{k=n-s}^{n+d_1} \epsilon_{n,k} p_k, \quad n > s, \quad \epsilon_{n,n-s} \neq 0. \tag{28}$$

Proof. (i) ⇒ (ii) Since the sequence $\{p_n\}_{n\geq 0}$ is a polynomial basis in the linear space of polynomials, we can write

$$\phi(x)\Delta p_{n+1} = \sum_{k=0}^{n+d_1} \epsilon_{n,k} p_k. \tag{29}$$

Using the orthogonality relation (13) and (27), we have

$$h_k \epsilon_{n,k} = L\left[\phi p_k \Delta p_{n+1}\right] = L\left[\phi \Delta\left(p_k p_{n+1}\right)\right] - L\left[\phi \Delta p_k p_{n+1}\right] - L\left[\phi \Delta p_k \Delta p_{n+1}\right].$$

Using (17) and orthogonality, we get

$$L\left[\phi \Delta\left(p_k p_{n+1}\right)\right] = L\left[\psi p_k p_{n+1}\right] = 0, \quad k < n+1-d_2,$$

and

$$L\left[\phi \Delta \varphi_k p_{n+1}\right] = kL\left[\phi \varphi_{k-1} p_{n+1}\right] = 0, \quad k < n+2-d_1,$$

from which it follows that

$$L\left[\phi \Delta\left(p_k p_{n+1}\right)\right] - L\left[\phi \Delta p_k p_{n+1}\right] = 0, \quad k < n-s.$$

Using (26) we see that

$$L\left[\phi \Delta p_k \Delta p_{n+1}\right] = 0, \quad k < n+1-s,$$

and therefore

$$\epsilon_{n,k} = 0, \quad k < n-s.$$

For $k = n-s$, we have

$$\epsilon_{n,n-s} = h_{n+1}\left(\frac{\psi_{d_2}}{h_{n+1-d_2}}\delta_{d_2,s+1} - \frac{n+2-d_1}{h_{n+2-d_1}}\delta_{d_1,s+2}\right),$$

and it follows that $\epsilon_{n,n-s} \neq 0$ as long as $d_1 - 2 \neq d_2 - 1$. If $d_1 = d_2 + 1$, we need the additional condition (21).

(ii) $\Rightarrow$ (i) Since $\left\{\frac{p_n}{h_n}L\right\}_{n\geq 0}$ is a basis of $\mathbb{P}^*$ satisfying

$$\frac{p_n}{h_n}L\left[p_k\right] = \frac{1}{h_n}L\left[p_k p_n\right] = \delta_{k,n},$$

we have the representation

$$\Delta^*\left(\phi L\right) = \sum_{n=0}^{\infty} c_n \frac{p_n}{h_n} L.$$

Using the structure relation (28) and orthogonality, we get

$$\begin{aligned} c_{n+1} &= \Delta^*\left(\phi L\right)\left[p_{n+1}\right] = -L\left[\phi \Delta p_{n+1}\right] \\ &= -\sum_{k=n-s}^{n+d_1} \epsilon_{n,k} L\left[p_k\right] = \begin{cases} 0, & n > s \\ -\mu_0 \epsilon_{n,0}, & n \leq s \end{cases}, \end{aligned}$$

where $\mu_0 = L[1]$. Thus, $\Delta^*(\phi L) + \psi L = 0$, with

$$\psi(x) = \mu_0 \sum_{n=1}^{s+1} \frac{\epsilon_{n-1,0}}{h_n} p_n(x).$$

Note that $\deg(\psi) - 1 \leq s$, in agreement with (20). ■

For more information about this kind of structure relations for Δ-semiclassical linear functionals, check [18]. Notice that in the Δ-classical case ($s = 0$) two structure relations characterizing such linear functionals are given in [11].

3 Coherent pairs

Let $\{L_0, L_1\}$ be a pair of quasi-definite functionals and $\left\{P_n^{(0)}(x)\right\}_{n\geq 0}, \left\{P_n^{(1)}(x)\right\}_{n\geq 0}$ be the corresponding sequences of monic orthogonal polynomials. We say that $\{L_0, L_1\}$ is a $\Delta-$*coherent pair of the second kind* (abbreviated Δc2) if there exists $\{\tau_n\}_{n\geq 0} \subset \mathbb{C} \setminus \{0\}$ such that

$$Q_n(x) = \frac{\Delta P_{n+1}^{(0)}(x)}{n+1} = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x), \quad n \in \mathbb{N}_0, \tag{30}$$

where $P_{-1}^{(0)}(x) = P_{-1}^{(1)}(x) = 0$. The sequences of polynomials $\left\{P_n^{(i)}(x)\right\}_{n\geq 0}$, $i = 0, 1$, satisfy the orthogonality conditions

$$L_i\left[P_k^{(i)} P_n^{(i)}\right] = h_n^{(i)} \delta_{n,k}, \quad i = 0, 1, \quad n, k \in \mathbb{N}_0. \tag{31}$$

Proposition 8 *If $\{L_0, L_1\}$ is a Δc2 and the functionals $\mathbf{v}_n^{(i)}$ are defined by*

$$\mathbf{v}_n^{(i)} = \frac{P_n^{(i)}}{h_n^{(i)}} L_i, \quad i = 0, 1, \quad n \in \mathbb{N}_0, \tag{32}$$

then

$$\Delta^* \mathbf{v}_n^{(1)} = \tau_{n+1}(n+2)\, \mathbf{v}_{n+2}^{(0)} - (n+1)\, \mathbf{v}_{n+1}^{(0)}, \quad n \in \mathbb{N}_0. \tag{33}$$

Proof. Using (31) and (32), we have

$$\mathbf{v}_n^{(i)}\left[P_k^{(i)}\right] = \delta_{n,k}, \quad i = 0, 1, \quad n, k \in \mathbb{N}_0. \tag{34}$$

Let $\mathbf{u}_n \in \mathbb{P}^*$ be defined by

$$\mathbf{u}_n [Q_k] = \delta_{n,k}, \quad n,k \in \mathbb{N}_0. \tag{35}$$

Since the sequence of linear functionals $\{\mathbf{u}_n\}_{n\geq 0}$ is a basis of $\mathbb{P}^*$, we can write

$$\mathbf{v}_n^{(1)} = \sum_{k=0}^{\infty} a_{n,k} \mathbf{u}_n.$$

Using (30) and (34), we get

$$a_{n,k} = \mathbf{v}_n^{(1)} [Q_k] = \mathbf{v}_n^{(1)} \left[P_k^{(1)} - \tau_k P_{k-1}^{(1)} \right] = \delta_{n,k} - \tau_k \delta_{n,k-1},$$

and therefore

$$\mathbf{v}_n^{(1)} = \mathbf{u}_n - \tau_{n+1} \mathbf{u}_{n+1}, \quad n \in \mathbb{N}_0. \tag{36}$$

If we write

$$\Delta^* \mathbf{u}_n = \sum_{k=0}^{\infty} b_{n,k} \mathbf{v}_k^{(0)},$$

then

$$b_{n,k} = \Delta^* \mathbf{u}_n \left[P_k^{(0)} \right] = -\mathbf{v}_n^{(0)} \left[\Delta P_k^{(0)} \right] = -k \mathbf{v}_n^{(0)} [Q_{k-1}] = -k \delta_{n,k-1},$$

and, as a consequence,

$$\Delta^* \mathbf{u}_n = -(n+1)\, \mathbf{v}_{n+1}^{(0)}. \tag{37}$$

Using (37) in (36), we obtain (33). ■

With the help of Proposition 8, we can prove a characterization of $\Delta-$coherent pairs of the second kind.

Theorem 9 *The following are equivalent:*

(1) $\{L_0, L_1\}$ *is a* Δc2.

(2) There exist $\Lambda_2, \Lambda_3 \in \mathbb{P}$ *defined by*

$$\Lambda_k (x) = \sum_{i=0}^{k} \lambda_i^{(k)} x^i, \quad k = 2, 3, \tag{38}$$

and satisfying

$$\lambda_2^{(2)} + (n-1)\, \lambda_3^{(3)} \neq 0, \quad n \in \mathbb{N}, \tag{39}$$

such that

$$\Delta^* L_1 = \Lambda_2 L_0, \quad L_1 = \Lambda_3 L_0. \tag{40}$$

Proof. (1) $\Rightarrow$ (2) Setting $n=0$ in (33), we have

$$\frac{1}{h_0^{(1)}}\Delta^* L_1 = 2\tau_1 \frac{P_2^{(0)}}{h_2^{(0)}} L_0 - \frac{P_1^{(0)}}{h_1^{(0)}} L_0,$$

and defining

$$\Lambda_2(x) = \frac{2\tau_1 h_0^{(1)}}{h_2^{(0)}} P_2^{(0)}(x) - \frac{h_0^{(1)}}{h_1^{(0)}} P_1^{(0)}(x),$$

we get $\Delta^* L_1 = \Lambda_2 L_0$. Since $\tau_1 \neq 0$, we see that $\deg(\Lambda_2) = 2$.

Similarly, setting $n=1$ in (33) gives

$$\frac{1}{h_1^{(1)}}\Delta^*\left(P_1^{(1)} L_1\right) = 3\tau_2 \frac{P_3^{(0)}}{h_3^{(0)}} L_0 - 2\frac{P_2^{(0)}}{h_2^{(0)}} L_0$$

and using (27) we have

$$\Delta^* P_1^{(1)} L_1 + P_1^{(1)} \Delta^* L_1 + \Delta^* P_1^{(1)} \Delta^* L_1 = \left(\frac{3\tau_2 h_1^{(1)}}{h_3^{(0)}} P_3^{(0)} - \frac{2h_1^{(1)}}{h_2^{(0)}} P_2^{(0)}\right) L_0.$$

Since $\Delta^* P_1^{(1)} = 1$ and $\Delta^* L_1 = \Lambda_2 L_0$, we conclude that $L_1 = \Lambda_3 L_0$ with

$$\Lambda_3(x) = \frac{3\tau_2 h_1^{(1)}}{h_3^{(0)}} P_3^{(0)}(x) - \frac{2h_1^{(1)}}{h_2^{(0)}} P_2^{(0)}(x) - \Lambda_2(x)\left[P_1^{(1)}(x) + 1\right].$$

Note that $\deg(\Lambda_3) \leq 3$.

(2) $\Rightarrow$ (1) Since the polynomial sequence $\{P_n^{(1)}(x)\}_{n\geq 0}$ is a basis of $\mathbb{P}$, we have

$$Q_n(x) = P_n^{(1)} + \sum_{k=0}^{n-1} c_{n,k} P_k^{(1)}(x).$$

Using orthogonality and (27), we get

$$\begin{aligned}
(n+1)\, h_k^{(1)} c_{n,k} &= (n+1)\, L_1\left[Q_n P_k^{(1)}\right] = L_1\left[\Delta P_{n+1}^{(0)} P_k^{(1)}\right] \\
&= L_1\left[\Delta\left(P_{n+1}^{(0)} P_k^{(1)}\right)\right] - L_1\left[P_{n+1}^{(0)} \Delta P_k^{(1)}\right] - L_1\left[\Delta P_{n+1}^{(0)} \Delta P_k^{(1)}\right] \\
&= -L_0\left[\Lambda_2 P_{n+1}^{(0)} P_k^{(1)}\right] - L_0\left[\Lambda_3 P_{n+1}^{(0)} \Delta P_k^{(1)}\right] - L_0\left[\Lambda_3 \Delta P_{n+1}^{(0)} \Delta P_k^{(1)}\right],
\end{aligned}$$

and hence for all $0 \leq k \leq n-1$

$$L_0\left[\Lambda_2 P_{n+1}^{(0)} P_k^{(1)}\right] = \lambda_2^{(2)} h_{n+1}^{(0)} \delta_{k,n-1},$$

$$L_0\left[\Lambda_3 P_{n+1}^{(0)} \Delta P_k^{(1)}\right] = (n-1)\, \lambda_3^{(3)} h_{n+1}^{(0)} \delta_{k,n-1}.$$

From (40) it follows that

$$\Delta^*\left(\Lambda_3 L_0\right) = \Lambda_2 L_0.$$

Therefore L_0 is $\Delta-$semiclassical of class at most 1. Using (26), we conclude that

$$L_0\left[\Lambda_3 \Delta P_{n+1}^{(0)} \Delta P_k^{(1)}\right] = 0, \quad k < n.$$

Thus, for all $0 \leq k \leq n-1$

$$(n+1)\, h_k^{(1)} c_{n,k} = -\left[\lambda_2^{(2)} + (n-1)\, \lambda_3^{(3)}\right] h_{n+1}^{(0)} \delta_{k,n-1},$$

and we obtain

$$Q_n(x) = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x),$$

where

$$\tau_n = \frac{\lambda_2^{(2)} + (n-1)\, \lambda_3^{(3)}}{n+1} \frac{h_{n+1}^{(0)}}{h_{n-1}^{(1)}}, \quad n \in \mathbb{N}. \tag{41}$$

Since the polynomials Λ_2, Λ_3 satisfy (39), we see that $\tau_n \neq 0$. ■

4 Classification

In this section, we will find all $\Delta-$coherent pairs of the second kind. In view of Theorem 9, it is enough to consider discrete semiclassical functionals of class $s \leq 1$ satisfying the Pearson equation

$$\Delta^*\left(\Lambda_3 L_0\right) = \Lambda_2 L_0, \quad \deg\left(\Lambda_2\right) = 2, \quad \deg\left(\Lambda_3\right) \leq 3. \tag{42}$$

In [8], we classified the discrete semiclassical functionals of class $s \leq 1$ satisfying (19), and obtained the following cases:

1. Charlier (class $s = 0$)

$$\rho(x) = \frac{z^x}{x!}, \tag{43}$$

$$\phi(x) = 1, \quad \psi(x) = \frac{x}{z} - 1. \tag{44}$$

2. Meixner (class $s = 0$)

$$\rho(x) = (a)_x \frac{z^x}{x!}, \tag{45}$$

$$\phi(x) = x + a, \quad \psi(x) = \frac{x}{z} - (x + a). \tag{46}$$

Subcase: Kravchuk polynomials

$$\rho(x) = (-N)_x \frac{z^x}{x!}, \quad N \in \mathbb{N}, \tag{47}$$

$$\phi(x) = x - N, \quad \psi(x) = \frac{x}{z} - (x - N). \tag{48}$$

3. Hahn (class $s = 0$)

$$\rho(x) = \frac{(-N)_x (a)_x}{(b+1)_x} \frac{1}{x!}, \quad N \in \mathbb{N}, \tag{49}$$

$$\phi(x) = (x - N)(x + a), \quad \psi(x) = (N - a + b)x + aN. \tag{50}$$

4. Generalized Charlier (class $s = 1$)

$$\rho(x) = \frac{1}{(b+1)_x} \frac{z^x}{x!}, \tag{51}$$

$$\phi(x) = 1, \quad \psi(x) = \frac{x(x+b)}{z} - 1. \tag{52}$$

5. Generalized Meixner (class $s = 1$)

$$\rho(x) = \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}, \tag{53}$$

$$\phi(x) = x + a, \quad \psi(x) = \frac{x(x+b)}{z} - (x + a). \tag{54}$$

6. Generalized Kravchuk (class $s = 1$)

$$\rho(x) = (-N)_x (a)_x \frac{z^x}{x!}, \quad N \in \mathbb{N}, \tag{55}$$

$$\phi(x) = (x - N)(x + a), \quad \psi(x) = \frac{x}{z} - (x - N)(x + a). \tag{56}$$

7. Generalized Hahn of type I (class $s = 1$)

$$\rho(x) = \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}, \tag{57}$$

$$\phi(x) = (x + a_1)(x + a_2), \quad \psi(x) = \frac{x(x+b)}{z} - (x + a_1)(x + a_2). \tag{58}$$

Note that if we set $z = 1, a_1 = a, a_2 = -N$, then we obtain the Hahn polynomials.

8. Generalized Hahn of type II (class $s = 1$)

$$\rho(x) = \frac{(-N)_x (a_1)_x (a_2)_x}{(b_1+1)_x (b_2+1)_x} \frac{1}{x!}, \quad N \in \mathbb{N}, \tag{59}$$

$$\phi(x) = (x - N)(x + a_1)(x + a_2), \tag{60}$$
$$\psi(x) = (N - a_1 - a_2 + b_1 + b_2) x^2 + (Na_1 + Na_2 - a_1 a_2 + b_1 b_2) x + N a_1 a_2$$

If we define $L_0, L_1 \in \mathbb{P}^*$ by

$$L_i[p] = \sum_{x=0}^{\infty} p(x) \rho_i(x), \quad i = 0, 1, \quad p \in \mathbb{P},$$

with $\rho_i(-1) = 0$, $i = 0, 1$, then we can use (2) and rewrite (40) as

$$\nabla(\Lambda_3 \rho_0) + \Lambda_2 \rho_0 = 0, \quad \rho_1 = \Lambda_3 \rho_0. \tag{61}$$

Proposition 10 *Let $q, \phi_0, \psi_0 \in \mathbb{P}$. If $L_0 \in \mathbb{P}^*$ satisfies*

$$\nabla(\phi_0 \rho_0) + \psi_0 \rho_0 = 0, \tag{62}$$

and

$$\Lambda_2 = (q - \nabla q)\psi_0 - \phi_0 \nabla q, \quad \Lambda_3 = q\phi_0, \tag{63}$$

then

$$\nabla(\Lambda_3 \rho_0) + \Lambda_2 \rho_0 = 0. \tag{64}$$

Proof. If we use the identity

$$\nabla (fg) = g\nabla f + f\nabla g - \nabla f \nabla g, \tag{65}$$

and (63), we have

$$\nabla (\Lambda_3 \rho_0) = (\nabla q)\, \phi_0 \rho_0 + (q - \nabla q)\, \nabla (\phi_0 \rho_0)\,.$$

Using (62), we get

$$\nabla (\Lambda_3 \rho_0) = (\nabla q)\, \phi_0 \rho_0 - (q - \nabla q)\, \psi_0 \rho_0 = -\Lambda_2 \rho_0,$$

and (64) follows. ■

We will now find all Δc2 pairs with $\deg(\Lambda_2) = 2$.

4.1 Case I: $\deg(\Lambda_3) = 0$

If we take $\Lambda_3 = 1$, we see from (61) that

$$\nabla \rho_0 + \Lambda_2 \rho_0 = 0, \quad \rho_1 = \rho_0,$$

and if $\rho_0(x)$ is the weight function corresponding to the generalized Charlier polynomials (51), we see from (52) that

$$\Lambda_2(x) = \frac{x(x+b)}{z} - 1, \quad z \neq 0.$$

Thus, the generalized Charlier polynomials are *self-coherent* of the second kind. Note that this was already observed in [10].

4.2 Case II (a): $\deg(\Lambda_3) = 1$

If we take $\Lambda_3 = x + \omega$, $\omega \in \mathbb{C}$, then we see from (61) that

$$\nabla [(x+\omega)\, \rho_0] + \Lambda_2 \rho_0 = 0, \quad \rho_1 = (x+\omega)\, \rho.$$

If $\rho_0(x)$ is the weight function corresponding to the Charlier polynomials (43), we can use Proposition 10 with $q = x + \omega$ and obtain

$$\Lambda_2(x) = \frac{x}{z}(x + \omega - 1) - (x+\omega)\,, \quad z \neq 0,$$

where we have also used (44). The functional L_1 is a *Christoffel transformation* of L_0 satisfying

$$\nabla\left[(x+\omega)\,\rho_1\right]+\Lambda_2\rho_1=0,$$

and using (54) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Meixner polynomials (53) with

$$a=\omega,\quad b=\omega-1.$$

4.3 Case II (b): $\deg(\Lambda_3)=1$

If we take $\Lambda_3=x+\omega$, $\omega\in\mathbb{C}$, then from (61) we get that

$$\nabla\left[(x+\omega)\,\rho_0\right]+\Lambda_2\rho_0=0,\quad \rho_1=(x+\omega)\,\rho.$$

If $\rho_0(x)$ is the weight function corresponding to the Kravchuk polynomials (47), we can use Proposition 10 with $q=x+\omega$ and obtain

$$\Lambda_2(x)=\frac{x}{z}(x+\omega-1)-(x+\omega)(x-N),\quad z\neq 0,$$

were we have also used (48). The functional L_1 is a Christoffel transformation of L_0 satisfying

$$\nabla\left[(x+\omega)\,\rho_1\right]+\Lambda_2\rho_1=0,$$

and using (56) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Kravchuk polynomials (55) with

$$a=\omega,\quad b=\omega-1.$$

4.4 Case III: $\deg(\Lambda_3)=2$

If we take $\Lambda_3=(x+\omega)(x+a)$, $\omega\in\mathbb{C}$, then from (61) it follows

$$\nabla\left[(x+\omega)(x+a)\,\rho_0\right]+\Lambda_2\rho_0=0,\quad \rho_1=(x+\omega)(x+a)\,\rho_0.$$

If $\rho_0(x)$ is the weight function corresponding to the Meixner polynomials (45), we can use Proposition 10 with $q=x+\omega$ and obtain

$$\Lambda_2(x)=\frac{x}{z}(x+\omega-1)-(x+\omega)(x+a),\quad z\neq 0,$$

were we have also used (46). The functional L_1 is a Christoffel transformation of L_0 satisfying

$$\nabla\left[(x+\omega)(x+a)\,\rho_1\right]+\Lambda_2\rho_1=0,$$

and using (58) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Hahn polynomials of type I (57) with

$$a_1=a,\quad a_2=\omega,\quad b=\omega-1.$$

4.5 Case IV: $\deg(\Lambda_3)=3$

If we take $\Lambda_3=(x+\omega)(x-N)(x+a)$, $\omega\in\mathbb{C}$, $N\in\mathbb{N}$, from (61) we deduce that

$$\nabla\left[(x+\omega)(x-N)(x+a)\,\rho_0\right]+\Lambda_2\rho_0=0,$$
$$\rho_1=(x+\omega)(x-N)(x+a)\,\rho_0.$$

If $\rho_0(x)$ is the weight function corresponding to the Hahn polynomials (49), we can use Proposition 10 with $q=x+\omega$ and obtain

$$\Lambda_2=(N-a+b-1)\,x^2+(N\omega+Na-a\omega+b\omega-b)\,x+Na\omega,$$

were we have also used (50). The functional L_1 is a Christoffel transformation of L_0 satisfying

$$\nabla\left[(x+\omega)(x-N)(x+a)\,\rho_1\right]+\Lambda_2\rho_1=0,$$

and using (60) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Hahn polynomials of type II (59) with

$$a_1=a,\quad a_2=\omega,\quad b_1=b,\quad b_2=\omega-1.$$

Note that Christoffel transforms of Δ-classical linear functionals have been analyzed in [23].

5 Conclusions and future directions

We have classified all Δ-coherent pairs of the second kind $\{L_0,L_1\}$ and derived the following results:

L_0	L_1
generalized Charlier	generalized Charlier
Charlier	generalized Meixner
Kravchuk	generalized Kravchuk
Meixner	generalized Hahn of type I
Hahn	generalized Hahn of type II

.

In all cases, the functional L_1 is a Christoffel transformation of L_0, in agreement with the general results obtained by Meijer [20] for the continuous case.

In a work in progress, we are dealing with analytic properties of discrete Sobolev inner products associated with a pair of coherent pairs of the second kind. We must point out that in [5] the authors deal with asymptotic properties and the location of zeros of discrete polynomials associated with a Sobolev inner product

$$\langle p, q\rangle_{\mathfrak{S}} = \langle u_0, pq\rangle + \lambda\langle u_1, \Delta p\Delta q\rangle,$$

where $\lambda \geq 0$, (u_0, u_1) is a Δ-coherent pair of positive-definite linear functionals, and u_1 is the Meixner linear functional. A limit relation between these orthogonal polynomials and the Laguerre-Sobolev orthogonal polynomials which is analogous to the one existing between Meixner and Laguerre polynomials in the Askey scheme is deduced. Notice that Meixner polynomials are Δ self-coherent, Thus, Mehler-Heine type formulas and zeros of such polynomials when $u_0 = u_1$ is the Meixner functional have been studied in [21]. Outer relative asymptotics and Plancherel-Rotach asymptotics as well as limit relations are analyzed in [4]. Algebraic properties of such polynomials as well as the behavior of their zeros appear in [2].

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References

[1] I. Area, E. Godoy, and F. Marcellán. Classification of all Δ-coherent pairs. *Integral Transform. Spec. Funct.* **9**(1), 1–18 (2000).

[2] I. Area, E. Godoy, and F. Marcellán. Inner products involving differences: the Meixner-Sobolev polynomials. *J. Differ. Equations Appl.* **6**(1), 1–31 (2000).

[3] I. Area, E. Godoy, and F. Marcellán. Δ-coherent pairs and orthogonal polynomials of a discrete variable. *Integral Transforms Spec. Funct.* **14**(1), 31–57 (2003).

[4] I. Area, E. Godoy, F. Marcellán, and J. J. Moreno-Balcázar. Ratio and Plancherel-Rotach asymptotics for Meixner-Sobolev orthogonal polynomials. *J. Comput. Appl. Math.* **116**(1), 63–75 (2000).

[5] I. Area, E. Godoy, F. Marcellán, and J. J. Moreno-Balcázar. Δ-Sobolev orthogonal polynomials of Meixner type: asymptotics and limit relation. *J. Comput. Appl. Math.* **178**(1-2), 21–36 (2005).

[6] T. S. Chihara. "An introduction to orthogonal polynomials". Mathematics and its Applications, Vol. 13. Gordon and Breach Science Publishers, New York-London-Paris (1978).

[7] A. M. Delgado and F. Marcellán. Companion linear functionals and Sobolev inner products: a case study. *Methods Appl. Anal.* **11**(2), 237–266 (2004).

[8] D. Dominici and F. Marcellán. Discrete semiclassical orthogonal polynomials of class one. *Pacific J. Math.* **268**(2), 389–411 (2014).

[9] D. Dominici and F. Marcellán. Discrete semiclassical orthogonal polynomials of class 2. In "Orthogonal polynomials: current trends and

applications", vol. 22 of "SEMA SIMAI Springer Ser.", pp. 103–169. Springer, Cham ([2021] ©2021).

[10] D. Dominici and J. J. Moreno-Balcázar. Asymptotic analysis of a family of sobolev orthogonal polynomials related to the generalized charlier polynomials. **arXiv:2210.00082** (2022).

[11] A. G. García, F. Marcellán, and L. Salto. A distributional study of discrete classical orthogonal polynomials. In "Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications (Evian-Les-Bains, 1992)", vol. 57, pp. 147–162 (1995).

[12] J. C. García-Ardila, F. Marcellán, and M. E. Marriaga. From standard orthogonal polynomials to Sobolev orthogonal polynomials: the role of semiclassical linear functionals. In "Orthogonal polynomials", Tutor. Sch. Workshops Math. Sci., pp. 245–292. Birkhäuser/Springer, Cham ([2020] ©2020).

[13] A. Iserles, P. E. Koch, S. P. Nø rsett, and J. M. Sanz-Serna. On polynomials orthogonal with respect to certain Sobolev inner products. *J. Approx. Theory* **65**(2), 151–175 (1991).

[14] A. Iserles, J. M. Sanz-Serna, P. E. Koch, and S. P. Nø rsett. Orthogonality and approximation in a Sobolev space. In "Algorithms for approximation, II (Shrivenham, 1988)", pp. 117–124. Chapman and Hall, London (1990).

[15] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. "Hypergeometric orthogonal polynomials and their q-analogues". Springer Monographs in Mathematics. Springer-Verlag, Berlin (2010). With a foreword by Tom H. Koornwinder.

[16] F. Marcellán, T. E. Pérez, and M. A. Piñar. Orthogonal polynomials on weighted Sobolev spaces: the semiclassical case. vol. 2, pp. 93–122. (1995). Special functions (Torino, 1993).

[17] F. Marcellán and J. Petronilho. Orthogonal polynomials and coherent pairs: the classical case. *Indag. Math. (N.S.)* **6**(3), 287–307 (1995).

[18] F. Marcellán and L. Salto. Discrete semi-classical orthogonal polynomials. *J. Differ. Equations Appl.* **4**(5), 463–496 (1998).

[19] F. Marcellán and Y. Xu. On Sobolev orthogonal polynomials. *Expo. Math.* **33**(3), 308–352 (2015).

[20] H. G. Meijer. Determination of all coherent pairs. *J. Approx. Theory* **89**(3), 321–343 (1997).

[21] J. J. Moreno-Balcázar. Δ-Meixner-Sobolev orthogonal polynomials: Mehler-Heine type formula and zeros. *J. Comput. Appl. Math.* **284**, 228–234 (2015).

[22] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. "Classical orthogonal polynomials of a discrete variable". Springer Series in Computational Physics. Springer-Verlag, Berlin (1991). Translated from the Russian.

[23] A. Ronveaux and L. Salto. Discrete orthogonal polynomials—polynomial modification of a classical functional. *J. Differ. Equations Appl.* **7**(3), 323–344 (2001).

[24] M. H. Suni, G. A. Marcato, F. Marcellán, and A. Sri Ranga. Coherent pairs of measures of the second kind and associated Sobolev orthogonal polynomials: a functional approach. **Submitted** (2022).