Asymptotics of binomially weighted sums

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- Different problems in combinatorics, analysis of algorithms or even physics involve binomially weighted sums

Example 1 (combinatorics): In paper *Evaluation of Binomial Double Sums Involving Absolute Values* of C. Krattenthaler and C. Schneider, sums of the following form appear when we are studying double sums with binomial coefficients:

$$-2^{2m+1}n\binom{2n}{n}\boxed{\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}}+2\binom{2m}{m}\binom{2n}{n}+2^{2m+2n}$$

We might be interested in an asymptotic expansion at $m\to+\infty$ for fixed m , which involves being able in particular to compute the expansion of the boxed sum

Example 2 (physics): Particle physics computations are often done in Mellin space, and for example in the paper *The $\mathcal{O}(\alpha_s^3 T_F^2)$ contributions to the gluonic operator matrix element* by J.Abligner, J. Blümlein, C. Schneider et al., we encounter sums of the form:

$$\frac{1}{4^n}\binom{2n}{n}\left(\boxed{\sum_{i=1}^{n}\frac{4^i}{i^2\binom{2i}{i}}S_1(i-1)}-7\zeta_3\right), \quad S_1(i-1):=\sum_{k=1}^{i-1}\frac{1}{k}$$

Sums can be nested, for example in *Iterated Binomial Sums and their Associated Iterated Integrals* by J.Ablinger, J.Blümlein, C.G. Raab and C. Schneider, we have also sums such as:

$$\sum_{i=1}^{n}\binom{2i}{i}S_2(i), \quad \sum_{i=1}^{n}\frac{1}{i\binom{2i}{i}}\sum_{j=1}^{i}\binom{2j}{j}(-2)^j$$

Aim: Being able to deal with those kind of sums in all generality, in particular **Mellin inversion** and **asymptotic expansion**

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Aim: Being able to deal with those kind of sums in all generality, in particular **Mellin inversion** and **asymptotic expansion**

- We define the binomially weighted sums as follows:

$$BS_{\{a_1,\ldots,a_k\}}(n) := \sum_{i_1=1}^{n} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

With

$$a_j(p) = a_j(p; b, c, m) = \binom{2p}{p}^b \frac{c^p}{p^m}, \quad b \in \{-1, 0, 1\},\, c \in \mathbb{R}^\star,\, m \in \mathbb{N}$$

- More generic summands can also be considered, such as:

$$\frac{c^n}{(2n+1)\binom{2n}{n}}$$

- We define respectively Mellin transform and Mellin convolution as:

$$M\left[f(x)\right](n) := \int_0^1 \mathrm{d}x\, x^n f(x) \quad f(x) {*} g(x) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, \delta(x - x_1 x_2) f(x_1) g(x_2)$$

Question: How to represent them as Mellin integrals?

- **First method** (used by `HarmonicSums` for general Mellin inversion): given $M\left[f(x)\right](n)$ as holonomic sequences, we obtain the associated holonomic **differential equation,** and by solving it we can obtain a closed form for $f(x)$

Pros: Very general and efficient
Cons: If the DE cannot be solved (not first-order factorizable or Kovacic method doesn't work), a Mellin representation cannot be obtained

- **Second method** (defined in [2]): compute it recursively from the BS using fundamental properties of Mellin transforms and "rule-theorems" that allow to compute in an automatic way Mellin convolutions

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Note: These Mellin representations will involve general polylogarithms

$$\mathrm{H}^*_{\emptyset}(x) := 1,\ \mathrm{H}^*_{\mathrm{b(t)},\vec{c}\,(\mathrm{t})}(x) = \mathrm{H}^*_{\mathrm{b},\vec{c}}(x) := \int_x^1 \mathrm{d}t\, b(t)\mathrm{H}^*_{\vec{c}}(t)$$

Defined over a 37 letter alphabet $\{f_0, f_{w_{32}}\}$ containing root singularities such that all iterated integrals are linearly independent over the algebraic functions

Examples:

$$\sum_{i=1}^{n} \frac{1}{i\binom{2i}{i}} \sum_{j=1}^{i} \binom{2j}{j}(-2)^j = \int_0^1 \mathrm{d}x\, \frac{(-2x)^n - 1}{x+\frac{1}{2}} \left(1 - \frac{1}{6\sqrt{2}\sqrt{x+\frac{1}{8}}}\right) - \int_0^1 \mathrm{d}x\, \frac{\left(\frac{x}{4}\right)^n - 1}{(x-4)} \frac{1}{\sqrt{1-x}}$$

$$\sum_{i=0}^{m} \frac{2^{-2i}\binom{2i}{i}}{i+n} = \frac{1}{n} + \frac{1}{\pi}\int_0^1 \mathrm{d}x\, \frac{x^m - 1}{x-1} x^n B_x\left(\frac{1}{2}-n, \frac{1}{2}\right) \quad (n > 0)$$

Where

$$B_x(a,b) := \int_0^x \mathrm{d}t\, t^{a-1}(1-t)^{b-1}$$

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Where

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We now want to obtain an asymptotic expansion for $n \to +\infty$ up to order p of a general expression of the form:

$$\tilde{M}_a[f(x)](n) := \int_0^1 \mathrm{d}x \, \frac{(ax)^n - 1}{x - \frac{1}{a}} f(x) = \int_0^1 \mathrm{d}x \, [(ax)^n - 1] \, \tilde{f}(x) \qquad (1)$$

Where

$$\tilde{f}(x) := \frac{f(x)}{x - \frac{1}{a}}$$

- There exist several method to compute this expansion, depending mostly on the regularity of f and if the integral can be splitted
- We will present three of them, all three of which have been implemented in a small package, `BinomialAsymptotics`

- **Hypothesis**: $\tilde{f}$ is at least p times differentiable at $x = 1$, $\mathcal{C}^0$ on $(0;1]$ and integrable on $[0;1]$

We proceed in the following way:

1. Split the integral in two parts:

$$\int_0^1 \mathrm{d}x[(ax)^n - 1]\tilde{f}(x) = a^n \int_0^1 \mathrm{d}x\, x^n \tilde{f}(x) - \int_0^1 \mathrm{d}x\, \tilde{f}(x) = a^n M[f(x)](n) - M[f(x)](0)$$

2. Compute the constant value $C = M[f(x)](0)$ (using `HarmonicSums` or directly Mathematica)

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3. Perform an integration by parts p times on the n-dependent part:

$$
\begin{aligned}
M[f(x)](n) &= \frac{x^{n+1}}{n+1}\tilde{f}(x)\Big|_0^1 - \frac{1}{n+1}\int_0^1 \mathrm{d}x\, x^{n+1}\tilde{f}'(x) \\
&= \frac{\tilde{f}(1)}{n+1} - \frac{1}{n+1}\left[\frac{x^{n+2}}{n+2}\tilde{f}'(x)\Big|_0^1 - \frac{1}{n+2}\int_0^1 \mathrm{d}x\, x^{n+2}\tilde{f}''(x)\right] \\
&\ \ \vdots \qquad\qquad\qquad \vdots \\
&= \sum_{i=0}^{p}(-1)^i \frac{\tilde{f}^{(i)}(1)}{(n+1)_{i+1}} + \underbrace{\frac{(-1)^p}{(n+1)_p}\int_0^1 \mathrm{d}x\, x^{n+p+1}\tilde{f}^{(p+1)}(x)}_{\mathcal{O}\left(\frac{1}{n^{p+1}}\right)} \qquad (2)
\end{aligned}
$$

Where $(a)_n := a(a+1)\cdots(a+n-1), (a)_0 := 1$

4. Expand (2) for n at $+\infty$ up to the order p and add the computed constant to finally get:

$$\tilde{M}_a[f(x)](n) \underset{n\to+\infty}{=} -C + a^n \left[\sum_{k=1}^{p} \frac{h_k}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right)\right]$$

Where for $k \in \{1, \dots, p\}$, $h_k \in \mathbb{R}$.

When the function $\tilde{f}$ is not regular enough at $x = 1$, we have to use another approach. The **geometric series resummation** is the first possible one.

- **Hypothesis**: $|f|$ integrable on [0;1], the constant a is such that $|a| < 1$

We proceed in the following way:

1. We split the integral and compute the constant $C = M[f(x)](0)$ as before
2. We start by expanding the geometric series:

$$\frac{1}{x - \frac{1}{a}} = (-a)\frac{1}{1 - ax} = (-a)\sum_{k=0}^{\infty}(ax)^k$$

3. We plug this result into the Mellin transform:

$$(-a)\int_0^1 \mathrm{d}x\, x^n \sum_{k=0}^{\infty}(ax)^k f(x) = (-a)\sum_{k=0}^{\infty} a^k \int_0^1 \mathrm{d}x\, x^{n+k} f(x)$$
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4. We compute the shifted Mellin transforms $k \to n + k$ using `GeneralMellin` from `HarmonicSums`
5. Once it's done the idea is to asymptotically expand $M[f(x)](n + k)$ for n around $+\infty$ up to order p:

$$M[f(x)](n+k) \underset{n\to+\infty}{=} \sum_{i=0}^{p} \frac{\alpha_i(k)}{n^i} + \mathcal{O}\left(\frac{1}{n^p}\right)$$

Where the $\alpha_i(k)$ are coefficients that depend on k.

6. Finally, we compute the infinite sum:

$$(-a)\sum_{k=0}^{\infty}\sum_{i=0}^{p} a^k \frac{\alpha_i(k)}{n^i} = \sum_{i=0}^{p} \frac{A_i}{n^i}, \quad A_i := \sum_{k=0}^{\infty} a^k \alpha_i(k)$$

And the full expasion is then:

$$M[f(x)](n) \underset{n\to+\infty}{=} -C + \sum_{i=0}^{p} \frac{A_i}{n^i} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right)$$

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When $a \geq 1$, the geometric series $\frac{1}{x-\frac{1}{a}}$ is divergent at $\frac{1}{a} \in (0;1]$. The singularity is actually only apparent:

$$\frac{(ax)^n - 1}{x - \frac{1}{a}} f(x) \underset{x \to \frac{1}{a}}{=} a\, n f\left(\frac{1}{a}\right) + \mathcal{O}\left(x - \frac{1}{a}\right)$$

Problem: We cannot split the integral to compute the constant part as usually. We have to use a new method that relies on a **change of variable** [2] and do an **integration by parts** to extract the constant part

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Problem: We cannot split the integral to compute the constant part as usually. We have to use a new method that relies on a **change of variable** [2] and do an **integration by parts** to extract the constant part

1. We make the change of variable:

$$x = \frac{e^{-z}}{a},\ \mathrm{d}x = -\frac{1}{a}e^{-z}\mathrm{d}z$$

This gives us:

$$\int_0^1 \mathrm{d}x\, \frac{(ax)^n - 1}{x - \frac{1}{a}} f(x) = -\frac{1}{a}\int_{+\infty}^{-\log a} \mathrm{d}z\, e^{-z} a \frac{e^{-zn} - 1}{e^{-z} - 1} f\left(\frac{e^{-z}}{a}\right)$$
$$= \int_{-\log a}^{+\infty} \mathrm{d}z\, (e^{-zn} - 1) \underbrace{\frac{e^{-z}}{e^{-z} - 1} f\left(\frac{e^{-z}}{a}\right)}_{=:g(z)}$$

In particular, expanding the function g defined above around $z = 0$, we get:

$$g(z) \underset{z\to 0}{=} \frac{\alpha_{-1}}{z} + \alpha_0 + \mathcal{O}(z)$$

Where $\alpha_{-1}, \alpha_0 \in \mathbb{C}$

2. We do an integration by parts to get rid of the $\frac{1}{z}$, since:

$$\int^z \mathrm{d}w\, g(w) \underset{z\to 0}{=} \alpha_{-1}\log z + \alpha_0 z + \mathcal{O}(z^2)$$

And $\log z$ is integrable at $z=0$. We define:

$$\begin{matrix} u'(z) = g(z) \\ v(z) = e^{-zn} - 1 \end{matrix} \to \begin{matrix} u(z) = \int^z \mathrm{d}w\, g(w) \\ v'(z) = -ne^{-zn} \end{matrix}$$

So that:

$$\int_{-\log a}^{+\infty} \mathrm{d}z\,(e^{-zn}-1)g(z) = (e^{-zn}-1)\int^z \mathrm{d}w\, g(w)\Big|_{-\log a}^{+\infty} + n\int_{-\log a}^{+\infty}\mathrm{d}z\, e^{-zn}\int^z \mathrm{d}w\, g(w)$$

3. We compute the constant terms:

$$C_2 := \lim_{z\to+\infty}(e^{-zn}-1)\int^z \mathrm{d}w\, g(w) = -\lim_{z\to+\infty}\int^z \mathrm{d}w\, g(w)$$

$$C_1 := \lim_{z\to-\log a}(e^{-zn}-1)\int^z \mathrm{d}w\, g(w) = (a^n-1)\lim_{z\to-\log a}\int^z \mathrm{d}w\, g(w)$$

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$$\int_{-\log a}^{+\infty} \mathrm{d}z\, (e^{-zn} - 1) g(z) = (e^{-zn} - 1) \int^{z} \mathrm{d}w\, g(w) \Big|_{-\log a}^{+\infty} + n \int_{-\log a}^{+\infty} \mathrm{d}z\, e^{-zn} \int^{z} \mathrm{d}w\, g(w)$$

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$$C_1 := \lim_{z\to-\log a} (e^{-zn} - 1) \int^{z} \mathrm{d}w\, g(w) = (a^n - 1) \lim_{z\to-\log a} \int^{z} \mathrm{d}w\, g(w)$$

4. We expand $\int^z \mathrm{d}w\, g(w)$ up to order p around $z = -\log a$:

$$\int^z \mathrm{d}w\, g(w) \underset{z\to -\log a}{\sim} \sum_{\alpha+\beta\leq p} g_{\alpha,\beta} z^{\alpha} (z+\log a)^{\beta}, \quad \alpha \in \mathbb{N},\ \beta \in \frac{1}{2}\mathbb{Z}_{\geq -1},\ g_{\alpha,\beta} \in \mathbb{R}$$

5. Finally, we integrate terms by terms, using the expansion above:

$$\int_{-\log a}^{+\infty} \mathrm{d}z\, e^{-zn} \sum_{\alpha+\beta\leq p} g_{\alpha,\beta} z^{\alpha} (z+\log a)^{\beta} = \sum_{\alpha+\beta\leq p} g_{\alpha,\beta} \int_{-\log a}^{+\infty} \mathrm{d}z\, e^{-zn} z^{\alpha} (z+\log a)^{\beta}$$

$$= \sum_{\alpha+\beta\leq p} \frac{h_{\alpha,\beta}}{n^{\alpha+\beta}} \quad h_{\alpha,\beta} \in \mathbb{R}$$

The full expansion is then:

$$\tilde{M}_a[\tilde{f}(x)](n) \underset{n\to+\infty}{=} C_2 - C_1 + \sum_{\alpha+\beta\leq p} \frac{h_{\alpha,\beta}}{n^{\alpha+\beta}} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right)$$

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$$\int^z \mathrm{d}w\, g(w) \underset{z \to -\log a}{\sim} \sum_{\alpha+\beta \leq p} g_{\alpha,\beta} z^\alpha (z+\log a)^\beta, \quad \alpha \in \mathbb{N},\ \beta \in \frac{1}{2}\mathbb{Z}_{\geq -1},\ g_{\alpha,\beta} \in \mathbb{R}$$

5. Finally, we integrate terms by terms, using the expansion above:

$$\int_{-\log a}^{+\infty} \mathrm{d}z\, e^{-zn} \sum_{\alpha+\beta \leq p} g_{\alpha,\beta} z^\alpha (z+\log a)^\beta = \sum_{\alpha+\beta \leq p} g_{\alpha,\beta} \int_{-\log a}^{+\infty} \mathrm{d}z\, e^{-zn} z^\alpha (z+\log a)^\beta$$

$$= \sum_{\alpha+\beta \leq p} \frac{h_{\alpha,\beta}}{n^{\alpha+\beta}} \quad h_{\alpha,\beta} \in \mathbb{R}$$

The full expansion is then:

$$\tilde{M}_a[\tilde{f}(x)](n) \underset{n \to +\infty}{=} C_2 - C_1 + \sum_{\alpha+\beta \leq p} \frac{h_{\alpha,\beta}}{n^{\alpha+\beta}} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right)$$

Examples

- First we load the package `BinomialAsymptotics`

```
<< "BinomialAsymptotics.m";
```

```
BinomialAsymptotics package by Nikolai Fadeev – © RISC – V 0.2 (November 2022)
```

$$\sum_{i=1}^{n} \frac{1}{i\binom{2i}{i}} \sum_{j=1}^{i} \binom{2j}{j} (-2)^j = \int_0^1 \mathrm{d}x \, \frac{(-2x)^n - 1}{x + \frac{1}{2}} \left(1 - \frac{1}{6\sqrt{2}\sqrt{x + \frac{1}{8}}}\right) - \int_0^1 \mathrm{d}x \, \frac{\left(\frac{x}{4}\right)^n - 1}{(x-4)} \frac{1}{\sqrt{1-x}}$$

$$\texttt{asymptoticExpansionInt}\left[\int_0^1 \frac{(-2x)^n - 1}{x + \frac{1}{2}} \left(1 - \frac{1}{6\sqrt{2}\sqrt{x + \frac{1}{8}}}\right) \mathrm{d}x - \frac{2}{3}\int_0^1 \frac{\left(\frac{x}{4}\right)^n - 1}{x - 4} * \frac{1}{\sqrt{1-x}} \mathrm{d}x, x, n, 4\right]$$

$$\frac{73 (-1)^n 2^{3+n}}{6561 n^4} - \frac{5 (-1)^n 2^{3+n}}{729 n^3} - \frac{7 (-1)^n 2^{3+n}}{243 n^2} + \frac{(-1)^n 2^{4+n}}{27 n} - \frac{9565 \times 2^{-9-2n} \sqrt{\pi}}{81 n^{7/2}} + \frac{227 \times 2^{-6-2n} \sqrt{\pi}}{27 n^{5/2}} - \frac{13 \times 2^{-2-2n} \sqrt{\pi}}{27 n^{3/2}} + \frac{2^{1-2n} \sqrt{\pi}}{9 \sqrt{n}} - \frac{2\pi}{9\sqrt{3}} + \frac{1}{27}\left(\sqrt{3}\pi - 27 \,\mathrm{Log}[3]\right)$$

Examples

$$\sum_{i=0}^{m} \frac{2^{-2i}\binom{2i}{i}}{i+n} = \frac{1}{n} + \frac{1}{\pi}\int_0^1 \mathrm{d}x\, \frac{x^m-1}{x-1} x^n B_x\left(\frac{1}{2}-n, \frac{1}{2}\right) \quad (n>0)$$

$$\frac{1}{n} + \mathtt{asymptoticExpansionInt}\left[-\frac{1}{\pi}\int_0^1 \frac{x^m-1}{x-1} * x^n * \mathrm{Beta}\left[x, \frac{1}{2}-n, \frac{1}{2}\right] \mathrm{d}x, x, m, 4, \mathrm{ComputeConstants}\to \mathrm{False}\right]$$

$$\frac{1}{n} - \frac{1}{\pi}\Bigg(\frac{2\sqrt{\pi}}{\sqrt{m}} - \frac{(7+8n)\sqrt{\pi}}{12\, m^{3/2}} + \frac{(61+176n+128n^2)\sqrt{\pi}}{320\, m^{5/2}} - \frac{(307+2936n+5760n^2+3072n^3)\sqrt{\pi}}{10752\, m^{7/2}} - \frac{(1+2n)\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}-n\right]}{m(-1+2n)\,\mathrm{Gamma}[1-n]} -$$

$$\frac{n(1+n)\times(1+2n)\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}-n\right]}{3m^3(-1+2n)\,\mathrm{Gamma}[1-n]} + \frac{(1+6n+6n^2)\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}-n\right]}{6m^2(-1+2n)\,\mathrm{Gamma}[1-n]} + \frac{(-1+30n^2+60n^3+30n^4)\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}-n\right]}{60m^4(-1+2n)\,\mathrm{Gamma}[1-n]} -$$

$$\frac{1}{(-1+2n)\,\mathrm{Gamma}[1-n]}\Bigg(2\,\mathrm{EulerGamma}\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}-n\right] + 2\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}-n\right]\mathrm{Log}[m] -$$

$$\mathrm{Gamma}[1-n]\lim_{z\to\infty}\int \frac{(e^{-z})^{1+n}\,\mathrm{Beta}\left[e^{-z}, \frac{1}{2}-n, \frac{1}{2}\right]}{-1+e^{-z}}\mathrm{d}z + 2n\,\mathrm{Gamma}[1-n]\lim_{z\to\infty}\int \frac{(e^{-z})^{1+n}\,\mathrm{Beta}\left[e^{-z}, \frac{1}{2}-n, \frac{1}{2}\right]}{-1+e^{-z}}\mathrm{d}z -$$

$$\mathrm{Gamma}[1-n]\,\mathrm{Limit}\left[(-1+e^{-mz})\int \frac{(e^{-z})^{1+n}\,\mathrm{Beta}\left[e^{-z}, \frac{1}{2}-n, \frac{1}{2}\right]}{-1+e^{-z}}\mathrm{d}z, z\to 0, \mathrm{Assumptions}\to m\in\mathbb{Z}\,\&\&\,m>0\right] +$$

$$2n\,\mathrm{Gamma}[1-n]\,\mathrm{Limit}\left[(-1+e^{-mz})\int \frac{(e^{-z})^{1+n}\,\mathrm{Beta}\left[e^{-z}, \frac{1}{2}-n, \frac{1}{2}\right]}{-1+e^{-z}}\mathrm{d}z, z\to 0, \mathrm{Assumptions}\to m\in\mathbb{Z}\,\&\&\,m>0\right]\Bigg)\Bigg)$$

- This expression can be simplified as, given that $n \in \mathbb{N}^\star$:

$$\sum_{i=0}^{m} \frac{2^{-2i}\binom{2i}{i}}{i+n} = -\frac{C_1(n)}{\pi} - \frac{C_2(n)}{\pi} + \frac{3072n^3 + 5760n^2 + 2936n + 307}{10752\sqrt{\pi}m^{7/2}} + \frac{-128n^2 - 176n - 61}{320\sqrt{\pi}m^{5/2}} + \frac{8n+7}{12\sqrt{\pi}m^{3/2}} - \frac{2}{\sqrt{\pi}\sqrt{m}} + \mathcal{O}\left(\frac{1}{m^{9/2}}\right)$$

- We have two constants coming from the integration by parts that need to be computed (in this case by hand)

$$C_1(n) = \lim_{z\to 0}\left(e^{-mz} - 1\right)\int^z \mathrm{d}w\, \frac{e^{-w(n+1)}B_{e^{-w}}\left(\frac{1}{2} - n, \frac{1}{2}\right)}{e^{-w} - 1} = 0$$

$$C_2(n) = -\lim_{z\to\infty}\int^z \mathrm{d}w\, \frac{e^{-w(n+1)}B_{e^{-w}}\left(\frac{1}{2} - n, \frac{1}{2}\right)}{e^{-w} - 1}$$

- For the second constant, the expression doesn't simplify for arbitrary n, one can compute its numerical value up to arbitrary precision by specialisin, e.g. for $n = 3$ and 100 digits:

$$C_2(n=3) = -2.30383461263251504153927181440496878174459089287506979019143969422829164345708078853487079754755787 9$$

$$C_2(n=1) = -\pi$$

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Conclusion

- All three methods have been implemented in a package that can compute asymptotic expansions in many cases of depth $d = 1$ and some cases of depth $d = 2$
- Some flexibility options are given: constants can be attempted to computed or not, time limitation
- Computations of the new constants is highly non-trivial and makes the algorithm get stuck: structure of binomial sums needs to be explored further (building a basis of binomial sums, unicity of root alphabet/relation between letters,...)
- Mathematica errors, technical improvements for package... work in progress!

Thank you for listening!

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Consider the general sum

$$BS_{\{a_1,\ldots,a_k\}}(n) := \sum_{i_1=1}^{n} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

We proceed outwards from the innermost sum:

- At step $1 \leq j < k$, suppose that we managed to represent

$$BS_{\{a_{j+1},\ldots,a_k\}}(i_j) = c_{0,j} + \sum_{p=1}^{k_j} c_{p,j+1}^{i_j} M[f_{p,j+1}(x)](i_j),\ c_{0,j+1}, c_{p,j+1} \in \mathbb{R} \tag{3}$$

- Compute an integral representation of the building block $a_j(i_j)$, possibly using Mellin convolutions

$$M[f(x) * g(x)](n) = M[f(x)](n) \cdot M[g(x)](n)$$

In order to obtain an expression

$$a_j(i_j) = c_{0,i_j} + \sum_{p=1}^{k_{i_j}} c_{p,i_j}^{i_j} M[f_{p,i_j}(x)](i_j) \tag{4}$$

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- Convolve $BS_{\{a_{j+1},\ldots,a_k\}}(i_j)$ with $a_j(i_j)$ to obtain again a sum
- Use the summation formula below to resum and obtain an integral representation for $BS_{\{a_j,a_{j+1},\ldots,a_k\}}(i_{j-1})$:

$$\sum_{i=1}^{n} c^i M[f(x)](i) = c^n M\left[\frac{x}{x-\frac{1}{c}} f(x)\right](n) - M\left[\frac{x}{x-\frac{1}{c}} f(x)\right](0) \qquad (5)$$

- Iterate until the outermost sum has been processed

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Notes:

- These Mellin representation will involve general polylogarithms

$$\mathrm{H}^*_\emptyset(x) := 1,\ \mathrm{H}^*_{\mathrm{b(t)},\overrightarrow{c}(\mathrm{t})}(x) = \mathrm{H}^*_{\mathrm{b},\overrightarrow{c}}(x) := \int_x^1 \mathrm{d}t\, b(t)\mathrm{H}^*_{\overrightarrow{c}}(t)$$

  Defined over a 37 letter alphabet $\{f_0, f_{w_{32}}\}$ containing root singularities such that all iterated integrals are linearly independent over the algebraic functions
- One can actually derive identities that allow to rewrite many convolution integrals in a direct way [2]

Example:

$$BS(n) = \sum_{k=1}^{n} \binom{2i}{i} S_2(i) = \sum_{k=1}^{n} \binom{2i}{i} \sum_{j=1}^{i} \frac{1}{i^2}$$

- First we compute the Mellin representation of $\frac{1}{i^2}$ by convolving $\frac{1}{i} = M\left[\frac{1}{x}\right](i)$ with itself. We get:

$$\frac{1}{i^2} = M\left[\frac{1}{x}\right](i) \cdot M\left[\frac{1}{x}\right](i) = M\left[\frac{1}{x} * \frac{1}{x}\right](i) = M\left[\frac{\mathrm{H}_0^*(x)}{x}\right](i)$$

Where

$$\mathrm{H}_0^*(x) := \int_x^1 \mathrm{d}t\, f_0(t) = \int_x^1 \mathrm{d}t\, \frac{1}{t} = -\log x$$

- Using the summation formula (5), we can then obtain:

$$S_2(i) = \sum_{k=1}^{i} M\left[\frac{\mathrm{H}_0^*(x)}{x}\right](i) = \underbrace{\int_0^1 \mathrm{d}x\, x^i \frac{\mathrm{H}_0^*(x)}{x-1}}_{M\left[\frac{x}{x-1}\frac{\mathrm{H}_0^*(x)}{x}\right](i)} - \underbrace{\int_0^1 \mathrm{d}x \frac{\mathrm{H}_0^*(x)}{x-1}}_{M\left[\frac{x}{x-1}\frac{\mathrm{H}_0^*(x)}{x}\right](0)} = \int_0^1 \mathrm{d}x\, x^i \frac{\mathrm{H}_0^*(x)}{x-1} + \zeta_2$$

- Now that the innermost sum has as integral representation, we shift to the next and last level. First, one can show (e.g. direct integration) that:

$$\binom{2i}{i} = \frac{4^i}{\pi} M\left[\frac{1}{\sqrt{x(1-x)}}\right](i)$$

So that

$$\sum_{i=1}^{k} \binom{2i}{i} S_2(i) = \frac{1}{\pi} \sum_{i=1}^{n} 4^i M\left[\frac{1}{\sqrt{x(1-x)}}\right](i) \cdot M\left[\frac{\mathrm{H}_0^*(x)}{x-1}\right](i) + \frac{\zeta_2}{\pi} \sum_{i=1}^{k} 4^i M\left[\frac{1}{\sqrt{x(1-x)}}\right]$$

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- We apply again the summation formula to obtain first the second part:

$$\frac{\zeta_2}{\pi}\sum_{i=1}^{k} 4^i M\left[\frac{1}{\sqrt{x(1-x)}}\right](i) = \frac{\zeta_2}{\pi}\int_0^1 \mathrm{d}x\, \frac{(4x)^n - 1}{x-\frac{1}{4}}\sqrt{\frac{x}{1-x}}$$

- Then we switch to the first part of the binomial and convolve the functions:

$$M\left[\frac{1}{\sqrt{x(1-x)}}\right](i)\cdot M\left[\frac{\mathrm{H}_0^*(x)}{x-1}\right](i) = M\left[\int_x^1 \mathrm{d}y\, \frac{\mathrm{H}_0^*(y)}{(y-1)\sqrt{y-x}}\right](i)$$

- A set of several "rule-theorems" have been proven in [2] to simplify further such expressions. One of them allows us to get:

$$\int_x^1 \mathrm{d}y\, \frac{\mathrm{H}_0^*(y)}{(y-1)\sqrt{y-x}} = \frac{\mathrm{H}_{\mathrm{b},\mathrm{w}_1}^*(x)}{\sqrt{x-1}},\ f_b(x) = \frac{1}{\sqrt{x(x-1)}},\ f_{w_1}(x) = \frac{1}{\sqrt{x(1-x)}}$$

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- Using the shuffle algebra, we can reduce the expression down to:

$$M\left[\frac{\mathrm{H}^*_{\mathrm{b},\mathrm{w}_1}(x)}{\sqrt{x(x-1)}}\right](i) = -M\left[\frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2\sqrt{x(1-x)}}\right](i)$$

- Finally, using once again the summation formula we get:

$$\sum_{i=1}^{n} 4^i M\left[\frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2\sqrt{x(1-x)}}\right](i) = \int_0^1 \mathrm{d}x\, \frac{(4x)^n - 1}{x - \frac{1}{4}} \sqrt{\frac{x}{1-x}} \frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2}$$

And resumming everything, we get:

$$\sum_{i=1}^{n} \binom{2i}{i} S_2(i) = -\frac{1}{\pi} \int_0^1 \mathrm{d}x\, \frac{(4x)^n - 1}{x - \frac{1}{4}} \sqrt{\frac{x}{1-x}} \left(\frac{\mathrm{H}^*_{\mathrm{w}_1}(x)^2}{2} - \zeta_2\right)$$

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- Remaining case: $a < -1$ and f not regular enough at $x = 1$: $\rightarrow \frac{1}{x-\frac{1}{a}}$ is not divergent on $(0;1)$, but the radius of convergence is still less than one, so geometric series approach doesn't apply.
- We simply use the **change of variable** approach with splitting of the integral this time

Note: This method works also when $a = 1$ and $f(x) \underset{x\to 1}{\sim} (x-1)^{\alpha}$, $\alpha \geq \frac{1}{2}$

We now consider a Mellin integral of the form:

$$\int_0^1 \mathrm{d}x\, \frac{(-ax)^n - 1}{x + \frac{1}{a}} f(x),\ a > 1$$

Hypothesis: $\tilde{f}(x) = \frac{f(x)}{x+\frac{1}{a}}$ integrable on $[0;1]$

1. Split the Mellin integral, factor out the $(-a)^n$, and compute the constant part:

$$\int_0^1 \mathrm{d}x \, \frac{(-ax)^n - 1}{x + \frac{1}{a}} f(x) = (-a)^n M\left[\frac{f(x)}{x+\frac{1}{2}}\right](n) - \underbrace{M\left[\frac{f(x)}{x+\frac{1}{a}}\right](0)}_{=:C}$$

2. In $M\left[\frac{f(x)}{x+\frac{1}{a}}\right](n)$, we make the original change of variable appearing in [2]:

$$x = e^{-z} \, \mathrm{d}x = -e^{-z}\mathrm{d}z$$

And end up with:

$$M\left[\frac{f(x)}{x+\frac{1}{2}}\right](n) = (-a)^n \int_0^{+\infty} \mathrm{d}z \, e^{-zn} \underbrace{\frac{e^{-z}}{e^{-z}+1} f(e^{-z})}_{=:g(z)}$$

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And end up with:

$$M\left[\frac{f(x)}{x+\frac{1}{2}}\right](n) = (-a)^n \int_0^{+\infty} \mathrm{d}z \, e^{-zn} \underbrace{\frac{e^{-z}}{e^{-z}+1} f(e^{-z})}_{=:g(z)}$$

3. We expand $g(z)$ around $z = 0$ up to the order p:

$$g(z) \underset{z\to 0}{=} \sum_{\alpha \le p} g_\alpha z^\alpha + \mathcal{O}(z^{\alpha+1}), \quad \alpha \in \frac{1}{2}\mathbb{Z}_{\ge -1},\ g_\alpha \in \mathbb{R}$$

4. Finally we integrate $M[\frac{f(x)}{x+\frac{1}{2}}](n)$ using the expansion above, and adding the constant and the $(-a)^n$ coefficient back,

$$\tilde{M}_{-a}[f(x)](n) \underset{n\to+\infty}{=} (-a)^n \sum_{\alpha \le p} \int_0^{+\infty} \mathrm{d}z\, e^{-zn} g_\alpha z^\alpha - C$$

$$= (-a)^n \sum_{\alpha \le p} \frac{h_\alpha}{n^\alpha} - C, \quad h_\alpha \in \mathbb{R}$$

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