AN UNIFIED FRAMEWORK TO PROVE MULTIPLICATIVE INEQUALITIES FOR THE PARTITION FUNCTION

KOUSTAV BANERJEE

Abstract. In this paper, we consider a certain class of inequalities for the partition function of the following form:

$$\prod_{i=1}^{T} p(n+s_i) \geq \prod_{i=1}^{T} p(n+r_i),$$

which we call multiplicative inequalities. Given a multiplicative inequality with the condition that $\sum_{i=1}^{T} s_i^m \neq \sum_{i=1}^{T} r_i^m$ for at least one $m \geq 1$, we shall construct an unified framework so as to decide whether such a inequality holds or not. As a consequence, we will see that study of such inequalities has manifold applications. For example, one can retrieve log-concavity property, strong log-concavity, and the inequalities for $p(n)$ considered by Bessenrodt and Ono, to name a few. Furthermore, we obtain the full asymptotic expansion for the finite difference of the logarithm of $p(n)$, denoted by $(-1)^{r-1}\Delta^r \log p(n)$, which extends a result by Chen, Wang, and Xie.

Keywords: Partition function, Hardy-Ramanujan-Rademacher formula, log-concavity, finite difference, partition inequalities

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Contents

1. Introduction

A partition of a positive integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. Hardy and Ramanujan [10] studied the asymptotic growth of $p(n)$ as follows:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{1.1}$$

Rademacher [16, 18, 17] improved the work of Hardy and Ramanujan and found a convergent series for $p(n)$ and Lehmer's [12, 11] study was on estimation for the remainder term of the series for $p(n)$. The Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k}\right] + R_2(n,N), \tag{1.2}$$

where

$$\mu(n)=\frac{\pi}{6}\sqrt{24n-1},\quad A_k(n)=\sum_{\substack{h \mod k\\(h,k)=1}} e^{-2\pi inh/k+\pi is(h,k)}$$

with

$$s(h,k)=\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\left\lfloor\frac{\mu}{k}\right\rfloor-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left\lfloor\frac{h\mu}{k}\right\rfloor-\frac{1}{2}\right),$$

and

$$|R_2(n,N)|<\frac{\pi^2N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3\sinh\frac{\mu(n)}{N}+\frac{1}{6}-\left(\frac{N}{\mu(n)}\right)^2\right]. \tag{1.3}$$

A sequence $\{a_n\}_{n\geq 0}$ is said to satisfy the Turán inequlaities or to be log-concave, if

$$a_n^2-a_{n-1}a_{n+1}\geq 0 \quad \text{for all} \quad n\geq 1. \tag{1.4}$$

Independently Nicolas [14] and DeSalvo and Pak [6, Theorem 1.1] proved that the partition function $p(n)$ is log-concave for all $n\geq 26$, conjectured by Chen [3]. DeSalvo and Pak [6, Theorem 4.1] also proved that for all $n\geq 2$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right)>\frac{p(n)}{p(n+1)}, \tag{1.5}$$

conjectured by Chen [3]. Further, they improved the rate of decay in (1.5) and proved that for all $n\geq 7$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right)>\frac{p(n)}{p(n+1)}, \tag{1.6}$$

see [6, p. 4.2]. DeSalvo and Pak [6] finally came up with the conjecture that the coefficient of $1/n^{3/2}$ in (1.6) can be improved to $\pi/\sqrt{24}$; i.e., for all $n\geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)>\frac{p(n)}{p(n+1)}, \tag{1.7}$$

which was proved by Chen, Wang and Xie [5, Sec. 2]. Recently, the author along with Paule, Radu, and Zeng [1, Theorem 7.6] confirmed that the coefficient of $1/n^{3/2}$ is indeed $\pi/\sqrt{24}$, which is the optimal; i.e., they proved that for all $n\geq 120$,

$$p(n)^2>\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\right)p(n-1)p(n+1). \tag{1.8}$$

DeSalvo and Pak [6, Theorem 5.1] also established that $p(n)$ satisfies the strong log-concavity property; i.e., for all $n>m>1$,

$$p(n)^2-p(n-m)p(n+m)>0. \tag{1.9}$$

Ono and Bessenrodt [2] extended (1.6) by considering the border case $m=n$. This leads to unveil multiplicative properties of the partition function encoded in the following theorem.

Theorem 1.1. *[2, Theorem 2.1] If a and b are integers with $a,b>1$ and $a+b>8$, then*

$$p(a)p(b)\geq p(a+b), \tag{1.10}$$

with equality holding only for $\{a,b\}=\{2,7\}$.

Let Δ be the forward difference operator define by $\Delta a(n) := a(n+1) - a(n)$ for a sequence $(a(n))_{n\geq 0}$. It is clear that the log-concavity property for $p(n)$ is equivalent to say that $-\Delta^2 \log p(n-1) > 0$ for all $n \geq 26$. Equations (1.7) and (1.8) show the asymptotic growth of $-\Delta^2 \log p(n-1)$. Chen, Wang, and Xie proved the positivity of $(-1)^{r-1}\Delta^r \log p(n)$ along with estimation of an upper bound.

Theorem 1.2. *[5, Thm. 3.1 and 4.1] For each $r \geq 1$, there exists a positive integer $n(r)$ such that for all $n \geq n(r)$,*

$$0 < (-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + \frac{\pi}{\sqrt{6}}\left(\frac{1}{2}\right)_{r-1} \frac{1}{(n+1)^{r-\frac{1}{2}}}\right). \tag{1.11}$$

The above inequalities can be rephrased in the following form:

$$\prod_{i=1}^{T} p(n+s_i) \geq \prod_{i=1}^{T} p(n+r_i), \tag{1.12}$$

which we call multiplicative inequalities for the partition function. Instead of applying the Hardy-Ramanujan-Rademacher formula (1.2) and Lehmer's error bound (1.3) but with different methodology for different inequalities for $p(n)$ as done in [2, 6, 14, 5], we will see how one can prove all such multiplicative inequalities under a unified framework so as to decide explicitly $N(T)$, such that for all $n \geq N(T)$, (1.12) holds. To prove (1.12), it is equivalent to show

$$\sum_{i=1}^{T} \log p(n+s_i) \geq \sum_{i=1}^{T} \log p(n+r_i), \tag{1.13}$$

and therefore, an infinite family of inequalities for logarithm of the shifted version of the partition function is a prerequisite, see Theorems 3.9 and 3.13. As an application of Theorem 3.9, we shall complete Theorem 1.2 (see Theorems 4.6 and 4.7 below) in the following aspects:

(1) by improving the lower bound in (1.11) to show that the rate of decay given in the upper bound is the optimal one,
(2) for each $r \geq 1$, computation of $n(r)$ by estimation of error bound based on the minimal choice of the truncation point w in Theorem 3.9,
(3) and a full asymptotic expansion for $(-1)^{r-1}\Delta^r \log p(n)$. This seems to be inaccessible from Theorem 1.2 because a key tool in the proof was on the relations between the higher order differences and derivatives (cf. Prop. 3.5, [5]) due to Odlyzko [15] which only contributes to the main term in the expansion; i.e., $\dfrac{\pi}{\sqrt{6}}\left(\dfrac{1}{2}\right)_{r-1} \dfrac{1}{(n+1)^{r-\frac{1}{2}}}$.

Even having Theorem 3.13 in hand, in order to decide whether (1.12) holds or not, there are two key factors remain unexplained. First, an explanation of the following assumption

$$\sum_{i=1}^{T} s_i^m \neq \sum_{i=1}^{T} r_i^m \quad \text{for at least one} \quad m \geq \mathbb{Z}_{\geq 1}. \tag{1.14}$$

and an appropriate choice of w, i.e., the truncation point as in Theorem 3.13. Now we move on to see how these two factors are intricately connected through a classical problem in Diophantine equations known as the Prouhet-Tarry-Escott problem [7, Chapter XXIV]. The problem originated in different guise from a letter of Goldbach [8] to Euler that dates back to 18 July, 1750. The Prouhet–Tarry–Escott problem asks for two distinct tuples of integers $(s_1, s_2, \dots, s_T)$ and $(r_1, r_2, \dots, r_T)$ such that

$$\sum_{i=1}^{T} s_i^k = \sum_{i=1}^{T} r_i^k, \quad \text{for all } 0 \leq k \leq m-1 \quad \text{and} \quad \sum_{i=1}^{T} s_i^m \neq \sum_{i=1}^{T} r_i^m.$$

We write $(s_1,\dots,s_T)\overset{m}{=}(r_1,\dots,r_T)$ to denote a solution of the Prouhet–Tarry–Escott problem. Recently, Merca and Katriel [13] connects the non-trivial linear homogeneous partition inequalities with the Prouhet–Tarry–Escott problem. In brevity, we shall explain why the optimal choice of truncation point $w=m+1$, with $(s_1,\dots,s_T)\overset{m}{=}(r_1,\dots,r_T)$ for a given (1.12) in Section 5.

The rest of the paper is organized as follows. In Section 2, we state preliminary lemmas and theorems from the work of Paule, Radu, Zeng, and the author [1]. Section 3 presents a detailed synthesis on derivation of inequalities for $\log p(n+s)$ for any non-negative integer s that leads to the main result of this paper, see Theorem 3.13. As an application of Theorem 3.13, we provide a full asymptotic expansion of $(-1)^{r-1}\Delta^r\log p(n)$ in Section 4. In Section 5, we work out the steps to verify multiplicative inequalities for the partition function. Section 6 is devoted to derive an infinite families of inequalities for $\prod_{i=1}^{T}p(n+s_i)$, given in Theorem 6.9. Finally we conclude this paper with a short discussion on the applications of Theorems 3.13 and 6.9.

2. Set up

Throughout this section, we follow the notations as in [1].

Definition 2.1 (Def. 5.1, [1]). *For $y\in\mathbb{R}$, $0<y^2<24$, we define*

$$G(y):=-\log\Big(1-\frac{y^2}{24}\Big)+\frac{\pi\sqrt{24}}{6y}\left(\sqrt{1-\frac{y^2}{24}}-1\right)+\log\left(1-\frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}\right), \tag{2.1}$$

and its sequence of Taylor coefficients by

$$G(y)=\sum_{u=1}^{\infty}g_uy^u. \tag{2.2}$$

Define $\alpha:=\dfrac{\pi^2}{36+\pi^2}$.

Lemma 2.2 (Lem. 5.4, [1]). *Let $G(y)=\sum_{u=1}^{\infty}g_uy^u$ be the Taylor expansion of $G(y)$ as in Definition 2.1. Then for $n\geq 1$,*

$$g_{2n}=\frac{1}{3^n2^{3n}n}-\frac{1}{3^n2^{3n+1}n}\Big(-1+\frac{1}{\alpha^n}\Big), \tag{2.3}$$

and for $n\geq 0$,

$$g_{2n+1}=\sqrt{6}\left[(-1)^{n+1}\binom{1/2}{n+1}\frac{\pi}{2^{3n+3}3^{n+2}}-\frac{1}{2^{3n+1}3^n\alpha^n(2n+1)\pi}\sum_{j=0}^{n}\alpha^j\binom{-\frac12+j}{j}\right]. \tag{2.4}$$

Lemma 2.3 (Lem. 5.8, [1]). *For $n\geq 0$, we have*

$$-\frac{\sqrt{6}}{2\pi2^{3n}3^n\alpha^n(2n+1)}\Big(\frac{\pi^2}{72}+1+\frac{\alpha}{2(1-\alpha)}\Big)\leq g_{2n+1}\leq-\frac{\sqrt{6}}{2\pi2^{3n}3^n\alpha^n(2n+1)}\Big(1+\frac{\alpha}{2}\Big). \tag{2.5}$$

Lemma 2.4 (Lem. 5.9, [1]). *For $n\geq 1$, we have*

$$-\frac{1}{3^n2^{3n+1}\alpha^nn}\leq g_{2n}\leq\frac{1}{3^n2^{3n}\alpha^nn}\Big(\frac{3\alpha}{2}-\frac{1}{2}\Big). \tag{2.6}$$

Definition 2.5 (Def. 4.3, [1]). *For $k\in\mathbb{Z}_{\geq 2}$, define*

$$g(k):=\frac{1}{24}\left(\frac{6^2}{\pi^2}\cdot\nu(k)^2+1\right),$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$.

Definition 2.6 (Def. 6.4, [1]). *For* $n, U \in \mathbb{Z}_{\geq 1}$, *we define*

$$P_n(U) := -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} g_u(1/\sqrt{n})^u.$$

Theorem 2.7 (Thm 6.6, [1]). *Let* $G(y) = \sum_{u=1}^{\infty} g_u y^u$ *as in Definition 2.1. Let* $g(k)$ *be as in Definition 2.5 and* $P_n(U)$ *as in Definition 2.6. If* $m \geq 1$ *and* $n > g(2m)$, *then*

$$P_n(2m-1) - \frac{2}{3^m 2^{3m}\alpha^m n^m m} < \log p(n) < P_n(2m-1) + \frac{1}{3^m 2^{3m}\alpha^m n^m m}; \tag{2.7}$$

if $m \geq 2$ *and* $n > g(2m-1)$, *then*

$$P_n(2m-2) - \frac{7}{3^m 2^{3m}\alpha^m n^{m-1/2}(2m-1)} < \log p(n) < P_n(2m-2) + \frac{2}{3^m 2^{3m}\alpha^m n^{m-1/2}(2m-1)}. \tag{2.8}$$

In other words, for $w \in \mathbb{Z}_{>0}$ with $\lceil w/2 \rceil \geq \gamma_0$ and $n > g(w)$, we have

$$P_n(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil}}\left(\frac{1}{\sqrt{n}}\right)^w < \log p(n) < P_n(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil}}\left(\frac{1}{\sqrt{n}}\right)^w, \tag{2.9}$$

where

$$(\gamma_0, \gamma_1, \gamma_2) = \begin{cases} (1, 4, 2), & \text{if } w \text{ is even} \\ (2, 7, 2), & \text{if } w \text{ is odd} \end{cases}. \tag{2.10}$$

Lemma 2.8 (Lem 7.3, [1]). *For* $n, s \in \mathbb{Z}_{\geq 1}, m \in \mathbb{N}$ *and* $n > 2s$, *let*

$$b_{m,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+m-1}}\binom{s+m-1}{s-1}\frac{1}{n^m},$$

then

$$-b_{m,n}(s) < \sum_{k=m}^{\infty}\binom{-\frac{2s-1}{2}}{k}\frac{1}{n^k} < b_{m,n}(s) \tag{2.11}$$

and

$$0 < \sum_{k=m}^{\infty}\binom{-\frac{2s-1}{2}}{k}\frac{(-1)^k}{n^k} < b_{m,n}(s). \tag{2.12}$$

Lemma 2.9 (Lem 7.4, [1]). *For* $n, s \in \mathbb{Z}_{\geq 1}, m \in \mathbb{N}$ *and* $n > 2s$, *let*

$$\beta_{m,n}(s) := \frac{2}{n^m}\binom{s+m-1}{s-1},$$

then

$$-\beta_{m,n}(s) < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{1}{n^k} < \beta_{m,n}(s) \tag{2.13}$$

and

$$0 < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{2.14}$$

Lemma 2.10 (Lem 7.5, [1]). *For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$, let*

$$c_{m,n}(s) := \frac{2}{m}\frac{s^m}{n^m},$$

then

$$-c_{m,n}(s) < \sum_{k=m}^{\infty} \frac{(-1)^{k+1}}{k}\frac{s^k}{n^k} < c_{m,n}(s) \text{ and } \quad -c_{m,n}(s) < -\sum_{k=m}^{\infty}\frac{1}{k}\frac{s^k}{n^k} < 0 \tag{2.15}$$

and

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty}\binom{1/2}{k}\frac{s^k}{n^k} < \frac{c_{m,n}(s)}{\sqrt{m}} \text{ and } \quad -\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty}\binom{1/2}{k}\frac{(-1)^k s^k}{n^k} < 0. \tag{2.16}$$

3. Inequalities for $\log p(n; \vec{\mathbf{s}})$

In this section, first we prove an infinite family of inequalities for $\log p(n+s)$ with s being a non-negative integer, see Theorem 3.9. Starting from Theorem 2.7, we will estimate $P_{n+s}(U)$ and the error terms given in (2.7) and (2.8), stated in Lemma 3.3-3.6. Finally, generalizing Theorem 3.9 by taking into consideration $\sum_{i=1}^{T}\log p(n+s_i)$ for $(s_1, s_2, \ldots, s_T) \in \mathbb{Z}_{\geq 0}^T$, we obtain Theorem 3.13.

Lemma 3.1. *Let the coefficient sequence $(g_n)_{n\geq 1}$ be as in Lemma 2.2. Then for all $n \geq 1$, we have*

$$|g_n| \leq \frac{1}{n}\frac{1}{(24\alpha)^{\lfloor n/2\rfloor}}. \tag{3.1}$$

Proof. Observe that for all $n \geq 0$, $\frac{\sqrt{6}}{2\pi}\left(1+\frac{\alpha}{2}\right)\frac{1}{(24\alpha)^n(2n+1)} > 0$ and $0 < \frac{\sqrt{6}}{2\pi}\left(\frac{\pi^2}{72}+1+\frac{\alpha}{2(1-\alpha)}\right) < 1$. Using (2.5), we obtain for all $n \geq 0$,

$$-\frac{1}{(24\alpha)^n(2n+1)} < g_{2n+1} < 0. \tag{3.2}$$

Since $\frac{3\alpha}{2} - \frac{1}{2} < 0$, from (2.6), it follows that for all $n \geq 1$,

$$-\frac{1}{(24\alpha)^n(2n)} \leq g_{2n} < 0. \tag{3.3}$$

From (3.2) and (2.4), we conclude that for all $n \geq 1$,

$$|g_n| \leq \frac{1}{(24\alpha)^{\lfloor n/2\rfloor}n}.$$

□

Definition 3.2. *For $s \in \mathbb{Z}_{\geq 0}$, define*

$$\delta_s := \begin{cases} 1, & \text{if } s \geq 1 \\ 0, & \text{if } s = 0 \end{cases}.$$

Lemma 3.3. *For $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, let*

$$P^1_{n,s}(w) := -\log n + \sum_{k=1}^{\lfloor\frac{w-1}{2}\rfloor}\frac{(-1)^k s^k}{k}\left(\frac{1}{\sqrt{n}}\right)^{2k} \text{ and } E^1_{n,s}(w) := \frac{2s^{\lceil\frac{w+1}{2}\rceil}}{\lceil w/2\rceil}\left(\frac{1}{\sqrt{n}}\right)^{w}\delta_s,$$

then

$$P^1_{n,s}(w) - E^1_{n,s}(w) \leq -\log(n+s) \leq P^1_{n,s}(w) + E^1_{n,s}(w). \tag{3.4}$$

Proof. For all $n, s \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, we split $\log(n+s)$ as follows

$$\log(n+s) = \log n + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k} = \log n + \sum_{k=1}^{\lfloor \frac{w-1}{2} \rfloor} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k} + \sum_{k=\lceil \frac{w}{2} \rceil}^{\infty} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k}. \tag{3.5}$$

Applying (2.15) with $m \mapsto \lceil \frac{w}{2} \rceil$, it follows that for all $n > 2s$,

$$-\frac{2}{\lceil w/2 \rceil} \left(\frac{s}{n}\right)^{\lceil w/2 \rceil} < \sum_{k=\lceil \frac{w}{2} \rceil}^{\infty} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k} < \frac{2}{\lceil w/2 \rceil} \left(\frac{s}{n}\right)^{\lceil w/2 \rceil}. \tag{3.6}$$

Since for all $s \in \mathbb{Z}_{\geq 0}$, $s^{\lceil w/2 \rceil} \leq s^{\lceil \frac{w+1}{2} \rceil}$, from (3.5) and (3.6), it follows that

$$P^1_{n,s}(w) - E^1_{n,s}(w) \leq -\log(n+s) \leq P^1_{n,s}(w) + E^1_{n,s}(w). \tag{3.7}$$

Observe that equality holds when $s = 0$. □

Lemma 3.4. *For $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, let*

$$P^2_{n,s}(w) := \pi\sqrt{\frac{2n}{3}} + \pi\sqrt{\frac{2}{3}} \sum_{k=1}^{\lfloor \frac{w}{2} \rfloor} \binom{1/2}{k} s^k \left(\frac{1}{\sqrt{n}}\right)^{2k-1} \text{and } E^2_{n,s}(w) := \frac{6 s^{\lceil \frac{w+1}{2} \rceil}}{\lceil w/2 \rceil} \left(\frac{1}{\sqrt{n}}\right)^w \delta_s,$$

then

$$P^2_{n,s}(w) - E^2_{n,s}(w) \leq \pi\sqrt{\frac{2n+2s}{3}} \leq P^2_{n,s}(w) + E^2_{n,s}(w). \tag{3.8}$$

Proof. For all $n, s \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, we split $\pi\sqrt{\frac{2n+2s}{3}}$ as follows

$$\pi\sqrt{\frac{2n+2s}{3}} = \pi\sqrt{\frac{2n}{3}} + \pi\sqrt{\frac{2}{3}} \sum_{k=1}^{\lfloor \frac{w}{2} \rfloor} \binom{1/2}{k} s^k \left(\frac{1}{\sqrt{n}}\right)^{2k-1} + \pi\sqrt{\frac{2n}{3}} \sum_{k=\lfloor \frac{w+2}{2} \rfloor}^{\infty} \binom{1/2}{k} \frac{s^k}{n^k}. \tag{3.9}$$

Applying (2.16) with $m \mapsto \lfloor \frac{w+2}{2} \rfloor$, it follows that for all $n > 2s$,

$$-\frac{2}{\left(\lfloor \frac{w+2}{2} \rfloor\right)^{3/2}} \left(\frac{s}{n}\right)^{\lfloor \frac{w+2}{2} \rfloor} < \sum_{k=\lfloor \frac{w+2}{2} \rfloor}^{\infty} \binom{1/2}{k} \frac{s^k}{n^k} < \frac{2}{\left(\lfloor \frac{w+2}{2} \rfloor\right)^{3/2}} \left(\frac{s}{n}\right)^{\lfloor \frac{w+2}{2} \rfloor}. \tag{3.10}$$

Therefore,

$$\begin{aligned} -2\pi\sqrt{\frac{2}{3}} \frac{s^{\lfloor \frac{w+2}{2} \rfloor}}{\left(\lfloor \frac{w+2}{2} \rfloor\right)^{3/2}} \left(\frac{1}{\sqrt{n}}\right)^{2\lfloor \frac{w+2}{2} \rfloor - 1} &< \pi\sqrt{\frac{2n}{3}} \sum_{k=\lfloor \frac{w+2}{2} \rfloor}^{\infty} \binom{1/2}{k} \frac{s^k}{n^k} \\ &< 2\pi\sqrt{\frac{2}{3}} \frac{s^{\lfloor \frac{w+2}{2} \rfloor}}{\left(\lfloor \frac{w+2}{2} \rfloor\right)^{3/2}} \left(\frac{1}{\sqrt{n}}\right)^{2\lfloor \frac{w+2}{2} \rfloor - 1}. \end{aligned} \tag{3.11}$$

Now for all $s \in \mathbb{Z}_{\geq 0}$,

$$\pi\sqrt{\frac{2}{3}} \frac{s^{\lfloor \frac{w+2}{2} \rfloor}}{\left(\lfloor \frac{w+2}{2} \rfloor\right)^{3/2}} \left(\frac{1}{\sqrt{n}}\right)^{2\lfloor \frac{w+2}{2} \rfloor - 1} < \frac{6 s^{\lceil \frac{w+1}{2} \rceil}}{\lceil w/2 \rceil} \left(\frac{1}{\sqrt{n}}\right)^w.$$

From (3.9) and (3.11), it follows that

$$P^2_{n,s}(w) - E^2_{n,s}(w) \leq \pi\sqrt{\frac{2n+2s}{3}} \leq P^2_{n,s}(w) + E^2_{n,s}(w), \tag{3.12}$$

with equality holds for $s = 0$. □

Lemma 3.5. *For $(n,s)\in\mathbb{Z}_{\ge 1}\times\mathbb{Z}_{\ge 0}$, $w\in\mathbb{Z}_{\ge 2}$, and $n>2s$, let*

$$\overline{g}_{\ell}(s;t):=g_{\ell}\binom{-\ell/2}{t-\lfloor \ell/2\rfloor}s^{t-\lfloor\frac{\ell}{2}\rfloor}\quad \text{for all}\quad \ell\in\mathbb{Z}_{\ge 1},$$

$$P^3_{n,s}(w):=\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^u+\sum_{t=1}^{\lfloor\frac{w-2}{2}\rfloor}\sum_{u=0}^{t-1}\overline{g}_{2u+1}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}+\sum_{t=2}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{u=1}^{t-1}\overline{g}_{2u}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t},$$

and

$$E^3_{n,s}(w):=\frac{29}{w}\Big(s+\frac{1}{24\alpha}\Big)^{\lceil\frac{w-1}{2}\rceil+1}\Big(\frac{1}{\sqrt{n}}\Big)^{w}\delta_s,$$

then

$$P^3_{n,s}(w)-E^3_{n,s}(w)\le\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n+s}}\Big)^u\le P^2_{n,s}(w)+E^3_{n,s}(w). \tag{3.13}$$

Proof. For all $n,s\in\mathbb{Z}_{\ge 1}$, $w\in\mathbb{Z}_{\ge 2}$, and $n>2s$, we split $\sum_{u=1}^{w-1}g_u(1/\sqrt{n+s})^u$ as

$$\begin{aligned}
\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n+s}}\Big)^u &= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^u\sum_{k=0}^{\infty}\binom{-u/2}{k}\frac{s^k}{n^k}\\
&= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^u+\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^u\sum_{k=1}^{\infty}\binom{-u/2}{k}\frac{s^k}{n^k}\\
&= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^u+\sum_{u=1}^{w-1}g_u\sum_{k=1}^{\infty}\binom{-u/2}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+u}\\
&= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^u+\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}g_{2u+1}\sum_{k=1}^{\infty}\binom{-\frac{2u+1}{2}}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u+1}\\
&\qquad+\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}g_{2u}\sum_{k=1}^{\infty}\binom{-u}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u}.
\end{aligned}\tag{3.14}$$

Now,

$$\begin{aligned}
&\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}g_{2u+1}\sum_{k=1}^{\infty}\binom{-\frac{2u+1}{2}}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u+1}\\
&=\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}g_{2u+1}\sum_{t=u+1}^{\infty}\binom{-\frac{2u+1}{2}}{t-u}s^{t-u}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\
&=\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}g_{2u+1}\sum_{t=u+1}^{\lfloor\frac{w-2}{2}\rfloor}\binom{-\frac{2u+1}{2}}{t-u}s^{t-u}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}+\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}g_{2u+1}\sum_{t=\lceil\frac{w-1}{2}\rceil}^{\infty}\binom{-\frac{2u+1}{2}}{t-u}s^{t-u}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\
&=\sum_{t=1}^{\lfloor\frac{w-2}{2}\rfloor}\sum_{u=0}^{t-1}\overline{g}_{2u+1}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}+\underbrace{\sum_{u=1}^{\lceil\frac{w-1}{2}\rceil}\sum_{t=\lceil\frac{w-1}{2}\rceil-u+1}^{\infty}g_{2u-1}\binom{-\frac{2u-1}{2}}{t}s^t\Big(\frac{1}{\sqrt{n}}\Big)^{2t+2u-1}}_{:=\mathcal{S}_o(w,n,s)}.
\end{aligned}\tag{3.15}$$

Next, we proceed to estimate the absolute value of the error sum $\mathcal{S}_o(w,n,s)$ for $s\in\mathbb{Z}_{\ge 1}$.

$|\mathcal{S}_o(w,n,s)|$

$$
\begin{aligned}
&\le \sum_{u=1}^{\lceil \frac{w-1}{2}\rceil} |g_{2u-1}| \Big(\frac{1}{\sqrt{n}}\Big)^{2u-1} \Bigg| \sum_{t=\lceil \frac{w-1}{2}\rceil-u+1}^{\infty} \binom{-\frac{2u-1}{2}}{t} \frac{s^t}{n^t} \Bigg| \\
&< 4 \sum_{u=1}^{\lceil \frac{w-1}{2}\rceil} |g_{2u-1}| \Big(\frac{1}{\sqrt{n}}\Big)^{2u-1} \sqrt{\frac{u}{\lceil \frac{w-1}{2}\rceil}} \binom{\lceil \frac{w-1}{2}\rceil}{u-1} \Big(\frac{s}{n}\Big)^{\lceil \frac{w-1}{2}\rceil-u+1} \\
&\qquad \Bigg(\text{by substitution } (m,s,n) \mapsto \Big(\Big\lceil \frac{w-1}{2}\Big\rceil - u+1, u, \frac{n}{s}\Big) \text{ in } (2.11)\Bigg) \\
&\le 4 \Bigg(\sum_{u=1}^{\lceil \frac{w-1}{2}\rceil} |g_{2u-1}| \binom{\lceil \frac{w-1}{2}\rceil}{u-1} \frac{1}{s^u} \Bigg) \Big(\frac{1}{\sqrt{n}}\Big)^{2\lceil \frac{w-1}{2}\rceil+1} s^{\lceil \frac{w-1}{2}\rceil+1} \\
&\le 4 \Bigg(\sum_{u=1}^{\lceil \frac{w-1}{2}\rceil} \frac{1}{(2u-1)(24\alpha)^{u-1}} \binom{\lceil \frac{w-1}{2}\rceil}{u-1} \frac{1}{s^u} \Bigg) \Big(\frac{1}{\sqrt{n}}\Big)^{w} s^{\lceil \frac{w-1}{2}\rceil+1} \quad \Big(\text{by Lemma } 3.1\Big) \\
&= 4 \Bigg(\sum_{u=0}^{\lceil \frac{w-1}{2}\rceil-1} \frac{1}{2u+1} \binom{\lceil \frac{w-1}{2}\rceil}{u} \frac{1}{(24\alpha s)^u} \Bigg) \Big(\frac{1}{\sqrt{n}}\Big)^{w} s^{\lceil \frac{w-1}{2}\rceil} \\
&\le \frac{16}{3} \Bigg(\sum_{u=0}^{\lceil \frac{w-1}{2}\rceil-1} \frac{1}{2u+2} \binom{\lceil \frac{w-1}{2}\rceil}{u} \frac{1}{(24\alpha s)^u} \Bigg) \Big(\frac{1}{\sqrt{n}}\Big)^{w} s^{\lceil \frac{w-1}{2}\rceil} \quad \Big(\text{as } \frac{4}{2u+1} \le \frac{8}{3(u+1)} \text{ for all } u \ge 1\Big) \\
&= \frac{16}{3} \frac{1}{2\Big(\lceil \frac{w-1}{2}\rceil + 1\Big)} \Bigg(\Bigg(\Big(1 + \frac{1}{24\alpha s}\Big)^{\lceil \frac{w-1}{2}\rceil+1} - 1 \Bigg) 24\alpha s - \Big(\frac{1}{24\alpha s}\Big)^{\lceil \frac{w-1}{2}\rceil} \Bigg) \Big(\frac{1}{\sqrt{n}}\Big)^{w} s^{\lceil \frac{w-1}{2}\rceil} \\
&< \frac{16}{3w} \Bigg(\Big(1 + \frac{1}{24\alpha s}\Big)^{\lceil \frac{w-1}{2}\rceil+1} - 1 \Bigg) 24\alpha s \Big(\frac{1}{\sqrt{n}}\Big)^{w} s^{\lceil \frac{w-1}{2}\rceil} < \frac{28}{w} \Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w-1}{2}\rceil+1} \Big(\frac{1}{\sqrt{n}}\Big)^{w}. \quad (3.16)
\end{aligned}
$$

Similar to (3.15), we get

$$
\begin{aligned}
&\sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor} g_{2u} \sum_{k=1}^{\infty} \binom{-u}{k} s^k \Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u} \\
&= \sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor} \sum_{t=u+1}^{\infty} g_{2u} \binom{-u}{t-u} s^{t-u} \Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\
&= \sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor} \sum_{t=u+1}^{\lfloor \frac{w-1}{2}\rfloor} g_{2u} \binom{-u}{t-u} s^{t-u} \Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor} \sum_{t=\lceil \frac{w}{2}\rceil}^{\infty} g_{2u} \binom{-u}{t-u} s^{t-u} \Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\
&= \sum_{t=2}^{\lfloor \frac{w-1}{2}\rfloor} \sum_{u=1}^{t-1} \overline{g}_{2u}(s;t) \Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \underbrace{\sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor} \sum_{t=\lceil \frac{w}{2}\rceil-u}^{\infty} g_{2u} \binom{-u}{t} s^t \Big(\frac{1}{\sqrt{n}}\Big)^{2t+2u}}_{:=\mathcal{S}_e(w,n,s)}. \quad (3.17)
\end{aligned}
$$

Consequently for $s \in \mathbb{Z}_{\geq 1}$,

$$
|\mathcal{S}_e(w,n,s)| \le \sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor} |g_{2u}| \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \Bigg| \sum_{t=\lceil \frac{w}{2}\rceil-u}^{\infty} \binom{-u}{t} \frac{s^t}{n^t} \Bigg|
$$

$$
\begin{aligned}
&< \ 2 \sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} |g_{2u}| \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \Big(\frac{s}{n}\Big)^{\lceil \frac{w}{2} \rceil - u} \\
&\qquad \Bigg(\text{by substitution } (m,s,n) \mapsto \Big(\Big\lceil \frac{w}{2} \Big\rceil - u, u, \frac{n}{s}\Big) \ \text{ in } \ (2.13)\Bigg) \\
&= 2 \Bigg(\sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} |g_{2u}| \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \frac{1}{s^u} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \\
&\leq 2 \Bigg(\sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} \frac{1}{2u} \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \frac{1}{(24\alpha s)^u} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \ \Big(\text{by Lemma } 3.1\Big) \\
&= \ 2 \Bigg(\sum_{u=1}^{\lceil \frac{w}{2} - 1 \rceil} \frac{1}{2u} \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \frac{1}{(24\alpha s)^u} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \\
&= \ \frac{1}{w} \Bigg(\Big(1 + \frac{1}{24\alpha s}\Big)^{\lceil \frac{w}{2} \rceil} - 1 - \Big(\frac{1}{24\alpha s}\Big)^{\lceil \frac{w}{2} \rceil} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \\
&< \frac{1}{w} \Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w-1}{2} \rceil + 1} \Big(\frac{1}{\sqrt{n}}\Big)^{w}.
\end{aligned} \tag{3.18}
$$

From (3.14), (3.15), and (3.17), we obtain

$$
\sum_{u=1}^{w-1} g_u \Big(\frac{1}{\sqrt{n+s}}\Big)^{u} - P^3_{n,s}(w) = \mathcal{S}_o(w,n,s) + \mathcal{S}_e(w,n,s), \tag{3.19}
$$

and taking absolute on both side of (3.19) and applying (3.16) and (3.18), it follows that

$$
\begin{aligned}
\Big| \sum_{u=1}^{w-1} g_u \Big(\frac{1}{\sqrt{n+s}}\Big)^{u} - P^3_{n,s}(w) \Big| &= \Big| \mathcal{S}_o(w,n,s) + \mathcal{S}_e(w,n,s) \Big| \\
&\leq \Big| \mathcal{S}_o(w,n,s) \Big| + \Big| \mathcal{S}_e(w,n,s) \Big| \\
&< \frac{29}{w} \Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w-1}{2} \rceil + 1} \Big(\frac{1}{\sqrt{n}}\Big)^{w}.
\end{aligned} \tag{3.20}
$$

Note that in (3.13), the equality holds for $s = 0$ because first, $P^3_{n,0}(w) = 0$ and secondly, the error term $\mathcal{S}_o(w,n,0)$ (resp. $\mathcal{S}_e(w,n,0)$) in (3.15) (resp. in (3.17)) is identically zero and therefore, we conclude that $E^3_{n,0}(w) = 0$. □

Lemma 3.6. *Let γ_1, γ_2 as in Equation* (2.10)*. For $(n,s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, and $w \in \mathbb{Z}_{\geq 2}$, then*

$$
- \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \leq - \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w} \Big(\frac{1}{\sqrt{n+s}}\Big)^{w} \tag{3.21}
$$

and

$$
\frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w} \Big(\frac{1}{\sqrt{n+s}}\Big)^{w} \leq \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w} \Big(\frac{1}{\sqrt{n}}\Big)^{w}. \tag{3.22}
$$

Proof. The proof of both (3.21) and (3.22) is immediate from the fact that $\frac{1}{\sqrt{n+s}} \leq \frac{1}{\sqrt{n}}$ for all $(n,s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$. □

Definition 3.7. *Let the coefficient sequence* $(g_n)_{n\geq 1}$ *be as in Lemma 2.2 and* $(\overline{g}_n(s;t))_{n\geq 1}$ *be as in Lemma 3.5. Then for* $(n,s)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$ *and* $U\in\mathbb{Z}_{\geq 1}$*, we define*

$$P_{n,s}(U) := -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} \widetilde{g}_{u,s}\Big(\frac{1}{\sqrt{n}}\Big)^{u}, \tag{3.23}$$

where

$$\widetilde{g}_{2u,s} := \frac{(-s)^u}{u} + g_{2u} + \sum_{k=1}^{u-1}\overline{g}_{2k}(s;u) \quad \text{for all } \ 1\leq u\leq \lfloor U/2\rfloor$$

and

$$\widetilde{g}_{2u+1,s} := \pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}s^{u+1} + g_{2u+1} + \sum_{k=0}^{u-1}\overline{g}_{2k+1}(s;u) \quad \text{for all } \ 0\leq u\leq \lfloor (U-1)/2\rfloor.$$

Definition 3.8. *Let* γ_1,γ_2 *be as in* (2.10)*. For* $(n,s)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$*,* $w\in\mathbb{Z}_{\geq 2}$*, and* $n>2s$*, we define*

$$E^{\mathcal{U}}_{n,s}(w) := \left(45\Big(s+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_s + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}$$

and

$$E^{\mathcal{L}}_{n,s}(w) := \left(45\Big(s+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_s + \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}.$$

Theorem 3.9. *Let* $P_{n,s}(U)$ *be as in Definition 3.7 and* $E^{\mathcal{L}}_{n,s}(w), E^{\mathcal{U}}_{n,s}(w)$ *be as in Definition 3.8. If* $(n,s)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$*,* $w\in\mathbb{Z}_{\geq 2}$*, and* $n>\max\{g(w)-s,2s\}$*, then*

$$P_{n,s}(w-1) - E^{\mathcal{L}}_{n,s}(w) < \log p(n+s) < P_{n,s}(w-1) + E^{\mathcal{U}}_{n,s}(w). \tag{3.24}$$

Proof. From (2.9), it follows that for $\lceil \frac{w}{2}\rceil \geq \gamma_0$ and $n>g(w)-s$,

$$P_{n+s}(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n+s}}\Big)^{w} < \log p(n+s) < P_{n+s}(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n+s}}\Big)^{w}, \tag{3.25}$$

where

$$P_{n+s}(w-1) = -\log 4\sqrt{3} - \log(n+s) + \pi\sqrt{\frac{2(n+s)}{3}} + \sum_{u=1}^{w-1} g_u\Big(\frac{1}{\sqrt{n+s}}\Big)^{u} \quad \text{(by Definition 2.6)}.$$

Applying Lemma 3.6 into (3.25), we obtain

$$P_{n+s}(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^{w} < \log p(n+s) < P_{n+s}(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}. \tag{3.26}$$

Invoking Lemma 3.3, 3.4, and 3.5 into (3.26), it follows that

$$\begin{aligned} -\log 4\sqrt{3} + \sum_{i=1}^{3} P^{i}_{n,s}(w) - \sum_{i=1}^{3} E^{i}_{n,s}(w) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^{w} &< \log p(n+s) \\ < -\log 4\sqrt{3} + \sum_{i=1}^{3} P^{i}_{n,s}(w) + \sum_{i=1}^{3} E^{i}_{n,s}(w) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}&. \end{aligned} \tag{3.27}$$

For $s \geq 1$,

$$\begin{aligned}
\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w &= \left(\frac{8s^{\lceil \frac{w+1}{2}\rceil}}{\lceil w/2\rceil} + \frac{29}{w}\Big(s+\frac{1}{24\alpha}\Big)^{\lceil \frac{w-1}{2}\rceil+1} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil} w}\right)\Big(\frac{1}{\sqrt{n}}\Big)^w \\
&= \left(\frac{8s^{\lceil \frac{w+1}{2}\rceil}}{\lceil w/2\rceil} + \frac{29}{w}\Big(s+\frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2}\rceil} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil} w}\right)\Big(\frac{1}{\sqrt{n}}\Big)^w \\
&\leq \left(\frac{16s^{\lceil \frac{w+1}{2}\rceil}}{w} + \frac{29}{w}\Big(s+\frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2}\rceil} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil} w}\right)\Big(\frac{1}{\sqrt{n}}\Big)^w \\
&< \left(45\Big(s+\frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2}\rceil} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^w, \qquad (3.28)
\end{aligned}$$

and for $s = 0$,

$$\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w = \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w. \qquad (3.29)$$

Similarly, for $s \geq 1$,

$$\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w < \left(45\Big(s+\frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2}\rceil} + \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^w, \qquad (3.30)$$

and for $s = 0$,

$$\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w = \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w. \qquad (3.31)$$

Putting (3.28)-(3.31) together into (3.27), we get

$$-\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) - E^{\mathcal{L}}_{n,s}(w) < \log p(n+s) < -\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) + E^{\mathcal{U}}_{n,s}(w). \quad (3.32)$$

From Lemma 3.3-3.5, it follows that

$$\begin{aligned}
-\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) &= -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{w-1} g_u\Big(\frac{1}{\sqrt{n}}\Big)^u \\
&+ \left(\sum_{k=1}^{\lfloor \frac{w-1}{2}\rfloor} \frac{(-1)^k s^k}{k}\Big(\frac{1}{\sqrt{n}}\Big)^{2k} + \sum_{t=2}^{\lfloor \frac{w-1}{2}\rfloor}\sum_{u=1}^{t-1} \overline{g}_{2u}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\right) \\
&+ \left(\pi\sqrt{\frac{2}{3}}\sum_{k=1}^{\lfloor \frac{w}{2}\rfloor}\binom{1/2}{k} s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k-1} + \sum_{t=1}^{\lfloor \frac{w-2}{2}\rfloor}\sum_{u=0}^{t-1} \overline{g}_{2u+1}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\right) \\
&= -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor}\left(\frac{(-s)^u}{u} + g_{2u} + \sum_{k=1}^{u-1}\overline{g}_{2k}(s;u)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
&+ \sum_{u=0}^{\lfloor \frac{w-2}{2}\rfloor}\left(\pi\sqrt{\frac{2}{3}}\binom{1/2}{k+1} s^{k+1} + g_{2u+1} + \sum_{k=0}^{u-1}\overline{g}_{2k+1}(s;u)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&= -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{\lfloor \frac{w-1}{2}\rfloor}\widetilde{g}_{2u,s}\Big(\frac{1}{\sqrt{n}}\Big)^{2u} + \sum_{u=0}^{\lfloor \frac{w-2}{2}\rfloor}\widetilde{g}_{2u+1,s}\Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&= P_{n,s}(w-1). \qquad (3.33)
\end{aligned}$$

From (3.32) and (3.33), we conclude the proof of (3.24). □

Next, we proceed to estimate $\sum_{i=1}^{T} \log p(n+s_i)$.

Definition 3.10. *For $n, T \in \mathbb{Z}_{\geq 1}$ and $\vec{\boldsymbol{s}} := (s_1, s_2, \ldots, s_T) \in \mathbb{Z}_{\geq 0}^{T}$, we define*

$$\log p(n; \vec{\boldsymbol{s}}) := \sum_{i=1}^{T} \log p(n+s_i).$$

Definition 3.11. *Let the coefficient sequence $(g_n)_{n\geq 1}$ be as in Lemma 2.2, $(\overline{g}_n(s;t))_{n\geq 1}$ be as in Lemma 3.5, and $\vec{\boldsymbol{s}}$ be as in Definition 3.10. For $n, T \in \mathbb{Z}_{\geq 1}$ and $U \in \mathbb{Z}_{\geq 1}$, we define*

$$P_{n,\vec{s}}(U) := -T \cdot \log 4\sqrt{3} - T \cdot \log n + T \cdot \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} \widetilde{g}_{u,\vec{s}} \Big(\frac{1}{\sqrt{n}}\Big)^{u}, \tag{3.34}$$

where

$$\widetilde{g}_{2u,\vec{s}} := \frac{1}{u}\sum_{i=1}^{T}(-s_i)^u + T \cdot g_{2u} + \sum_{i=1}^{T}\sum_{k=1}^{u-1} \overline{g}_{2k}(s_i; u) \quad \text{for all} \quad 1 \leq u \leq \lfloor U/2 \rfloor$$

and

$$\widetilde{g}_{2u+1,\vec{s}} := \pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}\sum_{i=1}^{T} s_i^{u+1} + T \cdot g_{2u+1} + \sum_{i=1}^{T}\sum_{k=0}^{u-1} \overline{g}_{2k+1}(s_i; u) \quad \text{for all} \quad 0 \leq u \leq \lfloor (U-1)/2 \rfloor.$$

Definition 3.12. *Let γ_1, γ_2 be as in* (2.10) *and $\vec{\boldsymbol{s}}$ be as in Definition 3.10. For each $\{s_i\}_{1\leq i\leq T}$, δ_{s_i} be as in Definition 3.2. For $n, T \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s_i$, we define*

$$E^{\mathcal{U}}_{n,\vec{s}}(w) := \left(45\sum_{i=1}^{T}\Big(s_i + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2} \rceil}\delta_{s_i} + \frac{T \cdot \gamma_2}{(24\alpha)^{\lceil w/2 \rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}$$

and

$$E^{\mathcal{L}}_{n,\vec{s}}(w) := \left(45\sum_{i=1}^{T}\Big(s_i + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2} \rceil}\delta_{s_i} + \frac{T \cdot \gamma_1}{(24\alpha)^{\lceil w/2 \rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}.$$

A generalized version of Theorem 3.9 is as follows:

Theorem 3.13. *Let $\log p(n; \vec{\boldsymbol{s}})$ be as in Definition 3.10, $P_{n,\vec{s}}(U)$ be as in Definition 3.11, and let $g(k)$ be as in Definition 2.5. Let $E^{\mathcal{L}}_{n,\vec{s}}(w)$ and $E^{\mathcal{U}}_{n,\vec{s}}(w)$ be as in Definition 3.12. If $n, T \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and*

$$n > \max_{1\leq i\leq T}\{g(w) - \min_{1\leq i\leq T}\{s_i\}, 2s_i\} := g(w; \vec{\boldsymbol{s}}),$$

then

$$P_{n,\vec{s}}(w-1) - E^{\mathcal{L}}_{n,\vec{s}}(w) < \log p(n; \vec{\boldsymbol{s}}) < P_{n,\vec{s}}(w-1) + E^{\mathcal{U}}_{n,\vec{s}}(w). \tag{3.35}$$

Proof. Applying (3.24) for each $\{s_i\}_{1\leq i\leq T}$ and summing up, we get (3.35). □

Remark 3.14. *A few applications of Theorem 3.13 are listed below.*

1. *Choosing $w = 5$ (resp. $w = 7$), we obtain $(p(n))_{n\geq 26}$ is* log-concave *(resp.* (1.7)*).*
2. *Define $u_n := \frac{p(n)p(n+2)}{p(n+1)^2}$ and let N be any positive integer. Then choosing $w = N$, we have a full asymptotic expansion of $\log u_n$ with a precise estimation of the error bound after truncation of the asymptotic expansion at a point N.*

(3) *Applying* $\vec{\boldsymbol{s}}=\{m,m\}$ *and* $\vec{\boldsymbol{r}}=\{0,2m\}$ *to* (3.35)*, and estimation of*

$$P_{n,\vec{s}}(4)+E^{\mathcal{L}}_{n,\vec{s}}(5)-P_{n,\vec{r}}(4)-E^{\mathcal{U}}_{n,\vec{s}}(5),$$

leads to the strong log*-concavity property of* $p(n)$*.*

(4) *Without loss of generality, assume* $b=\lambda a$ *with* $\lambda\geq 1$ *in Theorem 1.1. By making the substitutions* $(n,\vec{\boldsymbol{s}})=(a,0)$*,* $(n,\vec{\boldsymbol{s}})=(\lambda a,0)$*, and* $(n,\vec{\boldsymbol{r}})=(a(1+\lambda),0)$ *to* (3.35)*, we can retrieve* (1.10)*.*

4. Asymptotics of $(-1)^{r-1}\Delta^r\log p(n)$

Lemma 4.1. *Let* $P_{n,s}(w-1)$ *be as in Theorem 3.9. Then for all* $r\geq 2$*,*

$$\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}P_{n,i}(2r)=C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1}-(r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r}, \tag{4.1}$$

where $C_r=\dfrac{\pi}{\sqrt{6}}\Big(\dfrac{1}{2}\Big)_{r-1}$ *and* $(a)_k$ *is the standard notation for the rising factorial.*

Proof. From Definition 3.7, it follows that

$$\begin{aligned}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}P_{n,i}(2r) &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\Bigg(-\log 4\sqrt{3}-\log n+\sqrt{\frac{2n}{3}}+\sum_{u=1}^{2r}\widetilde{g}_{u,i}\Big(\frac{1}{\sqrt{n}}\Big)^{u}\Bigg)\\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\sum_{u=1}^{2r}\widetilde{g}_{u,i}\Big(\frac{1}{\sqrt{n}}\Big)^{u}\\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\sum_{u=1}^{2r-2}\widetilde{g}_{u,i}\Big(\frac{1}{\sqrt{n}}\Big)^{u}+\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\widetilde{g}_{2r-1,i}\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1}\\ &\quad+\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\widetilde{g}_{2r,i}\Big(\frac{1}{\sqrt{n}}\Big)^{2r}.\end{aligned} \tag{4.2}$$

Following the notation from [9], here $\left\{{n \atop m}\right\}$ denotes the Stirling number of second kind. For all integers $1\leq u\leq 2r-2$ and $u\equiv 0 \pmod 2$, we have

$$\begin{aligned}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\sum_{u=1}^{r-1}\widetilde{g}_{2u,i}\Big(\frac{1}{\sqrt{n}}\Big)^{2u} &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\sum_{u=1}^{r-1}\Bigg[\frac{(-i)^u}{u}+g_{2u}+\sum_{k=1}^{u-1}\overline{g}_{2k}(i;u)\Bigg]\Big(\frac{1}{\sqrt{n}}\Big)^{2u}\\ &= \sum_{u=1}^{r-1}\frac{(-1)^u}{u}(-1)^{r+1}r!\left\{{u \atop r}\right\}\Big(\frac{1}{\sqrt{n}}\Big)^{2u}\\ &\quad+\sum_{u=1}^{r-1}\sum_{k=1}^{u-1}g_{2k}\binom{-k}{u-k}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}i^{u-k}\Big(\frac{1}{\sqrt{n}}\Big)^{2u}\\ &= \sum_{u=1}^{r-1}\frac{(-1)^u}{u}(-1)^{r+1}r!\left\{{u \atop r}\right\}\Big(\frac{1}{\sqrt{n}}\Big)^{2u}\\ &\quad+\sum_{u=1}^{r-1}\sum_{k=1}^{u-1}g_{2k}\binom{-k}{u-k}(-1)^{r+1}r!\left\{{u-k \atop r}\right\}\Big(\frac{1}{\sqrt{n}}\Big)^{2u}\\ &= 0\quad\Bigg(\text{as }\left\{{n \atop m}\right\}=0\text{ for all }n<m\Bigg).\end{aligned} \tag{4.3}$$

Similarly for all integers $1 \leq u \leq 2r-2$ and $u \equiv 1 \pmod 2$, we obtain

$$\begin{aligned}
&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=0}^{r-2} \widetilde{g}_{2u+1,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=0}^{r-2} \left[\pi \sqrt{\frac{2}{3}} \binom{1/2}{u+1} i^{u+1} + g_{2u+1} + \sum_{k=0}^{u-1} \overline{g}_{2k+1}(i;u) \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&= \sum_{u=0}^{r-2} \pi \sqrt{\frac{2}{3}} \binom{1/2}{u+1} (-1)^{r+1} r! \left\{ {u+1 \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&\quad + \sum_{u=0}^{r-2} \sum_{k=0}^{u-1} g_{2k+1} \binom{-k-1/2}{u-k} (-1)^{r+1} r! \left\{ {u-k \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&= 0 \quad \left(\text{as } \left\{ {n \atop m} \right\} = 0 \text{ for all } n < m \right).
\end{aligned} \tag{4.4}$$

From (4.3) and (4.4), it follows that for all $1 \leq u \leq 2r-2$,

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=1}^{2r-2} \widetilde{g}_{u,i} \Big(\frac{1}{\sqrt{n}}\Big)^{u} = 0.$$

Now

$$\begin{aligned}
&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2r-1,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \left[\pi \sqrt{\frac{2}{3}} \binom{1/2}{r} i^{r} + g_{2r-1} + \sum_{k=0}^{r-2} \overline{g}_{2k+1}(i;r-1) \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&= \left[\pi \sqrt{\frac{2}{3}} \binom{1/2}{r} (-1)^{r+1} r! \left\{ {r \atop r} \right\} + \sum_{k=0}^{r-2} g_{2k+1} \binom{-k-1/2}{r-1-k} (-1)^{r+1} r! \left\{ {r-1-k \atop r} \right\} \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&= \frac{\pi}{\sqrt{6}} \Big(\frac{1}{2}\Big)_{r-1} \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \quad \left(\text{since } \left\{ {r-1-k \atop r} \right\} = 0 \ \text{ for all } \ 0 \leq k \leq r-2 \right).
\end{aligned} \tag{4.5}$$

We finish the proof by showing that

$$\begin{aligned}
\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2r,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2r} &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \left[\frac{(-i)^r}{r} + g_{2r} + \sum_{k=1}^{r-1} \overline{g}_{2k}(i;r) \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2r} \\
&= \left[-(r-1)! + \sum_{k=1}^{r-1} g_{2k} \binom{-k}{r-k} (-1)^{r+1} r! \left\{ {r-k \atop r} \right\} \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2r} \\
&= -(r-1)! \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1}.
\end{aligned} \tag{4.6}$$

□

Definition 4.2. *Let γ_1 be as in* (2.10) *and C_r be as in Lemma 4.1. Then for all $r \geq 2$, define*

$$\begin{aligned}
L_1(r) &:= \left(\frac{\gamma_1}{(12\alpha)^{r+1}} + 45 \sum_{i=1}^{r} \binom{r}{i} \Big(i + \frac{1}{24\alpha}\Big)^{r+1} \right) \frac{1}{2r+1}, \\
L(r) &:= (r-1)! + L_1(r),
\end{aligned}$$

and

$$N_L(r) := \max\left\{\Big(\frac{L(r)}{C_r}\Big)^2, g(2r+1)\right\}.$$

Lemma 4.3. *Let $L(r), N_L(r)$ be as in Definition 4.2 and C_r be as in Lemma 4.1. Then for all $n > N_L(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) > \log\left(1 + C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r}\right). \tag{4.7}$$

Proof. We split $(-1)^{r-1}\Delta^r \log p(n)$ as follows:

$$\begin{aligned}(-1)^{r-1}\Delta^r \log p(n) &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\log p(n+i)\\ &= \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}\log p(n+2i+1) - \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}\log p(n+2i). \end{aligned}\tag{4.8}$$

Applying Theorem 3.9 with $w = 2r+1$ to (4.8), we have for all $n > \max\limits_{0\le i\le r}\{g(2r+1)-i, 2i\} = g(2r+1)$,

$$\begin{aligned}&(-1)^{r-1}\Delta^r \log p(n)\\ &> \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}P_{n,i}(2r) - \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{L}}_{n,2i+1}(2r+1) - \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{U}}_{n,2i}(2r+1)\\ &= C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} - \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{L}}_{n,2i+1}(2r+1)\\ &\qquad - \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{U}}_{n,2i}(2r+1) \quad \Big(\text{by Lemma 4.1}\Big).\end{aligned}\tag{4.9}$$

From Definition 3.8, it is clear that $E^{\mathcal{U}}_{n,s}(w) < E^{\mathcal{L}}_{n,s}(w)$ because $\gamma_2 < \gamma_1$. Therefore,

$$\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{L}}_{n,2i+1}(2r+1) + \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{U}}_{n,2i}(2r+1) < \sum_{i=0}^{r}\binom{r}{i}E^{\mathcal{L}}_{n,i}(2r+1), \tag{4.10}$$

and

$$\sum_{i=0}^{r}\binom{r}{i}E^{\mathcal{L}}_{n,i}(2r+1) = L_1(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r+1}. \tag{4.11}$$

From (4.9) and (4.11), it follows that

$$\begin{aligned}(-1)^{r-1}\Delta^r \log p(n) &> C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} - L_1(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r+1}\\ &> C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r},\end{aligned}\tag{4.12}$$

and consequently for all $n > N_L(r)$, we get

$$(-1)^{r-1}\Delta^r \log p(n) > \log\left(1 + C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r}\right).$$

□

Definition 4.4. *Let $L_1(r)$ be as in Definition 4.2 and C_r be as in Lemma 4.1. Then for all $r \geq 2$, define*

$$N_U(r) := \max\left\{ \left(\frac{L_1(r)+1}{(r-1)!}\right)^2, \left(\frac{C_r^2}{2}\right)^{2/2r-3}, g(2r+1) \right\}.$$

Lemma 4.5. *Let $L_1(r)$ be as in Definition 4.2, C_r be as in Lemma 4.1, and $N_U(r)$ be as in Definition 4.4. Then for all $n > N_U(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + C_r\left(\frac{1}{\sqrt{n}}\right)^{2r-1}\right). \tag{4.13}$$

Proof. Applying Theorem 3.9 with $w = 2r+1$ to (4.8), we have for all $n > g(2r+1)$,

$$\begin{aligned}
&(-1)^{r-1}\Delta^r \log p(n) \\
&< \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}P_{n,i}(2r) + \sum_{i=0}^{\lfloor \frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{U}}_{n,2i+1}(2r+1) + \sum_{i=0}^{\lfloor \frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{L}}_{n,2i}(2r+1) \\
&< C_r\left(\frac{1}{\sqrt{n}}\right)^{2r-1} - (r-1)!\left(\frac{1}{\sqrt{n}}\right)^{2r} + \sum_{i=0}^{r}\binom{r}{i}E^{\mathcal{L}}_{n,i}(2r+1) \quad \big(\text{by Lemma 4.1}\big) \\
&= C_r\left(\frac{1}{\sqrt{n}}\right)^{2r-1} - (r-1)!\left(\frac{1}{\sqrt{n}}\right)^{2r} + L_1(r)\left(\frac{1}{\sqrt{n}}\right)^{2r+1}.
\end{aligned} \tag{4.14}$$

For all $n > N_U(r)$, it follows that

$$-(r-1)!\left(\frac{1}{\sqrt{n}}\right)^{2r} + L_1(r)\left(\frac{1}{\sqrt{n}}\right)^{2r+1} < -\frac{C_r^2}{2\, n^{2r-1}}. \tag{4.15}$$

From (4.14) and (4.15), it follows that for all $n > N_U(r)$,

$$(-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + C_r\left(\frac{1}{\sqrt{n}}\right)^{2r-1}\right).$$

□

Theorem 4.6. *Let $L(r), N_L(r)$ be as in Definition 4.2 and $N_U(r)$ be as in Definition 4.4. Let C_r be as in Lemma 4.1. Then for all $n > N(r) := \max\left\{N_L(r), N_U(r)\right\}$,*

$$\log\left(1+C_r\left(\frac{1}{\sqrt{n}}\right)^{2r-1} - L(r)\left(\frac{1}{\sqrt{n}}\right)^{2r}\right) < (-1)^{r-1}\Delta^r \log p(n) < \log\left(1+C_r\left(\frac{1}{\sqrt{n}}\right)^{2r-1}\right). \tag{4.16}$$

Proof. Lemmas 4.1 and 4.3 together imply (4.16). □

Theorem 4.7. *For all $r \geq 2$,*

$$(-1)^{r-1}\Delta^r \log p(n) \underset{n\to\infty}{\sim} \sum_{u=2r-1}^{\infty} G_u\left(\frac{1}{\sqrt{n}}\right)^u, \tag{4.17}$$

with

$$\begin{aligned}
G_{2u} &= \left[\frac{(-1)^u}{u}\left\{{u \atop r}\right\} + \sum_{k=1}^{u-r} g_{2k}\binom{-k}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r! \quad \textit{for all } u \geq r, \\
G_{2u+1} &= \left[\pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}\left\{{u+1 \atop r}\right\} + \sum_{k=0}^{u-r} g_{2k+1}\binom{-k-1/2}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r! \quad \textit{for all } u \geq r-1.
\end{aligned} \tag{4.18}$$

Proof. Following (4.2) and letting $w \to \infty$, we obtain

$$(-1)^{r-1}\Delta^r \log p(n) \underset{n\to\infty}{\sim} \sum_{u=2r-1}^{\infty}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\widetilde{g}_{u,i}\Big(\frac{1}{\sqrt{n}}\Big)^u. \tag{4.19}$$

For all $u \geq 2r-1$ and $u \equiv 0 \pmod 2$, we get

$$\begin{aligned}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\widetilde{g}_{2u,i} &= \left[\frac{(-1)^u}{u}\left\{{u \atop r}\right\} + \sum_{k=1}^{u-1} g_{2k}\binom{-k}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r!\\ &= \left[\frac{(-1)^u}{u}\left\{{u \atop r}\right\} + \sum_{k=1}^{u-r} g_{2k}\binom{-k}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r!.\end{aligned}$$

Similarly, for all $u \geq 2r-1$ and $u \equiv 1 \pmod 2$, it follows that

$$\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\widetilde{g}_{2u+1,i} = \left[\pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}\left\{{u+1 \atop r}\right\} + \sum_{k=0}^{u-r} g_{2k+1}\binom{-k-1/2}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r!.$$

□

5. A framework to verify multiplicative inequalities for $p(n)$

Here we list down the steps in order make a decision whether a given multiplicative inequality holds or not.

- (Step 0): Given $\prod_{i=1}^{T} p(n+s_i)$ and $\prod_{i=1}^{T} p(n+r_i)$ with $T \geq 1$. Without loss of generality, assume that s_i, r_i are non-negative integers for all $1 \leq i \leq T$. Transform the products into additive ones by applying the natural logarithm; i.e., $\sum_{i=1}^{T}\log p(n+s_i)$ and $\sum_{i=1}^{T}\log p(n+r_i)$.
- (Step 1): Choose $w = m+1$, where $(s_1,\ldots,s_T) \overset{m}{=} (r_1,\ldots,r_T)$. From (3.35), we observe that for each $1 \leq i \leq T$, $\log p(n+s_i)$ and $\log p(n+r_i)$ has the main term $P_{n,\vec{\mathbf{s}}}(w-1)$ and $P_{n,\vec{\mathbf{r}}}(w-1)$ respectively. Consequently, each of these main terms are dominated by $T\cdot c\sum_{i=1}^{T}\sqrt{n+s_i}$ and $T\cdot c\sum_{i=1}^{T}\sqrt{n+r_i}$ with $c = \pi\sqrt{2/3}$ respectively. Therefore, in order to choose w, it is enough to compute the Taylor expansion of $\sum_{i=1}^{T}\big(\sqrt{n+s_i}-\sqrt{n+s_i}\big)$ which is given by:
$$\sum_{i=1}^{T}\big(\sqrt{n+s_i}-\sqrt{n+s_i}\big) = \sum_{m=1}^{\infty}\frac{\binom{1/2}{m}}{\sqrt{n}^{2m-1}}\sum_{i=1}^{T}(s_i^m - r_i^m). \tag{5.1}$$
So our optimal choice is such minimal $m \geq 1$ so that $\sum_{i=1}^{T}(s_i^m - r_i^m) \neq 0$.
- (Step 2): Applying $w = m+1$ as in the previous step to Theorem 3.35, it remains to verify whether
$$P_{n,\vec{\mathbf{s}}}(m) - E^{\mathcal{L}}_{n,\vec{\mathbf{s}}}(m+1) > P_{n,\vec{\mathbf{r}}}(m) + E^{\mathcal{U}}_{n,\vec{\mathbf{r}}}(m+1) \tag{5.2}$$
or
$$P_{n,\vec{\mathbf{r}}}(m) - E^{\mathcal{L}}_{n,\vec{\mathbf{r}}}(m+1) > P_{n,\vec{\mathbf{s}}}(m) + E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(m+1), \tag{5.3}$$

in order to decide whether $\sum_{i=1}^{T} \log p(n+s_i) \geq \sum_{i=1}^{T} \log p(n+r_i)$ or $\sum_{i=1}^{T} \log p(n+r_i) \geq \sum_{i=1}^{T} \log p(n+s_i)$ respectively.

6. Inequalities for $p(n;\vec{\mathbf{s}})$

Definition 6.1. *Let $\widetilde{g}_{u,\vec{\mathbf{s}}}$ be as in Definition 3.11, and $\vec{\mathbf{s}}$ be as in Definition 3.10. For $n, T, U \in \mathbb{Z}_{\geq 1}$, define*

$$\mathcal{M}(n;T) := \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{T},$$

and

$$\widetilde{P}_{n,\vec{\mathbf{s}}}(U) := \exp\left(\sum_{u=1}^{U} \widetilde{g}_{u,\vec{\mathbf{s}}}\Big(\frac{1}{\sqrt{n}}\Big)^{u}\right).$$

Definition 6.2. *Let γ_1, γ_2 be as in* (2.10) *and $\vec{\mathbf{s}}$ be as in Definition 3.10. For each $\{s_i\}_{1\leq i\leq T}$, δ_{s_i} be as in Definition 3.2. For $n, T \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s_i$, we define*

$$C_{\mathcal{U}}(w;\vec{\mathbf{s}}) := \left(45\sum_{i=1}^{T}\Big(s_i+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_{s_i} + \frac{T\cdot\gamma_2}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}$$

and

$$C_{\mathcal{L}}(w;\vec{\mathbf{s}}) := \left(45\sum_{i=1}^{T}\Big(s_i+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_{s_i} + \frac{T\cdot\gamma_1}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}.$$

Lemma 6.3. *Let $\log p(n;\vec{\mathbf{s}})$ be as in Definition 3.10, and let $g(k)$ be as in Definition 2.5. Let $\mathcal{M}(n;T)$ and $\widetilde{P}_{n,\vec{\mathbf{s}}}(U)$ be as in Definition 6.1. Let $g(w;\vec{\mathbf{s}})$ be as in Theorem 3.13, and $C_{\mathcal{L}}(w;\vec{\mathbf{s}}), C_{\mathcal{U}}(w;\vec{\mathbf{s}})$ be as in Definition 6.2. If $n, T \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and*

$$n > \max\left\{g(w;\vec{\mathbf{s}}), \Big(C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big)^{2/w}, \Big(C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big)^{2/w}\right\} := N_1(w;\vec{\mathbf{s}}),$$

then

$$\mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)\left(1-C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right) < p(n;\vec{\mathbf{s}}) < \mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)\left(1+2\,C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right). \tag{6.1}$$

Proof. Applying the exponential function on both side of the inequality (3.35), we get for all $n > g(w;\vec{\mathbf{s}})$,

$$\mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)e^{-E^{\mathcal{L}}_{n,\vec{\mathbf{s}}}(w)} < p(n;\vec{\mathbf{s}}) < \mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)e^{E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w)}. \tag{6.2}$$

Now for all $n > \max\left\{\Big(C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big)^{2/w}, C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big)^{2/w}\right\}$, it follows that

$$0 < E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w) < 1 \quad\text{and}\quad 0 < E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w) < 1. \tag{6.3}$$

For all $0 < x < 1$, we know that $e^{x} < 1+2x$ and $e^{-x} > 1-x$. Therefore from (6.3) and following Definition 3.12, we finally have

$$e^{E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w)} < 1 + 2\,C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w} \quad\text{and}\quad e^{-E^{\mathcal{L}}_{n,\vec{\mathbf{s}}}(w)} > 1 - C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}. \tag{6.4}$$

Equations (6.2) and (6.4) together imply (6.1). □

Definition 6.4. *For $k \in \mathbb{Z}_{\geq 0}$, $w \geq 2$, and $\vec{\ell} := (\ell_1, \ldots, \ell_{w-1})$, define*

$$X(k) := \Big\{ \vec{\ell} \in \mathbb{Z}_{\geq 0}^{w-1} : \sum_{u=1}^{w-1} \ell_u = k \Big\},$$

$$X_{\mathcal{M}}(k) := \Big\{ \vec{\ell} \in X(k) : 0 \leq \sum_{u=1}^{w-1} u\ell_u \leq w-1 \Big\},$$

and

$$X_{\mathcal{E}}(k) := \Big\{ \vec{\ell} \in X(k) : \sum_{u=1}^{w-1} u\ell_u \geq w \Big\}.$$

Definition 6.5. *Let $X(k)$ and $X_{\mathcal{M}}(k)$ be as in Definition 6.4 and $\widetilde{g}_{u,\vec{s}}$ be as in Definition 3.11. Then for all $w \geq 2$, define*

$$\widehat{P}_{n,\vec{s}}(w-1) := \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{M}}(k)} F(k; w; \vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u},$$

and

$$\widehat{E}_{n,\vec{s}}(w-1) := \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{E}}(k)} F(k; w; \vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u}.$$

where

$$F(k; w; \vec{\boldsymbol{s}}) := \binom{k}{\ell_1, \ldots, \ell_{w-1}} \prod_{u=1}^{w-1} \Big(\widetilde{g}_{u,\vec{s}}\Big)^{\ell_u},$$

with $\binom{k}{\ell_1,\ldots,\ell_{w-1}} = \frac{k!}{\ell_1! \cdots \ell_{w-1}!}$ *is a multinomial coefficient.*

Definition 6.6. *Let $X_{\mathcal{E}}(k)$ be as in Definition 6.4 and $F(k; w; \vec{\boldsymbol{s}})$ be as in Definition 6.5 and $\widetilde{g}_{u,\vec{s}}$ be as in Definition 3.11. For $w \geq 2$, define*

$$E(w; \vec{\boldsymbol{s}}) := \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{E}}(k)} \Big| F(k; w; \vec{\boldsymbol{s}}) \Big| + 3\Big(\big|\widetilde{g}_{1,\vec{s}}\big| + 1 \Big)^{w}.$$

Lemma 6.7. *Let $\widetilde{P}_{n,\vec{s}}(U)$ be as in Definition 6.1 and $X_{\mathcal{E}}(k)$ be as in Definition 6.4. Let $\widehat{P}_{n,\vec{s}}(w-1), \widehat{P}_{n,\vec{s}}(w-1)$, and $F(k; w; \vec{\boldsymbol{s}})$ be as in Definition 6.5. Let $E(w; \vec{\boldsymbol{s}})$ be as in Definition 6.6. Then for all $w \geq 2$ and*

$$n > \max_{1 \leq u \leq w-1} \Big\{ \Big((w-1)\big|\widetilde{g}_{u,\vec{s}}\big| \Big)^{2/u} \Big\} := N_2(w; \vec{\boldsymbol{s}}),$$

we have

$$\Big| \widetilde{P}_{n,\vec{s}}(w-1) - \widehat{P}_{n,\vec{s}}(w-1) \Big| < E(w; \vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}}\Big)^{w}. \tag{6.5}$$

Proof. Expanding $\widetilde{P}_{n,\vec{s}}(w-1)$ and splitting it as follows:

$$\widetilde{P}_{n,\vec{s}}(w-1) \;=\; \widehat{P}_{n,\vec{s}}(w-1) + \widehat{E}_{n,\vec{s}}(w-1) + \sum_{k=w}^{\infty} \frac{1}{k!} \sum_{\vec{\ell} \in X(k)} F(k; w; \vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u}$$

$$= \widehat{P}_{n,\vec{\mathbf{s}}}(w-1) + \widehat{E}_{n,\vec{\mathbf{s}}}(w-1) + \sum_{k=w}^{\infty} \frac{1}{k!}\left(\sum_{u=1}^{w-1} \frac{\widetilde{g}_{u,\vec{\mathbf{s}}}}{\sqrt{n}^{u}}\right)^{k}. \tag{6.6}$$

Therefore

$$\begin{aligned}
&\left|\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1) - \widehat{P}_{n,\vec{\mathbf{s}}}(w-1)\right| \\
&\le \left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| + \left(\sum_{u=1}^{w-1} \frac{\left|\widetilde{g}_{u,\vec{\mathbf{s}}}\right|}{\sqrt{n}^{u}}\right)^{w} \sum_{k=0}^{\infty} \frac{1}{(k+w)!}\left(\sum_{u=1}^{w-1} \frac{\left|\widetilde{g}_{u,\vec{\mathbf{s}}}\right|}{\sqrt{n}^{u}}\right)^{k} \\
&= \left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| + \left(\frac{1}{\sqrt{n}}\right)^{w}\left(\left|\widetilde{g}_{1,\vec{\mathbf{s}}}\right| + \sum_{u=1}^{w-2} \frac{\left|\widetilde{g}_{u+1,\vec{\mathbf{s}}}\right|}{\sqrt{n}^{u}}\right)^{w} \sum_{k=0}^{\infty} \frac{1}{(k+w)!}\left(\sum_{u=1}^{w-1} \frac{\left|\widetilde{g}_{u,\vec{\mathbf{s}}}\right|}{\sqrt{n}^{u}}\right)^{k} \\
&< \left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| + \left(\frac{1}{\sqrt{n}}\right)^{w}\left(\left|\widetilde{g}_{1,\vec{\mathbf{s}}}\right| + 1\right)^{w} \sum_{k=0}^{\infty} \frac{1}{(k+w)!} \quad \Big(\text{since } \ n > N_2(w;\vec{\mathbf{s}})\Big) \\
&\le \left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| + \frac{\left(\left|\widetilde{g}_{1,\vec{\mathbf{s}}}\right| + 1\right)^{w}}{w!}\left(\frac{1}{\sqrt{n}}\right)^{w} \sum_{k=0}^{\infty} \frac{1}{k!} < \left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| + 3\frac{\left(\left|\widetilde{g}_{1,\vec{\mathbf{s}}}\right| + 1\right)^{w}}{w!}\left(\frac{1}{\sqrt{n}}\right)^{w}.
\end{aligned} \tag{6.7}$$

Now

$$\begin{aligned}
\left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| &\le \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{E}}(k)} \left|F(k;w;\vec{\mathbf{s}})\right| \left(\frac{1}{\sqrt{n}}\right)^{\sum_{u=1}^{w-1} u\ell_u} \\
&\le \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{E}}(k)} \left|F(k;w;\vec{\mathbf{s}})\right| \left(\frac{1}{\sqrt{n}}\right)^{w} \quad \Big(\text{since } \ \vec{\ell} \in X_{\mathcal{E}}(k)\Big).
\end{aligned} \tag{6.8}$$

Combining (6.7) and (6.8), we get (6.5). □

Definition 6.8. *Let $C_{\mathcal{U}}(w;\vec{\boldsymbol{s}})$ and $C_{\mathcal{L}}(w;\vec{\boldsymbol{s}})$ be as in Definition 6.2. Let $E(w;\vec{\boldsymbol{s}})$ be as in Definition 6.6. Then for all $w \ge 2$, define*

$$E_L(w;\vec{\boldsymbol{s}}) := 3\ C_{\mathcal{L}}(w;\vec{\boldsymbol{s}}) + E(w;\vec{\boldsymbol{s}}),$$

and

$$E_U(w;\vec{\boldsymbol{s}}) := 6\ C_{\mathcal{U}}(w;\vec{\boldsymbol{s}}) + E(w;\vec{\boldsymbol{s}})\Big(2\ C_{\mathcal{U}}(w;\vec{\boldsymbol{s}}) + 1\Big).$$

Theorem 6.9. *Let $\mathcal{M}(n;T)$ be as in Definition 6.1 and $\widehat{P}_{n,\vec{\mathbf{s}}}(w-1)$ be as in Definition 6.5. Let $E^L_{n,\vec{\mathbf{s}}}(w)$ and $E^U_{n,\vec{\mathbf{s}}}(w)$ be as in Definition 6.8. Let $N_1(w;\vec{\boldsymbol{s}})$ and $N_2(w;\vec{\boldsymbol{s}})$ be as in Lemmas 6.3 and 6.7. Then for all $w \ge 2$ and*

$$N > \max\Big\{N_1(w;\vec{\boldsymbol{s}}), N_2(w;\vec{\boldsymbol{s}})\Big\} := N(w;\vec{\boldsymbol{s}}),$$

we have

$$\mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{\mathbf{s}}}(w-1) - E_L(w;\vec{\boldsymbol{s}})\left(\frac{1}{\sqrt{n}}\right)^{w}\right) < p(n;\vec{\boldsymbol{s}}) < \mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{\mathbf{s}}}(w-1) + E_U(w;\vec{\boldsymbol{s}})\left(\frac{1}{\sqrt{n}}\right)^{w}\right). \tag{6.9}$$

Proof. From Lemmas 6.3 and 6.7, for $n > N(w;\vec{\mathbf{s}})$, it follows that

$$p(n;\vec{\mathbf{s}}) < \mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{\mathbf{s}}}(w-1) + E(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right)\left(1 + 2\ C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right), \tag{6.10}$$

and

$$p(n;\vec{\mathbf{s}}) > \mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{\mathbf{s}}}(w-1) - E(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right)\left(1 -\ C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right). \tag{6.11}$$

Now

$$\begin{aligned}
\Big|\widehat{P}_{n,\vec{\mathbf{s}}}(w-1)\Big| &= \left|\sum_{k=0}^{w-1}\frac{1}{k!}\sum_{\vec{\ell}\in X_{\mathcal{M}}(k)} F(k;w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u}\right| \\
&\le \left|\sum_{k=0}^{w-1}\frac{1}{k!}\sum_{\vec{\ell}\in X(k)} F(k;w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u}\right| \quad \Big(\text{as } \ X_{\mathcal{M}}(k)\subseteq X(k)\Big) \\
&= \left|\sum_{k=0}^{w-1}\frac{1}{k!}\left(\sum_{u=1}^{w-1}\frac{\widetilde{g}_{u,\vec{\mathbf{s}}}}{\sqrt{n}^{u}}\right)^{k}\right| \le \sum_{k=0}^{w-1}\frac{1}{k!}\left(\sum_{u=1}^{w-1}\frac{\big|\widetilde{g}_{u,\vec{\mathbf{s}}}\big|}{\sqrt{n}^{u}}\right)^{k} < \sum_{k=0}^{w-1}\frac{1}{k!} \quad \Big(\text{as } n > N_2(w;\vec{\mathbf{s}})\Big) \\
&< 3.
\end{aligned} \tag{6.12}$$

Applying (6.12) to (6.10), we arrive at the upper bound of (6.9). We get the lower bound of (6.9) by applying (6.12) to (6.11) and from the fact that $C_{\mathcal{L}}(w;\vec{\mathbf{s}})\cdot E(w;\vec{\mathbf{s}}) > 0$ for all $w \ge 2$. $\square$

7. Conclusion

We conclude this paper by pointing out the following aspects in which Theorem 6.9 remains incomplete.

(1) Suppose we are given the following two functions defined by shifts of $p(n)$:

$$SP(n;S) := \sum_{j=1}^{M}\prod_{i=1}^{T} p(n+s_{i,j}) \ \text{ and } \ SP(n;R) := \sum_{j=1}^{M}\prod_{i=1}^{T} p(n+r_{i,j}),$$

where $S = (s_{i,j})_{1\le i\le T, 1\le j\le M}$ and $R = (r_{i,j})_{1\le i\le T, 1\le j\le M}$. Now in order to decide whether $SP(n;S) \ge SP(n;R)$ for all $n \ge N(S,R)$, we need to estimate $\prod_{i=1}^{T} p(n+s_{i,j})$ and $\prod_{i=1}^{T} p(n+r_{i,j})$ individually for each $1 \le j \le M$. In view of Theorem 6.9, estimation of two factors come into the prominence: computation of the term $\sum_{j=1}^{M}\Big(\widehat{P}_{n,\vec{\mathbf{s}}_j}(w-1) - \widehat{P}_{n,\vec{\mathbf{r}}_j}(w-1)\Big)$ with $\vec{\mathbf{s}}_j := (s_{1,j},\dots,s_{T,j}), \vec{\mathbf{r}}_j := (r_{1,j},\dots,r_{T,j})$, and approximation of the error term.

(2) Depending on the truncation point w, one can compute the main term $\sum_{j=1}^{M}\Big(\widehat{P}_{n,\vec{\mathbf{s}}_j}(w-1) - \widehat{P}_{n,\vec{\mathbf{r}}_j}(w-1)\Big)$. But computational complexity will arise in the estimation of the error term because in order to approximate $\widehat{E}(w;\vec{\mathbf{s}}_j)$ for each j, one needs to have a good control over $X_{\mathcal{E}}(k)$ for $0 \le k \le w-1$. This seems to be difficult as w tends to infinity, growth of $|X_{\mathcal{E}}(k)|$ is exponential.

(3) For example, in order to prove the higher order Turán inequality for $p(n)$, the minimal choice for w is 10 and consequently, by Theorem 6.9 with appropriate choices for $\vec{\mathbf{s}}$, it follows that

$$4\big(1-u_{n-1}\big)\big(1-u_n\big)-\big(1-u_nu_{n-1}\big)^2=\frac{\pi^3}{12\sqrt{6}}\frac{1}{n^{9/2}}+O\Big(\frac{1}{n^5}\Big).$$

This concludes that $p(n)$ satisfies the higher order Turán inequality for sufficiently large n although due to Chen, Jia, and Wang [4], we know that the inequality holds for all $n \geq 95$. So, from the aspect of error bound computation in order to confirm such inequalities from a certain explicit point onward, our method is inaccessible.

(4) Last, but not the least, the above discussions naively suggest that for making a decision whether a given inequalities for the partition function (of the above types) holds or not, we need to have a full asymptotic expansion for shifted value of the partition function and an explicit computation of the error bound after truncation the expansion at any positive integer w.

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Research Institute for Symbolic Computation, Johannes Kepler University, Altenberger Strasse 69, A-4040 Linz, Austria

Email address: Koustav.Banerjee@risc.uni-linz.ac.at