NEW INEQUALITIES FOR $p(n)$ AND $\log p(n)$

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Abstract. Let $p(n)$ denote the number of partitions of n. A new infinite family of inequalities for $p(n)$ is presented. This generalizes a result by William Chen et al. From this infinite family, another infinite family of inequalities for $\log p(n)$ is derived. As an application of the latter family one, for instance, obtains that for $n \geq 120$,

$$p(n)^2 > \Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\Big)p(n-1)p(n+1).$$

1. Introduction

We denote by $p(n)$ the number of partitions of n. The first 50 values of $p(n)$ starting from $n = 0$ read as follows,

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1,1,2,3,5,7,11,15,22,30,42,56,77,101,135,176,231,297,385,490,
627,792,1002,1255,1575,1958,2436,3010,3718,4565,5604,6842,
8349,10143,12310,14883,17977,21637,26015,31185,37338,44583,
53174,63261,75175,89134,105558,124754,147273,173525.
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A well-known asymptotic formula for $p(n)$ was found by G.H. Hardy and Srinivasa Ramanujan [10] in 1918 and independently by James Victor Uspensky in 1920 [19]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{1.1}$$

An elementary proof of (1.1) was given by Paul Erdős [8] in 1942. At MICA 2016 (Milestones in Computer Algebra) held in Waterloo in July 2016, Zhenbing Zeng et al. [18] reported that using numerical analysis they found a better asymptotic formula[1] for $p(n)$ by searching for constants $C_{i,j}$ to fit the following formula,

$$\log p(n) = \pi\sqrt{\frac{2}{3}}\sqrt{n} - \log(n) - 4\log\sqrt{3} + \frac{C_{0,-1}}{\log n} + \frac{C_{1,0}}{\sqrt{n}}$$

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[1] In the literature, the Hardy-Ramanujan-Rademacher is also called an asymptotic formula/approximation. However, it is built by an expression of substantially more complicated type. For example, the log concavity of $p(n)$ follows nontrivially from it, as shown in the work of DeSalvo and Pak [7].

$$+\frac{C_{1,-1}}{\sqrt{n}\log(n)}+\frac{C_{2,1}\log(n)}{n}+\frac{C_{2,0}}{n}+\dots. \tag{1.2}$$

By substituting for $n = 2^{10}, 2^{11}, \dots, 2^{20}$ into (1.2) they obtained,

$$C_{0,-1}=0, \quad C_{1,0}=-0.4432\dots, \quad C_{1,-1}=0, \quad C_{2,1}=0, \quad C_{2,0}=-0.0343\dots.$$

The OEIS [14] for A000041 shows that a similarly refined asymptotic formula for $p(n)$ was discovered by Jon E. Schoenfield in 2014,

$$p(n)\sim\frac{1}{4n\sqrt{3}}e^{\pi\cdot\left(\frac{2n}{3}+c_0+\frac{c_1}{\sqrt{n}}+\frac{c_2}{n}+\frac{c_3}{n\sqrt{n}}+\frac{c_4}{n^2}+\dots\right)^{\frac{1}{2}}}, \tag{1.3}$$

where the coefficients are approximately

$$c_0=-0.230420\dots, \quad c_1=-0.017841\dots, \quad c_2=0.005132\dots,$$

$$c_3=-0.001112\dots, \quad c_4=0.000957\dots,$$

Later Vaclav Kotesovec according to OEIS [14] for A000041 got the precise value of $c_0, c_1, \dots, c_4$ as follows:

$$c_0=-\frac{1}{36}-\frac{2}{\pi^2}, \quad c_1=\frac{1}{6\sqrt{6}\pi}-\frac{\sqrt{6}}{2\pi^3}, \quad c_2=\frac{1}{2\pi^4},$$

$$c_3=-\frac{5}{16\sqrt{6}\pi^3}+\frac{3\sqrt{6}}{8\pi^5}, \quad c_4=\frac{1}{576\pi^2}-\frac{1}{24\pi^4}+\frac{93}{80\pi^6}.$$

To the best of our knowledge, the details of the methods of Schoenfield and Kotesovec have not yet been published.

In this article, using symbolic-numeric computation, we present our method to derive (1.2) together with a closed form formula for the $C_{i,j}$ in (1.2). Namely we show that

$$\log p(n)\sim\pi\sqrt{\frac{2n}{3}}-\log(n)-4\log\sqrt{3}+\sum_{u=1}^{\infty}\frac{g_u}{\sqrt{n}^u},$$

where the g_u are as in Definition 5.1. In particular $C_{i,j}=0$, if $j\neq 0$, and $C_{i,0}=g_i$, otherwise. This result is obtained as a consequence of an infinite family of inequalities for $\log p(n)$, Theorem 6.6 (main theorem). We also apply our method to conjecture an analogous formula to (1.2) for $a(n)$, the cubic partitions of n, with $a(n)$ given by

$$\sum_{n=0}^{\infty}a(n)q^n=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-q^{2n})}. \tag{1.4}$$

In the OEIS, this sequence is registered as A002513. The first 50 values of $a(n)$, $n\geq 0$, are

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1,1,3,4,9,12,23,31,54,73,118,159,246,329,489,651,940,
1242,1751,2298,3177,4142,5630,7293,9776,12584,16659,
21320,27922,35532,46092,58342,75039,94503,120615,
151173,191611,239060,301086,374026,468342,579408.
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This sequence appears in a letter from Richard Guy to Morris Newman [9]. In [5], William Chen and Bernard Lin proved that the sequence $a(n)$ satisfies several congruence properties. For example, $a(3n+2) \equiv 0 \pmod 3$, $a(25n+22) \equiv 0 \pmod 5$. An asymptotic formula for $a(n)$ was obtained by Kotesovec [11] in 2015 as follows:

$$a(n) \sim \frac{e^{\pi\sqrt{n}}}{8n^{5/4}}. \tag{1.5}$$

In [20] the fourth author investigated the combinatorial properties of the sequence $a(n)$ by using Maple.

We summarize some of our main results:

Theorem 1.1. *For the usual partition function $p(n)$ we have*

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log(4\sqrt{3}) - \frac{0.44\ldots}{\sqrt{n}},\ n \to \infty. \tag{1.6}$$

The proof of this theorem will be given in Section 6.

Conjecture 1.2. *For the cubic partitions $a(n)$ we have*

$$\log a(n) \sim \pi\sqrt{n} - \frac{5}{4}\log n - \log 8 - \frac{0.79\ldots}{\sqrt{n}},\ n \to \infty. \tag{1.7}$$

Theorem 1.3. *For the partition numbers $p(n)$ we have the inequalities*

$$\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1 - \frac{1}{2\sqrt{n}}\Big) < p(n) < \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1 - \frac{1}{3\sqrt{n}}\Big),\ n \geq 1.$$

The proof of this is given in Section 3.

This paper is organized as follows. In Section 2 we present the methods used in the mathematical experiments that led us Theorem 1.1 and Conjecture 1.2. In Section 3 we prove Theorem 1.3 by adapting methods used by Chen et al. to fit our purpose. In Section 4 we generalize an inequality by Chen et al. by extending it to an infinite family of inequalities for $p(n)$. In Section 5 we introduce preparatory results required to prove Theorem 6.6. In Section 6 we prove our main result, Theorem 6.6, by using the main result from Section 4, Theorem 4.4. This gives an infinite family of inequalities for $\log p(n)$. Finally in Section 7 we give an application of the results in Section 5 which extends DeSalvo's and Pak's log concavity theorem for $p(n)$. In Section 8 (the Appendix) we give additional information on the method used to discover the asymptotic formulas. We remark explicitly that to finalize the proof of Theorem 6.6, we use the Cylindrical Algebraic Decomposition in Mathematica; the details of this are also put to Section 8.

2. Mathematical Experiments for Better Asymptotics for $a(n)$ and $p(n)$

Before proving our theorems, in this section we briefly describe the experimental mathematics which led us to their discovery. Our strategy is as follows. If we have sufficiently many instances of a given sequence, how can we find an asymptotic formula for this sequence? Take the cubic partitions $a(n)$ and the partition numbers $p(n)$ as examples.

We have

$p(10) = 42, \ldots, p(100) = 190569292, \ \ldots, p(1000) = 24061467864032622473692149727991,$
$a(10) = 118, \ldots, a(100) = 16088094127, \ldots,$
$a(1000) = 302978131076521633719614157876165279276.$

A plot of the two curves through the points $(n, a(n))$, resp. $(n, p(n))$, for $n \in \{1, \ldots, 1000\}$ is shown in the Fig. 1(a) and 1(b). According to the Hardy-Ramanujan Theorem 1.1 and the asymptotic formula of Kotesovec (1.5), the curves are increasing with "sub-exponential" speeds. Thus, we may plot two curves using data points $(n, \log a(n))$ and $(n, \log p(n))$ as in Fig. 1(c). One observes that the new curves look like parabolas $y = \sqrt{x}$. This is also very natural in view of,

$$\log p(n) \sim \sqrt{\frac{2}{3}}\pi \cdot \sqrt{n} - \log(n) - \log(4\sqrt{3}),$$
$$\log a(n) \sim \pi \cdot \sqrt{n} - \frac{5}{4} \cdot \log(n) - \log(8). \tag{2.1}$$

So if we modify further with $(\sqrt{n}, \log a(n))$ and $(\sqrt{n}, \log p(n))$ to plot the curves, we get two almost-straight lines as shown in the Fig. 1(d).

This provides the starting point for finding the improved asymptotic formulas (1.6) for $p(n)$ and (1.7) for $a(n)$ from their data sets. We restrict our description to the latter case. Motivated by (2.1), we compute the differences of $\log(a(n))$ with the estimation values $a_e(n) := \frac{e^{\pi\sqrt{n}}}{8n^{5/4}}$:

$$\Delta(n) := \log(a_e(n)) - \log(a(n)) = \pi\sqrt{n} - \frac{5}{4}\log(n) - \log 8 - \log(a(n)).$$

Then we can plot curves from the data points $(n, \Delta(n))$ in Fig. 2(a) and 2(b), and $(n, n\cdot\Delta(n))$ and $(n, \sqrt{n} \cdot \Delta(n))$ in Fig. 2(c) and 2(d), in order to confirm the next dominant term approximately. We can see in Fig. 2(d) that after multiplying $\Delta(n)$ by $\sqrt{n}$ the curve is almost constant, confirming that the next term is $\frac{C}{\sqrt{n}}$. Also multiplying $\Delta(n)$ by n, in Fig. 2(c) we see that the behaviour is like $\sqrt{n}$ as expected. By using least square regression on the original data set $(n, a(n))$ for $1 \leq n \leq 10000$, we aimed at finding the best constant C that minimizes [2]

$$-\log a(n) + \alpha \cdot \sqrt{n} - \beta \cdot \log(n) - \log(\gamma) + \frac{C}{\sqrt{n}},$$

[2]The fourth author of this paper told the result to V. Kotesovec in May 2016 and got a reply in January 2017 that the precise value of C could be `Pi/16+15/(8*Pi)=0.7931...`

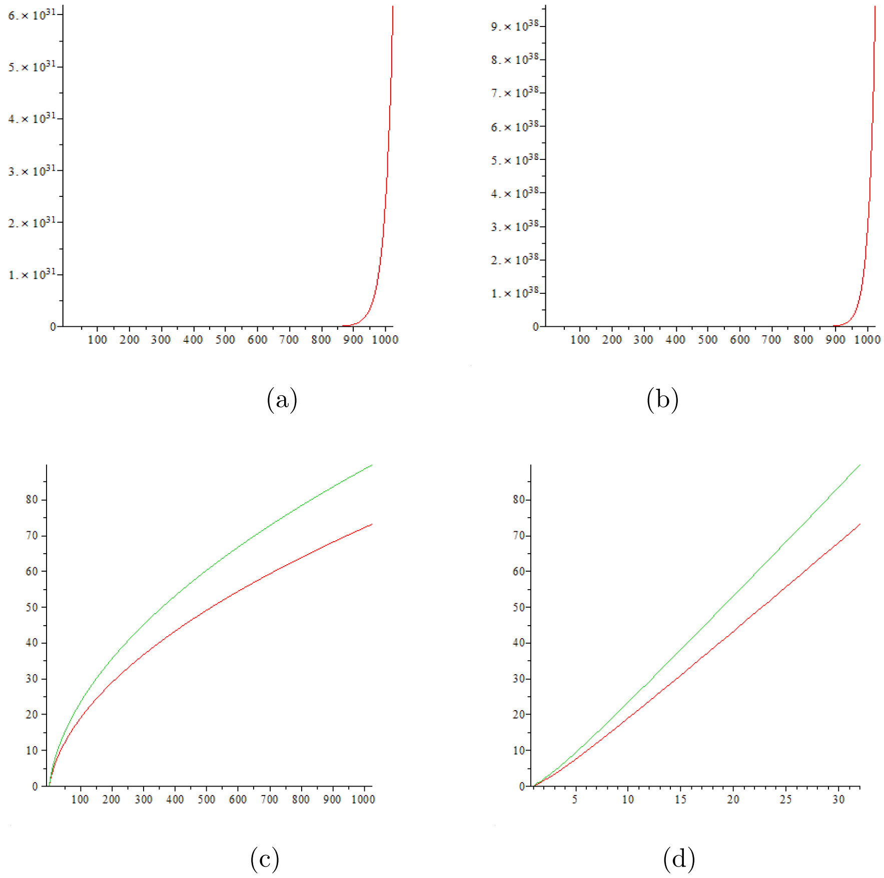

FIGURE 1. In (c) the upper curve is $\{(n, \log a(n)) | 1 \leq n \leq 1000\}$, and the lower curve is $\{(n, \log p(n)) | 1 \leq n \leq 1000\}$. The two curves are like the parabola $y = \sqrt{x}$. In (d) the two lines are for $\{(\sqrt{n}, \log a(n)) | 1 \leq n \leq 1000\}$ (upper) and $\{(\sqrt{n}, \log p(n)) | 1 \leq n \leq 1000\}$ (lower).

where we fixed $\alpha = \pi, \beta = 5/4, \gamma = 8$ according to (1.5). As a result, we obtained that $C \approx 0.7925$.

In the Appendix, Section 8, we explain that the constants α, β, γ can also be found via regression analysis with MAPLE instead of getting them from (1.5) directly.

3. PROOF OF THEOREM 1.3

We separate the proof into two lemmas. The first lemma is the upper bound for $p(n)$ and second lemma is the lower bound. In order to prove these lemmas we will state several facts which are routine to prove.

Lemma 3.1. *For all* $n \in \mathbb{Z}_{\geq 1}$,

$$p(n) < \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1 - \frac{1}{3\sqrt{n}}\Big).$$

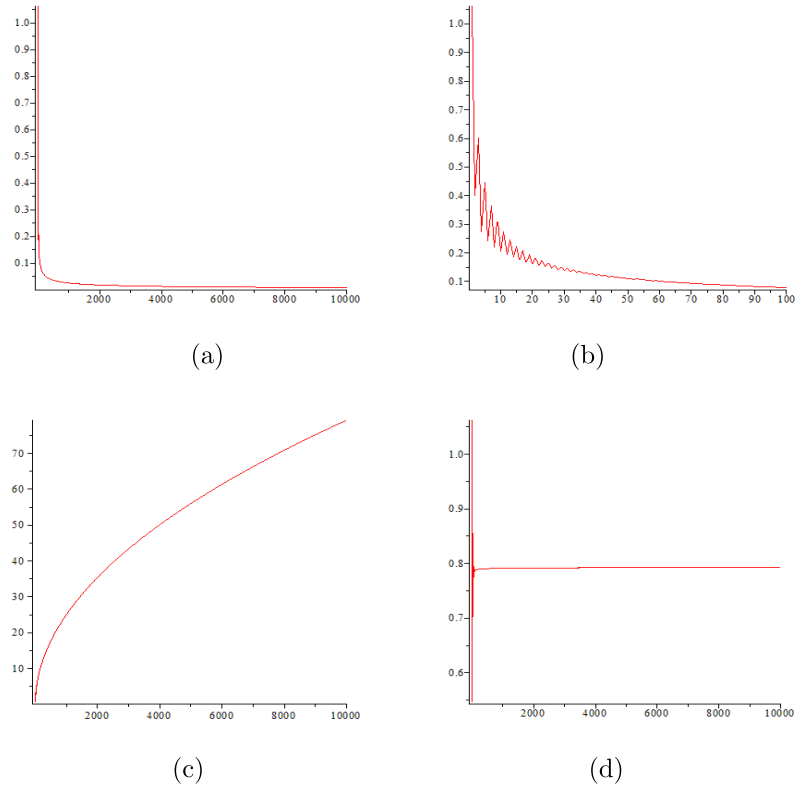

FIGURE 2. The curve in (a) is for $(n, \Delta(n))$ where $1 \le n \le 10000$, (b) is for $(n, \Delta(n)$ where $1 \le n \le 100$. The curve in (c) is for $n \cdot \Delta(n)$, and (d) is for $\sqrt{n} \cdot \Delta(n)$ where $1 \le n \le 10000$.

Proof. By [2, (2.7)-(2.8)] and with $A_k(n)$ and $R(n, N)$ as defined there, we have,

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \Big[\Big(1 - \frac{k}{\mu(n)}\Big) e^{\frac{\mu(n)}{k}} + \Big(1 + \frac{k}{\mu(n)}\Big) e^{-\frac{\mu(n)}{k}} \Big] + R(n, N), \ n \ge 1, \quad (3.1)$$

where

$$\mu(n) := \frac{\pi}{6}\sqrt{24n-1}.$$

We will exploit the case $N = 2$ together with $A_1(n) = 1$ and $A_2(n) = (-1)^n$ for any positive integer n. For $N \ge 1$, Lehmer [12, (4.14), p. 294] gave the following error bound:

$$|R(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \Big[\big(\frac{N}{\mu(n)}\big)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \big(\frac{N}{\mu(n)}\big)^2 \Big], \ n \ge 1, \quad (3.2)$$

and for $N = 2$ (cf. [2, (2.9)-(2.10)]):

$$p(n) = \frac{\sqrt{12} e^{\mu(n)}}{24n-1} \Big(1 - \frac{1}{\mu(n)} + T_1(n)\Big), \ n \ge 1, \quad (3.3)$$

where

$$T_1(n) := \frac{(-1)^n}{\sqrt{2}}\Big(\Big(1-\frac{2}{\mu(n)}\Big)e^{-\frac{\mu(n)}{2}} + \Big(1+\frac{2}{\mu(n)}\Big)e^{-\frac{3\mu(n)}{2}}\Big) + \Big(1+\frac{1}{\mu(n)}\Big)e^{-2\mu(n)} + \frac{(24n-1)R(n,2)}{\sqrt{12}e^{\mu(n)}}. \tag{3.4}$$

We first estimate the absolute value of $T_1(n)$; for convenience we denote subexpressions by a_1, b_1, c_1 and d_1:

$$|T_1(n)| \leq \underbrace{\frac{1}{\sqrt{2}}\Big(1-\frac{2}{\mu(n)}\Big)e^{-\frac{\mu(n)}{2}}}_{=:a_1} + \underbrace{\frac{1}{\sqrt{2}}\Big(1+\frac{2}{\mu(n)}\Big)e^{-\frac{3\mu(n)}{2}}}_{=:b_1} + \underbrace{\Big(1+\frac{1}{\mu(n)}\Big)e^{-2\mu(n)}}_{=:c_1} + \underbrace{\Big|\frac{(24n-1)R(n,2)}{\sqrt{12}e^{\mu(n)}}\Big|}_{=:d_1}.$$

The following facts are easily verified.

Fact A. For all $n \geq 1$, $a_1 < e^{-\frac{\mu(n)}{2}}$.

Fact B. For all $n \geq 1$, $b_1 < e^{-\frac{\mu(n)}{2}}$.

Fact C. For all $n \geq 1$, $c_1 < e^{-\frac{\mu(n)}{2}}$.

Now,

$$\begin{aligned} d_1 =& \frac{36}{\pi^2\sqrt{12}}\frac{\mu(n)^2}{e^{\mu(n)}}|R(n,2)| \\ <& \frac{\mu(n)^2e^{-\mu(n)}}{2^{2/3}} + \frac{12\sqrt[3]{2}e^{-\frac{\mu(n)}{2}}}{\mu(n)} - \frac{12\sqrt[3]{2}e^{-3\frac{\mu(n)}{2}}}{\mu(n)} - 12\sqrt[3]{2}e^{-\mu(n)} \quad \text{(by (3.2))} \\ <& \underbrace{\frac{\mu(n)^2e^{-\mu(n)}}{2^{2/3}}}_{=:d_1^*} + \underbrace{\frac{12\sqrt[3]{2}e^{-\frac{\mu(n)}{2}}}{\mu(n)}}_{=:d_2^*}. \end{aligned}$$

Fact D. For all $n \geq 7$, $d_1^* < e^{-\frac{\mu(n)}{2}}$.

Fact E. For all $n \geq 35$, $d_2^* < e^{-\frac{\mu(n)}{2}}$.

By Fact D and Fact E, we have

Fact F. $d_1 = d_1^* + d_2^* < 2e^{-\frac{\mu(n)}{2}}$ for all $n \geq 35$.

Now, by Fact A, B, C and Fact F we conclude that for all $n \geq 35$;

$$|T_1(n)| \leq a_1 + b_1 + c_1 + d_1 < 5e^{-\frac{\mu(n)}{2}}. \tag{3.5}$$

By (3.5), we have for all $n \geq 35$;

$$1 - \frac{1}{\mu(n)} - 5e^{-\frac{\mu(n)}{2}} < 1 - \frac{1}{\mu(n)} + T_1(n) < 1 - \frac{1}{\mu(n)} + 5e^{-\frac{\mu(n)}{2}}. \tag{3.6}$$

Fact G. For all $n \geq 3$, $1 - \frac{1}{\mu(n)} - 5e^{-\frac{\mu(n)}{2}} > 0$.

Therefore from (3.3) and Fact G, we have for all $n \geq 35$,

$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)} + T_1(n)\Big) < \underbrace{\frac{\sqrt{12}e^{\mu(n)}}{24n-1}}_{=:e_1}\underbrace{\Big(1 - \frac{1}{\mu(n)} + 5e^{-\frac{\mu(n)}{2}}\Big)}_{=:f_1}. \tag{3.7}$$

Fact H. $f_1 < 1 - \frac{1}{3\sqrt{n}}$ for all $n \geq 23$.

Fact I. $e_1 < \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$ for all $n \geq 1$.

Therefore by Fact H, I and (3.7) we have for all $n \geq 35$,

$$p(n) < \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}\Big(1 - \frac{1}{3\sqrt{n}}\Big).$$

This completes the proof of the stated upper bound in Lemma 3.1. □

Lemma 3.2. *For all* $n \in \mathbb{Z}_{\geq 1}$,

$$\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}\Big(1 - \frac{1}{2\sqrt{n}}\Big) < p(n). \tag{3.8}$$

Proof. By [7, Prop 2.4] for all $n \geq 1$,

$$p(n) > T_2(n)\Big(1 - \frac{|R(n)|}{T_2(n)}\Big), \tag{3.9}$$

where

$$T_2(n) := \frac{\sqrt{12}}{24n-1}\Big[\Big(1 - \frac{1}{\frac{\pi}{6}\sqrt{24n-1}}\Big)e^{\frac{\pi}{6}\sqrt{24n-1}} + \frac{(-1)^n}{\sqrt{2}}e^{\frac{\pi}{12}\sqrt{24n-1}}\Big]$$

and $R(n)$ is as in [7, (7)]. The exact expression of $R(n)$ will not be needed here but rather a bound on $|R(n)|$; see below.

From the definition of $T_2(n)$ one verifies:

Fact J. $T_2(n) > 0$ for all $n \in \mathbb{Z}_{\geq 1}$.

The following bound holds for $|R(n)|$ (see [7, (13)]),

$$0 < \frac{|R(n)|}{T_2(n)} < e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}},\ n \geq 2. \tag{3.10}$$

Hence by Fact J,

$$T_2(n)\Big(1 - \frac{|R(n)|}{T_2(n)}\Big) > T_2(n)\Big(1 - e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}\Big),\ n \geq 2. \tag{3.11}$$

Plugging the definition of $T_2(n)$ into (3.11) gives for $n \geq 2$,

$$p(n) > \frac{\sqrt{12}}{24n-1}\Big[\underbrace{\Big(1-\frac{1}{\frac{\pi}{6}\sqrt{24n-1}}\Big)}_{=:a_2}e^{\frac{\pi}{6}\sqrt{24n-1}} + \frac{(-1)^n}{\sqrt{2}}e^{\frac{\pi}{12}\sqrt{24n-1}}\Big]\underbrace{(1-e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}})}_{=:d_2}$$

$$> \frac{\sqrt{12}}{24n}e^{\pi\sqrt{\frac{2n}{3}}}\Big[a_2 \times \underbrace{e^{\frac{\pi}{6}\sqrt{24n-1}-\frac{\pi}{6}\sqrt{24n}}}_{=:b_2} + \underbrace{\frac{(-1)^n}{\sqrt{2}}e^{\frac{\pi}{12}\sqrt{24n-1}-\frac{\pi}{6}\sqrt{24n}}}_{=:c_2}\Big] \times d_2$$

$$= \frac{1}{4\sqrt{3}n}e^{\pi\sqrt{\frac{2n}{3}}}(a_2b_2+c_2)d_2.$$

Fact K. $a_2 > 1-\frac{2}{5\sqrt{n}} > 0$ for all $n \geq 1$.

Fact L. $b_2 > 1-\frac{2}{37\sqrt{n}} > 0$ for all $n \geq 1$.

Fact M. $c_2 > -\frac{1}{225\sqrt{n}}$ for all $n \geq 29$.

By Fact K, L and M we have,

Fact N. $a_2b_2+c_2 > (1-\frac{2}{5\sqrt{n}})(1-\frac{2}{37\sqrt{n}}) - \frac{1}{225\sqrt{n}} > 0$ for all $n \in \mathbb{Z}_{\geq 1}$.

Fact O. $d_2 > 1-\frac{1}{25\sqrt{n}} > 0$ for all $n \geq 631$.

From Fact N and O we have for all $n \geq 631$,

$$(a_2b_2+c_2)d_2 > \underbrace{\Big[\Big(1-\frac{2}{5\sqrt{n}}\Big)\Big(1-\frac{2}{37\sqrt{n}}\Big)-\frac{1}{225\sqrt{n}}\Big]\Big(1-\frac{1}{25\sqrt{n}}\Big)}_{=:I(n)}.$$

Fact P. $I(n) > 1-\frac{1}{2\sqrt{n}} > 0, \quad$ for all $n \geq 1$.

From all the above facts we can conclude that (3.8) holds for all $n \geq 631$. Using Mathematica we checked that (3.8) also holds for all $1 \leq n \leq 630$. This concludes the proof of Lemma 3.2. □

Finally, combining Lemma 3.1 and Lemma 3.2, we have Theorem 1.3.

4. A Generalization of a result by Chen, Jia and Wang

In this section we have again that $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$; this should not be confused with the real variable μ which we will use below. The main goal of this section is to generalize [2, Lem. 2.2] which says that for $n \geq 1206$:

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\Big).$$

Our improvement is Theorem 4.4 where we replace the 10 in this formula by k and the 1206 by a parametrized bound $g(k)$. In order to achieve this, for a fixed k one needs to find an explicit constant $\nu(k) \in \mathbb{R}$ such that $\frac{1}{6}e^{\mu/2} > \mu^k$ for all $\mu \in \mathbb{R}$ with $\mu > \nu(k)$. One can show that

$$\tilde{\nu}(k) := \min\Big\{h \in \mathbb{R} \big| \forall_{\mu\in\mathbb{R}} \Big(\mu > h \Rightarrow \frac{1}{6}e^{\mu/2} > \mu^k\Big)\Big\}$$

satisfies $\frac{1}{6}e^{\tilde{\nu}(k)/2} = \tilde{\nu}(k)^k$. Theorem 4.4 is crucial for proving our main result, Theorem 6.6, presented in the next section. In Lemma 4.1 we find such a constant $\nu(k)$ for all $k \geq 7$. In Lemma 4.2 we find a lower bound on $\tilde{\nu}(k)$. In this way, we see that what is delivered by Lemma 4.1, is best possible in the sense that our $\nu(k)$ from Lemma 4.1 and the minimal possible $\tilde{\nu}(k)$ satisfies $|\nu(k) - \tilde{\nu}(k)| < \frac{3k\log\log k}{\log k}$ for all $k \geq 7$.

Lemma 4.1. *For $k \in \mathbb{Z}_{\geq 7}$ let*

$$\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \frac{5k\log\log k}{\log k},$$

then

$$\frac{1}{6} \cdot e^{\nu(k)/2} > \nu(k)^k, \qquad k \geq 7. \tag{4.1}$$

Moreover, if $\mu > \nu(k)$ for some $k \geq 7$, then

$$\frac{1}{6} \cdot e^{\mu/2} > \mu^k, \qquad k \geq 7. \tag{4.2}$$

Proof. Let $f(\mu) := -\log 6 + \mu/2 - k\log\mu$. By $f'(\mu) = 1/2 - k/\mu$, f is increasing for $\mu > 2k$. Hence the fact $\nu(k) > 2k$ gives $f(\mu) > f(\nu(k))$, and (4.2) follows from (4.1) which is equivalent to $f(\nu(k)) > 0$, $k \geq 7$. Setting,

$$\tilde{\nu}(k) := -1 + \frac{\nu(k)}{2k\log k} = \frac{\log 6}{k\log k} + \frac{\log 2}{\log k} + \frac{\log\log k}{\log k} + \frac{5\log\log k}{2(\log k)^2},$$

the positivity of $f(\nu(k))$ is shown as follows:

$$\begin{aligned}
f(\nu(k)) &= -\log 6 + \nu(k)/2 - k\log(2k\log k) - k\log(1+\tilde{\nu}(k)) \\
&= \frac{5k\log\log k}{2\log k} - k\log(1+\tilde{\nu}(k)) \\
&> k\Big(\frac{5\log\log k}{2\log k} - \tilde{\nu}(k)\Big) \qquad (\text{by } \log(1+x) < x \ \text{ for } \ 0 < x) \\
&= \frac{k}{2\log k}\Big(3\log\log k - \frac{2\log 6}{k} - 2\log 2 - \frac{5\log\log k}{\log k}\Big) \\
&> \frac{k}{2\log k}\Big(3\log\log k - \frac{1}{5} - \frac{7}{5} - 2\Big) = \frac{k}{2\log k}\Big(3\log\log k - \frac{18}{5}\Big),
\end{aligned}$$

where the last inequality holds for all $k \geq 18$, because for such k:

$$\frac{2\log 6}{k} < \frac{1}{5}, \frac{5\log\log k}{\log k} < 2, \ \text{ and } \ 2\log 2 < \frac{7}{5}.$$

It is also straight-forward to prove $\log\log k > 6/5$ for all $k \geq 28$. For the the remaining cases $7 \leq k \leq 27$ the inequality (4.1) is verified by numerical computation, which completes the proof of Lemma 4.1. ∎

Lemma 4.2. *For $k \in \mathbb{Z}_{\geq 7}$ let*

$$\kappa(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \frac{2k\log\log k}{\log k},$$

then

$$\frac{1}{6}e^{\kappa(k)/2} < \kappa(k)^k.$$

Proof. Let f defined as in Lemma 4.1, then the statement is equivalent to proving that

$$f(\kappa(k)) = 2\log 6 + 2k\log\kappa(k) - \kappa(k) < 0.$$

Setting

$$\tilde{\kappa}(k) := -1 + \frac{\kappa(k)}{2k\log k} = \frac{\log(6)}{k\log k} + \frac{\log 2}{\log k} + \frac{\log\log k}{\log k} + \frac{\log\log k}{(\log k)^2}$$

we observe that

$$\begin{aligned} f(\kappa(k)) =& -\log 6 + \kappa(k)/2 - k\log(2k\log k) - k\log(1+\tilde{\kappa}(k)) \\ =& \frac{2k\log\log k}{2\log k} - k\log(1+\tilde{\kappa}(k)) \\ <& \frac{k\log\log k}{\log k} - k(\tilde{\kappa}(k) - \tilde{\kappa}(k)^2/2), \text{ because of } \log(1+x) > x - x^2/2 \text{ for } x \in \mathbb{R}_{>0}. \end{aligned}$$

This inequality is equivalent to

$$\begin{aligned} &2\frac{\log 6\log k + (\log 2)k\log k + k\log\log k}{k(\log k)^2} \\ &> \Big(\frac{\log 6\log k + (\log 2)k\log k + k(\log\log k)\log k + k\log\log k}{k(\log k)^2}\Big)^2, \end{aligned}$$

which is equivalent to the inequality

$$2\log k\Big(\frac{\log 6}{k} + \log 2 + \frac{\log\log k}{\log k}\Big) > (\log\log k)^2\Big(\frac{\log 6}{k\log\log k} + \frac{\log 2}{\log\log k} + 1 + \frac{1}{\log k}\Big)^2.$$

Since

$$2\log k\Big(\frac{\log 6}{k} + \log 2 + \frac{\log\log k}{\log k}\Big) > 2\log 2\log k > \frac{5}{4}\log k,\ k \geq 3,$$

it suffices to show

$$\begin{aligned} &\frac{5}{4}\log k > (\log\log k)^2\Big(\frac{\log 6}{k\log\log k} + \frac{\log 2}{\log\log k} + 1 + \frac{1}{\log k}\Big)^2 \\ \Leftrightarrow &\frac{5}{4}\frac{\log k}{(\log\log k)^2} > \Big(\frac{\log 6}{k\log\log k} + \frac{\log 2}{\log\log k} + 1 + \frac{1}{\log k}\Big)^2. \end{aligned}$$

Now note that $\frac{\log k}{(\log\log k)^2}$ is increasing for $k \geq \lceil e^{e^2}\rceil = 1619$. For the same choice of k the right-hand side of "$\Leftrightarrow$" is decreasing. Evaluating both sides at $k = e^{e^2}$ gives $\frac{5}{4}\frac{e^2}{4} > \frac{23}{10}$ for the left, and $\left(1 + \frac{1}{e^2} + \frac{\log 2}{2} + \frac{\log 6}{2e^{e^2}}\right)^2 < \frac{22}{10}$ for the right side. This proves the inequality for $k \geq 1619$. For $7 \leq k \leq 1618$ the result follows by numerical evaluation. $\square$

Definition 4.3. *For $k \in \mathbb{Z}_{\geq 2}$ define*

$$g(k) := \frac{1}{24}\Big(\frac{6^2}{\pi^2}(\nu(k)^2+1)\Big),$$

where $\nu(k)$ is as in Lemma 4.1.

Theorem 4.4. *For all $k \in \mathbb{Z}_{\geq 2}$ and $n > g(k)$ such that $(n,k) \neq (6,2)$ we have*

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^k}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^k}\Big). \tag{4.3}$$

Proof. From [2, p. 8, (2.9)] we find that

$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+T(n)\Big),\ n \geq 1$$

where $T(n)$ is defined in [2, (2.10)]. And in [2, (2.22)] it is proven that

$$|T(n)| < 6e^{-\frac{\mu(n)}{2}} \text{ for } n > 350. \tag{4.4}$$

By Lemma 4.1 we have that $\mu(n)^k < \frac{1}{6}e^{\frac{\mu(n)}{2}}$ for $k \geq 7$ and $\mu(n) > \nu(k)$, which is equivalent to

$$6e^{-\frac{\mu(n)}{2}} < \frac{1}{\mu(n)^k}, \text{ for } \mu(n) > \nu(k), \tag{4.5}$$

Since $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$, it follows that $\mu(n) > \nu(k)$ iff $n > g(k)$. Furthermore for $k \geq 7$, we have $g(k) > 350$, this means that (4.4) is satisfied for $n > g(k)$.

By (4.4) and (4.5) we obtain that $|T(n)| < \frac{1}{\mu(n)^k}$ for $n > g(k)$ which proves that statement for $k \geq 7$. To prove the statement for $k \in \{2,\ldots,6\}$ we use the statement for $k = 7$ which says that for all $n \geq \lceil g(7) \rceil = 581$ we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^7}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^7}\Big). \tag{4.6}$$

However

$$p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^7}\Big) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^k}\Big) \tag{4.7}$$

for $k \in \{2,\ldots,6\}$ and $n \geq 581$. To prove (4.7) for $g(k) < n < 581$ it is enough to do a numerical evaluation of (4.7) for these values of n with the exception $n = 6$ when $k = 2$. We did this using computer algebra. Analogously, we see that for $k \in \{2,\ldots,6\}$ and $n \geq 581$ we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^k}\Big) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^7}\Big) < p(n). \tag{4.8}$$

In the same way we prove (4.8) for $g(k) < n < 581$.

□

5. Preparing the proof of the Main Theorem 6.6

In this section we prepare for the proof of our Main Theorem, Theorem 6.6, which is presented in Section 6. To this end, we need to introduce a variety of lemmas.

Definition 5.1. *For $y \in \mathbb{R}$, $0 < y^2 < 24$, we define*

$$G(y) := -\log\Big(1-\frac{y^2}{24}\Big) + \frac{\pi}{6y}\sqrt{24}\Big(\sqrt{1-\frac{y^2}{24}}-1\Big) + \log\Big(1-\frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}\Big), \tag{5.1}$$

and its sequence of Taylor coefficients by

$$G(y) = \sum_{u=1}^{\infty} g_u y^u. \tag{5.2}$$

Definition 5.2. *For $0 < y^2 < 24$ and $i \in \{-1, 1\}$ define*

$$G_{i,k}(y) := G(y) + \log\Big(1 + \frac{i\Big(\frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}\Big)^k}{1-\frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}}\Big).$$

Lemma 5.3. *Let $g(k)$ be as in Definition 4.3. Then for all $k \geq 2$ and $n > g(k)$ with $(k, n) \neq (2, 6)$ we have*

$$-\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + G_{-1,k}(1/\sqrt{n}) < \log p(n) < -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + G_{1,k}(1/\sqrt{n}).$$

Proof. Taking log of both sides of (4.3) gives:

$$\log E_{-1,k}(n) < \log p(n) < \log E_{1,k}(n)$$

where

$$E_{i,k}(n) := \log\sqrt{12} - \log(24n-1) + \mu(n) + \log\Big(1 - \frac{1}{\mu(n)} + \frac{i}{\mu(n)^k}\Big).$$

Now

$$\begin{aligned} E_{i,k}(n) &= \log\frac{\sqrt{12}}{24} - \log n - \log\Big(1-\frac{1}{24n}\Big) + \pi\sqrt{\frac{2n}{3}} + \mu(n) \\ &\quad - \frac{\pi}{6}\sqrt{24n} + \log\Big(1-\frac{1}{\mu(n)} + \frac{A}{\mu(n)^k}\Big) \\ &= -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + R_{i,k}(n), \end{aligned} \tag{5.3}$$

where

$$R_{i,k}(x) := -\log\Big(1-\frac{1}{24x}\Big) + \mu(x) - \frac{\pi}{6}\sqrt{24x} + \log\Big(1-\frac{1}{\mu(x)}+\frac{i}{\mu(x)^k}\Big).$$

Finally one verifies that $R_{i,k}(x) = G_{i,k}(1/\sqrt{x})$. □

The entity

$$\alpha := \frac{\pi^2}{36+\pi^2} \tag{5.4}$$

will play an important role in this and the next section.

Lemma 5.4. *Let $G(y)=\sum_{u=1}^{\infty} g_u y^u$ be the Taylor series expansion of $G(y)$ as in Definition 5.1. Then*

$$g_{2n}=\frac{1}{3^n 2^{3n} n}-\frac{1}{2^{3n+1}3^n n}\Big(-1+\frac{1}{\alpha^n}\Big),\ n\geq 1, \tag{5.5}$$

and for $n\geq 0$,

$$g_{2n+1}=\sqrt{6}\Big[(-1)^{n+1}\binom{1/2}{n+1}\frac{\pi}{2^{3n+3}3^{n+2}}-\frac{1}{2^{3n+1}3^n\alpha^n(1+2n)\pi}\sum_{j=0}^{n}\alpha^j\binom{-\frac{1}{2}+j}{j}\Big]. \tag{5.6}$$

Proof. By using

$$\log\Big(1-\frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}\Big)=-\sum_{k=1}^{\infty}y^k k^{-1}\pi^{-k}6^k 24^{-k/2}\Big(1-\Big(\frac{y}{\sqrt{24}}\Big)^2\Big)^{-k/2},$$

together with

$$\Big(1-\Big(\frac{y}{\sqrt{24}}\Big)^2\Big)^{-k/2}=\sum_{n=0}^{\infty}(-1)^n\binom{-k/2}{n}\Big(\frac{y}{\sqrt{24}}\Big)^{2n},$$

we obtain

$$g_{2n}=\frac{1}{3^n 2^{3n} n}-\sum_{u=0}^{n-1}\frac{1}{3^{2u-n}2^{n+2u}\pi^{2n-2u}(2n-2u)}(-1)^u\binom{u-n}{u},\ n\geq 1,$$

and for $n\geq 0$,

$$\begin{aligned}g_{2n+1}=&\sqrt{6}\Big[(-1)^{n+1}\binom{1/2}{n+1}\frac{\pi}{2^{3n+3}3^{n+2}}\\&-\sum_{u=0}^{n}\frac{1}{3^{2u-n}2^{n+1+2u}\pi^{2n+1-2u}(2n+1-2u)}(-1)^u\binom{u-n-1/2}{u}\Big].\end{aligned}$$

Inputting this into the package Sigma developed by Carsten Schneider [17], we obtain (5.5) and (5.6). □

We need various additional facts about the Taylor coefficients g_u of $G(y)$.

Corollary 5.5.

$$\lim_{n\to\infty}3^n 2^{3n}\alpha^n n g_{2n}=-\frac{1}{2}$$

and

$$\lim_{n\to\infty}3^n 2^{3n}\alpha^n(1+2n)g_{2n+1}=-\frac{\sqrt{6}}{2\pi\sqrt{1-\alpha}}.$$

Proof. The first statement is immediate because $\alpha>1$. The second statement follows from $\frac{1}{\sqrt{1-\alpha}}=\sum_{j=0}^{\infty}\alpha^j\binom{-\frac{1}{2}+j}{j}$, because of $\binom{j-1/2}{j}=(-1)^j\binom{-\frac{1}{2}}{j}$. □

Lemma 5.6. *For $0\leq a<1$,*

$$\frac{a}{2}\leq\sum_{j=1}^{n}a^j\binom{j-1/2}{j}\leq\frac{a}{2(1-a)}.$$

Proof. First we note that $\binom{j-1/2}{j} = (-1)^j\binom{-\frac{1}{2}}{j} > 0$. Hence

$$\sum_{j=1}^{n} a^j \binom{j-\frac{1}{2}}{j} = \sum_{j=1}^{n}(-a)^j\binom{-\frac{1}{2}}{j} = \sum_{j=0}^{n}(-a)^j\binom{-\frac{1}{2}}{j} - 1$$
$$< \sum_{j=0}^{\infty}(-a)^j\binom{-\frac{1}{2}}{j} - 1 = \frac{1}{\sqrt{1-a}} - 1 \le \frac{a}{2(1-a)}.$$

This proves the upper bound. To prove the lower bound note that the first term of the sum is $\frac{a}{2}$ and the other terms are all positive. □

Lemma 5.7. *Let $s_n := (-1)^n\binom{1/2}{n+1}$. For $n \ge 0$ we have $s_n \ge 0$ and s_n is a decreasing sequence, that is $s_n > s_{n+1}$ for all $n \ge 0$.*

Lemma 5.8. *For $n \ge 0$ we have*

$$-\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}(1+\frac{\alpha}{2}) \ge g_{2n+1} \ge -\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}\Big(\frac{\pi^2}{72}+1+\frac{\alpha}{2(1-\alpha)}\Big).$$

Proof. From Lemma 5.4 and Lemma 5.7 we obtain

$$-\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}(1+\frac{\alpha}{2}) \ge g_{2n+1}.$$

Again by Lemma 5.4 and Lemma 5.7 we have:

$$\begin{aligned} g_{2n+1} &\ge -\frac{\sqrt{6}}{2^{3n}3^n}\Big(\frac{\pi}{72}(-1)^{0+1}\binom{1/2}{0+1} + \frac{1+\frac{\alpha}{2(1-\alpha)}}{2\pi\alpha^n(1+2n)}\Big) \\ &= -\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}\Big(\frac{\pi^2\alpha^n(1+2n)}{72}+1+\frac{a}{2(1-\alpha)}\Big) \qquad (5.7) \\ &\ge -\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}\Big(\frac{\pi^2\alpha^0(1+2\cdot 0)}{72}+1+\frac{\alpha}{2(1-\alpha)}\Big) \end{aligned}$$

The last line is because $\alpha^n(1+2n)$ is a decreasing sequence of n for $n \ge 0$. □

Lemma 5.9. *For $n \ge 1$ we have*

$$-\frac{1}{3^n 2^{3n+1}n\alpha^n} \le g_{2n} \le \frac{1}{3^n 2^{3n}n\alpha^n}\Big(\frac{3\alpha}{2}-\frac{1}{2}\Big).$$

Proof. By Lemma 5.4 the statement follows from

$$g_{2n} = \frac{1}{3^n 2^{3n}n} - \frac{1}{2^{3n+1}3^n n}\Big(-1+\frac{1}{\alpha^n}\Big) \qquad (5.8)$$

and

$$g_{2n} = \frac{1}{3^n 2^{3n}n} - \frac{1}{2^{3n+1}3^n n}\Big(-1+\frac{1}{\alpha^n}\Big) = \frac{1}{3^n 2^{3n}\alpha^n n}\Big(\frac{3\alpha^n}{2}-\frac{1}{2}\Big). \qquad (5.9)$$

□

Lemma 5.10. *Define*

$$\mu_1 := \frac{\sqrt{6}}{2\pi}\Big(\frac{\pi^2}{72}+1+\frac{\alpha}{2(1-\alpha)}\Big) \quad \text{and} \quad \mu_2 := \frac{\sqrt{6}}{2\pi}\Big(1+\frac{\alpha}{2}\Big).$$

Then for $m \geq 0$ and $0 < y \leq \epsilon < 2\sqrt{6\alpha}$,

$$-\frac{\mu_2}{2^{3m}3^m\alpha^m(1+2m)}y^{2m+1} \geq \sum_{n=m}^{\infty} g_{2n+1}y^{2n+1} \geq -\frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}}y^{2m+1}.$$

Proof. By Lemma 5.8 we have

$$\begin{aligned}\sum_{n=m}^{\infty} g_{2n+1}y^{2n+1} &\geq -\mu_1\sum_{n=m}^{\infty}\frac{1}{2^{3n}3^n\alpha^n(1+2n)}y^{2n+1} \geq -\frac{\mu_1 y^{2m+1}}{1+2m}\sum_{n=0}^{\infty}\frac{1}{2^{3(n+m)}3^{n+m}\alpha^{n+m}}y^{2n}\\ &= -\frac{\mu_1 y^{2m+1}}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{y^2}{3\alpha\cdot 2^3}} \geq -\frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}}y^{2m+1},\end{aligned} \tag{5.10}$$

and again by Lemma 5.8 we have

$$\sum_{n=m}^{\infty} y^{2n+1}g_{2n+1} \leq -\mu_2\sum_{n=m}^{\infty}\frac{y^{2n+1}}{2^{3n}3^n\alpha^n(1+2n)} \leq -\mu_2\frac{y^{2m+1}}{2^{3m}3^m\alpha^m(1+2m)}. \tag{5.11}$$

□

Lemma 5.11. *For $m \geq 1$ and $0 < y \leq \epsilon < 2\sqrt{6\alpha}$,*

$$\frac{3\alpha-1}{3^m 2^{3m+1}m\alpha^m}y^{2m} \geq \sum_{n=m}^{\infty} g_{2n}y^{2n} \geq -y^{2m}\frac{1}{3^m 2^{3m+1}m\alpha^m}\frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\cdot\alpha}}.$$

Proof. By Lemma 5.9,

$$\begin{aligned}\sum_{n=m}^{\infty} g_{2n}y^{2n} &\geq -\frac{1}{2}\sum_{n=m}^{\infty}\frac{1}{3^n 2^{3n}n\alpha^n}y^{2n} \geq -y^{2m}\frac{1}{2}\sum_{n=m}^{\infty}\frac{1}{3^n 2^{3n}m\alpha^n}y^{2n-2m}\\ &= -y^{2m}\frac{1}{3^m 2^{3m+1}m\alpha^m}\frac{1}{1-\frac{y^2}{3\cdot 2^3\cdot\alpha}} \geq -y^{2m}\frac{1}{3^m 2^{3m+1}m\alpha^m}\frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\cdot\alpha}}.\end{aligned} \tag{5.12}$$

Again by Lemma 5.9,

$$\sum_{n=m}^{\infty} g_{2n}y^{2n} \leq \frac{3\alpha-1}{2}\sum_{n=m}^{\infty}\frac{1}{3^n 2^{3n}n\alpha^n}y^{2n} \leq \frac{3\alpha-1}{2}\frac{1}{3^m 2^{3m}m\alpha^m}y^{2m}. \tag{5.13}$$

□

Definition 5.12. *For $0 < y \leq \epsilon < 1$ define*

$$B(y) := \frac{y}{\frac{\pi}{6}\sqrt{24-y^2}} \quad \text{and} \quad B_{\epsilon,k} := \epsilon^{-k}\frac{B(\epsilon)^k}{1-B(\epsilon)}. \tag{5.14}$$

Lemma 5.13. *If $0 < y \leq \epsilon < 1$, then*

$$\log\Big(1+\frac{B(y)^k}{1-B(y)}\Big) \leq \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}y^k, \; k \geq 1.$$

Proof. First note that for $0 < y < \sqrt{24}$ the function $B(y)$ is increasing and also that $\frac{B(y)^k}{1-B(y)} \leq \frac{B(y)^k}{1-B(\epsilon)}$ and $B(y) < \frac{y}{\frac{\pi}{6}\sqrt{24-\epsilon^2}}$. Hence

$$\frac{B(y)^k}{1-B(\epsilon)} < \frac{\epsilon^{-k}y^k B(\epsilon)^k}{1-B(\epsilon)} = B_{\epsilon,k}y^k.$$

Consequently,

$$\log\Big(1+\frac{B(y)^k}{1-B(y)}\Big) \leq \log\Big(1+B_{\epsilon,k}y^k\Big) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}B_{\epsilon,k}^n y^{kn}$$
$$= -\sum_{n=1}^{\infty}\frac{1}{2n}B_{\epsilon,k}^{2n}y^{2kn} + \sum_{n=0}^{\infty}\frac{1}{2n+1}B_{\epsilon,k}^{2n+1}y^{k(2n+1)}$$
$$\leq \sum_{n=0}^{\infty}\frac{1}{2n+1}B_{\epsilon,k}^{2n+1}y^{k(2n+1)} \leq \sum_{n=0}^{\infty}B_{\epsilon,k}^{2n+1}y^{k(2n+1)}$$
$$= \frac{B_{\epsilon,k}y^k}{1-(B_{\epsilon,k}y^k)^2} \leq \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}y^k.$$

□

Lemma 5.14. *If* $0 < y \leq \epsilon < 1$, *then*

$$\log\Big(1-\frac{B(y)^k}{1-B(y)}\Big) \geq -\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}y^k, \;\; k \geq 1.$$

Proof.

$$\log\Big(1-\frac{B(y)^k}{1-B(y)}\Big) \geq \log\Big(1-B_{\epsilon,k}y^k) = -\sum_{n=1}^{\infty}\frac{1}{n}B_{\epsilon,k}^n y^{kn} \geq -\sum_{n=1}^{\infty}B_{\epsilon,k}^n y^{kn}$$
$$= -\frac{B_{\epsilon,k}y^k}{1-B_{\epsilon,k}y^k} \geq -\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}y^k.$$

□

Lemma 5.15. *For all* $k \geq 2$ *and* $0 < \epsilon \leq \frac{1}{\sqrt{7}}$ *we have*

$$\frac{6^k}{5^k\pi^k} < B_{\epsilon,k} \leq \frac{b_0 \cdot 6^k}{\pi^k(\sqrt{24-\frac{1}{7}})^k},$$

where $b_0 := \frac{1}{1-\frac{6}{\sqrt{7}\pi\sqrt{24-\frac{1}{7}}}}$ *and again* $B_{\epsilon,k}$ *as in* (5.14).

Proof. Define

$$s := \sqrt{24-\epsilon^2},\; l_s := \sqrt{24-\frac{1}{7}},\; u_s := 4.9,\; l_\epsilon := 0, \text{ and } u_\epsilon := \frac{1}{\sqrt{7}}. \tag{5.15}$$

For all $k \geq 2$ and $0 < \epsilon \leq \frac{1}{\sqrt{7}}$, we have

$$l_s \leq s < u_s \text{ and } l_\epsilon < \epsilon \leq u_\epsilon.$$

The following conventions for the letters "l" and "u" will be useful: l_a denotes a lower bound for the quantity a, and u_a will denote an upper bound for the quantity a. And again we use $B(y)$ as defined in Definition 5.12.

Then

$$0 = \frac{l_\epsilon}{\frac{\pi}{6}u_s} < B(\epsilon) = \frac{\epsilon}{\frac{\pi}{6}s} \leq \frac{u_\epsilon}{\frac{\pi}{6}l_s}.$$

Let us define $l_B := 0$ and $u_B := \frac{u_\epsilon}{\frac{\pi}{6}l_s}$. Then

$$l_B < B(\epsilon) \leq u_B \Rightarrow 1 - u_B \leq 1 - B(\epsilon) < 1 - l_B = 1 \Rightarrow \frac{1}{1-l_B} = 1 < \frac{1}{1-B(\epsilon)} \leq \frac{1}{1-u_B},$$

and $\frac{1}{(\frac{\pi}{6}u_s)^k} < \frac{1}{(\frac{\pi}{6}s)^k} \leq \frac{1}{(\frac{\pi}{6}l_s)^k}$. Hence

$$\begin{aligned}\frac{6^k}{5^k\pi^k} <& \frac{6^k}{(4.9)^k\pi^k} = \frac{1}{(1-l_B)(\frac{\pi}{6}u_s)^k} < B_{\epsilon,k} \leq \frac{1}{(1-u_B)(\frac{\pi}{6}l_s)^k} \\ =& \frac{1}{(1 - \frac{\frac{1}{\sqrt{7}}}{\frac{\pi}{6}\sqrt{24-\frac{1}{7}}})(\frac{\pi^k}{6^k}(\sqrt{24-\frac{1}{7}})^k)} = \frac{b_0}{\frac{\pi^k}{6^k}(\sqrt{24-\frac{1}{7}})^k}.\end{aligned}$$

□

Definition 5.16. *Define*

$$\beta := \sqrt{24 - \frac{1}{7}}$$

and for $k \geq 0$,

$$C_k := \frac{6^k}{(\pi\beta)^k}.$$

Lemma 5.17. *Let* $0 < \epsilon \leq \frac{1}{\sqrt{7}}$ *and* $B_{\epsilon,k}$ *be as in* (5.14). *Then for* $k \geq 2$,

$$\frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} \leq b_1 B_{\epsilon,k} \ \text{and}\ \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} \leq b_2 B_{\epsilon,k},$$

with

$$b_1 := \frac{1}{1-\frac{1}{49}b_0^2\left(\frac{6}{\pi\beta}\right)^4},\ b_2 := \frac{1}{1-\frac{1}{7}b_0\left(\frac{6}{\pi\beta}\right)^2},\ \text{and}\ b_0 := \frac{1}{1-\frac{6}{\sqrt{7}\pi\sqrt{24-\frac{1}{7}}}}.$$

Proof. We obtain, using Lemma 5.15,

$$\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{7}B_{\epsilon,k}} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{7}\frac{b_0\cdot 6^2}{\pi^2(\beta)^2}} = b_2 B_{\epsilon,k},$$

and

$$\frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{49}B_{\epsilon,k}^2} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{49}b_0^2\left(\frac{6}{\pi\beta}\right)^4} = b_1 B_{\epsilon,k}.$$

□

Lemma 5.18. *Let* C_k *be as in Definition 5.16, then*

$$C_{2m} < \frac{1}{3^m 2^{3m}\alpha^m m},\ m \geq 10,\ \text{and}\ C_{2m-1} < \frac{69}{25}\frac{1}{2^{3m}3^m\alpha^m(2m-1)},\ m \geq 14.$$

Proof. We start with the first inequality:

$$C_m = \Big(\frac{252}{167\pi^2}\Big)^m < \frac{(36+\pi^2)^m}{3^m 2^{3m} m\pi^{2m}} \Leftrightarrow \Big(\frac{6048}{6012+167\pi^2}\Big)^m m < 1.$$

To prove the inequality in the rewritten form, define $\ell := \frac{6048}{6012+167\pi^2}$ and note that $\ell < 1$. Moreover, for $m \geq 10$:

$$m\ell^m < 1 \Leftrightarrow \log m + m\log\ell < 0.$$

Define $f(m) := m\log\ell + \log m$. That is, we have to show $f(m) < 0$ for all $m \geq 10$. We first show that $f(m)$ is decreasing for $m \geq 10$. This is equivalent to $f'(m) = \log\ell + \frac{1}{m} < 0$ for $m \geq 10$:

$$\log\ell + \frac{1}{m} < 0 \Leftrightarrow \frac{1}{m} < \log\frac{1}{\ell} \Leftrightarrow e^{1/m} < \frac{1}{\ell} \Leftrightarrow \ell e^{1/m} < 1.$$

Now for $m \geq 10$ we have

$$\frac{1}{m} \leq \frac{1}{10} \Leftrightarrow e^{1/m} \leq e^{1/10} \Leftrightarrow \ell e^{1/m} \leq e^{1/10}\ell.$$

By numerics, $e^{1/10}\ell < 1$ and $f(10) < 0$. Since $f(m)$ is decreasing, $f(m) \leq f(10) < 0$, and the first inequality is proven. Now for the second inequality, first note that

$$C_{2m-1} = \Big(\frac{6}{\pi\beta}\Big)^{2m-1} = \Big(\frac{252}{167\pi^2}\Big)^m \Big(\frac{\pi}{6}\sqrt{\frac{167}{7}}\Big).$$

Hence we have to show

$$\Big(\frac{252}{167\pi^2}\Big)^m \Big(\frac{\pi}{6}\sqrt{\frac{167}{7}}\Big) < \frac{69}{25}\frac{1}{2^{3m}3^m\alpha^m(2m-1)},$$

which is equivalent to

$$\Big(\frac{6048}{6012+167\pi^2}\Big)^m (2m-1) < \frac{414}{25\pi}\sqrt{\frac{7}{167}} \Leftrightarrow (2m-1)\ell^m < \frac{414}{25\pi}\sqrt{\frac{7}{167}}$$
$$\Leftrightarrow m\log\ell + \log(2m-1) - \log\Big(\frac{414}{25\pi}\sqrt{\frac{7}{167}}\Big) < 0.$$

We define $g(m) := m\log\ell + \log(2m-1) - \log\Big(\frac{414}{25\pi}\sqrt{\frac{7}{167}}\Big)$, and for $m \geq 14$ we show that $g(m)$ is decreasing:

$$g'(m) < 0 \Leftrightarrow \log\ell + \frac{2}{2m-1} < 0 \Leftrightarrow \frac{2}{2m-1} < \log\frac{1}{\ell} \Leftrightarrow \ell e^{\frac{2}{2m-1}} < 1.$$

Indeed, for $m \geq 14$, we have

$$\frac{2}{2m-1} \leq \frac{2}{27} \Leftrightarrow e^{\frac{2}{2m-1}} \leq e^{\frac{2}{27}} \Leftrightarrow \ell e^{\frac{2}{2m-1}} \leq \ell e^{\frac{2}{27}}.$$

By numerics, $\ell e^{\frac{2}{27}} < 1$ and $g(14) < 0$, hence $g(m) \leq g(14) < 0$. □

6. Proofs of Theorem 6.6 and Theorem 1.1

After the preparations made in Section 5, in this section we prove our Main Theorem, Theorem 6.6, which implies Theorem 1.1 as a corollary. Again

$$\alpha = \frac{\pi^2}{36+\pi^2}.$$

Definition 6.1. *Let $B_{\epsilon,k}$ be as in Definition 5.12 and μ_1, μ_2 as in Lemma 5.10 and $\nu := \frac{3\alpha-1}{2}$. Moreover, let $0 < \epsilon \leq \frac{1}{\sqrt{7}}$. For $m, k \in \mathbb{Z}_{\geq 1}$ we define*

$$\begin{aligned}
A_{1,k}(2m) &:= \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}\epsilon^{k-2m} + \nu\frac{1}{3^m 2^{3m} m\alpha^m},\\
A_{-1,k}(2m) &:= \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}\epsilon^{k-2m} + \frac{1}{3^m 2^{3m+1} m\alpha^m}\frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\alpha}} + \frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}},\\
A_{1,k}(2m-1) &:= \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}\epsilon^{k-2m+1} - \frac{\mu_2}{2^{3m-3}3^{m-1}\alpha^{m-1}(2m-1)},\\
A_{-1,k}(2m-1) &:= \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}\epsilon^{k-2m+1} + \frac{1}{3^m 2^{3m+1} m\alpha^m}\frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\alpha}},\\
&+ \frac{\mu_1}{2^{3m-3}3^{m-1}\alpha^{m-1}(2m-1)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}}.
\end{aligned}$$

Lemma 6.2. *Let $G(y) = \sum_{n=1}^{\infty} g_n y^n$ as in Definition 5.1 and $G_{i,k}(y)$ as in Definition 5.2 Moreover let $0 < \epsilon \leq \frac{1}{\sqrt{7}}$. Then for $k \geq 2m \geq 2$:*

$$\sum_{n=1}^{2m-1} g_n y^n - A_{-1,k}(2m)y^{2m} \leq G_{-1,k}(y) \text{ and } G_{1,k}(y) \leq \sum_{n=1}^{2m-1} g_n y^n + A_{1,k}(2m)y^{2m},$$

and for $k \geq 2m-1 \geq 1$,

$$\sum_{n=1}^{2m-2} g_n y^n - A_{-1,k}(2m-1)y^{2m-1} \leq G_{-1,k}(y) \text{ and } G_{1,k}(y) \leq \sum_{n=1}^{2m-2} g_n y^n + A_{1,k}(2m-1)y^{2m-1}.$$

Proof. For $k \geq 2m \geq 2$, by using the Lemmas 5.10 to 5.13, we obtain

$$\begin{aligned}
G_{1,k}(y) &\leq \sum_{n=1}^{2m-1} g_n y^n + \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}y^k + \nu\frac{1}{3^m 2^{3m} m\alpha^m}y^{2m} - \frac{\mu_2}{2^{3m}3^m\alpha^m(1+2m)}y^{2m+1}\\
&\leq \sum_{n=1}^{2m-1} g_n y^n + \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}\epsilon^{k-2m}y^{2m} + \nu\frac{1}{3^m 2^{3m} m\alpha^m}y^{2m}\\
&= \sum_{n=1}^{2m-1} g_n y^n + A_{1,k}(2m)y^{2m}.
\end{aligned} \tag{6.1}$$

By using the Lemmas 5.10 to 5.11 together with Lemma 5.14 we obtain

$$\begin{aligned} G_{-1,k}(y) \geq& \sum_{n=1}^{2m-1} g_n y^n - \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} y^k - \frac{1}{3^m 2^{3m+1} m\alpha^m} \frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\alpha}} y^{2m} \\ &- \frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)} \frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}} y^{2m+1} \\ \geq& \sum_{n=1}^{2m-1} g_n y^n - \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} \epsilon^{k-2m} y^{2m} - \frac{1}{3^m 2^{3m+1} m\alpha^m} \frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\alpha}} y^{2m} \\ &- \frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)} \frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}} y^{2m} \\ =& \sum_{n=1}^{2m-1} g_n y^n - A_{-1,k}(2m) y^{2m}. \end{aligned} \tag{6.2}$$

The statement for $A_{-1,k}(2m-1)$ is proven analogously. □

Lemma 6.3. *We have*

$$A_{1,k}(2m) < \frac{1}{3^m 2^{3m} m\alpha^m}, \ A_{-1,k}(2m) < \frac{2}{3^m 2^{3m} m\alpha^m}, \ m \geq 10$$

and

$$A_{1,k}(2m-1) < \frac{2}{3^m 2^{3m}(2m-1)\alpha^m}, \ A_{-1,k}(2m-1) < \frac{7}{3^m 2^{3m}(2m-1)\alpha^m} \ m \geq 14.$$

Proof. For $m \geq 10$ we have,

$$\begin{aligned} A_{1,k}(2m) =& \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} \epsilon^{k-2m} + \nu \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 6.2)} \\ <& b_1 B_{\epsilon,k}\epsilon^{k-2m} + \nu \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 5.17)} \\ <& b_1 b_0 \frac{6^k}{(\pi\beta)^k}\epsilon^{k-2m} + \nu \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 5.15)} \\ =& b_0 b_1 C_k \epsilon^{k-2m} + \nu \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by using Definition 5.16)} \\ \leq& b_0 b_1 C_{2m} + \nu \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (because } f(k) := C_k\epsilon^{k-2m} \text{ is decreasing for all } k \geq 2m) \\ <& b_0 b_1 \frac{2}{3^m 2^{3m}\alpha^m m} + \nu \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 5.18)} \\ =& \Big(b_0 b_1 + \nu\Big) \frac{1}{3^m 2^{3m}\alpha^m m} < \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by evaluating } b_0 b_1 + \nu \text{ numerically).} \end{aligned}$$

$$
\begin{aligned}
&A_{-1,k}(2m)\\
=&\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}\epsilon^{k-2m}+\frac{1}{3^m2^{3m+1}\alpha^m m}\frac{1}{1-\frac{\epsilon^2}{24\alpha}}+\frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} \text{ (by Lemma 6.2)}\\
<&b_2B_{\epsilon,k}\epsilon^{k-2m}+\frac{1}{2}\frac{1}{3^m2^{3m}\alpha^m m}\frac{1}{1-\frac{\epsilon^2}{24\alpha}}+\frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} \text{ (by Lemma 5.17)}\\
<&b_2b_0\frac{6^k}{(\pi\beta)^k}\epsilon^{k-2m}+\frac{1}{2}\frac{1}{3^m2^{3m}\alpha^m m}\frac{1}{1-\frac{\epsilon^2}{24\alpha}}+\frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} \text{ (by Lemma 5.15)}\\
\le&b_0b_2\cdot C_{2m}+\frac{1}{2}\frac{1}{3^m2^{3m}\alpha^m m}\frac{1}{1-\frac{1}{168\alpha}}+\frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{1}{168\alpha}}\\
<&b_0b_2\frac{1}{3^m2^{3m}\alpha^m m}+\frac{1}{2}\frac{1}{3^m2^{3m}\alpha^m m}\frac{1}{1-\frac{1}{168\alpha}}+\frac{1}{2}\frac{\mu_1}{2^{3m}3^m\alpha^m m}\frac{1}{1-\frac{1}{168\alpha}} \text{ (by Lemma 5.18)}\\
=&\Big(b_0b_2+\frac{1}{2}\frac{1}{1-\frac{1}{168\alpha}}(1+\mu_1)\Big)\frac{1}{3^m2^{3m}\alpha^m m}<\frac{2}{3^m2^{3m}\alpha^m m},
\end{aligned}
$$

where the last equality is by evaluating $b_0b_2+\frac{1}{2}\frac{1}{1-\frac{1}{168\alpha}}(1+\mu_1)$ numerically.

The statements for $A_{1,k}(2m-1)$ and $A_{-1,k}(2m-1)$ are proven analogously.

□

Definition 6.4. *For $n,U\in\mathbb{Z}_{\geq 1}$ we define*

$$P_n(U):=-\log 4\sqrt{3}-\log n+\pi\sqrt{\frac{2n}{3}}+\sum_{u=1}^{U}g_u(1/\sqrt{n})^u.$$

Lemma 6.5. *Let $g(k)$ be as in Definition 4.3 and $P_n(U)$ as in Definition 6.4. If $m\geq 1$, $k\geq 2m$ and*

$$n>\begin{cases} 6, & \text{if } m=1\\ g(k), & \text{if } m\geq 2.\end{cases},$$

then

$$-A_{-1,k}(2m)\frac{1}{n^m}<\log p(n)-P_n(2m-1)<A_{1,k}(2m)\frac{1}{n^m}; \tag{6.3}$$

if $m\geq 2$, $k\geq 2m-1$, and $n>g(k)$, then

$$-A_{-1,k}(2m-1)\frac{1}{\sqrt{n}n^{m-1}}<\log p(n)-P_n(2m-2)<A_1(2m-1)\frac{1}{\sqrt{n}n^{m-1}}. \tag{6.4}$$

Proof. We start with the inequality from Lemma 5.3. Next we use Lemma 6.2 to bound $G_{1,k}(y)$. Finally we set $y=\frac{1}{\sqrt{n}}$ and obtain the desired result. □

Theorem 6.6. *Let $G(y)=\sum_{n=1}^{\infty}g_ny^n$ be as in Definition 5.1. Let $g(k)$ be as in Definition 4.3 and $P_n(U)$ as in Definition 6.4. If $m\geq 1$ and $n>g(2m)$, then*

$$P_n(2m-1)-\frac{2}{3^m2^{3m}\alpha^m mn^m}<\log p(n)<P_n(2m-1)+\frac{1}{3^m2^{3m}\alpha^m mn^m}; \tag{6.5}$$

if $m \geq 2$ *and* $n > g(2m-1)$*, then*

$$P_n(2m-2) - \frac{7}{3^m 2^{3m} \alpha^m (2m-1) n^{m-1/2}} < \log p(n) < P_n(2m-2) + \frac{2}{3^m 2^{3m} \alpha^m (2m-1) n^{m-1/2}}. \tag{6.6}$$

Proof. We start by setting $k = 2m$ in (6.3) of Lemma 6.5, and $k = 2m-1$ in (6.4). In this inequality we bound $A_{1,k}(m)$ resp $A_{-1,k}(m)$ by using Lemma 6.3. This gives (6.6) for all $m \geq 14$ and $n > g(2m-1)$, and (6.5) for $m \geq 10$ and $n > g(2m)$.

In order to prove (6.6) and (6.5) for the remaining values of m, firstly we will prove that

if (6.5) holds for $m \geq 2$ and all $n \geq y \geq 1$, then (6.5) holds for $m-1$ and all $n \geq y$. (6.7)

In particular, if we subtract from the lower bound for m in (6.5) the lower bound for $m-1$, we obtain $f(2m,-4) - g(2m-2,-4)$, where

$$f(w,x) := \sum_{u=w-2}^{w-1} g_u \Big(\frac{1}{\sqrt{n}}\Big)^u + \frac{x}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w} \Big(\frac{1}{\sqrt{n}}\Big)^w$$

and

$$g(w,x) := \frac{x}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w} \Big(\frac{1}{\sqrt{n}}\Big)^w.$$

Similarly, if we subtract from the upper bound for $m \to m-1$ in (6.5) the upper bound for m, we obtain $g(2m-2,2) - f(2m,2)$. Hence in order to prove (6.7), it suffices to prove

$$f(2m,-4) > g(2m-2,-4) \text{ and } f(2m,2) < g(2m-2,2). \tag{6.8}$$

Analogously, in order to prove that if (6.6) holds for all $m \geq 3$ and all $n \geq y \geq 1$, then (6.6) holds for $m-1$ and all $n \geq y$, it suffices to prove

$$f(2m-1,-7) > g(2m-3,-7) \text{ and } f(2m-1,2) < g(2m-3,2). \tag{6.9}$$

For proving (6.8) and (6.9), we shall prove

$$f(w,x_0(w)) > g(w-2,x_0(w)) \text{ with } x_0(w) := \begin{cases} -4, & \text{if } w \text{ is even} \\ -7, & \text{if } w \text{ is odd} \end{cases} \tag{6.10}$$

and

$$f(w,y_0) < g(w-2,y_0) \text{ with } y_0 \in \mathbb{Z}_{>0}. \tag{6.11}$$

From Lemma 5.8 and Lemma 5.9 we have

$$\frac{\ell_w}{(24\alpha)^{\lfloor \frac{w}{2} \rfloor} w} \leq g_w \leq \frac{u_w}{(24\alpha)^{\lfloor \frac{w}{2} \rfloor} w} \tag{6.12}$$

with

$$\ell_w := \begin{cases} -\mu_1, & \text{if } w \text{ is odd} \\ -1, & \text{if } w \text{ is even} \end{cases} \text{ and } u_w := \begin{cases} -\mu_2, & \text{if } w \text{ is odd} \\ 2\nu, & \text{if } w \text{ is even} \end{cases},$$

where μ_1 and μ_2 are as in Lemma 5.10. Consequently,

$$\begin{aligned}f(w,x_0)=&\sum_{u=w-2}^{w-1} g_u\Big(\frac{1}{\sqrt{n}}\Big)^u+\frac{x_0}{(24\alpha)^{\lceil\frac{w}{2}\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w\\ \geq&\frac{\ell_{w-2}}{(24\alpha)^{\lfloor\frac{w-2}{2}\rfloor}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2}+\frac{\ell_{w-1}}{(24\alpha)^{\lfloor\frac{w-1}{2}\rfloor}(w-1)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-1}\\ &+\frac{x_0}{(24\alpha)^{\lceil\frac{w}{2}\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w.\end{aligned}$$

In order to prove (6.10), it is enough to prove

$$\begin{aligned}&\frac{\ell_{w-2}}{(24\alpha)^{\lfloor\frac{w-2}{2}\rfloor}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2}+\frac{\ell_{w-1}}{(24\alpha)^{\lfloor\frac{w-1}{2}\rfloor}(w-1)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-1}+\frac{x_0}{(24\alpha)^{\lceil\frac{w}{2}\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w\\ &>\frac{x_0}{(24\alpha)^{\lceil\frac{w-2}{2}\rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2}.\end{aligned}$$

This inequality is equivalent to

$$\frac{\ell_{w-2}}{w-2}+\frac{\ell_{w-1}}{(24\alpha)^{\alpha_w}(w-1)}\frac{1}{\sqrt{n}}+\frac{x_0}{(24\alpha)^{\beta_w}w}\frac{1}{n}>\frac{x_0}{(24\alpha)^{\delta_w}(w-2)} \tag{6.13}$$

where

$$\alpha_w=\lfloor\frac{w-1}{2}\rfloor-\lfloor\frac{w-2}{2}\rfloor=\begin{cases}0, & \text{if } w \text{ is even}\\ 1, & \text{if } w \text{ is odd}\end{cases},\ \beta_w=\lceil\frac{w}{2}\rceil-\lfloor\frac{w-2}{2}\rfloor=\begin{cases}1, & \text{if } w \text{ is even}\\ 2, & \text{if } w \text{ is odd}\end{cases},$$

$$\text{and } \delta_w=\lceil\frac{w-2}{2}\rceil-\lfloor\frac{w-2}{2}\rfloor=\begin{cases}0, & \text{if } w \text{ is even}\\ 1, & \text{if } w \text{ is odd}\end{cases}.$$

Inequality (6.13) is equivalent to

$$\Big(\ell_{w-2}-\frac{x_0}{(24\alpha)^{\alpha_w}}\Big)\frac{1}{w-2}>-\frac{\ell_{w-1}}{(24\alpha)^{\delta_w}(w-1)}\frac{1}{\sqrt{n}}+\frac{x_0}{(24\alpha)^{\beta_w}w}\frac{1}{n},$$

which is implied by

$$\Big(\ell_{w-2}-\frac{x_0}{(24\alpha)^{\alpha_w}}\Big)\frac{1}{w-2}>-\Big(\frac{\ell_{w-1}}{(24\alpha)^{\alpha_w}(w-1)}+\frac{x_0}{(24\alpha)^{\beta_w}w}\Big)\frac{1}{\sqrt{n}} \tag{6.14}$$

since $\delta_w=\alpha_w$, $x_0\in\mathbb{Z}_{<0}$ and $\frac{1}{\sqrt{n}}\geq\frac{1}{n}$ for all $n\geq 1$. Inequality (6.14) is equivalent to

$$n\geq\left\lceil\frac{(w-2)^2\Big(\frac{\ell_{w-1}}{(24\alpha)^{\alpha_w}(w-1)}+\frac{x_0}{(24\alpha)^{\beta_w}w}\Big)^2}{\Big(\ell_{w-2}-\frac{x_0}{(24\alpha)^{\alpha_w}}\Big)^2}\right\rceil=:N_1(w,x_0).$$

We checked with Mathematica that $N_1(w,x_0(w))\leq 1$; see the Appendix, Section 8.3.

Similarly to above, for $y_0\in\mathbb{Z}_{>0}$ one has,

$$\begin{aligned}f(w,y_0)=&\sum_{u=w-2}^{w-1} g_u\Big(\frac{1}{\sqrt{n}}\Big)^u+\frac{y_0}{(24\alpha)^{\lceil\frac{w}{2}\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w\\ \leq&\frac{u_{w-2}}{(24\alpha)^{\lceil\frac{w-2}{2}\rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2}+\frac{u_{w-1}}{(24\alpha)^{\lceil\frac{w-1}{2}\rceil}(w-1)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-1}+\frac{y_0}{(24\alpha)^{\lceil\frac{w}{2}\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w.\end{aligned}$$

In order to prove (6.11), it is enough to show

$$\frac{u_{w-2}}{(24\alpha)^{\lceil \frac{w-2}{2} \rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2} + \frac{u_{w-1}}{(24\alpha)^{\lceil \frac{w-1}{2} \rceil}(w-1)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-1} + \frac{y_0}{(24\alpha)^{\lceil \frac{w}{2} \rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}$$
$$< \frac{y_0}{(24\alpha)^{\lceil \frac{w-2}{2} \rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2}.$$

This last inequality can be rewritten as the following equivalent inequality,

$$\frac{u_{w-2}}{w-2} + \frac{u_{w-1}}{(24\alpha)^{\alpha_w}(w-1)}\frac{1}{\sqrt{n}} + \frac{y_0}{(24\alpha)^{\beta_w}w}\frac{1}{n} < \frac{y_0}{(24\alpha)^{\alpha_w}(w-2)},$$

which is implied by:

$$\Big(\frac{y_0}{(24\alpha)^{\alpha_w}} - u_{w-2}\Big)\frac{1}{w-2} > \Big(\frac{u_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} + \frac{y_0}{(24\alpha)^{\beta_w}w}\Big)\frac{1}{\sqrt{n}} \tag{6.15}$$

since $y_0 \in \mathbb{Z}_{>0}$ and $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$. Inequality (6.15) is equivalent to

$$n \geq \left\lceil \frac{(w-2)^2\Big(\frac{u_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} + \frac{y_0}{(24\alpha)^{\beta_w}w}\Big)^2}{\Big(\frac{y_0}{(24\alpha)^{\alpha_w}} - u_{w-2}\Big)^2} \right\rceil =: N_2(w, y_0).$$

We checked using Mathematica that $N_2(w, y_0) \leq 1$ for all $y_0 \geq 1$; see the Appendix, Section 8.3.

We have checked with Mathematica that (6.5) holds for $m \in \{2, \dots, 10\}$ and $n \in \mathbb{N}$ such that

$$g(2m-2) < n \leq g(2m). \tag{6.16}$$

Now (6.5) is true for $m = 10$ and $n > g(2m)$. Next, assume that (6.5) is true for $m = N$ with $2 \leq N \leq 10$ and $n > g(2N)$. Then, as shown above, (6.5) is true for $m = N-1$ if $n > g(2N)$. By (6.16), (6.5) is true for $m = N-1$ if $g(2N-2) < n \leq g(2N)$. This implies that (6.5) is true for $m = N-1$ and $n > g(2N-2)$. Hence the result follows inductively. The proof of (6.6) is done analogously.

□

Finally, we are put into the position to prove Theorem 1.1.

Proof of Theorem 1.1: We apply (6.5) in Theorem 6.6, with $m = 1$. Then for $n \geq 1$, we have,

$$-\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} - \sqrt{6}\Big(\frac{\pi}{144} + \frac{1}{2\pi}\Big)\frac{1}{\sqrt{n}} - \frac{2}{24\alpha}\frac{1}{n}$$
$$< \log p(n) < -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} - \sqrt{6}\Big(\frac{\pi}{144} + \frac{1}{2\pi}\Big)\frac{1}{\sqrt{n}} + \frac{1}{24\alpha}\frac{1}{n}.$$

Noting that $\sqrt{6}\Big(\frac{\pi}{144} + \frac{1}{2\pi}\Big) = 0.44\dots$ finishes the proof. □

7. An Application to Chen-DeSalvo-Pak Log Concavity Result

In 2010 at FPSAC [4], William Chen conjectured that $\{p(n)\}_{n\geq 26}$ is log-concave and for $n \geq 1$,

$$p(n)^2 < \Big(1+\frac{1}{n}\Big)p(n-1)p(n+1). \tag{7.1}$$

DeSalvo and Pak [7] proved these two conjectures. Moreover, they refined (7.1) by proposing the following conjecture for $n \geq 45$:

$$p(n)^2 < \Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}\Big)p(n-1)p(n+1). \tag{7.2}$$

Chen, Wang and Xie [3] gave an affirmative answer to (7.2). In this section, using Theorem 6.6, we continue this research by obtaining the following inequality,

$$\Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\Big)p(n-1)p(n+1) < p(n)^2 < \Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}\Big)p(n-1)p(n+1);$$

for a more precise statement see Theorem 7.6. Note that the right inequality is just (7.2), but we give here our proof in order to show that, alternatively, one can obtain this from Theorem 6.6. In order to achieve our goal we also need to prove the Lemmas 7.3 to 7.5 in this section. These Lemmas deal with estimating the tail of an infinite series involving standard binomials.

Proposition 7.1. *For $s \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{N}$ we have*

$$\binom{-\frac{2s-1}{2}}{k} = \frac{(-1)^k}{4^k}\frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}}$$

and

$$\binom{-s}{k} = (-1)^k\binom{s+k-1}{s-1}.$$

Proof. By simplifying quotients formed by taking each expression in $k+1$ divided by the original expression in k. □

Lemma 7.2. *For $k, m \geq 0$ and $s \geq 1$,*

$$\binom{s-1+m+k}{s-1} \leq \binom{s-1+m}{s-1}s^k. \tag{7.3}$$

Proof. From

$$\binom{s-1+m+k}{s-1} = \frac{(s-1+m+k)!}{(s-1)!(m+k)!} = \binom{s-1+m}{s-1}\frac{(s+m)\cdots(s+m+k-1)}{(m+1)\cdots(m+k)}$$

we have $\frac{s+m+j}{m+j+1} < s$ for each $0 \leq j \leq k-1$; this is because

$$s+m+j < s(m+j+1) \Leftrightarrow m(s-1)+j(s-1) > 0.$$

This proves (7.3). □

Lemma 7.3. *For $n, s \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{N}$, and $n > 2s$ let*

$$b_{m,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+m-1}} \binom{s+m-1}{s-1} \frac{1}{n^m},$$

then

$$-b_{m,n}(s) < \sum_{k=m}^{\infty} \binom{-\frac{2s-1}{2}}{k} \frac{1}{n^k} < b_{m,n}(s) \tag{7.4}$$

and

$$0 < \sum_{k=m}^{\infty} \binom{-\frac{2s-1}{2}}{k} \frac{(-1)^k}{n^k} < b_{m,n}(s). \tag{7.5}$$

Proof. For $s \geq 1$:

$$\begin{aligned}
\Big|\sum_{k=m}^{\infty} \binom{-\frac{2s-1}{2}}{k} \frac{1}{n^k}\Big| &= \Big|\sum_{k=m}^{\infty} \frac{(-1)^k}{4^k} \frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}} \frac{1}{n^k}\Big| \text{ (by Proposition 7.1)} \\
&\leq \sum_{k=m}^{\infty} \frac{1}{4^k} \frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}} \frac{1}{n^k} \\
&\leq \sum_{k=m}^{\infty} \frac{2\sqrt{s-1}}{\sqrt{\pi(s+k-1)}} \binom{s+k-1}{s-1} \frac{1}{n^k} \text{ (using } \frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}}) \\
&< \frac{2\sqrt{s-1}}{\sqrt{s+m-1}} \sum_{k=m}^{\infty} \binom{s-1+k}{s-1} \frac{1}{n^k} \\
&\quad \text{(using } \frac{1}{\sqrt{\pi}} < 1 \text{ and } \frac{1}{\sqrt{s+k-1}} \leq \frac{1}{\sqrt{s+m-1}} \text{ for all } k \geq m) \\
&= \frac{2\sqrt{s-1}}{\sqrt{s+m-1}} \sum_{k=0}^{\infty} \binom{s-1+m+k}{s-1} \frac{1}{n^{m+k}} \\
&= \frac{2\sqrt{s-1}}{\sqrt{s+m-1}} \frac{1}{n^m} \sum_{k=0}^{\infty} \binom{s-1+m+k}{s-1} \frac{1}{n^k}.
\end{aligned}$$

Now we apply Lemma 7.2 to obtain,

$$\begin{aligned}
\Big|\sum_{k=m}^{\infty} \binom{-\frac{2s-1}{2}}{k} \frac{1}{n^k}\Big| &\leq \frac{2\sqrt{s-1}}{\sqrt{s-1+m}} \frac{1}{n^m} \binom{s-1+m}{s-1} \sum_{k=0}^{\infty} \frac{s^k}{n^k} \\
&= \frac{2\sqrt{s-1}}{\sqrt{s+m-1}} \binom{s-1+m}{s-1} \frac{1}{n^m} \frac{n}{n-s} < b_{m,n}(s),
\end{aligned}$$

where the latter inequality is by $n > 2s$. This proves (7.4). Moreover, the bound we obtained also works for

$$\sum_{k=m}^{\infty} \frac{1}{4^k} \frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}} \frac{1}{n^k}.$$

Hence applying Proposition 7.1 implies (7.5). □

Lemma 7.4. *For $n, s \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{N}$, and $n > 2s$ let*

$$\beta_{m,n}(s) := \frac{2}{n^m}\binom{s+m-1}{s-1},$$

then

$$-\beta_{m,n}(s) < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{1}{n^k} < \beta_{m,n}(s) \tag{7.6}$$

and

$$0 < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{7.7}$$

Proof.

$$\begin{aligned}\Big|\sum_{k=m}^{\infty}\binom{-\frac{2s}{2}}{k}\frac{1}{n^k}\Big| =& \Big|\sum_{k=m}^{\infty}(-1)^k\binom{s+k-1}{s-1}\frac{1}{n^k}\Big| \quad \text{(by Proposition 7.1)}\\ \leq& \sum_{k=m}^{\infty}\binom{s+k-1}{s-1}\frac{1}{n^k} = \frac{1}{n^m}\sum_{k=0}^{\infty}\binom{s+k-1+m}{s-1}\frac{1}{n^k}\\ <& \frac{1}{n^m}\binom{s-1+m}{s-1}\sum_{k=0}^{\infty}\frac{s^k}{n^k} \quad \text{(by Lemma 7.2)},\end{aligned}$$

and geometric series summation implies (7.6). Applying Proposition 7.1 implies (7.7). □

Finally, we need another similar lemma which is easy to prove.

Lemma 7.5. *For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$c_{m,n}(s) := \frac{2}{m}\frac{s^m}{n^m},$$

then

$$-c_{m,n}(s) < \sum_{k=m}^{\infty}\frac{(-1)^{k+1}}{k}\frac{s^k}{n^k} < c_{m,n}(s) \text{ and } -c_{m,n}(s) < -\sum_{k=m}^{\infty}\frac{1}{k}\frac{s^k}{n^k} < 0 \tag{7.8}$$

and

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty}\binom{1/2}{k}\frac{s^k}{n^k} < \frac{c_{m,n}(s)}{\sqrt{m}} \text{ and } -\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty}\binom{1/2}{k}\frac{(-1)^k s^k}{n^k} < 0. \tag{7.9}$$

The following theorem was announced in the abstract; its proof is the goal of this section. To arrive at the intermediate inequality (7.14), we need our main result, Theorem 6.6. For the remainder of the proof, one spends some time on simplifying (7.14) in order to arrive at the desired form. In order to do, one needs the Lemmas 7.3 to 7.5 which we have proven above in this section.

Theorem 7.6. *For $n \geq 45$,*

$$p(n)^2 < \Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\Big)p(n-1)p(n+1),$$

and for $n \geq 120$

$$p(n)^2 > \Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\Big)p(n-1)p(n+1).$$

Proof. We set $m = 3$ in the first equation of Theorem 6.6, which gives for all $n \geq \left\lceil g(6) \right\rceil$:

$$\underbrace{P_n(5) - \frac{2}{3(24\alpha)^3}\frac{1}{n^3}}_{=:l(n)} < \log p(n) < \underbrace{P_n(5) + \frac{1}{3(24\alpha)^3}\frac{1}{n^3}}_{=:u(n)},$$

where we used the notation from Definition 6.4. This inequality has the form

$$l(n) < \log p(n) < u(n). \tag{7.10}$$

By substituting n by $n+1$ and multiplying by -1 into (7.10) we obtain

$$-u(n+1) < -\log p(n+1) < -l(n+1), \tag{7.11}$$

and by substituting n by $n-1$ and multiplying by -1 again into (7.10) gives

$$-u(n-1) < -p(n-1) < -l(n-1). \tag{7.12}$$

Multiplying (7.10) by 2, and by adding (7.11) and (7.12), results in

$$2l(n)-u(n-1)-u(n+1) < 2\log p(n)-\log p(n-1)-\log p(n+1) < 2u(n)-l(n-1)-l(n+1). \tag{7.13}$$

We define

$$A_1(n) := \log\Big(1+\frac{1}{n}\Big) + \log\Big(1-\frac{1}{n}\Big),$$

$$A_2(n) := -\pi\sqrt{\frac{2n}{3}}\Big(\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{(-1)^k}{n^k} + \sum_{k=1}^{\infty}\binom{1/2}{k}\frac{1}{n^k}\Big)$$

and for $t \geq 3$:

$$A_t(n) := -\frac{g_{t-2}}{(\sqrt{n})^{t-2}}\Big(\sum_{k=1}^{\infty}\binom{-\frac{t-2}{2}}{k}\frac{(-1)^k}{n^k} + \sum_{k=1}^{\infty}\binom{-\frac{t-2}{2}}{k}\frac{1}{n^k}\Big),$$

where g_n is as in Definition 5.1. Then from (7.13), by substituting $l(n)$ and $u(n)$ according to their definitions, we obtain,

$$-\frac{7}{(24\alpha)^3\cdot 3}\frac{1}{n^3} + \sum_{t=1}^{7}A_t(n) < 2\log p(n) - \log p(n-1) - \log p(n+1) < \sum_{t=1}^{7}A_t(n) + \frac{8}{(24\alpha)^3}\frac{1}{n^3},$$

which implies

$$-\frac{3}{(24\alpha)^3}\frac{1}{n^3} + \sum_{t=1}^{7}A_t(n) < 2\log p(n) - \log p(n-1) - \log p(n+1) < \sum_{t=1}^{7}A_t(n) + \frac{3}{(24\alpha)^3}\frac{1}{n^3}. \tag{7.14}$$

Finally, we establish bounds for the $A_t(n)$. For $t = 1$,

$$A_1(n) = \log(1+\frac{1}{n}) + \log(1-\frac{1}{n}) = -\frac{1}{n^2} - \frac{1}{2n^4} + \sum_{k=5}^{\infty}\frac{(-1)^{k+1}}{kn^k} - \sum_{k=5}^{\infty}\frac{1}{kn^k}.$$

Taking $s = 1$ and $m = 5$ in Lemma 7.5 we have

$$-\frac{1}{n^2} - \frac{1}{2n^4} - \frac{4}{5n^5} < A_1(n) < -\frac{1}{n^2} - \frac{1}{2n^4} + \frac{2}{5n^5}$$

which implies,

$$-\frac{1}{n^2}-\frac{2}{n^3}<A_1(n)<-\frac{1}{n^2}. \tag{7.15}$$

For $t=2$, note that

$$A_2(n)=-\pi\sqrt{\frac{2n}{3}}\Big(-\frac{5}{64n^4}-\frac{1}{4n^3}+\sum_{k=5}^{\infty}\binom{1/2}{k}\frac{(-1)^k}{n^k}+\sum_{k=5}^{\infty}\binom{1/2}{k}\frac{1}{n^k}\Big).$$

Applying Lemma 7.5, (7.9), with $s=1$ and $m=5$ gives

$$-\pi\sqrt{\frac{2n}{3}}\Big(-\frac{1}{4n^2}-\frac{5}{64n^4}-\frac{4}{5\sqrt{5}}\frac{1}{n^5}\Big)<A_2(n)<-\pi\sqrt{\frac{2n}{3}}\Big(-\frac{1}{4n^2}-\frac{5}{64n^4}+\frac{2}{5\sqrt{5}}\frac{1}{n^5}\Big),$$

which implies,

$$\frac{\pi}{\sqrt{24}n^{3/2}}<A_2(n)<\frac{\pi}{\sqrt{24}n^{3/2}}+\frac{2}{n^{5/2}}. \tag{7.16}$$

Next we consider odd indices; i.e., for $1\le t\le 3$,

$$A_{2t+1}(n)=-\frac{g_{2t-1}}{(\sqrt{n})^{2t-1}}\Big(\frac{\left(\frac{2t-1}{2}\right)_2}{n^2}+\frac{\left(\frac{2t-1}{2}\right)_4}{12n^4}+\sum_{k=5}^{\infty}\binom{-\frac{2t-1}{2}}{k}\frac{(-1)^k}{n^k}+\sum_{k=5}^{\infty}\binom{-\frac{2t-1}{2}}{k}\frac{1}{n^k}\Big),$$

where $(a)_k:=a(a-1)\dots(a-k+1)$. Applying Lemma 7.3 with $s=t$ and $m=5$ gives

$$\begin{aligned}&-\frac{g_{2t-1}}{(\sqrt{n})^{2t-1}}\Big(\frac{\left(\frac{2t-1}{2}\right)_2}{n^2}+\frac{\left(\frac{2t-1}{2}\right)_4}{12n^4}-\frac{4\sqrt{t}}{\sqrt{t+4}}\binom{t+4}{t-1}\frac{1}{n^5}\Big)<A_{2t+1}(n)\\&<-\frac{g_{2t-1}}{(\sqrt{n})^{2t-1}}\Big(\frac{\left(\frac{2t-1}{2}\right)_2}{n^2}+\frac{\left(\frac{2t-1}{2}\right)_4}{12n^4}+\frac{8\sqrt{t}}{\sqrt{t+4}}\binom{t+4}{t-1}\frac{1}{n^5}\Big),\end{aligned}$$

which implies

$$-\frac{3g_1}{4n^{5/2}}+\frac{4g_1}{\sqrt{5}}\frac{1}{n^3}<A_3(n)<-\frac{5g_1}{n^{5/2}}, \tag{7.17}$$

$$\frac{4\sqrt{6}g_3}{n^3}<A_5(n)<-\frac{29g_3}{n^{5/2}}, \tag{7.18}$$

$$\frac{4\sqrt{2}}{\sqrt{7}}\binom{7}{2}\frac{g_5}{n^3}<A_7(n)<-\frac{117g_5}{n^{5/2}}. \tag{7.19}$$

Finally, we consider even indices; i.e., for $1\le t\le 2$,

$$A_{2t+2}(n)=-\frac{g_{2t}}{(\sqrt{n})^{2t}}\Big(\frac{\left(-\frac{2t}{2}\right)_2}{n^2}+\frac{\left(-\frac{2t}{2}\right)_4}{12n^4}+\sum_{k=5}^{\infty}\binom{-\frac{2t}{2}}{k}\frac{(-1)^k}{n^k}+\sum_{k=5}^{\infty}\binom{-\frac{2t}{2}}{k}\frac{1}{n^k}\Big).$$

Applying Lemma 7.4 with $s=t$ and $m=5$, we obtain

$$\begin{aligned}&-\Big(\frac{(-t)_2}{n^2}+\frac{(-t)_4}{12n^4}-\frac{2}{n^5}\binom{t+4}{t-1}\Big)\frac{g_{2t}}{(\sqrt{n})^{2t}}<A_{2t+2}(n)\\&<-\Big(\frac{(-t)_2}{n^2}+\frac{(-t)_4}{12n^4}+\frac{4}{n^5}\binom{t+4}{t-1}\Big)\frac{g_{2t}}{(\sqrt{n})^{2t}}.\end{aligned}$$

From this,

$$\frac{2g_2}{n^3}<A_4(n)<-\frac{8g_2}{n^{5/2}}, \tag{7.20}$$

$$\frac{12g_4}{n^3} < A_6(n) < -\frac{40g_4}{n^{5/2}}. \tag{7.21}$$

Now, substituting (7.15) to (7.21) into (7.14) gives,

$$\begin{aligned}
&\frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} - \frac{1}{n^2} - \frac{3g_1}{4}\frac{1}{n^{5/2}} + \Big(-2 + \frac{4g_1}{\sqrt{5}} + 2g_2 + 4\sqrt{6}g_3 + 12g_4 + \frac{4\sqrt{2}}{\sqrt{7}}\binom{7}{2}g_5 - \frac{3}{(24\alpha)^3}\Big)\frac{1}{n^3} \\
<&2\log p(n) - \log p(n-1) - \log p(n+1) \\
<&\frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} - \frac{1}{n^2} + \Big(2 - 5g_1 - 8g_2 - 29g_3 - 40g_4 - 117g_5 + \frac{3}{(24\alpha)^3}\Big)\frac{1}{n^{5/2}}.
\end{aligned}$$

By using numerical estimations of the coefficient of $1/n^{5/2}$ and of the coefficient of $1/n^3$ in the lower bound, and of the coefficient of $1/n^{5/2}$ in the upper bound above, we are led to

$$\begin{aligned}
&\frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} - \frac{1}{n^2} + \frac{1}{4}\frac{1}{n^{5/2}} - \frac{4}{n^3} < 2\log p(n) - \log p(n-1) - \log p(n+1) \\
<&\frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} - \frac{1}{n^2} + \frac{7}{n^{5/2}}.
\end{aligned}$$

Next we observe that

$$-\frac{1}{n^2} + \frac{7}{n^{5/2}} < -\frac{\pi^2}{48n^3} \text{ for all } n \geq 50$$

and

$$-\frac{1}{n^2} + \frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} + \frac{1}{4}\frac{1}{n^{5/2}} - \frac{4}{n^3} > -\frac{1}{n^2} + \frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} \text{ for all } n \geq 257.$$

Therefore, for $n \geq 257$,

$$\frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2} < 2\log p(n) - \log p(n-1) - \log p(n+1) < \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{\pi^2}{48n^3}. \tag{7.22}$$

Because of $\log(1+x) < x$ for $x > 0$, we have

$$\log\Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\Big) < \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}, \tag{7.23}$$

and because of $x - \frac{x^2}{2} < \log(1+x)$ for all $x > 0$, we have

$$\frac{\pi}{\sqrt{24}n^{3/2}} - \frac{\pi^2}{48n^3} < \log\Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\Big). \tag{7.24}$$

Applying (7.23) and (7.24) to (7.22) gives

$$\log\Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\Big) < 2\log p(n) - \log p(n-1) - \log p(n+1) < \log\Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\Big),$$

which after exponentiation gives the desired result for $n \geq 257$. To extend the proofs of the statements for $n \geq 45$, resp. $n \geq 120$, is done by straight forward numerics. □

8. Appendix

8.1. **Methods to discover the results.** We will describe very briefly the mathematical experiments used in this research. We want to point out that without these experiments the theoretical results of this paper would never have been found. For this reason we feel that it is important to give at least a brief sketch of what led us to the final formulas and how we were led to conjecture special cases of related asymptotics. The final asymptotic formulas can easily be derived from our main result, Theorem 6.6 presented in Section 6.

In Section 3 we proved the following inequality

$$\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1-\frac{1}{2\sqrt{n}}\Big) < p(n) < \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1-\frac{1}{3\sqrt{n}}\Big) \tag{8.1}$$

which was found by mathematical experiments. Our proof uses methods similar to those used in [7] and [2]. In our attempt to prove the following formula for the asymptotics of $\log p(n)$,

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log(4\sqrt{3}) - \frac{0.44\ldots}{\sqrt{n}}, \tag{8.2}$$

we first tried to prove the log-version of (8.1). However, we soon realised that this inequality is not sharp enough in order to prove (8.2). We noted that the inequality for $p(n)$ in [2, Lemma 2.2] can be used instead. This formula says that for $n \geq 1206$,

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\Big), \tag{8.3}$$

where $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$. We observed that after taking the log of both sides, with some extra work, (8.2) can be proven. When we saw the asymptotics (1.3), discovered by Schoenfield and Kotesovec, we naturally wondered whether these asymptotics can also be proven by taking the log of an appropriate inequality. We observed that (8.3) is enough also to prove these asymptotics, and we observed that (8.3) can be used to prove an even more refined asymptotic formula that takes the form

$$\log p(n) \sim \pi\sqrt{\frac{2}{3}n} - \log n - \log 4\sqrt{3} + b_1\Big(\frac{1}{\sqrt{n}}\Big) + b_2\Big(\frac{1}{\sqrt{n}}\Big)^2 + \cdots + b_9\Big(\frac{1}{\sqrt{n}}\Big)^9,$$

where

$$\begin{aligned}
b_1 &= -\frac{\pi\sqrt{6}}{2^4 3^2} - \frac{\sqrt{6}}{2\pi} \approx -0.44328...,\\
b_2 &= \frac{1}{3\cdot 2^3} - \frac{3}{2^2\pi^2} \approx -0.034324...,\\
b_3 &= -\frac{\pi\sqrt{6}}{2^9 3^3} - \frac{\sqrt{6}}{2^5 3\pi} - \frac{\sqrt{6}}{2^2\pi^3} \approx -0.028428...,\\
b_4 &= \frac{1}{2^7 3^2} - \frac{1}{2^5\pi^2} - \frac{9}{2^4\pi^4} \approx -0.0080728...,\\
b_5 &= -\frac{\pi\sqrt{6}}{2^{13}3^4} - \frac{\sqrt{6}}{2^{10}3\pi} - \frac{\sqrt{6}}{2^6\pi^3} - \frac{9\sqrt{6}}{5\cdot 8\pi^5} \approx -0.0033007...,\\
b_6 &= \frac{1}{2^9 3^4} - \frac{1}{2^8 3\pi^2} - \frac{3}{2^6\pi^4} - \frac{9}{2^4\pi^6} \approx -0.001174124716...,\\
b_7 &= -\frac{5\pi\sqrt{6}}{2^{19}3^5} - \frac{5\sqrt{6}}{2^{14}3^3\pi} - \frac{5\sqrt{6}}{2^{11}3\pi^3} - \frac{3\sqrt{6}}{2^7\pi^5} - \frac{3^3\sqrt{6}}{2^4 7\pi^7} \approx -0.00045651...,\\
b_8 &= \frac{1}{2^{14}3^4} - \frac{1}{2^{11}3^2\pi^2} - \frac{3}{2^{10}\pi^4} - \frac{3^2 2}{2^7\pi^6} - \frac{3^4}{2^7\pi^8} \approx -0.00017464...,\\
b_9 &= -\frac{7\pi\sqrt{6}}{2^{23}3^6} - \frac{35\sqrt{6}}{2^{20}3^4\pi} - \frac{35\sqrt{6}}{2^{15}3^3\pi^3} - \frac{7\sqrt{6}}{2^{12}\pi^5} - \frac{9\sqrt{6}}{2^8\pi^7} - \frac{9\sqrt{6}}{2^5\pi^9} \approx -0.000068757...,\\
&\vdots
\end{aligned}$$

Of course we wondered whether one can get an even better formula. The only obstacle that seemed to limit us was the 10 in the formula (8.3) above. This led us to look into the details of the proof of (8.3), and we observed that the 10 can be replaced by a k. This then led us to the discovery of the complete asymptotics. That is, we also got $b_{10}, b_{11}, \ldots$, etc. At this point we still were not fully satisfied. Even though we observed that the formula (8.3) could be generalised, it was not a proper generalization because we could not say explicitly for which precise range of n the generalized inequality (4.3) for $p(n)$ holds. We only could say that there is some sufficiently big constant $C(k)$ such that (4.3) for all $n > C(k)$.

We felt that this is not a proper generalization because (8.3) gives $C(10)$ explicitly, namely $C(10) = 1206$. After some work, we realized that we can obtain an explicit expression for $C(k)$, which is very close to the optimal value, according to mathematical experiments. This $C(k)$ is our $g(k)$ of Section 4 where we gave a generalization of (8.3).

Because (8.3) could be generalized, we suspected that also (8.1) could be generalized. The difference between the two inequalities is that (8.1) is in terms of $\sqrt{n}$, while (8.3) is in terms of $\mu(n)$. We again took the log of both sides of the generalized version of (8.3) and aimed not only at getting a refined asymptotic but rather a new type of inequality. This was achieved in Section 6. However, even after we found a preliminary version of Theorem 6.6, still something was missing. We wondered whether we can guarantee that this inequality is optimal in some sense, and not overestimated. After various experiments, we got control in the form (6.5) and (6.6), where the error term in the inequality cannot be improved to a smaller integer in

```
with(combinat);
rt := proc (n) local rtn, k;
    rtn := combinat:-numbpart(n);
    for k to (1/2)*n do
        rtn := rtn+combinat:-numbpart(k)*combinat:-numbpart(n-2*k)
    end do;
    rtn
end proc
```

FIGURE 3. Procedure for computing the number of cubic partitions of n.

the numerator—the same time keeping the statement unaltered. This is the point where we stopped.

8.2. **Discovery of Kotesovec's formula** (1.5) **by regression analysis.** We used the procedure shown in Figure 3 to compute the sequence $a(n)$ defined in (1.4). This procedure works fine for computing $a(n)$ in the range $1 \le n \le 2^{15}$. The computation took 24 hours on a notebook computer with Intel Core i7 CPU.

To find the approximate relation between $\log a(n), \sqrt{n}$ and $\log(n)$, substitute the values $n = 2^k,\ 2^{k+1},\ 2^{k+2}$ into the target expression,

$$\log a(n) \sim \alpha \cdot \sqrt{n} - \beta \cdot \log(n) - \log(\gamma),$$

to obtain a system with three equations:

$$\begin{cases} \log_2 a(2^k) = a_k \log_2(e) \cdot \sqrt{2^k} - b_k \cdot k - c_k + \varepsilon_k, \\ \log_2 a(2^{k+1}) = a_k \log_2(e) \cdot \sqrt{2^{k+1}} - b_k \cdot (k+1) - c_k + \varepsilon_{k+1}, \\ \log_2 a(2^{k+2}) = a_k \log_2(e) \cdot \sqrt{2^{k+2}} - b_k \cdot (k+2) - c_k + \varepsilon_{k+2}, \end{cases}$$

and solve it successively for k from 1 to 13. Let (a_k, b_k, c_k) be the solution of the above equation system under the assumption $\varepsilon_k = \varepsilon_{k+1} = \varepsilon_{k+2} = 0$ for all $k \in \{1, \dots, 13\}$. The numerical values of the (a_k, b_k, c_k) are presented in Figure 4. In the limit $k \to \infty$,

$$\begin{aligned} a_k &= \tfrac{\log_2 a(2^k) + \log_2 a(2^{k+2}) - 2\log_2 a(2^{k+1})}{(3-2\sqrt{2})\sqrt{2^k}\log_2(e)} \to \alpha, \\ b_k &= \tfrac{\log_2 a(2^k) + \log_2 a(2^{k+2}) - 2\log_2 a(2^{k+1})}{\sqrt{2}-1} - \{\log_2 a(2^{k+1}) - \log_2 a(2^k)\} \to \beta, \\ c_k &= \tfrac{2^{a_k \log_2(e)\sqrt{2^k} - k\, b_k}}{a(2^k)} \to \log_2(\gamma). \end{aligned}$$

The numerical values in Figure 4 clearly support the precise values

$$\alpha = \pi, \quad \beta = \frac{5}{4}, \quad \gamma = 2^3 = 8.$$

Note that we have used a sub-sequence $a(2^k)$, $k = 1, 2, \dots, 15$. The regression analysis to obtain the numerical data for Fig. 1 and Fig. 2 are rather routine, so we will not list any further details here.

$$
\begin{aligned}
a_1 &= 2.856681587, & b_1 &= .8292510713, & c_1 &= 3.414213592,\\
a_2 &= 3.104034810, & b_2 &= 1.124879663, & c_2 &= 3.536666941,\\
a_3 &= 3.138359816, & b_3 &= 1.182896591, & c_3 &= 3.502681525,\\
a_4 &= 3.134634608, & b_4 &= 1.173992098, & c_4 &= 3.516802205,\\
a_5 &= 3.133095462, & b_5 &= 1.168789090, & c_5 &= 3.530255390,\\
a_6 &= 3.135881324, & b_6 &= 1.182107364, & c_6 &= 3.482499147,\\
a_7 &= 3.138560309, & b_7 &= 1.200219526, & c_7 &= 3.399441064,\\
a_8 &= 3.140063351, & b_8 &= 1.214590204, & c_8 &= 3.319170509,\\
a_9 &= 3.140825268, & b_9 &= 1.224893620, & c_9 &= 3.251316705,\\
a_{10} &= 3.141207944, & b_{10} &= 1.232211776, & c_{10} &= 3.195805627,\\
a_{11} &= 3.141399944, & b_{11} &= 1.237403601, & c_{11} &= 3.151230912,\\
a_{12} &= 3.141496152, & b_{12} &= 1.241082894, & c_{12} &= 3.115957155,\\
a_{13} &= 3.141544378, & b_{13} &= 1.243690699, & c_{13} &= 3.088371824.
\end{aligned}
$$

FIGURE 4. Numerical values of the (a_k, b_k, c_k).

8.3. **Mathematica computations used in the proof of Theorem 6.6.** We present Mathematica computations needed in the proof of Theorem 6.6. Note that in order to complete the proof of Theorem 6.6 we needed to bound four terms by 1; however, in each inequality proven with Mathematica as shown below, we checked that each inequality holds in fact for bounds smaller than 1, namely $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{26}$ and $\frac{1}{26}$. The Mathematica computations are based on Cylindrical Algebraic Decomposition [6].

In[1]:= $\mathbf{a := \frac{\pi^2}{36+\pi^2}}$

In[2]:= $\mathbf{(mu1, mu2, nu) := \left(\frac{\sqrt{6}}{2\pi}\left(\frac{\pi^2}{72}+1+\frac{a}{2(1-a)}\right), \frac{\sqrt{6}}{2\pi}\left(1+\frac{a}{2}\right), 3\frac{a}{2}-\frac{1}{2}\right)}$

In[3]:= $\mathbf{CylindricalDecomposition[\{(2w-2)^2\frac{\left(\frac{-\mu 1}{2w-1}+\frac{-x}{(24a)2w}\right)^2}{(-1+x)^2} < \frac{1}{5}, w \geq 1, x \geq 4\}, \{w, x\}]}$

Out[3]= $w \geq 1\ \&\&\ x \geq 4$

In[4]:= $\mathbf{CylindricalDecomposition[\{\left(\left(\frac{2w-3}{x - 24\ a\ mu1}\right)\left(\frac{1}{2w-2}+\frac{x}{24a(2w-1)}\right)\right)^2 < \frac{1}{3}, w \geq 2, x \geq 7\}, \{w, x\}]}$

Out[4]= $w \geq 2\ \&\&\ x \geq 7$

In[5]:= $\mathbf{CylindricalDecomposition[\{\left((2w-2)\frac{\left(\frac{-mu2}{2\ w-1}+\frac{y}{24a(2w)}\right)}{y-2nu}\right)^2 < \frac{1}{26}, w \geq 1, y \geq 1\}, \{w, y\}]}$

Out[5]= $w \geq 1\ \&\&\ y \geq 1$

In[6]:= $\mathbf{CylindricalDecomposition[\{\left(\left(\frac{2w-3}{y + 24\ a\ mu2}\right)\left(\frac{2\ nu}{2w-2}+\frac{y}{24a(2w-1)}\right)\right)^2 < \frac{1}{26}, w \geq 2, y \geq 1\}, \{w, y\}]}$

Out[6]= $w \geq 2\ \&\&\ y \geq 1$

REFERENCES

[1] C. Bessenrodt and K. Ono. Maximal multiplicative properties of partitions. *Ann. Comb.*, 20(1):59–64, 2016.

[2] W. Y. C. Chen, D. X. Q. Jia and L. X. Wang. Higher order Turán inequalities for the partition function. *Trans. Amer. Math. Soc.*, 372(3):2143–2165, 2019.

[3] W. Y. C. Chen, L. X. Wang and G. Y. B. Xie. Finite differences of the logarithm of the partition function. *Math. Comput.*, 85:825–847, 2016.

[4] W. Y. C. Chen. Recent developments on log-concavity and q-log-concavity of combinatorial polynomials. *22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*, 2010. http://www.billchen.org/talks/2010-FPSAC.pdf

[5] W. Y. C. Chen and B. L. S. Lin. Congruences for the number of cubic partitions derived from modular forms. https://archive.org/details/arxiv-0910.1263.

[6] G. E. Collins. Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition. *Lecture Notes in Computer Science*, 33:134–183, 1975.

[7] S. DeSalvo and I. Pak. Log-concavity of the partition function. *Ramanujan J.*, 38(1):61–73, 2015.

[8] P. Erdős. On an elementary proof of some asymptotic formulas in the theory of partitions. *Ann. Math.*, 43(2):437–450, 1942, doi:10.2307/1968802.

[9] R. K. Guy. Letter to Morris Newman, Aug 21 1986, concerning A2513 [E]. https://oeis.org/A002513/a002513.pdf

[10] G. H. Hardy and S. Ramanujan. Asymptotic formulae in combinatory analysis. *Proc. London Math. Soc.*, 17:75–115, 1918.

[11] V. Kotesovec. A method of finding the asymptotics of q-series based on the convolution of generating functions. https://arxiv.org/abs/1509.08708.

[12] D. H. Lehmer. On the remainders and convergence of the series for the partition function. *Trans. Amer. Math. Soc.*, 46:362–373, 1939.

[13] M. Newman. Construction and application of a class of modular function. *Proc. London Math. Soc.*, 9(3):373–387, 1959.

[14] OEIS Foundation Inc. (2021). *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org

[15] W. A. Pribitkin. Simple upper bounds for partition functions. *Ramanujan J.*, 18(1):113–119, 2009.

[16] H. Rademacher. *Topics in Analytic Number Theory*, Springer, 1973.

[17] C. Schneider. Symbolic summation assists combinatorics. *Sem. Lothar. Combin.*, 56:1–36, 2007, Article B56b.

[18] M. Shen, Z. Yang and Z. Zeng. Some applications of sparse interpolation via compressive sensing. *MICA2016, Milestones in computer algebra*, July 16-18, Waterloo, 2016. https://cs.uwaterloo.ca/conferences/mica2016/MICA_abstract.pdf

[19] J. V. Uspensky. Les expressions asymntotiques des fonctions numériques, coincidant dans les problémes de dislocation des nombres en composés. *Bulletin de l'Académie des Sciences de Russie*, VI série, 14:199–218, 1920.

[20] W.-H. Zeng. Some Properties of Morris Newman Sequence. Thesis for Bachelor Degree in Mathematics. East China Normal University, Shanghai, China, 2016.

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