HOOK TYPE ENUMERATION AND PARITY OF PARTS IN PARTITIONS

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ABSTRACT. This paper is devoted to study an association between hook type enumeration and counting integer partitions subject to parity of its parts. We shall primarily focus on a result of Andrews in two possible direction. First, we confirm a conjecture of Rubey and secondly, we extend the theorem of Andrews in a more general set up.

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1. INTRODUCTION

The field of hypergeometric series and partitions have been closely connected ever since Euler's primarily work on the subject. Since then it became a standard method to use results in q-series in order to aid the proofs of numerous partition identities. Study on parity of parts in partitions sprout in work of Euler, Sylvester, Franklin, Glaisher, Fine and Andrews among others. In recent years, parity study in partitions became significant. Andrews' study in [2] continued canonically with work of Bringmann and Jennings-Shaffer [4] which connects to partial theta functions and often with modular forms. Another beautiful aspect of studying parity in partitions has been done in [3] by looking through the lens of Ramanujan's theta functions.

We have endeavoured to show the impact of hook type enumeration involved in our construction by proving a conjecture made by Rubey (cf. Theorem 1.3) on the nature of partition statistics studied by Andrews in Theorem 1.1. In the same spirit of enquiry we also provide generalization of Theorem 1.1 which equates two different partition functions based on parts separated by parity. Our aim has been to show how simple and elegant combinatorial arguments can be in a sense unify disparate areas of the subject by a common thread.

A partition of $n \geq 0$ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\ell)$ of positive integers whose sum is n, denoted by $\lambda \vdash n$. $p(n)$ denotes the number of partitions of n and $P(n)$ is the set of all partitions of n. Define $\ell = \ell(\lambda)$ to be the number of parts in λ, $a(\lambda)$ to be the largest part of λ and $\mathrm{mult}(\lambda_i)$ to be the multiplicity of the part λ_i. We also use $\lambda = (\lambda_1^{m_1} \dots \lambda_\ell^{m_\ell})$ as an alternative notation for partition. For partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ define the sum $\lambda + \mu$ to be the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$. Similarly, define the union $\lambda \cup \mu$ to be the partition with parts $\{\lambda_i, \mu_j\}$, arranged in non-increasing order. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_m) \vdash n$ (where m is the largest part of λ) by choosing $\lambda_i^{'}$ as the number of parts of λ that are $\geq i$. The partition $\lambda^{'}$ is called the conjugate of λ. Notice that the graphical representation of the conjugate is obtained by reflecting the Young diagram (which we will define in few lines) in the main diagonal. For example, if $\lambda = (6,3,3,2,1)$, then conjugate of λ is $\lambda^{'} = (5,4,3,1,1,1)$.

To each partition $\lambda \vdash n$ we associate Y_λ, the celebrated graphical representation called the Young diagram of λ. For each box v in Y_λ, define the hook length of v, denoted by $h_v(\lambda)$, to be the number of boxes u such that $u = v$ or u lies in the same column as v and above v or in the same row as v and to the right of v. The hook length multiset of λ, denoted by $\mathcal{H}_\lambda$, is the multiset of all hook lengths of λ. Each hook length h can be split into $h = a + l + 1$, where a is the arm length (the no. of boxes to the right in the same row) and l the leg length (the no. of boxes above in the same column). The ordered pair (a, l) is called hook type of the chosen box in the Young tableau. A hook length tableau (resp. hook type tableau) is obtained by filling in the boxes of the Young diagram with hook length (resp. hook type) of each box. The boxes will be colored according to the index of the parts in the color partition considered.

For $\lambda = (6,3,3,2) \vdash 14$, the hook length tableau is

2	1				
4	3	1			
5	4	2			
9	8	6	3	2	1

Figure 1: Hook length tableau for the partition $\lambda = (6,3,3,2)$

and the hook type tableau of Y_λ is

$(1,0)$	$(0,0)$				
$(2,1)$	$(1,1)$	$(0,0)$			
$(2,2)$	$(1,2)$	$(0,1)$			
$(5,3)$	$(4,3)$	$(3,2)$	$(2,0)$	$(1,0)$	$(0,0)$

Figure 2: Hook type tableau for the partition $\lambda = (6,3,3,2)$.

This paper principally aims to study problems on parity of parts in integer partitions originated from the work of Andrews [2]. Rubey conjectured that there may be a possible bijective proof of Andrews' theorem other than the given one in proof of Theorem 1.1 through which we can observe that there exists an one to one correspondence between total number of parts strictly greater than 1 and total number of boxes with a specific hook type respectively in partitions enumerated by $\mathcal{O}_d(n)$ and $p_{eu}^{od}(n)$, given in Theorem 1.3. Whereas, studying the structure of Young diagram we will see how one can extend the principle of bijection given by

Andrews in [2, Theorem 1.3] to get partition identity by relaxing some constraints on parts in Theorem 1.5.

Theorem 1.1. [2, Theorem 1.3] *Let $\mathcal{O}_d(n)$ denote the number of partitions of n in which the odd parts are distinct and each positive odd integer smaller than the largest odd part must appear as a part. Then*

$$p_{eu}^{od}(n) = \mathcal{O}_d(n),$$

where $p_{eu}^{od}(n)$ denotes the number of partitions of n in which each even part is less than each odd part and odd parts are distinct.
For example, the 7 partitions enumerated by $\mathcal{O}_d(8)$ are 8, $6+2$, $4+4$, $4+2+2$, $4+3+1$, $3+2+2+1$, $2+2+2+2$ and those enumerated by $p_{eu}^{od}(8)$ are 8, $7+1$, $5+3$, $6+2$, $4+4$, $4+2+2$, $2+2+2+2$.

Definition 1.2. *Let $O_d(n)$ (resp. $P_{eu}^{od}(n)$) denote the set of all partitions counted by $\mathcal{O}_d(n)$ (resp. $p_{eu}^{od}(n)$).*
$Q_{>1}(\lambda)$ is defined to be the total number of parts strictly greater than 1 in $\lambda \in O_d(n)$ and

$$Q_{>1}(n) = \sum_{\lambda \in O_d(n)} Q_{>1}(\lambda).$$

$B_{(\mathfrak{o},0)}(\lambda)$ denote the total number of boxes with hook type $(\mathfrak{o}, 0)$ for odd positive integers $\mathfrak{o}$ in Y_λ where $\lambda \in P_{eu}^{od}(n)$ and

$$B_{(\mathfrak{o},0)}(n) = \sum_{\lambda \in P_{eu}^{od}(n)} B_{(\mathfrak{o},0)}(\lambda).$$

As an instance, for $n = 8$

$O_d(8)$	$Q_{>1}(\lambda)$	$P_{eu}^{od}(8)$	$B_{(\mathfrak{o},0)}(\lambda)$
8	1	8	4
$6+2$	2	$7+1$	3
$4+4$	2	$6+2$	3
$4+3+1$	2	$5+3$	2
$4+2+2$	3	$4+4$	2
$3+2+2+1$	3	$4+2+2$	2
$2+2+2+2$	4	$2+2+2+2$	1
Total	$Q_{>1}(8) = 17$	*Total*	$B_{(\mathfrak{o},0)}(8) = 17$

Theorem 1.3. [5, Rubey's Conjecture] *For all positive integers n,*

$$Q_{>1}(n) = B_{(\mathfrak{o},0)}(n)$$

and for $\mu \in P_{eu}^{od}(n)$,

$$B_{(\mathfrak{o},0)}(\mu) = \left\lfloor \frac{a(\mu)}{2} \right\rfloor.$$

Let $p_{eu}^{ou}(n)$ denote the number of partitions such that odd parts of a partition are unrestricted and each even part is less that each odd part of the considered partition and the set of all such partitions is denoted by $P_{eu}^{ou}(n)$. For example, there are 12 partitions enumerated by $p_{eu}^{ou}(9)$ are $9, 7+2,\ 7+1+1,\ 5+4,\ 5+3+1,\ 5+2+2,\ 5+1+1+1+1,\ 3+3+3,\ 3+3+1+1+1,\ 3+2+2+2,\ 3+1+1+1+1+1+1,\ 1+1+1+1+1+1+1+1+1.$

Definition 1.4. *For $\lambda \vdash n$,*

$$\begin{aligned}
\mathit{OMax}\ (\lambda) &:= \text{ greatest odd part of } \lambda \text{ and } \mathit{OMax}(\lambda) = 0 \text{ if } \lambda \text{ has no odd part},\\
\mathit{EMax}\ (\lambda) &:= \begin{cases} \text{greatest even part of } \lambda, & \text{if even parts occur in } \lambda,\\ 0, & \text{otherwise,}\end{cases}\\
\mathit{OEMaxSum}\ (\lambda) &:= \mathit{OMax}(\lambda) + \mathit{EMax}(\lambda),\\
\mathit{OEMaxDiff}\ (\lambda) &:= |\mathit{OMax}(\lambda) - \mathit{EMax}(\lambda)|,\\
O_{\overline{u}}(n) &:= \{\lambda \vdash n : \text{for any odd } k < \mathit{OMax}(\lambda); k \text{ must appear as a part of } \lambda\},\\
O_{\overline{u}}^{*}(n) &:= \{\lambda \in O_{\overline{u}}(n) : \mathit{OEMaxDiff}\ (\lambda) = \min\{\mathit{OEMaxDiff}\ (\lambda^{'}) : \lambda^{'} \in O_{\overline{u}}(n)\}\},\\
o_{\overline{u}}^{*}(n) &:= \Big|\{\lambda \vdash n : \lambda \in O_{\overline{u}}^{*}(n)\}\Big|.
\end{aligned}$$

For example, there are 12 partitions enumerated by $o_{\overline{u}}^{*}(9)$ are $8+1,\ 6+2+1,\ 5+3+1,\ 4+4+1,\ 4+3+1+1,\ 4+2+2+1,\ 3+2+1+1+1+1,\ 3+3+1+1+1+1+1,\ 3+1+1+1+1+1+1,\ 2+2+2+2+1,\ 2+2+2+1+1+1,\ 1+1+1+1+1+1+1+1+1$; we see that according to our definition, the partition $\lambda = (6,1,1,1) \notin O_{\overline{u}}^{*}(9)$ but $(4,3,1,1) \in O_{\overline{u}}^{*}(9)$.

Theorem 1.5. *$o_{\overline{u}}^{*}(n) = p_{eu}^{ou}(n)$.*

Remark 1.6. *From the above Theorem 1.5, it is clear that if we restrict ourselves to the distinct odd parts, then Theorem 1.1 follows as a corollary. Moreover, following the bijection given in the proof of Theorem 1.5, it follows that total number of odd parts in $O_{\overline{u}}^{*}(n)$ is equal to the total number of odd parts in $P_{eu}^{ou}(n)$, whenever odd parts occur in $O_{\overline{u}}^{*}(n)$. Let $\lambda \in O_{\overline{u}}^{*}(n)$ with λ_o consists of only odd parts in λ, say $\ell(\lambda_o) = r$ and then following three Cases in the Proof of Theorem 1.5, we see that the resulting partition $\mu \in P_{eu}^{ou}(n)$ has also r odd parts. We can also observe that the same odd statistics follows in Theorem 1.1.*

2. Proof of Theorem 1.3 and 1.5

Proof of Theorem 1.3: First, we note that for $\lambda = (\lambda_1^{k_1} \dots \lambda_m^{k_m}) \vdash n$, we have

$$B_{(\mathfrak{o},0)}(n) = \Big\lfloor \frac{\lambda_m}{2} \Big\rfloor + \sum_{i=1}^{m-1} \Big\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \Big\rfloor. \tag{2.1}$$

We construct a bijection $\phi : O_d(n) \longrightarrow P_{eu}^{od}(n)$ by defining it on odd (resp. even) component of a partition $\lambda \in O_d(n)$.

Let $\lambda \in O_d(n)$ with all parts odd. Following the definition of $O_d(n)$, we can write $\lambda := \lambda_o = (2t-1, 2t-3, \ldots, 3, 1) \vdash t^2$ for a non-negative integer t. We define

$$\phi(\lambda_o) = \lambda_o \tag{2.2}$$

and clearly, $\lambda_o \in P_{eu}^{od}(n)$. So, ϕ is a one to one map. Consequently, $Q_{>1}(\lambda_o) = t-1$ and by (2.1), it follows that

$$B_{(\mathfrak{o},0)}(\phi(\lambda_o) = t-1.$$

As a trivial remark, we note that $\left\lfloor \frac{a(\phi(\lambda_o))}{2} \right\rfloor = \left\lfloor \frac{2t-1}{2} \right\rfloor = t-1$.

Now, for $\lambda \in O_d(n)$ with even parts only, say, $\lambda := \lambda_e = (\mu_1^{m_{\mu_1}} \ldots \mu_s^{m_{\mu_s}})$. Define $\phi(\lambda_e) = 2(\frac{1}{2}\ \lambda_e)'$ with multiplication and division by 2 is component-wise. Following Definition **??**, for all $1 \le i \le s$

$$\phi(\lambda_e) = \left(\tilde{\mu_1}^{\mu_s/2}\ \tilde{\mu_2}^{(\mu_{s-1}-\mu_s)/2} \ldots \tilde{\mu_s}^{(\mu_1-\mu_2)/2}\right) \text{ with } \tilde{\mu_i} = 2\sum_{j=1}^{s-i+1} m_{\mu_j} \tag{2.3}$$

and therefore, $\phi(\lambda_e) \in P_{eu}^{od}(n)$.

Let assume $\phi(\lambda_e) = \phi(\lambda_e^*)$ with $\lambda_e,\ \lambda_e^* \in O_d(n)$. For $\lambda_e = (\mu_1^{m_{\mu_1}} \ldots \mu_s^{m_{\mu_s}})$ and $\lambda_e^* = (\mu_1^{*m_{\mu_1}^*} \ldots \mu_s^{*m_{\mu_s}^*})$, from (2.3), we can observe that $s = t$, $m_{\mu_i} = m_{\mu_i}^*$ and $\mu_i = \mu_i^*$. This implies ϕ is a one to one map.

So, $Q_{>1}(\lambda_e) = \sum_{i=1}^{s} m_{\mu_i}$ and by (2.1), it follows that

$$B_{(\mathfrak{o},0)}(\phi(\lambda_e)) = \sum_{i=1}^{s} m_{\mu_i}.$$

Moreover, $\left\lfloor \frac{a(\phi(\lambda_e))}{2} \right\rfloor = \left\lfloor \frac{\tilde{\mu_1}}{2} \right\rfloor = \sum_{i=1}^{s} m_{\mu_i}$.

In a more general set up, consider $\lambda \in O_d(n)$ with both odd and even component, namely, λ_o with $\ell(\lambda_o) = t$ for some strictly positive integer t and $\lambda_e = (\mu_1^{m_{\mu_1}} \ldots \mu_s^{m_{\mu_s}})$. We define $\phi(\lambda) = \phi(\lambda_e \cup \lambda_o) = \phi(\lambda_e) + \phi(\lambda_o)$ where addition of parts is being done component-wise. From (2.2) and (2.3), it follows that $\phi(\lambda) \in P_{eu}^{od}(n)$ and by similar argument as explained in the odd and even cases, ϕ is one to one.

By definition of λ, it follows that

$$Q_{>1}(\lambda) = t - 1 + \sum_{i=1}^{s} m_{\mu_i} \tag{2.4}$$

and by (2.1), we have

$$B_{(\mathfrak{o},0)}(\lambda) = t - 1 + \sum_{i=1}^{s} m_{\mu_i}. \tag{2.5}$$

Furthermore,

$$\left\lfloor \frac{a(\phi(\lambda))}{2} \right\rfloor = t - 1 + \sum_{i=1}^{s} m_{\mu_i}. \tag{2.6}$$

Now, it remains to show that ϕ is onto. Let $\mu \in P_{eu}^{od}(n)$ and μ has t odd parts with $t \in \mathbb{Z}_{\ge 0}$. Let us consider the partition $\lambda_o \vdash t^2$ with $\lambda_o = (2t-1, 2t-3, \ldots, 3, 1)$. Let $\mu_e :=$

$\frac{1}{2}(\mu - \lambda_o)$ where multiplication (by scalars) and subtraction of parts is component-wise. Define $\phi^{-1}(\mu) = 2\mu_e^{'} \cup \lambda_o$ where $\mu_e^{'}$ is the conjugate partition of μ_e and we observe that $\nu := 2\mu_e^{'} \cup \lambda_o \in O_d(n)$. We split ν into its even component (resp. odd) by ν_e (resp. ν_o). Following the definition of ϕ in context of even component, we get the transformed even component of ν, $\phi(\nu_e) = 2\nu_e^{'}$ where $\nu_e^{'}$ is the conjugate partition of ν_e and $\nu_o = \lambda_o$. Now, $\phi(\nu) = \phi(2\nu_e^{'} \cup \lambda_o) = 2\nu_e^{'} + \lambda_o = 2\mu_e + \lambda_o = 2(\frac{1}{2}(\mu - \lambda_o)) + \lambda_o = \mu$ because for $\lambda \in O_d(n)$, $\phi(\lambda) = 2\lambda_e^{'} + \lambda_o$ where λ_e are even parts of λ divided by 2.
From (2.6), we can observe that for $\mu \in P_{eu}^{od}(n)$,

$$B_{(\mathfrak{o},0)}(\mu) = \left\lfloor \frac{a(\mu)}{2} \right\rfloor.$$

Before we conclude the proof, will provide an explicit example to show how the bijection had been work out. To be consistent with the example given in the Theorem 1.3, for $n = 8$, the map $\phi : O_d(8) \longrightarrow P_{eu}^{od}(8)$ described as follows

$$\begin{gathered}
8 \longrightarrow 4 \longrightarrow 1+1+1+1 \longrightarrow 2+2+2+2 \\
6+2 \longrightarrow 3+1 \longrightarrow 2+1+1 \longrightarrow 4+2+2 \\
4+4 \longrightarrow 2+2 \longrightarrow 2+2 \longrightarrow 4+4 \\
4+2+2 \longrightarrow 2+1+1 \longrightarrow 3+1 \longrightarrow 6+2 \\
4+3+1 \longrightarrow \{4\} \cup \{3,1\} \longrightarrow \{2\} \cup \{3,1\} \longrightarrow \{1,1\} \cup \{3,1\} \longrightarrow \{2,2\} \cup \{3,1\} \longrightarrow 5+3 \\
3+2+2+1 \longrightarrow \{2,2\} \cup \{3,1\} \longrightarrow \{1,1\} \cup \{3,1\} \longrightarrow \{2\} \cup \{3,1\} \longrightarrow \{4\} \cup \{3,1\} \longrightarrow 7+1 \\
2+2+2+2 \longrightarrow 1+1+1+1 \longrightarrow 4 \longrightarrow 8
\end{gathered}$$

and for the inverse map $\psi : P_{eu}^{od}(8) \longrightarrow O_d(8)$

$$\begin{gathered}
8 \longrightarrow 4 \longrightarrow 1+1+1+1 \longrightarrow 4 \\
7+1 \longrightarrow \{2\} \cup \{3,1\} \longrightarrow \{1,1\} \cup \{3,1\} \longrightarrow \{2,2\} \cup \{3,1\} \rightarrow 3+2+2+1 \\
6+2 \longrightarrow 3+1 \longrightarrow 2+1+1 \longrightarrow 4+2+2 \\
5+3 \longrightarrow \{1,1\} \cup \{3,1\} \longrightarrow \{2\} \cup \{3,1\} \longrightarrow \{4\} \cup \{3,1\} \rightarrow 4+3+1 \\
4+4 \longrightarrow 2+2 \longrightarrow 2+2 \longrightarrow 4+4 \\
4+2+2 \longrightarrow 2+1+1 \longrightarrow 3+1 \longrightarrow 6+2 \\
2+2+2+2 \longrightarrow 1+1+1+1 \longrightarrow 4 \longrightarrow 8.
\end{gathered}$$

□

Proof of Theorem 1.5: Consider the Young diagram Y_λ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in O_{\overline{u}}^*(n)$. We separate λ into $\lambda_o = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$ where $1 \leq o_i \leq l$ and $\lambda_e = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_t})$ where $1 \leq e_j \leq l$ according to the odd and even parts, respectively. Let Y_{λ_o} and Y_{λ_e} be the corresponding Young diagrams of λ_o and λ_e. Next, we join Y_{λ_o} and Y_{λ_e} by successively adjoining their rows with respect to the ordering of the parts in λ_o, λ_e, respectively, starting with the largest one and ending with the smallest one. Call the restricting Young diagram Y_μ. Now, we consider the following three cases
1. If the number of odd parts is equal to the number of even parts in a partition $\lambda \in O_{\overline{u}}^*(n)$, then Y_μ is with $\mu \in P_{eu}^{ou}(n)$. Since $\lambda_o = (\lambda_{o_1}, \ldots, \lambda_{o_r})$ and $\lambda_e = (\lambda_{e_1}, \ldots, \lambda_{e_r})$ have equal number of parts, the resulting partition $\mu = (\lambda_{o_1} + \lambda_{e_1}, \ldots, \lambda_{o_r} + \lambda_{e_r})$. Correspondingly no

row remains left neither in Y_{λ_o} nor in Y_{λ_e} after adjoining.
2. Suppose the number of odd parts is greater than the number of even parts in a partition $\lambda \in O^*_{\overline{u}}(n)$; let the difference be t. Then a similar argument shows that the t rows in Y_{λ_o} remain left after adjoining of rows of Y_{λ_o} and Y_{λ_e}. Therefore, in the resulting Y_μ with $\mu \in P^{ou}_{eu}(n)$, t rows will be positioned in the same order as in Y_{λ_o}.
3. Suppose the number of even parts is greater than the number of odd parts in a partition $\lambda \in O^*_{\overline{u}}(n)$; let the difference be u. Similar to the argument given in (1) we see that u rows in Y_{λ_e} remain left after adjoining the rows of Y_{λ_o} and Y_{λ_e}. Here u rows will be inserted into Y_{λ_o} so that the resulting Y_μ with $\mu \in P^{ou}_{eu}(n)$ does not violate the structure of the Young diagram.
Let $\mu = (\mu_1, \ldots, \mu_s) \in P^{ou}_{eu}(n)$. Separate μ into $\mu_o = (\mu_{o_1}, \ldots, \mu_{o_i})$ with the odd parts, $\mu_{o_i} \le \mu_{o_{i-1}} \le \cdots \le \mu_{o_1}$ where $\mu_{o_i} \ge \mu_s$, $\mu_{o_1} \le \mu_1$ and into μ_e with the even parts. We keep aside the even component Y_{μ_e} of Y_μ. First, we assume that all odd parts of μ are distinct; i.e., there are i distinct odd values with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$. Now, for all j $(1 \le j \le i)$, we extract $2j-1$ boxes from the jth row of Y_{μ_o} and attach $2j-1$ boxes to Y_{μ_o} without violating the structure of the Young diagram Y_{μ_o}. Now, we break an odd part μ_{o_t} of the partition μ_o into $(\mu_{o_t} - (2v-1), 2v-1)$ with $v = i - t + 1$, where the part μ_{o_t} corresponds to the number of boxes in the vth row of Y_{μ_o}. The Young diagram $Y_{\tilde{\mu}}$ obtained from Y_{μ_o} by the above construction and adjoining Y_{μ_e} with it to get the unique resulting Young diagram, say Y_π with $\pi \in O^*_{\overline{u}}(n)$. This is because all the odd parts are distinct and their corresponding position in Y_μ is unique, and hence the resulting partition $\pi \in O^*_{\overline{u}}(n)$ is the unique pre-image of the partition $\mu \in P^{ou}_{eu}(n)$. Explicitly, for $\mu = (\mu_1, \ldots, \mu_s) \in P^{ou}_{eu}(n)$ with $\mu_o = (\mu_{o_1}, \ldots, \mu_{o_i})$ into odd parts and μ_e into even parts, then by the construction discussed before, we have $\lambda_o = (2i-1, 2i-3, \ldots, 3, 1)$ and $\lambda_e = (\mu_{o_1} - (2i-1), \ldots, \mu_{o_t} - 1) \cup \mu_e$. Now, define $\lambda = \lambda_o \cup \lambda_e \in O^*_{\overline{u}}(n)$ and considering three cases stated before, we see that λ transformed to μ.
Next, we consider $\mu_o = (\mu_{o_1}, \ldots, \mu_{o_i})$ with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$ with the assumption that $\mu_{o_1}, \ldots, \mu_{o_i}$ occurs with multiplicity $k_1, k_2, \ldots, k_i$, respectively; i.e., a part μ_{o_t} $(1 \le t \le i)$ occurs with multiplicity k_t. Then we break the k_t tuple $(\mu_{o_t}, \ldots, \mu_{o_t})$ into $((\mu_{o_t} - (2v-1), 2v-1), \ldots, (\mu_{o_t} - (2v-1), 2v-1))$ where the part μ_{o_t} corresponds to the number of boxes in the vth row of Y_{μ_o}. Similar argument as before shows that the resulting partition, say $\pi \in O^*_{\overline{u}}(n)$. This the concludes the proof with examples for $n \in \{6, 7\}$:

$P^{ou}_{eu}(6)$	$\longrightarrow$	$O^*_{\overline{u}}(6)$	$\longrightarrow$	$P^{ou}_{eu}(6)$
6		6		6
$5+1$		$3+2+1$		$5+1$
$4+2$		$4+2$		$4+2$
$3+3$		$2+2+1+1$		$3+3$
$3+1+1+1$		$2+1+1+1+1$		$3+1+1+1$
$2+2+2$		$2+2+2$		$2+2+2$
$1+1+1+1+1+1$		$1+1+1+1+1+1$		$1+1+1+1+1+1$

$$
\begin{array}{ccccc}
P_{eu}^{ou}(7) & \longrightarrow & O_{\overline{u}}^{*}(7) & \longrightarrow & P_{eu}^{ou}(7) \\
7 & & 6+1 & & 7 \\
5+2 & & 4+2+1 & & 5+2 \\
5+1+1 & & 3+2+1+1 & & 5+1+1 \\
3+3+1 & & 3+3+1 & & 3+3+1 \\
3+2+2 & & 2+2+2+1 & & 3+2+2 \\
3+1+1+1+1 & & 3+1+1+1+1 & & 3+1+1+1+1 \\
1+1+1+1+1+1+1 & & 1+1+1+1+1+1+1 & & 1+1+1+1+1+1+1
\end{array}
$$

$\square$

3. Conclusion

It might be interesting to study Theorem 1.3 for other partition functions whose parts are in consideration with parity found in [2] and [4].

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