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Recurrence relations for the moments of discrete semiclassical orthogonal polynomials.

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Abstract

We study recurrence relations satisfied by the moments $\nu_n(z)$ of a linear functional L. We consider the class of functionals whose first moment satisfies a differential equation (in z) with polynomial coefficients.

We give examples for all cases where the order of the ODE is less or equal than 3.

Dedicated to Dick Askey (1933 – 2019), Grandmaster of Special Functions!

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1 Introduction

Let $\mathbb{K}$ be a commutative ring (for our purposes we mostly think of $\mathbb{K}$ as the set of complex numbers $\mathbb{C}$) and $\mathbb{N}_0$ be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

We will denote by $\delta_{k,n}$ the *Kronecker delta*, defined by

$$\delta_{k,n} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}, \quad k, n \in \mathbb{N}_0.$$

Suppose that $L : \mathbb{K}[x] \to \mathbb{K}$ is a *linear functional* (acting on the variable x), $\{\Lambda_n(x)\}_{n\geq 0}$ is a *basis* of $\mathbb{K}[x]$ with $\deg(\Lambda_n) = n$, and we choose a nonzero sequence of *norms* $\{h_n\}_{n\geq 0} \subset \mathbb{K}\setminus\{0\}$. If the system of linear equations

$$\sum_{i=0}^{n} L[\Lambda_k \Lambda_i] \xi_{n,i} = h_n \delta_{k,n}, \quad 0 \leq k \leq n, \tag{1}$$

has a **unique solution** $\{\xi_{n,i}\}_{0\leq i\leq n}$, we can define a polynomial $P_n(x)$ by

$$P_n(x) = \sum_{i=0}^{n} \xi_{n,i} \Lambda_i(x).$$

We say that $\{P_n(x)\}_{n\geq 0}$ is an *orthogonal polynomial sequence* with respect to the functional L.

The system (1) can be written as

$$L[\Lambda_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,$$

and using linearity we see that the sequence $\{P_n(x)\}_{n\geq 0}$ satisfies the *orthogonality conditions*

$$L[P_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n. \tag{2}$$

If we define the (symmetric) *Gram matrix* $\mathfrak{G}$ by

$$\mathfrak{G}_{i,k} = L[\Lambda_i \Lambda_k], \quad i, k \in \mathbb{N}_0, \tag{3}$$

one can show [24] that the condition

$$\det_{0\leq i,k\leq n} (\mathfrak{G}_{i,k}) \neq 0, \quad n \geq 0,$$

is equivalent to the existence of a unique family of orthogonal polynomial satisfying (2) and $\deg(P_n) = n$.

The theory of orthogonal polynomials is vast and rich, extending all the way back to the groundbreaking work of Legendre [69], where he introduced the family of polynomials that now bears his name. We direct the interested reader to (some of!) the fundamental treatises on the field [9], [12], [50], [54], [57], [65], [103].

A particular fruitful approach that has received a lot of attention in recent years, is to work with the (infinite) matrix (3) acting on the (infinite) vector $\overrightarrow{P} = (P_0, P_1, \ldots)$. One can then view orthogonal polynomials as elements of an infinite dimensional vector space [28], [37], [52], [70], [112], [113], [114], [115].

Of course, in its full generality, it's difficult to get results that apply to *any* family of orthogonal polynomials. Thus, one chooses, for example:

i.) an operator (difference, differential, functional, integral) that annihilates $P_n(x)$.

ii.) a degree-reducing operator relating $P_n(x)$ and $P_{n-1}(x)$ (Sheffer classification, umbral calculus, generating functions).

iii.) a particular form of the linear functional L (continuous, discrete, matrix valued, q-series).

iv.) a particular domain of L ($\mathbb{C}$, $\mathbb{N}_0$, $\mathbb{R}$, linear and quadratic lattices, unit circle).

Another possibility, is to ask L to satisfy a relation of the form

$$L[\sigma p] = L[\tau U[p]], \quad p \in \mathbb{K}[x],$$

where $\sigma(x), \tau(x)$ are *fixed* polynomials, and $U : \mathbb{K}[x] \to \mathbb{K}[x]$ is a given operator. In this case, we say that L is a *semiclassical functional* with respect to U. The *class* of the functional L is defined by

$$s = \max\{\deg(\sigma) - 2, \deg(\tau) - 1\},$$

and semiclassical functional of class $s = 0$ are called *classical*.

This type of functionals was introduced by Shohat [100], and studied in detail by P. Maroni and collaborators [80], [82], [83], particularly when $U[p] = p'(x)$ is the derivative operator [72], [78], [79], and also for the operator

$$U_\omega[p] = \frac{p(x+\omega) - p(x)}{\omega},$$

which contains the finite difference operators Δ, ∇ as special cases $(\omega = \pm 1)$, and the derivative operator as a limiting case [1]. Other examples include the $q-$semiclassical polynomials [63], [85], associated with the operator

$$U_q[p] = \frac{p(qx) - p(x)}{(q-1)x}, \quad q \neq 1.$$

In this paper, we will focus on the so-called *discrete semiclassical orthogonal polynomials* [8], [42], [76], [87], [117], where U is the shift operator $U[p] = p(x+1)$. In this case, the linear functional L is of the form

$$L[p] = \sum_{x=0}^{\infty} p(x)\rho(x), \quad p \in \mathbb{K}[x],$$

where $\rho(x)$ is a given *weight function*. The traditional starting point is the *Pearson equation* satisfied by $\rho(x)$

$$U[\sigma\rho] = \tau(x)\rho(x), \tag{4}$$

but after trying this approach in [39], we found it very dissatisfying, especially when one considers spectral transformations of L.

For example, applying an Uvarov transformation to L at a point ω (see Section 3.3) will lead to the Pearson equation

$$\frac{\widetilde{\rho}(x+1)}{\widetilde{\rho}(x)} = \frac{(x-\omega)(x+1-\omega)\tau(x)}{(x-\omega)(x+1-\omega)\sigma(x+1)},$$

and this begs the question of when one is allowed (or not) to simplify the above expression. A possibility to avoid this problem is to study the difference equation satisfied by the *Stieltjes transform* of L

$$S(t) = L\left[\frac{1}{t-x}\right], \quad t \notin \mathbb{N}_0,$$

and we did this in [40], where we classified the discrete semiclassical orthogonal polynomials of class $s \leq 2$.

Now suppose that the weight function $\rho(x)$ also contains an *independent variable* z, $\rho = \widetilde{\rho}(x;z)$. Although this may seem like an extra assumption, we note that one could always introduce such a variable as a *Toda deformation* [10], [92], [105],

$$\widetilde{\rho}(x;z) = \rho(x)e^{xf(z)}, \quad f(z_0) = 0,$$

and recover the original functional L by setting $z = z_0$. We studied this type of weight functions in [38], and observed that the operator ϑ defined by

$$\vartheta [u] = z \frac{du}{dz}$$

is naturally associated to the shift operator.

As we will see in Section 2, this allow us to replace the Pearson equation (4) with the ODE satisfied by the first moment $\lambda_0 (z) = L [1]$,

$$\sigma (\vartheta) [\lambda_0] = z\tau (\vartheta) [\lambda_0] . \tag{5}$$

We note in passing that the ODE (5) is the true starting point of the theory, and by considering alternative equations satisfied by $\lambda_0 (z)$, one could study semiclassical orthogonal polynomials associated with different operators U.

The structure of the paper is as follows: in Section 2, we introduce the operator ϑ and the ODE satisfied by the moments of a discrete linear functional

$$\sigma (\vartheta) \Lambda_n (\vartheta) [\lambda_0] = z\tau (\vartheta) \Lambda_n (\vartheta + 1) [\lambda_0] , \quad n \in \mathbb{N}_0. \tag{6}$$

This will naturally lead to the class of functionals whose first moment $\lambda_0 (z)$ can be represented as a (generalized) hypergeometric function.

Since the ODE (6) contains a shift, we need to choose a convenient basis $\{\Lambda_n (x)\}_{n\geq 0}$. In Section 2.1, we study the monomial basis and derive a linear recurrence $\underline{\text{of order } n + s + 1}$ for the (standard) moments $\mu_n (z)$. We also find a representation for $\mu_n (z)$ as a linear combination involving a family of polynomials that satisfies a differential-difference equation.

In Section 2.2, we consider the basis of falling factorial polynomials defined by $\phi_0 (x) = 1$,

$$\phi_{n+1} (x) = x\phi_n (x - 1) , \quad n \in \mathbb{N},$$

which allow us to easily work on the lattice $\mathbb{Z}$. We use Newton's interpolation formula and obtain a linear recurrence $\underline{\text{of order } s + 1}$ for the (modified) moments $\nu_n (z)$. The linear functionals of class $s = 1$ are particularly interesting, since in this case the moments $\nu_n (z)$ are themselves a family of orthogonal polynomials. This is an area that has been studied in detail by M. Ismail and D. Stanton, see [58], [59], and [60].

Both the monomials and the falling factorial polynomials are examples of *Newton basis polynomials* defined by $\mathfrak{n}_0 (x) = 1$ and

$$\mathfrak{n}_k (x) = \prod_{j=0}^{k-1} (x - \kappa_j) ,$$

where $\{\kappa_j\}_{j\geq 0}$ is a fixed sequence. This type of polynomials satisfy $2-$term recurrence relations, which we study in Section 2.3. Among other results, we look at the connection between the monomial and falling factorial bases (through Stirling numbers), and find the (formal) representation for the Stieltjes transform

$$S(\omega; z) = \sum_{k=0}^{\infty} \frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)}. \tag{7}$$

In [41], we used (7) to derive recurrence relations for the modified moments $\nu_n(z)$.

In Section 3, we consider transformations Ω_β^α between different families of discrete semiclassical orthogonal polynomials. We introduce a uniform notation to label objects belonging to different families, and show how the recurrence relations for the moments change as we apply a transformation.

In Sections 3.1, 3.2, 3.3, and (3.4) we consider the special cases $\alpha = \beta + 1$ (*Christoffel transformation*) [18], [45], [97], $\alpha = \beta - 1$ (*Geronimus transformation)* [30], [31], [67], [81], their composition (*Uvarov transformation)* [5], [7], [25], [64], [75], and $\alpha = \beta = -N$, $N \in \mathbb{N}$ (*truncation transformation).* These *rational spectral transformations* have been studied by many authors, [4], [91], [119]. The relation between these transformations and the so-called *Darboux transformation*, has also been considered [19], [74], [118].

Finally, Section 4 applies the results obtained in the paper to all the families of class $s \leq 2$. We see how many special subcases can be obtained as single and multiple spectral transformations of polynomials in a lower class.

2 Differential operators and moment functionals

Let $\mathbb{F}$ denote the ring of *formal power series* in the variable z

$$\mathbb{F} = \mathbb{K}[[z]] = \left\{ \sum_{n=0}^{\infty} c_n z^n : \quad c_n \in \mathbb{K} \right\},$$

and $\vartheta : \mathbb{F} \to \mathbb{F}$ be the differential operator defined by [89, 16.8.2]

$$\vartheta = z\partial_z,$$

where ∂_z is the *derivative operator*

$$\partial_z = \frac{\partial}{\partial z}.$$

The action of ϑ on the monomials is given by

$$\vartheta^k [z^x] = x^k z^x,$$

where we always assume that x and z are *independent variables.* Using linearity, it follows that

$$u(\vartheta)[z^x] = u(x) z^x, \quad u \in \mathbb{K}[x]. \tag{8}$$

Note that ϑ is multiplicative

$$\vartheta^{n+m}[z^x] = x^{n+m} z^x = x^n x^m z^x = x^n \vartheta^m [z^x] = \vartheta^m [x^n z^x] = [\vartheta^m \vartheta^n][z^x],$$

and therefore

$$(uv)(\vartheta) = u(\vartheta) v(\vartheta), \quad u, v \in \mathbb{K}[x].$$

On the other hand, if one of the terms is multiplied by a power of z, we have

$$\begin{aligned} u(\vartheta)\left[z^k v(\vartheta)[z^x]\right] &= u(\vartheta)\left[z^k v(x) z^x\right] = u(\vartheta)\left[v(x) z^{x+k}\right] \\ &= v(x) u(\vartheta)\left[z^{x+k}\right] = v(x) u(x+k) z^{x+k} \\ &= z^k v(x) u(x+k) z^x = z^k [v(\vartheta) u(\vartheta + k)][z^x], \quad u, v \in \mathbb{K}[x], \end{aligned}$$

and therefore

$$u(\vartheta)\left[z^k v(\vartheta)\right] = z^k \mathfrak{S}_\vartheta^k [u] v(\vartheta), \quad k \geq 0, \tag{9}$$

where $\mathfrak{S}_x$ denotes the *shift operator* in the variable x

$$\mathfrak{S}_x [u] = u(x+1). \tag{10}$$

Suppose that $L : \mathbb{K}[x] \to \mathbb{F}$ is the *linear functional* (acting on the variable x) defined by

$$L[u] = \sum_{x=0}^{\infty} u(x) \rho(x) z^x, \quad u \in \mathbb{K}[x],$$

where $\rho : \mathbb{N}_0 \to \mathbb{K}$ is a given function. Note that if $f \in \mathbb{K}[[x]]$, we can extend (8) to

$$f(\vartheta)[z^x] = \sum_{n=0}^{\infty} c_n \vartheta^n [z^x] = z^x \sum_{n=0}^{\infty} c_n x^n = f(x) z^x,$$

and therefore we can consider L as a functional on $\mathbb{K}\left[\left[x\right]\right]$, satisfying

$$f\left(\vartheta\right)\left[L\left[u\right]\right]=\sum_{x=0}^{\infty}u\left(x\right)\rho\left(x\right)f\left(x\right)z^{x}=L\left[fu\right],\quad f\in\mathbb{K}\left[\left[x\right]\right],u\in\mathbb{K}[x]. \quad (11)$$

Let $\left\{\Lambda_{n}\right\}_{n\geq0}$ be a *monic polynomial basis*, i.e., $\Lambda_{n}\left(x\right)$ is a monic polynomial with $\deg\left(\Lambda_{n}\right)=n$. If we define a sequence of *moments* [2], [3], [101] by

$$\lambda_{n}\left(z\right)=L\left[\Lambda_{n}\right]\in\mathbb{F},$$

then from (11) we obtain

$$f\left(\vartheta\right)\left[\lambda_{0}\right]=f\left(\vartheta\right)\left[L\left[1\right]\right]=L\left[f\right],\quad f\in\mathbb{K}\left[\left[x\right]\right], \quad (12)$$

and in particular

$$\lambda_{n}\left(z\right)=L\left[\Lambda_{n}\right]=\Lambda_{n}\left(\vartheta\right)\left[\lambda_{0}\right]. \quad (13)$$

Note that if $E\left(t,x\right)$ is the *exponential generating function* [116] of the polynomials $\Lambda_{n}\left(x\right)$

$$E\left(t;x\right)=\sum_{n=0}^{\infty}\Lambda_{n}\left(x\right)\frac{t^{n}}{n!},$$

we can use (13) and get

$$\sum_{n=0}^{\infty}\lambda_{n}\left(z\right)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\Lambda_{n}\left(\vartheta\right)\left[\lambda_{0}\right]\frac{t^{n}}{n!}=E\left(t;\vartheta\right)\left[\lambda_{0}\right].$$

Using (11), we obtain the exponential generating function of the moments

$$\sum_{n=0}^{\infty}\lambda_{n}\left(z\right)\frac{t^{n}}{n!}=L\left[E\left(t;x\right)\right],$$

and in the special case of a power function $E\left(t;x\right)=\left[f\left(t\right)\right]^{x}$, we have

$$L\left[E\left(t;x\right)\right]=L\left[f^{x}\right]=\sum_{x=0}^{\infty}\rho\left(x\right)\frac{\left[zf\left(t\right)\right]^{x}}{x!}=\lambda_{0}\left[zf\left(t\right)\right]. \quad (14)$$

If the first moment $\lambda_{0}\left(z\right)$ satisfies a *differential equation with polynomial coefficients*

$$\left[\sigma\left(\vartheta\right)-z\tau\left(\vartheta\right)\right]\left[\lambda_{0}\right]=0,\quad\sigma,\tau\in\mathbb{K}\left[x\right], \quad (15)$$

then we see from (9) that

$$u(\vartheta)\,\sigma(\vartheta)\,[\lambda_0] = u(\vartheta)\,[z\tau(\vartheta)\,[\lambda_0]] = z\tau(\vartheta)\,u(\vartheta+1)\,[\lambda_0]\,, \quad u \in \mathbb{K}[x]\,. \tag{16}$$

Using (12), we conclude that L is a *semiclassical functional* with respect to the shift operator $\mathfrak{S}_x$

$$L[\sigma u] = L[z\tau\mathfrak{S}_x[u]]\,, \quad u \in \mathbb{K}[x]\,. \tag{17}$$

Suppose that $\sigma(0) = 0$. Using (17), we have

$$\sum_{x=1}^{\infty}\sigma(x)\,u(x)\,\rho(x)\,z^x = L[\sigma u] = L[z\tau u(x+1)]$$

$$= \sum_{x=0}^{\infty}\tau(x)\,u(x+1)\,\rho(x)\,z^{x+1} = \sum_{x=1}^{\infty}\tau(x-1)\,u(x)\,\rho(x-1)\,z^x,$$

and we conclude that $\rho(x)$ satisfies the *Pearson equation [90]*

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\tau(x)}{\sigma(x+1)}, \quad x \in \mathbb{N}_0. \tag{18}$$

Solving (18), we get

$$\rho(n) = \prod_{k=0}^{n-1}\frac{\tau(k)}{\sigma(k+1)}, \quad n \in \mathbb{N}, \tag{19}$$

where we set $\rho(0) = 1$.

We define the *Pochhammer symbol* $(c)_x$ by [93]

$$(c)_x = \lim_{k\to\infty} k^x \prod_{j=0}^{k}\frac{c+j}{c+x+j}, \quad -(c+x) \notin \mathbb{N}_0,$$

and note that if $x = n \in \mathbb{N}_0$, the Pochhammer symbol becomes a *polynomial* in z of degree n

$$(c)_n = \prod_{j=0}^{n-1}(c+j)\,, \quad n \in \mathbb{N}, \quad (c)_0 = 1. \tag{20}$$

We will use the notation [89, 16.1]

$$(\mathbf{c})_n = (c_1)_n \cdots (c_m)_n\,, \quad \mathbf{c} \in \mathbb{K}^m,$$

and also

$$(x + \mathbf{c}) = (x + c_1) \cdots (x + c_m)\,, \quad \mathbf{c} \in \mathbb{K}^m.$$

In the special case $m = 0$, we understand that

$$\mathbf{c} \in \mathbb{K}^0 = \emptyset, \quad (\emptyset)_n = 1.$$

Writing

$$\sigma(x) = x(x + \mathbf{b})\,, \quad \tau(x) = (x + \mathbf{a})\,, \quad \mathbf{a} \in \mathbb{K}^p, \quad \mathbf{b} \in \mathbb{K}^q, \tag{21}$$

and using (20), we can rewrite (19) as

$$\rho(x) = \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{1}{x!}, \quad \mathbf{a} \in \mathbb{K}^p, \quad \mathbf{b} \in \mathbb{K}^q.$$

The ODE

$$[\vartheta(\vartheta + \mathbf{b}) - z(\vartheta + \mathbf{a})][\lambda_0] = 0, \quad \mathbf{a} \in \mathbb{K}^p, \quad \mathbf{b} \in \mathbb{K}^q,$$

is the (generalized) *hypergeometric differential equation* [89, 16.8.3] of order

$$\mathfrak{o} = \max\{p, q+1\}\,,$$

and the first moment $\lambda_0(z)$ can be represented as

$$\lambda_0(z) = {}_pF_q\left(\begin{array}{c} \mathbf{a} \\ \mathbf{b}+\mathbf{1} \end{array}; z\right),$$

where the (generalized) *hypergeometric function* ${}_pF_q$ is defined by [89, 16.2.1]

$${}_pF_q\left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array}; z\right) = \sum_{x=0}^{\infty} \frac{(\mathbf{a})_x}{(\mathbf{b})_x} \frac{z^x}{x!}, \quad \mathbf{a} \in \mathbb{K}^p, \quad \mathbf{b} \in \mathbb{K}^q.$$

We define the *class* s of the semiclassical functional L by

$$s = \mathfrak{o} - 1 = \max\{p-1, q\}\,,$$

and the functionals of class $s=0$ are called *classical*.

Combining (16) and (21), we conclude that

$$\vartheta\left(\vartheta+\mathbf{b}\right)\Lambda_{n}\left(\vartheta\right)\left[\lambda_{0}\right]=z\left(\vartheta+\mathbf{a}\right)\Lambda_{n}\left(\vartheta+1\right)\left[\lambda_{0}\right],\quad n\in\mathbb{N}_{0}, \tag{22}$$

and expanding the polynomials coefficients on the basis $\left\{\Lambda_{n}\right\}_{n\geq 0}$,

$$\begin{aligned}\vartheta\left(\vartheta+\mathbf{b}\right)\Lambda_{n}\left(\vartheta\right)&=\sum_{k=0}^{n+q+1}c_{n,k}\Lambda_{k}\left(\vartheta\right),\\ \left(\vartheta+\mathbf{a}\right)\Lambda_{n}\left(\vartheta+1\right)&=\sum_{k=0}^{n+p}\widetilde{c}_{n,k}\Lambda_{k}\left(\vartheta\right),\end{aligned}$$

we get a recurrence relation of order $n+s+1$ for the moments

$$\sum_{k=-n}^{q+1}c_{n,n+k}\lambda_{n+k}-z\sum_{k=-n}^{p}\widetilde{c}_{n,n+k}\lambda_{n+k}=0. \tag{23}$$

The question is: can we do better than this? In other words, can one choose a convenient basis Λ_{n} so that the recurrence (23) will have minimal order $s+1$? The answer is yes, as we will see in Section 2.2. In the meantime, we will study the simplest basis: the monomials.

2.1 Standard moments

If $\mu_{n}\left(z\right)\in\mathbb{F}$ denote the *standard moments* of L on the monomial basis $\Lambda_{n}\left(x\right)=x^{n}$

$$\mu_{n}\left(z\right)=L\left[x^{n}\right],\quad n\in\mathbb{N}_{0},$$

it follows from (22) that

$$\vartheta^{n+1}\left(\vartheta+\mathbf{b}\right)\left[\mu_{0}\right]=z\left(\vartheta+1\right)^{n}\left(\vartheta+\mathbf{a}\right)\left[\mu_{0}\right]. \tag{24}$$

If we use the *umbral notation [95]*

$$f^{k}\leftrightarrow f_{k},$$

then we see from (13) and (24) that the standard moments $\mu_{n}\left(z\right)$ satisfy the recurrence

$$\left(\mu+\mathbf{b}\right)\mu^{n+1}-z\left(\mu+\mathbf{a}\right)\left(\mu+1\right)^{n}=0. \tag{25}$$

The coefficients of the polynomials $(x + \mathbf{c})$ can be written in terms of the *elementary symmetric polynomials* $e_n(\mathbf{c})$, defined by the generating function [71]

$$\sum_{n=0}^{\infty} e_n(\mathbf{c})\, t^n = \prod_{i=1}^{m} (1 + tc_i), \quad \mathbf{c} \in \mathbb{K}^m. \tag{26}$$

It follows from (26) that

$$(x + \mathbf{c}) = \sum_{k=0}^{m} e_{m-k}(\mathbf{c})\, x^k, \quad \mathbf{c} \in \mathbb{K}^m,$$

and using this formula in (25), we obtain the explicit recurrence

$$\sum_{k=0}^{q} e_{q-k}(\mathbf{b})\, \mu_{n+k+1} - z \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{p} e_{p-j}(\mathbf{a})\, \mu_{k+j} = 0. \tag{27}$$

In particular, for $n = 0$

$$\sum_{k=0}^{q} e_{q-k}(\mathbf{b})\, \mu_{k+1} - z \sum_{j=0}^{p} e_{p-j}(\mathbf{a})\, \mu_j = 0. \tag{28}$$

It is clear from (27) that elements of the set

$$\{\mu_k : \quad k > s\}, \quad s = \max\{p-1, q\},$$

are linear combinations of the first $s+1$ standard moments. Thus, we have a representation of the form

$$\mu_n(z) = \sum_{k=0}^{s} g_{n,k}(z)\, \mu_k(z), \quad n \in \mathbb{N}_0, \tag{29}$$

where the coefficients must satisfy

$$g_{n,k}(z) = \delta_{n,k}, \quad 0 \le n, k \le s. \tag{30}$$

If we introduce the vectors $\overrightarrow{\mu}, \overrightarrow{g}_n \in \mathbb{F}^{s+1}$ defined by

$$(\overrightarrow{\mu})_k = \mu_k, \quad (\overrightarrow{g}_n)_k = g_{n,k}, \quad 0 \le k \le s,$$

we can write (29) as an inner product

$$\mu_n = \overrightarrow{g}_n \cdot \overrightarrow{\mu}. \tag{31}$$

To satisfy the initial conditions (30), we need

$$\overrightarrow{g}_n = \overrightarrow{\varepsilon}_n, \quad 0 \leq n \leq s,$$

where the *standard unit vectors* $\overrightarrow{\varepsilon}_n \in \mathbb{K}^{s+1}$ are defined by

$$(\overrightarrow{\varepsilon}_n)_k = \delta_{n,k}, \quad 0 \leq k \leq s, \quad n \in \mathbb{N}_0.$$

From (31). we get

$$\mu_{n+1} = \vartheta[\mu_n] = \vartheta[\overrightarrow{g}_n \cdot \overrightarrow{\mu}] = \vartheta[\overrightarrow{g}_n] \cdot \overrightarrow{\mu} + \overrightarrow{g}_n \cdot \vartheta[\overrightarrow{\mu}]$$

and since

$$\vartheta[\overrightarrow{\mu}] = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{s+1} \end{pmatrix} = \begin{pmatrix} \overrightarrow{g}_1 \\ \overrightarrow{g}_2 \\ \vdots \\ \overrightarrow{g}_{s+1} \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_s \end{pmatrix},$$

we have $\vartheta[\overrightarrow{\mu}] = M^T \overrightarrow{\mu}$, with

$$M^T = \begin{pmatrix} \overrightarrow{g}_1 \\ \overrightarrow{g}_2 \\ \vdots \\ \overrightarrow{g}_{s+1} \end{pmatrix} \in \mathbb{F}^{(s+1)\times(s+1)},$$

where vectors form the *rows* of the matrix M^T. Thus,

$$\overrightarrow{g}_{n+1} \cdot \overrightarrow{\mu} = \mu_{n+1} = \vartheta[\overrightarrow{g}_n] \cdot \overrightarrow{\mu} + \overrightarrow{g}_n \cdot \left(M^T \overrightarrow{\mu}\right) = (\vartheta[\overrightarrow{g}_n] + M \overrightarrow{g}_n) \cdot \overrightarrow{\mu}$$

from which we conclude that the vector $\overrightarrow{g}_n(z)$ satisfies the *differential-difference equation*

$$\overrightarrow{g}_{n+1} = (\vartheta + M)\, \overrightarrow{g}_n, \quad n \geq 0, \quad \overrightarrow{g}_0 = \overrightarrow{\varepsilon}_0, \tag{32}$$

with

$$M = (\overrightarrow{\varepsilon}_1, \overrightarrow{\varepsilon}_2, \cdots, \overrightarrow{\varepsilon}_s, \overrightarrow{g}_{s+1}) \in \mathbb{F}^{(s+1)\times(s+1)},$$

where now vectors form the *columns* of the matrix M.

It follows from (27) that we have three cases to consider:

1) If $p > q + 1$, then the standard moments will satisfy a recurrence of the form

$$z\mu_{n+p} = \sum_{k=0}^{n+p-1} c_{n,k}(z)\,\mu_k,$$

and setting

$$\overrightarrow{g}_n(z) = z^{-n}\overrightarrow{Q}_n(z), \quad n \geq 0,$$

the *vector polynomials* $\overrightarrow{Q}_n \in (\mathbb{K}[z])^{s+1}$ will satisfy the differential-difference equation

$$\overrightarrow{Q}_{n+1} = z(\vartheta + M - nI)\overrightarrow{Q}_n, \quad n \geq 0, \quad \overrightarrow{Q}_0 = \overrightarrow{\varepsilon}_0, \tag{33}$$

where I is the $(s + 1 \times s + 1)$ identity matrix.

2) If $p = q + 1$, then the standard moments will satisfy a recurrence of the form

$$(1 - z)\,\mu_{n+p} = \sum_{k=0}^{n+p-1} c_{n,k}(z)\,\mu_k,$$

and if we set

$$\overrightarrow{g}_n(z) = (1 - z)^{-n}\,\overrightarrow{Q}_n(z), \quad n \geq 0,$$

the *vector polynomials* $\overrightarrow{Q}_n \in (\mathbb{K}[z])^{s+1}$ will satisfy the differential-difference equation

$$\overrightarrow{Q}_{n+1} = [(1 - z)(\vartheta + M) + nzI]\,\overrightarrow{Q}_n, \quad n \geq 0, \quad \overrightarrow{Q}_0 = \overrightarrow{\varepsilon}_0. \tag{34}$$

3) If $p < q + 1$, then the standard moments will satisfy a recurrence of the form

$$\mu_{n+q+1} = z\sum_{k=0}^{n+q} c_{n,k}\mu_k,$$

and the functions $\overrightarrow{g}_n(z)$ already are polynomials in z.

Remark 1 *In [35], we derived (32) using a different method.*

The exponential generating function of the monic basis is the exponential function

$$\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = e^{xt},$$

and using (14) we obtain

$$\sum_{n=0}^{\infty} \mu_n(z) \frac{t^n}{n!} = L\left[e^{xt}\right] = \mu_0\left(ze^t\right). \tag{35}$$

Since

$$\partial_t \left[y\left(ze^t\right)\right] = ze^t y'\left(ze^t\right) = z\partial_z \left[y\left(ze^t\right)\right] = \vartheta\left[y\left(ze^t\right)\right],$$

it follows from (15) that $\mu_0(ze^t)$ is a solution of the linear ODE (in the t variable)

$$\left[\sigma\left(\partial_t\right) - ze^t \tau\left(\partial_t\right)\right][y] = 0. \tag{36}$$

If we define

$$G_k(t, z) = \sum_{n=0}^{\infty} g_{n,k}(z) \frac{t^n}{n!}, \quad 0 \leq k \leq s,$$

it follows from (29) that

$$\mu_0\left(ze^t\right) = \sum_{k=0}^{s} G_k(t, z)\, \mu_k(z),$$

and therefore the functions $G_k(t, z)$, $0 \leq k \leq s$ form a basis of solutions of the ODE (36) with initial conditions

$$\left[\partial_t^n G_k\right]_{t=0} = \delta_{n,k}, \quad 0 \leq n, k \leq s,$$

since from (35) we see that

$$\left[\partial_t^n \mu_0\left(ze^t\right)\right]_{t=0} = \mu_n(z).$$

2.2 Modified moments

Let $\phi_n(x)$ denote the *falling factorial polynomials* defined by $\phi_0(x) = 1$ and

$$\phi_n(x) = \prod_{k=0}^{n-1} (x - k), \quad n \in \mathbb{N}. \tag{37}$$

Sometimes, the polynomials $\phi_n(x)$ are called "binomial polynomials", since

$$\frac{\phi_n(x)}{n!} = \binom{x}{n}, \quad n \in \mathbb{N}_0. \tag{38}$$

From the definition (37), we see that

$$\phi_{n+1}(x) = (x-n)\,\phi_n(x) = x\phi_n(x-1), \quad n \geq 0, \tag{39}$$

and from (20) it follows that the falling factorial polynomials and the Pochhammer polynomials are related by

$$\phi_n(x) = (-1)^n (-x)_n = (x+1-n)_n .$$

The falling factorial polynomials are eigenfuncions of the differential operator $z^n \partial_z^n$ since

$$z^n \partial_z^n [z^x] = z^n \phi_n(x)\, z^{x-n} = \phi_n(x)\, z^x. \tag{40}$$

Remark 2 *Caution must be exercised when using the operators* $z^n \partial_z^n$ *and* ϑ^n *since*

$$\vartheta^n = (z\partial_z)^n \neq z^n (\partial_z)^n, \quad n > 1.$$

Using (40) and the formula [89, 16.3.1]

$$\partial_z^n \left[{}_pF_q \left(\begin{matrix} \mathbf{a} \\ \mathbf{b}+1 \end{matrix} ; z \right) \right] = \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \, {}_pF_q \left(\begin{matrix} \mathbf{a}+n \\ \mathbf{b}+n+1 \end{matrix} ; z \right),$$

we conclude that the *modified moments*

$$\nu_n(z) = L[\phi_n], \quad n \in \mathbb{N}_0,$$

admit the hypergeometric representation

$$\nu_n(z) = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \, {}_pF_q \left(\begin{matrix} \mathbf{a}+n \\ \mathbf{b}+n+1 \end{matrix} ; z \right).$$

Using (22) with $\Lambda_n(\vartheta) = \phi_n(\vartheta - 1)$, we get

$$\vartheta(\vartheta + \mathbf{b})\,\phi_n(\vartheta - 1)\,\sigma(\vartheta)[\nu_0] = z(\vartheta + \mathbf{a})\,\phi_n(\vartheta)\,\tau(\vartheta)[\nu_0],$$

and from (39) we conclude that

$$\left[\left(\vartheta + \mathbf{b} \right) \phi_{n+1} \left(\vartheta \right) - z \left(\vartheta + \mathbf{a} \right) \phi_n \left(\vartheta \right) \right] \left[\nu_0 \right] = 0. \tag{41}$$

Unlike the monomial case, there is no immediate formula that would express products of the form $\left(\vartheta + \mathbf{c} \right) \phi_n \left(\vartheta \right)$ in terms of the polynomials $\phi_n \left(\vartheta \right)$. Thus, we will find one next.

Any polynomial $u\left(x \right)$ can be represented in the basis of falling factorials using *Newton's interpolation formula [29]*

$$u\left(x \right) = \sum_{k=0}^{\deg(u)} \frac{\Delta^k \left[u \right] \left(c \right)}{k!} \phi_k \left(x - c \right), \tag{42}$$

where the *forward difference operator* Δ^n (acting on x) is defined by

$$\Delta^n \left[f \right] \left(x \right) = \sum_{k=0}^{n} \binom{n}{k} \left(-1 \right)^{n-k} f\left(x + k \right). \tag{43}$$

We start with a result that may be already known, but we have not been able to find in the literature.

Lemma 3 *For any function $f\left(x \right)$, we have*

$$\Delta^j \left[f \phi_n \right] \left(0 \right) = 0, \quad 0 \leq j < n, \tag{44}$$

and

$$\frac{\Delta^{n+j} \left[f \phi_n \right] \left(0 \right)}{(n+j)!} = \frac{\Delta^j \left[f \right] \left(n \right)}{j!}, \quad n, j \geq 0. \tag{45}$$

Proof. Using the definition (43),

$$\Delta^j \left[f \phi_n \right] \left(0 \right) = \sum_{i=0}^{j} \binom{j}{i} \left(-1 \right)^{j-i} f\left(i \right) \phi_n \left(i \right),$$

and since $\phi_n \left(i \right) = 0$, for $i < n$, we see that

$$\Delta^j \left[f \phi_n \right] \left(0 \right) = 0, \quad 0 \leq j < n,$$

If $j \geq 0$, then

$$\begin{aligned}\Delta^{n+j}\left[f\phi_n\right](0) &= \sum_{i=n}^{n+j}\binom{n+j}{i}(-1)^{n+j-i} f(i)\,\phi_n(i)\\ &= \sum_{i=0}^{j}\binom{n+j}{n+i}(-1)^{j-i} f(n+i)\,\phi_n(n+i).\end{aligned}$$

Using (38), we have

$$\binom{n+j}{n+i}\phi_n(n+i) = \frac{(n+j)!}{j!}\binom{j}{i},$$

and therefore

$$\begin{aligned}\Delta^{n+j}\left[f\phi_n\right](0) &= \frac{(n+j)!}{j!}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i} f(n+i)\\ &= \frac{(n+j)!}{j!}\Delta^{j}[f](n).\end{aligned}$$

■

Using (45), we obtain the following Corollary.

Corollary 4 *If $u(x)$ is a polynomial of degree k, then*

$$u(x)\,\phi_n(x) = \sum_{j=0}^{k}\frac{\Delta^j[u](n)}{j!}\phi_{n+j}(x).$$

Proof. Using (42) and (44), we have

$$\begin{aligned}u(x)\,\phi_n(x) &= \sum_{j=0}^{n+k}\frac{\Delta^j[u\phi_n](0)}{j!}\phi_j(x) = \sum_{j=n}^{n+k}\frac{\Delta^j[u\phi_n](0)}{j!}\phi_j(x)\\ &= \sum_{j=0}^{k}\frac{\Delta^{n+j}[u\phi_n](0)}{(n+j)!}\phi_{n+j}(x),\end{aligned}$$

and the results follows from (45). ■

From the previous Corollary, we finally obtain the representation we were looking for.

Corollary 5 *We have*

$$(x+\mathbf{c})\,\phi_n(x) = \sum_{j=0}^{m} \frac{\Delta^j\left[(x+\mathbf{c})\right](n)}{j!}\phi_{n+j}(x), \quad \mathbf{c}\in\mathbb{K}^m.$$

Let the *recurrence operators* $\Upsilon_n(\mathbf{c})$ be defined by

$$\Upsilon_n(\mathbf{c})[f] = \sum_{j=0}^{m} \frac{\Delta^j\left[(x+\mathbf{c})\right](n)}{j!} f_{n+j}, \quad \mathbf{c}\in\mathbb{K}^m, \tag{46}$$

We have $\Upsilon_n(\emptyset)[f] = f_n$, and from (46), we get

$$\Upsilon_n(c)[f] = f_{n+1} + (n+c)\,f_n.$$

In general, we have the following result.

Lemma 6 *The recurrence operators Υ_n satisfy the basic recurrence*

$$\Upsilon_n(\mathbf{c},\gamma) = \Upsilon_{n+1}(\mathbf{c}) + (n+\gamma)\,\Upsilon_n(\mathbf{c}). \tag{47}$$

Proof. From the definition of Υ_n, we have

$$\Upsilon_n(\mathbf{c},\gamma)[f] = \sum_{j=0}^{m+1} \frac{\Delta^j\left[(x+\mathbf{c})(x+\gamma)\right](n)}{j!} f_{n+j}.$$

If we use Leibniz rule [61]

$$\Delta^j[uv](n) = \sum_{i=0}^{j}\binom{j}{i}\Delta^{j-i}[u](n+i)\,\Delta^i[v](n),$$

we get

$$\Delta^j\left[(x+\mathbf{c})(x+\gamma)\right](n) = (n+\gamma)\,\Delta^j\left[(x+\mathbf{c})\right](n) + j\Delta^{j-1}\left[(x+\mathbf{c})\right](n+1).$$

Since

$$\sum_{j=0}^{m+1} \frac{j\Delta^{j-1}\left[(x+\mathbf{c})\right](n+1)}{j!} f_{n+j}$$
$$= \sum_{j=1}^{m+1} \frac{\Delta^{j-1}\left[(x+\mathbf{c})\right](n+1)}{(j-1)!} f_{n+j} = \sum_{j=0}^{m} \frac{\Delta^{j}\left[(x+\mathbf{c})\right](n+1)}{j!} f_{n+j+1},$$

we conclude that

$$\sum_{j=0}^{m+1} \frac{\Delta^j \left[(x+\mathbf{c})(x+\gamma)\right](n)}{j!} f_{n+j}$$
$$= (n+\gamma) \sum_{j=0}^{m} \frac{\Delta^j \left[(x+\mathbf{c})\right](n)}{j!} f_{n+j} + \sum_{j=0}^{m} \frac{\Delta^j \left[(x+\mathbf{c})\right](n+1)}{j!} f_{n+1+j}$$

and the result follows. ∎

If $m=2$, (47) gives

$$\begin{aligned}\Upsilon_n(c_1,c_2)[f] &= \Upsilon_{n+1}(c_1)[f] + (n+c_2)\Upsilon_n(c_1)[f] \\ &= f_{n+2} + (n+1+c_1) f_{n+1} + (n+c_2)\left[f_{n+1} + (n+c_1) f_n\right],\end{aligned}$$

and hence

$$\Upsilon_n(c_1,c_2) = \mathfrak{S}_n^2 + (2n+c_1+c_2+1)\mathfrak{S}_n + (n+c_1)(n+c_2).$$

Note that

$$\Upsilon_n(c_1,c_2) = (\mathfrak{S}_n + n + c_1)(\mathfrak{S}_n + n + c_2) = \Upsilon_n(c_1) \circ \Upsilon_n(c_2),$$

where clearly

$$\Upsilon_n(c_1) \circ \Upsilon_n(c_2) = \Upsilon_n(c_2) \circ \Upsilon_n(c_1).$$

Using induction, it follows that

$$\Upsilon_n(\mathbf{c}) = (\mathfrak{S}_n + n + \mathbf{c}), \quad \mathbf{c} \in \mathbb{K}^m,$$

and

$$\Upsilon_n(\mathbf{c}) = \Upsilon_n(c_1) \circ \Upsilon_n(c_2) \circ \cdots \circ \Upsilon_n(c_m), \quad \mathbf{c} \in \mathbb{K}^m. \tag{48}$$

Remark 7 *Note that*

$$[a_1\mathfrak{S}_n + b_1 n + c_1, a_2\mathfrak{S}_n + b_2 n + c_2] = (a_1 b_2 - a_2 b_1)\mathfrak{S}_n,$$

so in general caution must be exercised when composing linear terms involving $\mathfrak{S}_n$.

Using (41) and (46), we see that the modified moments $\nu_n(z)$ satisfy the recurrence

$$[\Upsilon_{n+1}(\mathbf{b}) - z\Upsilon_n(\mathbf{a})][\nu] = 0, \tag{49}$$

and using (46) we have the explicit recurrence of order $s+1 = \max\{p, q+1\}$

$$\sum_{j=0}^{q} \frac{\Delta^j [(x+\mathbf{b})](n+1)}{j!} \nu_{n+1+j} - z \sum_{j=0}^{p} \frac{\Delta^j [(x+\mathbf{a})](n)}{j!} \nu_{n+j} = 0. \tag{50}$$

It is clear from (50) that the elements of the set $\{\nu_k : k \geq s+1\}$, are linear combinations of the first $s+1$ modified moments. Thus, we have a representation of the form

$$\nu_n(z) = \sum_{k=0}^{s} f_{n,k}(z)\, \nu_k(z), \tag{51}$$

where the coefficients must satisfy the initial conditions

$$f_{j,k}(z) = \delta_{j,k}, \quad 0 \leq j, k \leq s.$$

Using (38) and the binomial theorem, we obtain the exponential generating function

$$\sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{x}{n} t^n = (1+t)^x,$$

and using (14), we get

$$\sum_{n=0}^{\infty} \nu_n(z) \frac{t^n}{n!} = L\left[(1+t)^x\right] = \nu_0(z+zt). \tag{52}$$

Since

$$\begin{aligned}(1+t)\,\partial_t [y((1+t)z)] &= z(1+t)\, y'((1+t)z) \\ &= z\partial_z [y((1+t)z)] = \vartheta [y((1+t)z)],\end{aligned}$$

it follows from (15) that $\nu_0(z+zt)$ is a solution of the linear ODE

$$\sigma((1+t)\,\partial_t)[y] = z\tau((1+t)\,\partial_t)[y], \tag{53}$$

Remark 8 *The differential equation (53) needs to be understood in an operational sense, since the coefficients are not constant. For instance, we have*

$$(1+t)\,\partial_t\left[(1+t)\,\partial_t\right] = (1+t)\left(\partial_t + (1+t)\,\partial_t^2\right) = (1+t)^2\,\partial_t^2 + (1+t)\,\partial_t,$$

and therefore

$$\left[(1+t)\,\partial_t + a_1\right]\left[(1+t)\,\partial_t + a_2\right] = (1+t)^2\,\partial_t^2 + (1+a_1+a_2)\,(1+t)\,\partial_t + a_1a_2.$$

If we define

$$F_k(t,z) = \sum_{n=0}^{\infty} f_{n,k}(z)\,\frac{t^n}{n!}, \quad 0 \le k \le s,$$

where $f_{n,k}(z)$ are the coefficients in (51), we see that

$$\nu_0(z+zt) = \sum_{k=0}^{s} F_k(t,z)\,\nu_k(z),$$

and therefore the functions $F_k(t,z)$, $0 \le k \le s$ form a basis of solutions of the ODE (53) with initial conditions

$$\left[\partial_t^n F_k\right]_{t=0} = \delta_{n,k}, \quad 0 \le n,k \le s,$$

since from (52) we see that

$$\left[\partial_t^n \nu_0(z+zt)\right]_{t=0} = \nu_n(z).$$

In the next section, we will look at more general polynomial bases that contain the monomials and falling factorial as particular cases.

2.3 Two-term recurrence relations

Both the monomial polynomials and the falling factorial polynomials satisfy a *2-term recurrence relation* of the form

$$x\Lambda_n(x) = \Lambda_{n+1}(x) + \kappa_n\Lambda_n(x), \tag{54}$$

where for the monomials $\kappa_n = 0$ and for the falling factorial polynomials $\kappa_n = n$. From (54), we have

$$\begin{aligned} x\Lambda_n(x)\,\Lambda_n(\omega) &= \Lambda_{n+1}(x)\,\Lambda_n(\omega) + \kappa_n\Lambda_n(x)\,\Lambda_n(\omega) \\ \omega\Lambda_n(x)\,\Lambda_n(\omega) &= \Lambda_n(x)\,\Lambda_{n+1}(\omega) + \kappa_n\Lambda_n(x)\,\Lambda_n(\omega), \end{aligned}$$

and therefore

$$(x-\omega)\,\Lambda_n(x)\,\Lambda_n(\omega) = \Lambda_{n+1}(x)\,\Lambda_n(\omega) - \Lambda_n(x)\,\Lambda_{n+1}(\omega)\,.$$

Dividing by $\Lambda_n(\omega)\,\Lambda_{n+1}(\omega)\,,$

$$(x-\omega)\,\frac{\Lambda_n(x)}{\Lambda_{n+1}(\omega)} = \frac{\Lambda_{n+1}(x)}{\Lambda_{n+1}(\omega)} - \frac{\Lambda_n(x)}{\Lambda_n(\omega)},$$

and summing from 0 to $n-1$, we obtain

$$\begin{aligned}(x-\omega)\sum_{k=0}^{n-1}\frac{\Lambda_k(x)}{\Lambda_{k+1}(\omega)} &= \sum_{k=0}^{n-1}\left[\frac{\Lambda_{k+1}(x)}{\Lambda_{k+1}(\omega)} - \frac{\Lambda_k(x)}{\Lambda_k(\omega)}\right] \\ &= \frac{\Lambda_n(x)}{\Lambda_n(\omega)} - \frac{\Lambda_0(x)}{\Lambda_0(\omega)}.\end{aligned}$$

Hence,

$$\frac{1}{\Lambda_n(\omega)}\frac{\Lambda_n(x)}{x-\omega} = \frac{1}{x-\omega} + \sum_{k=0}^{n-1}\frac{\Lambda_k(x)}{\Lambda_{k+1}(\omega)}, \tag{55}$$

since $\Lambda_0(x) = 1$.

Applying L to (55), we see that

$$\frac{1}{\Lambda_n(\omega)} L\left[\frac{\Lambda_n}{x-\omega}\right] = L\left[\frac{1}{x-\omega}\right] + \sum_{k=0}^{n-1}\frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)},$$

and therefore

$$S(\omega;z) = \frac{1}{\Lambda_n(\omega)} L\left[\frac{\Lambda_n}{\omega-x}\right] + \sum_{k=0}^{n-1}\frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)},$$

where

$$S(\omega;z) = L\left[\frac{1}{\omega-x}\right]$$

is the *Stieltjes transform* of the functional L [104].

Remark 9 *Since*

$$\lim_{n\to\infty}\frac{\Lambda_n(x)}{\Lambda_n(\omega)} = 1,$$

we have (at least formally)

$$S(\omega; z) = \sum_{k=0}^{\infty} \frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)}.$$

The falling factorial case was already considered in [17].

Suppose that

$$x^n = \sum_{i=0}^{n} \xi_{n,i} \Lambda_i(x). \tag{56}$$

Since $\Lambda_n(x)$ is monic, we need $\xi_{n,n} = 1$. Using (54), we have

$$\begin{aligned}\sum_{i=0}^{n+1} \xi_{n+1,i} \Lambda_i(x) = x^{n+1} &= \sum_{i=0}^{n} \xi_{n,i} x \Lambda_i(x) \\ &= \sum_{i=0}^{n} \xi_{n,i} \left[\Lambda_{i+1}(x) + \kappa_i \Lambda_i(x)\right] \\ &= \sum_{i=1}^{n+1} \xi_{n,i-1} \Lambda_i(x) + \sum_{i=0}^{n} \xi_{n,i} \kappa_i \Lambda_i(x).\end{aligned}$$

Comparing coefficients, we obtain the recurrence

$$\xi_{n+1,i} = \xi_{n,i-1} + \kappa_i \xi_{n,i}, \quad \xi_{n,n} = 1,$$

and the boundary conditions

$$\xi_{n,i} = 0, \quad i \notin [0, n].$$

In a similar way, if we define the inverse coefficients by

$$\Lambda_n(x) = \sum_{i=0}^{n} \overline{\xi}_{n,i} x^i, \tag{57}$$

then

$$\begin{aligned}\sum_{i=0}^{n+1} \overline{\xi}_{n+1,i} x^i + \sum_{i=0}^{n} \kappa_n \overline{\xi}_{n,i} x^i &= \Lambda_{n+1}(x) + \kappa_n \Lambda_n(x) \\ = x\Lambda_n(x) &= \sum_{i=0}^{n} \overline{\xi}_{n,i} x^{i+1} = \sum_{i=1}^{n+1} \overline{\xi}_{n,i-1} x^i,\end{aligned}$$

and therefore

$$\overline{\xi}_{n+1,i} = \overline{\xi}_{n,i-1} - \kappa_n \overline{\xi}_{n,i}, \quad \overline{\xi}_{n,n} = 1,$$

with

$$\overline{\xi}_{n,i} = 0, \quad i \notin [0, n] .$$

In particular, if $\Lambda_n (x) = \phi_n (x)$, we get

$$\begin{aligned} \xi_{n+1,i} &= \xi_{n,i-1} + i\xi_{n,i}, \quad \xi_{n,n} = 1, \\ \overline{\xi}_{n+1,i} &= \overline{\xi}_{n,i-1} - n\overline{\xi}_{n,i}, \quad \overline{\xi}_{n,n} = 1. \end{aligned}$$

In this case, the coefficients $\xi_{n,i}$ are known as *Stirling numbers of the second kind*, and the coefficients $\overline{\xi}_{n,i}$ are known as *Stirling numbers of the first kind [94]*.

Using Newton's interpolation formula (42), we have

$$x^n = \sum_{k=0}^{n} \frac{\Delta^k [x^n] (0)}{k!} \phi_k (x) ,$$

and therefore the Stirling numbers of the second kind have the representation [89, 26.8.6]

$$\left\{ {n \atop k} \right\} = \frac{\Delta^k [x^n] (0)}{k!} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

Applying L to (56) and (57), we see that

$$\mu_n = \sum_{i=0}^{n} \xi_{n,i} \lambda_i, \quad \lambda_n = \sum_{i=0}^{n} \overline{\xi}_{n,i} \mu_i,$$

and in particular

$$\mu_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \nu_k. \tag{58}$$

3 Transformations of functionals

Let $p, q \in \mathbb{N}_0$, $\mathbf{a} \in \mathbb{K}^p$ and $\mathbf{b} \in \mathbb{K}^q$. In the remaining of the paper, we will use the notation $\left\langle {p \atop q} \cdot \right\rangle$ to stress the "location" of an object $(\rho, L, \mu_n, \nu_n, \text{etc.})$ with respect to the number of parameters in the hypergeometric representation of the first moment

$$\left\langle {}^p_q\lambda \right\rangle_0 (z) = {}_pF_q \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b}+1 \end{array} ; z \right).$$

For example, we have

$$\left\langle {}^p_q\rho \right\rangle (x) = \frac{(\mathbf{a})_x}{(\mathbf{b}+\mathbf{1})_x} \frac{1}{x!},$$

and $\left\langle {}^p_q\lambda \right\rangle_n (z)$ denotes the moments of the linear functional

$$\left\langle {}^p_qL \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \left\langle {}^p_q\rho \right\rangle (x)\, z^x, \quad u \in \mathbb{K}[x], \tag{59}$$

on the $\Lambda_n(x)$ basis, i.e.,

$$\left\langle {}^p_q\lambda \right\rangle_n (z) = \sum_{x=0}^{\infty} \Lambda_n(x) \left\langle {}^p_q\rho \right\rangle (x)\, z^x.$$

The first moment $\left\langle {}^p_q\lambda \right\rangle_0$ satisfies the hypergeometric ODE

$$\left[\left\langle {}^p_q\sigma \right\rangle (\vartheta) - z \left\langle {}^p_q\tau \right\rangle (\vartheta) \right] \left[\left\langle {}^p_q\lambda \right\rangle_0 \right] = 0,$$

where

$$\left\langle {}^p_q\sigma \right\rangle (\vartheta) = \vartheta\left(\vartheta + \mathbf{b}\right), \quad \left\langle {}^p_q\tau \right\rangle (\vartheta) = \left(\vartheta + \mathbf{a}\right).$$

We will also use the notation

$$\left\langle {}^p_q\Phi \right\rangle_n [\mu] = \left(\mu + \mathbf{b}\right) \mu^{n+1} - z \left(\mu + \mathbf{a}\right) \left(\mu + 1\right)^n,$$

and

$$\left\langle {}^p_q\Psi \right\rangle_n = \Upsilon_{n+1}\left(\mathbf{b}\right) - z \Upsilon_n \left(\mathbf{a}\right),$$

which allow us to write the recurrences for the standard and modified moments as $\left\langle {}^p_q\Phi \right\rangle_n [\mu] = 0$ and $\left\langle {}^p_q\Psi \right\rangle_n [\nu] = 0$ respectively.

We define the *upper moment transformation* Ω^α by

$$\Omega^\alpha \left[\left\langle {}^p_q\lambda \right\rangle_0 \right] = {}^\alpha\widetilde{\lambda}_0,$$

where ${}^{\alpha}\widetilde{\lambda}_0\left(z\right)$ is a solution of the hypergeometric ODE

$$\left[\vartheta\left(\vartheta+\mathbf{b}\right)-z\left(\vartheta+\alpha\right)\left(\vartheta+\mathbf{a}\right)\right]\left[y\right]=0. \tag{60}$$

From (25) and (60), we see that the transformed standard moments ${}^{\alpha}\widetilde{\mu}_n$ satisfy the recurrence

$$\left(\mu+\mathbf{b}\right)\mu^{n+1}-z\left(\mu+\alpha\right)\left(\mu+\mathbf{a}\right)\left(\mu+1\right)^n=0,$$

and from (49) and (60), we see that the transformed modified moments ${}^{\alpha}\widetilde{\nu}_n$ satisfy the recurrence

$$\left[\Upsilon_{n+1}\left(\mathbf{b}\right)-z\Upsilon_n\left(\mathbf{a},\alpha\right)\right]\left[\nu\right]=0. \tag{61}$$

Using (47), we can rewrite (61) as

$$\left[\Upsilon_{n+1}\left(\mathbf{b}\right)-z\Upsilon_{n+1}\left(\mathbf{a}\right)-z\left(n+\alpha\right)\Upsilon_n\left(\mathbf{a}\right)\right]\left[\nu\right]=0.$$

In a similar way, we can define the *lower moment transformation* Ω_β by

$$\Omega_\beta\left[\left\langle {}^p_q\lambda\right\rangle_0\right]={}_{\beta}\widetilde{\lambda}_0,$$

where ${}_{\beta}\widetilde{\lambda}_0\left(z\right)$ is a solution of the hypergeometric ODE

$$\left[\left(\vartheta+\beta-1\right)\vartheta\left(\vartheta+\mathbf{b}\right)-z\left(\vartheta+\mathbf{a}\right)\right]\left[y\right]=0. \tag{62}$$

From (25) and (62), we see that the transformed standard moments ${}_{\beta}\widetilde{\mu}_n$ satisfy the recurrence

$$\left(\mu+\beta-1\right)\left(\mu+\mathbf{b}\right)\mu^{n+1}-z\left(\mu+\mathbf{a}\right)\left(\mu+1\right)^n=0,$$

and from (49) and (62), we see that the transformed modified moments ${}_{\beta}\widetilde{\nu}_n$ satisfy the recurrence

$$\left[\Upsilon_{n+1}\left(\mathbf{b},\beta-1\right)-z\Upsilon_n\left(\mathbf{a}\right)\right]\left[\nu\right]=0. \tag{63}$$

Using (47), we can rewrite (63) as

$$\left[\Upsilon_{n+2}\left(\mathbf{b}\right)+\left(n+\beta\right)\Upsilon_{n+1}\left(\mathbf{b}\right)-z\Upsilon_n\left(\mathbf{a}\right)\right]\left[\nu\right]=0.$$

Of course, we can compose two (or more) of the basic transformations Ω^{α} and Ω_{β}, and in this case we will write

$$\Omega^{\alpha} \circ \Omega_{\beta} = \Omega_{\beta} \circ \Omega^{\alpha} = \Omega_{\beta}^{\alpha},$$

and

$$\Omega_{\beta}^{\alpha} \left[\left\langle {}_{q}^{p}\lambda \right\rangle_{0} \right] = {}_{\beta}^{\alpha}\widetilde{\lambda}_{0},$$

where ${}_{\beta}^{\alpha}\widetilde{\lambda}_{0}(z)$ is a solution of the hypergeometric ODE

$$\left[(\vartheta + \beta - 1) \vartheta (\vartheta + \mathbf{b}) - z (\vartheta + \alpha) (\vartheta + \mathbf{a}) \right] [\lambda] = 0, \tag{64}$$

the transformed standard moments ${}_{\beta}^{\alpha}\widetilde{\mu}_{n}$ satisfy the recurrence

$$(\mu + \beta - 1) (\mu + \mathbf{b}) \mu^{n+1} - z (\mu + \alpha) (\mu + \mathbf{a}) (\mu + 1)^{n} = 0, \tag{65}$$

and the transformed modified moments ${}_{\beta}^{\alpha}\widetilde{\nu}_{n}$ satisfy the recurrence

$$\left[\Upsilon_{n+2} (\mathbf{b}) + (n + \beta) \Upsilon_{n+1} (\mathbf{b}) - z \Upsilon_{n+1} (\mathbf{a}) - z (n + \alpha) \Upsilon_{n} (\mathbf{a}) \right] [\nu] = 0. \tag{66}$$

Let's consider the composition $(\mathfrak{S}_{n} + c) \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right]$. We have

$$\begin{aligned} (\mathfrak{S}_{n} + c) \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] [\mu] &= (\mathfrak{S}_{n} + c) \left[(\mu + \mathbf{b}) \mu^{n+1} - z (\mu + \mathbf{a}) (\mu + 1)^{n} \right] \\ &= (\mu + \mathbf{b}) \mu^{n+2} - z (\mu + \mathbf{a}) (\mu + 1)^{n+1} + c (\mu + \mathbf{b}) \mu^{n+1} - zc (\mu + \mathbf{a}) (\mu + 1)^{n} \\ &= (\mu + c) (\mu + \mathbf{b}) \mu^{n+1} - z (\mu + c + 1) (\mu + \mathbf{a}) (\mu + 1)^{n} = \Omega_{c+1}^{c+1} \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] [\mu], \end{aligned}$$

and thus

$$(\mathfrak{S}_{n} + c - 1) \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] = \Omega_{c}^{c} \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right]. \tag{67}$$

If we compose two of these linear factors, we get

$$\begin{aligned} &(\mathfrak{S}_{n} + c - 1) (\mathfrak{S}_{n} + c) \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] [\mu] = \Omega_{c,c+1}^{c,c+1} \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] [\mu] \\ &= (\mu + c - 1) (\mu + c) (\mu + \mathbf{b}) \mu^{n+1} - z (\mu + c) (\mu + c + 1) (\mu + \mathbf{a}) (\mu + 1)^{n} \\ &= \left[(\mu + c - 1) (\mu + \mathbf{b}) \mu^{n+1} - z (\mu + c + 1) (\mu + \mathbf{a}) (\mu + 1)^{n} \right] (\mu + c), \end{aligned}$$

and hence,

$$(\mathfrak{S}_{n} + c - 1) (\mathfrak{S}_{n} + c) \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] = \Omega_{c}^{c+1} \left[\left\langle {}_{q}^{p}\Phi \right\rangle_{n} \right] \circ (\mathfrak{S}_{n} + c). \tag{68}$$

Finally, we note that

$$\begin{aligned}\Omega_{c+1}^{c}\left[\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\right][\mu] &= (\mu+c)\,(\mu+\mathbf{b})\,\mu^{n+1}-z\,(\mu+c)\,(\mu+\mathbf{a})\,(\mu+1)^{n}\\ &= \left[(\mu+\mathbf{b})\,\mu^{n+1}-z\,(\mu+\mathbf{a})\,(\mu+1)^{n}\right](\mu+c)\,,\end{aligned}$$

and therefore

$$\Omega_{c+1}^{c}\left[\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\right]=\left[\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\right]\circ(\mathfrak{S}_{n}+c)\,. \tag{69}$$

Similarly, we have

$$\begin{aligned}&(\mathfrak{S}_{n}+n+c)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]\\ &=\Upsilon_{n+2}(\mathbf{b})-z\Upsilon_{n+1}(\mathbf{a})+(n+c)\,\Upsilon_{n+1}(\mathbf{b})-z\,(n+c)\,\Upsilon_{n}(\mathbf{a})\\ &=\Upsilon_{n+1}(\mathbf{b},c-1)-z\Upsilon_{n}(\mathbf{a},c)\,,\end{aligned}$$

and comparing with (66) we conclude that

$$(\mathfrak{S}_{n}+n+c)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]=\Omega_{c}^{c}\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]. \tag{70}$$

Also, using (48), we obtain

$$\begin{aligned}(\mathfrak{S}_{n}+n+c+1)\,(\mathfrak{S}_{n}+n+c)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right] &= \Omega_{c,c+1}^{c,c+1}\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]\\ &=\Upsilon_{n+1}(\mathbf{b},c-1,c)-z\Upsilon_{n}(\mathbf{a},c,c+1)\\ &=\left[\Upsilon_{n+1}(\mathbf{b},c-1)-z\Upsilon_{n}(\mathbf{a},c+1)\right]\circ\Upsilon_{n}(\mathbf{a},c)\,,\end{aligned}$$

and hence

$$(\mathfrak{S}_{n}+n+c+1)\,(\mathfrak{S}_{n}+n+c)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]=\Omega_{c}^{c+1}\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]\circ(\mathfrak{S}_{n}+n+c)\,. \tag{71}$$

Finally,

$$\Omega_{c+1}^{c}\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]=\Upsilon_{n+1}(\mathbf{b},c)-z\Upsilon_{n}(\mathbf{a},c)=\left[\Upsilon_{n+1}(\mathbf{b})-z\Upsilon_{n}(\mathbf{a})\right]\circ\Upsilon_{n}(c)\,,$$

and therefore

$$\Omega_{c+1}^{c}\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]=\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\circ(\mathfrak{S}_{n}+n+c)\,. \tag{72}$$

It follows that the special cases $\alpha=\beta$ and $\alpha=\beta\pm 1$ lead to some interesting transformations. We will study them in detail in the next sections.

3.1 The Christoffel transformation

The *Christoffel transformation* is defined by

$$\left\langle {}_{q}^{p}\lambda^{C}\right\rangle_{0}=\Omega_{-\omega}^{-\omega+1}\left[\left\langle {}_{q}^{p}\lambda\right\rangle_{0}\right].$$

From (64), we see that $\left\langle {}_{q}^{p}\lambda^{C}\right\rangle_{0}$ is a solution of the ODE

$$\left[\left(\vartheta-\omega-1\right)\vartheta\left(\vartheta+\mathbf{b}\right)-z\left(\vartheta-\omega+1\right)\left(\vartheta+\mathbf{a}\right)\right][\lambda]=0, \tag{73}$$

and admits the hypergeometric representation

$$\left\langle {}_{q}^{p}\lambda^{C}\right\rangle_{0}(z)=-\omega\ {}_{p+1}F_{q+1}\left(\begin{array}{c}\mathbf{a},-\omega+1\\ \mathbf{b}+1,-\omega\end{array};z\right). \tag{74}$$

The reason for choosing this particular solution is the identity

$$-\omega\frac{\left(-\omega+1\right)_{x}}{\left(-\omega\right)_{x}}=x-\omega \tag{75}$$

which shows that

$$\left\langle {}_{q}^{p}L^{C}\right\rangle[u]=\left\langle {}_{q}^{p}L\right\rangle\left[\left(x-\omega\right)u\left(x\right)\right],\quad u\in\mathbb{K}\left[x\right]. \tag{76}$$

This transformation was introduced by Elwin Bruno Christoffel (1829–1900) in his pioneering work [26].

Clearly we must have

$$\left\langle {}_{q}^{p}\lambda^{C}\right\rangle_{0}=\left\langle {}_{q}^{p}L\right\rangle\left[x-\omega\right]=\left(\vartheta-\omega\right)\left[\left\langle {}_{q}^{p}\lambda\right\rangle_{0}\right]\neq0,$$

and since the operator $\vartheta-\omega$ annihilates any multiple of z^{ω}, we need

$$\left\langle {}_{q}^{p}\lambda\right\rangle_{0}(z)\neq\eta z^{\omega},\quad\eta\in\mathbb{K}.$$

From (54) and (76), we get

$$\left\langle {}_{q}^{p}\lambda_{n}\right\rangle=\left\langle {}_{q}^{p}L^{C}\right\rangle\left[\Lambda_{n}\right]=\left\langle {}_{q}^{p}L\right\rangle\left[\left(x-\omega\right)\Lambda_{n}\right]=\left\langle {}_{q}^{p}\lambda\right\rangle_{n+1}+\left(\kappa_{n}-\omega\right)\left\langle {}_{q}^{p}\lambda_{n}\right\rangle,$$

and in particular

$$\left\langle {}_{q}^{p}\mu^{C}\right\rangle_{n}=\left\langle {}_{q}^{p}\mu\right\rangle_{n+1}-\omega\left\langle {}_{q}^{p}\mu\right\rangle_{n}, \tag{77}$$

and
$$\left\langle {}_{q}^{p}\nu^{C}\right\rangle_{n}=\left\langle {}_{q}^{p}\nu\right\rangle_{n+1}+\left(n-\omega\right)\left\langle {}_{q}^{p}\nu\right\rangle_{n}. \tag{78}$$
Note that,
$$\left\langle {}_{q}^{p}\lambda^{C}\right\rangle_{0}=\left\langle {}_{q}^{p}\mu\right\rangle_{1}-\omega\left\langle {}_{q}^{p}\mu\right\rangle_{0}=\left\langle {}_{q}^{p}\nu\right\rangle_{1}-\omega\left\langle {}_{q}^{p}\nu\right\rangle_{0}. \tag{79}$$

From (65), we see that the standard moments $\left\langle {}_{q}^{p}\mu^{C}\right\rangle_{n}$ satisfy the recurrence $\left\langle {}_{q}^{p}\Phi^{C}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\mu^{C}\right\rangle\right]=0$, where
$$\left\langle {}_{q}^{p}\Phi^{C}\right\rangle_{n}[\mu]=\left(\mu-\omega-1\right)\left(\mu+\mathbf{b}\right)\mu^{n+1}-z\left(\mu-\omega+1\right)\left(\mu+\mathbf{a}\right)\left(\mu+1\right)^{n},$$
and from (66), we see that the modified moments $\left\langle {}_{q}^{p}\nu^{C}\right\rangle_{n}$ satisfy the recurrence $\left\langle {}_{q}^{p}\Psi^{C}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu^{C}\right\rangle\right]=0$, where
$$\left\langle {}_{q}^{p}\Psi^{C}\right\rangle_{n}=\Upsilon_{n+2}\left(\mathbf{b}\right)+\left(n-\omega\right)\Upsilon_{n+1}\left(\mathbf{b}\right)-z\Upsilon_{n+1}\left(\mathbf{a}\right)-z\left(n-\omega+1\right)\Upsilon_{n}\left(\mathbf{a}\right). \tag{80}$$

Remark 10 *Using (68), we obtain*
$$\left(\mathfrak{S}_{n}-\omega-1\right)\left(\mathfrak{S}_{n}-\omega\right)\left[\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\right]=\left\langle {}_{q}^{p}\Phi^{C}\right\rangle_{n}\circ\left(\mathfrak{S}_{n}-\omega\right),$$
and therefore
$$\begin{aligned}\left\langle {}_{q}^{p}\Phi^{C}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\mu\right\rangle_{n+1}-\omega\left\langle {}_{q}^{p}\mu\right\rangle_{n}\right]&=\left\langle {}_{q}^{p}\Phi^{C}\right\rangle_{n}\left[\left(\mathfrak{S}_{n}-\omega\right)\left[\left\langle {}_{q}^{p}\mu\right\rangle\right]\right]\\ =\left(\mathfrak{S}_{n}-\omega-1\right)\left(\mathfrak{S}_{n}-\omega\right)\left[\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\right]\left[\left\langle {}_{q}^{p}\mu\right\rangle\right]&=0=\left\langle {}_{q}^{p}\Phi^{C}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\mu^{C}\right\rangle_{n}\right],\end{aligned}$$
in agreement with (77).

Similarly, using (71), we see that
$$\left(\mathfrak{S}_{n}+n-\omega+1\right)\left(\mathfrak{S}_{n}+n-\omega\right)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]=\left\langle {}_{q}^{p}\Psi^{C}\right\rangle_{n}\circ\left(\mathfrak{S}_{n}+n-\omega\right),$$
and hence
$$\begin{aligned}\left\langle {}_{q}^{p}\Psi^{C}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu\right\rangle_{n+1}+\left(n-\omega\right)\left\langle {}_{q}^{p}\nu\right\rangle_{n}\right]&=\left\langle {}_{q}^{p}\Psi^{C}\right\rangle_{n}\left[\left(\mathfrak{S}_{n}+n-\omega\right)\left[\left\langle {}_{q}^{p}\nu\right\rangle\right]\right]\\ =\left(\mathfrak{S}_{n}+n-\omega+1\right)\left(\mathfrak{S}_{n}+n-\omega\right)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right]\left[\left\langle {}_{q}^{p}\nu\right\rangle\right]&=0=\left\langle {}_{q}^{p}\Psi^{C}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu^{C}\right\rangle_{n}\right],\end{aligned}$$
in agreement with (80).

Using (35) and (77), we obtain the exponential generating function of the transformed standard moments

$$\sum_{n=0}^{\infty} \left\langle {}_{q}^{p}\mu^{C} \right\rangle_{n}(z) \frac{t^{n}}{n!} = \left\langle {}_{q}^{p}\lambda^{C} \right\rangle_{0}\left(ze^{t}\right) = \left\langle {}_{q}^{p}\mu \right\rangle_{1}\left(ze^{t}\right) - \omega \left\langle {}_{q}^{p}\mu \right\rangle_{0}\left(ze^{t}\right),$$

while from (52) and (78) we get the exponential generating function of the transformed modified moments

$$\sum_{n=0}^{\infty} \left\langle {}_{q}^{p}\nu^{C} \right\rangle_{n}(z) \frac{t^{n}}{n!} = \left\langle {}_{q}^{p}\lambda^{C} \right\rangle_{0}(z+zt) = \left\langle {}_{q}^{p}\nu \right\rangle_{1}(z+zt) - \omega \left\langle {}_{q}^{p}\nu \right\rangle_{0}(z+zt).$$

3.2 The Geronimus transformation

The *Geronimus transformation* is defined by

$$\left\langle {}_{q}^{p}\lambda^{G} \right\rangle_{0} = \Omega_{-\omega+1}^{-\omega}\left[\left\langle {}_{q}^{p}\lambda \right\rangle_{0}\right], \quad \omega \notin \mathbb{N}_{0}.$$

From (64), we see that $\left\langle {}_{q}^{p}\lambda^{G} \right\rangle_{0}(z)$ is a solution of the ODE

$$\vartheta(\vartheta+\mathbf{b})(\vartheta-\omega)\left[\left\langle {}_{q}^{p}\lambda^{G} \right\rangle_{0}\right] = z(\vartheta+\mathbf{a})(\vartheta-\omega)\left[\left\langle {}_{q}^{p}\lambda^{G} \right\rangle_{0}\right], \tag{81}$$

and admits the hypergeometric representation

$$\left\langle {}_{q}^{p}\lambda^{G} \right\rangle_{0}(z) = -\omega^{-1}\ {}_{p+1}F_{q+1}\left(\begin{array}{c} \mathbf{a}, -\omega \\ \mathbf{b}+1, -\omega+1 \end{array}; z\right).$$

Remark 11 *The function z^{ω} is also a solution of (81), and therefore we could define (as some authors do)*

$$\left\langle {}_{q}^{p}\lambda^{G} \right\rangle_{0}(z) = -\omega^{-1}\ {}_{p+1}F_{q+1}\left(\begin{array}{c} \mathbf{a}, -\omega \\ \mathbf{b}+1, -\omega+1 \end{array}; z\right) + \eta z^{\omega}$$

where η is an arbitrary constant.

The identity (75) shows that

$$\left\langle {}_{q}^{p}L^{G} \right\rangle[u] = \left\langle {}_{q}^{p}L \right\rangle\left[\frac{u(x)}{x-\omega}\right], \quad u \in \mathbb{K}[x], \tag{82}$$

and

$$\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{0}(z)=\left\langle {}_{q}^{p}L\right\rangle \left[\frac{1}{x-\omega}\right](z)=-\left\langle {}_{q}^{p}S\right\rangle (\omega;z), \tag{83}$$

where $\left\langle {}_{q}^{p}S\right\rangle (\omega;z)$ is the Stieltjes transform of the functional $\left\langle {}_{q}^{p}L\right\rangle$. Since

$$(\vartheta-\omega)\left[\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{0}\right]=\left\langle {}_{q}^{p}L\right\rangle \left[(x-\omega)\frac{1}{x-\omega}\right]=\left\langle {}_{q}^{p}L\right\rangle [1]=\left\langle {}_{q}^{p}\lambda\right\rangle_{0},$$

we need

$$\left\langle {}_{q}^{p}S\right\rangle (\omega;z)\neq\eta z^{\omega},\quad \eta\in\mathbb{K}.$$

This transformation was introduced by Yakov Lazarevich Geronimus (1898–1984) in his groundbreaking article [55].

Remark 12 *If we use the integral representation [89, 16.5.2]*

$$\begin{aligned}&{}_{p+1}F_{q+1}\left(\begin{matrix}\mathbf{a},\alpha\\ \mathbf{b},\beta\end{matrix};z\right)\\ &=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_{0}^{1}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\ {}_{p}F_{q}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix};zt\right)dt,\end{aligned}$$

we obtain

$$\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{0}(z)=\int_{0}^{1}t^{-\omega-1}\left\langle {}_{q}^{p}\lambda\right\rangle_{0}(zt)\,dt. \tag{84}$$

If we use (84) and formally integrate term by term, we get

$$\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{0}(z)=\sum_{x=0}^{\infty}\frac{(\mathbf{a})_{x}}{(\mathbf{b}+\mathbf{1})_{x}}\frac{z^{x}}{x!}\int_{0}^{1}t^{x-\omega-1}dt=\sum_{x=0}^{\infty}\frac{1}{x-\omega}\frac{(\mathbf{a})_{x}}{(\mathbf{b}+\mathbf{1})_{x}}\frac{z^{x}}{x!},$$

in agreement with (82). Extending (84), we conclude that

$$\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{n}(z)=\int_{0}^{1}t^{-\omega-1}\left\langle {}_{q}^{p}\lambda\right\rangle_{n}(zt)\,dt,\quad n\in\mathbb{N}_{0}. \tag{85}$$

From (54) and (82), we see that

$$\begin{aligned}&\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{n+1}+(\kappa_{n}-\omega)\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{n}\\&=\left\langle {}_{q}^{p}L^{G}\right\rangle\left[(x-\omega)\Lambda_{n}(x)\right]=\left\langle {}_{q}^{p}L\right\rangle\left[\Lambda_{n}(x)\right]=\left\langle {}_{q}^{p}\lambda\right\rangle_{n},\end{aligned}$$

and in particular

$$\left\langle {}_{q}^{p}\mu^{G}\right\rangle_{n+1}-\omega\left\langle {}_{q}^{p}\mu^{G}\right\rangle_{n}=\left\langle {}_{q}^{p}\mu\right\rangle_{n}, \tag{86}$$

and

$$\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n+1}+(n-\omega)\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n}=\left\langle {}_{q}^{p}\nu\right\rangle_{n}. \tag{87}$$

Using (55), we get

$$\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{n}=\Lambda_{n}(\omega)\left(\left\langle {}_{q}^{p}\lambda\right\rangle_{0}+\sum_{k=0}^{n-1}\frac{\left\langle {}_{q}^{p}\lambda\right\rangle_{k}}{\Lambda_{k+1}(\omega)}\right),$$

where care needs to be exercised if $\Lambda_k(\omega)=0$ for some k.

From (69), we have

$$\left\langle {}_{q}^{p}\Phi^{G}\right\rangle_{n}\;[\mu]=\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\circ(\mathfrak{S}_{n}-\omega)\,[\mu]\;, \tag{88}$$

in agreement with (86), since

$$\begin{aligned}\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\left[\left\langle {}_{q}^{p}\mu^{G}\right\rangle_{n+1}-\omega\left\langle {}_{q}^{p}\mu^{G}\right\rangle_{n}\right]&=\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\circ(\mathfrak{S}_{n}-\omega)\left[\left\langle {}_{q}^{p}\mu^{G}\right\rangle\right]\\&=\left\langle {}_{q}^{p}\Phi^{G}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\mu^{G}\right\rangle\right]=0=\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\left[\left\langle {}_{q}^{p}\mu\right\rangle\right].\end{aligned}$$

From (72), we get $\left\langle {}_{q}^{p}\Psi^{G}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu^{G}\right\rangle\right]=0$, where

$$\left\langle {}_{q}^{p}\Psi^{G}\right\rangle_{n}=\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\circ(\mathfrak{S}_{n}+n-\omega), \tag{89}$$

in agreement with (87), since

$$\begin{aligned}\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n+1}+(n-\omega)\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n}\right]&=\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\circ(\mathfrak{S}_{n}+n-\omega)\left[\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n}\right]\\&=\left\langle {}_{q}^{p}\Psi^{G}\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n}\right]=0=\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\left[\left\langle {}_{q}^{p}\nu\right\rangle_{n}\right].\end{aligned}$$

Using (35) and (83), we obtain the exponential generating function of $\left\langle {}_{q}^{p}\mu^{G}\right\rangle_{n}$

$$\sum_{n=0}^{\infty}\left\langle {}_{q}^{p}\mu^{G}\right\rangle_{n}(z)\,\frac{t^{n}}{n!}=\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{0}\left(ze^{t}\right)=-\left\langle {}_{q}^{p}S\right\rangle\left(\omega;ze^{t}\right),$$

and for the transformed modified moments $\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n}$ we get

$$\sum_{n=0}^{\infty}\left\langle {}_{q}^{p}\nu^{G}\right\rangle_{n}\frac{t^{n}}{n!}=\left\langle {}_{q}^{p}\lambda^{G}\right\rangle_{0}(z+zt)=-\left\langle {}_{q}^{p}S\right\rangle(\omega;z+zt).$$

3.3 The Uvarov transformation

Let's consider the composite transformations (Christoffel-Geronimus)

$$\left(\Omega_{1-\omega}^{-\omega} \circ \Omega_{-\omega}^{1-\omega}\right)\left[\left\langle {}_{q}^{p}\lambda \right\rangle_{0}\right],$$

and (Geronimus-Christoffel)

$$\left(\Omega_{-\omega}^{1-\omega} \circ \Omega_{1-\omega}^{-\omega}\right)\left[\left\langle {}_{q}^{p}\lambda \right\rangle_{0}\right].$$

We see that in either case, the transformed first moment $\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}$ is a solution of the ODE

$$\begin{gathered} \left(\vartheta-\omega\right)\left(\vartheta-\omega-1\right)\vartheta\left(\vartheta+\mathbf{b}\right)\left[\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}\right] \\ = z\left(\vartheta-\omega\right)\left(\vartheta-\omega+1\right)\left(\vartheta+\mathbf{a}\right)\left[\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}\right] \end{gathered}, \tag{90}$$

which can be written as

$$\left(\vartheta-\omega\right)\left(\vartheta-\omega-1\right)\left[\left\langle {}_{q}^{p}\sigma \right\rangle\left(\vartheta\right) - z\left\langle {}_{q}^{p}\tau \right\rangle\left(\vartheta\right)\right]\left[\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}\right] = 0. \tag{91}$$

Clearly, $\left\langle {}_{q}^{p}\lambda \right\rangle_{0}$ is a solution of (91). If we set $\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}(z) = z^{\omega}$, we have

$$\left[\left\langle {}_{q}^{p}\sigma \right\rangle\left(\vartheta\right) - z\left\langle {}_{q}^{p}\tau \right\rangle\left(\vartheta\right)\right]\left[z^{\omega}\right] = \left\langle {}_{q}^{p}\sigma \right\rangle\left(\omega\right) z^{\omega} - \left\langle {}_{q}^{p}\tau \right\rangle\left(\omega\right) z^{\omega+1}, \tag{92}$$

and therefore

$$\begin{aligned} &\left(\vartheta-\omega\right)\left(\vartheta-\omega-1\right)\left[\left\langle {}_{q}^{p}\sigma \right\rangle\left(\vartheta\right) - z\left\langle {}_{q}^{p}\tau \right\rangle\left(\vartheta\right)\right]\left[\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}\right] \\ &= \left(\vartheta-\omega\right)\left(\vartheta-\omega-1\right)\left[\left\langle {}_{q}^{p}\sigma \right\rangle\left(\omega\right) z^{\omega} - \left\langle {}_{q}^{p}\tau \right\rangle\left(\omega\right) z^{\omega+1}\right] = 0. \end{aligned}$$

Thus, the linear combination

$$\left\langle {}_{q}^{p}\lambda_{\omega}^{U} \right\rangle_{0}(z) = \left\langle {}_{q}^{p}\lambda \right\rangle_{0}(z) + \eta z^{\omega}, \quad \eta \in \mathbb{K}, \tag{93}$$

is a solution of (91).

We define the *Uvarov transformation* by

$$\left\langle {}_{q}^{p}L_{\omega}^{U} \right\rangle[u] = \left\langle {}_{q}^{p}L \right\rangle[u] + \eta u\left(\omega\right) z^{\omega}, \quad u \in \mathbb{K}\left[x\right],$$

which is well defined as long as

$$\left\langle {}_{q}^{p}\lambda \right\rangle_{0}(z) \neq -\eta z^{\omega}.$$

This transformation was introduced by Vasiliĭ Borisovich Uvarov (1929–1997) in his monumental paper [110].

From (67), we see that the standard moments $\left\langle {}^{p}_{q}\mu^{U}_{\omega}\right\rangle_{n}$ satisfy

$$\left\langle {}^{p}_{q}\Phi^{U}_{\omega}\right\rangle_{n} = \left(\mathfrak{S}_{n}-\omega-1\right)\left(\mathfrak{S}_{n}-\omega\right)\left[\left\langle {}^{p}_{q}\Phi\right\rangle_{n}\right], \tag{94}$$

and from (70), we have

$$\left\langle {}^{p}_{q}\Psi^{U}_{\omega}\right\rangle_{n} = \left(\mathfrak{S}_{n}+n-\omega+1\right)\left(\mathfrak{S}_{n}+n-\omega\right)\left[\left\langle {}^{p}_{q}\Psi\right\rangle_{n}\right]. \tag{95}$$

If $\left\langle {}^{p}_{q}\sigma\right\rangle(\omega)=0$, then we see from (92) that

$$\left[\left\langle {}^{p}_{q}\sigma\right\rangle(\vartheta) - z\left\langle {}^{p}_{q}\tau\right\rangle(\vartheta)\right][z^{\omega}] = -\left\langle {}^{p}_{q}\tau\right\rangle(\omega)\, z^{\omega+1}$$

and therefore the transformed moment $\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0}$ satisfies the *reduced ODE*

$$\left(\vartheta-\omega-1\right)\left[\left\langle {}^{p}_{q}\sigma\right\rangle(\vartheta) - z\left\langle {}^{p}_{q}\tau\right\rangle(\vartheta)\right]\left[\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0}\right]=0. \tag{96}$$

Similarly, If $\left\langle {}^{p}_{q}\tau\right\rangle(\omega)=0$, then we see from (92) that

$$\left[\left\langle {}^{p}_{q}\sigma\right\rangle(\vartheta) - z\left\langle {}^{p}_{q}\tau\right\rangle(\vartheta)\right][z^{\omega}] = \left\langle {}^{p}_{q}\sigma\right\rangle(\omega)\, z^{\omega}$$

and therefore the transformed moment $\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0}$ satisfies the *reduced ODE*

$$\left(\vartheta-\omega\right)\left[\left\langle {}^{p}_{q}\sigma\right\rangle(\vartheta) - z\left\langle {}^{p}_{q}\tau\right\rangle(\vartheta)\right]\left[\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0}\right]=0. \tag{97}$$

Comparing with (64), we can interpret $\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0}$ as

$$\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0} = \Omega^{-\omega}_{-\omega}\left[\left\langle {}^{p}_{q}\lambda\right\rangle_{0}\right], \quad \left\langle {}^{p}_{q}\sigma\right\rangle(\omega)=0, \tag{98}$$

and

$$\left\langle {}^{p}_{q}\lambda^{U}_{\omega}\right\rangle_{0} = \Omega^{1-\omega}_{1-\omega}\left[\left\langle {}^{p}_{q}\lambda\right\rangle_{0}\right], \quad \left\langle {}^{p}_{q}\tau\right\rangle(\omega)=0. \tag{99}$$

From (65) and (98), we get

$$\left\langle {}^{p}_{q}\Phi^{U}_{\omega}\right\rangle_{n} = \left(\mathfrak{S}_{n}-\omega-1\right)\left[\left\langle {}^{p}_{q}\Phi\right\rangle_{n}\right], \quad \left\langle {}^{p}_{q}\sigma\right\rangle(\omega)=0, \tag{100}$$

while (66) gives

$$\left\langle {}_{q}^{p}\Psi_{\omega}^{U}\right\rangle_{n}=\left(\mathfrak{S}_{n}+n-\omega\right)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right],\quad \left\langle {}_{q}^{p}\sigma\right\rangle(\omega)=0. \tag{101}$$

From (65) and (99), we have

$$\left\langle {}_{q}^{p}\Phi_{\omega}^{U}\right\rangle_{n}=\left(\mathfrak{S}_{n}-\omega\right)\left[\left\langle {}_{q}^{p}\Phi\right\rangle_{n}\right],\quad \left\langle {}_{q}^{p}\tau\right\rangle(\omega)=0, \tag{102}$$

while (66) gives

$$\left\langle {}_{q}^{p}\Psi_{\omega}^{U}\right\rangle_{n}=\left(\mathfrak{S}_{n}+n-\omega+1\right)\left[\left\langle {}_{q}^{p}\Psi\right\rangle_{n}\right],\quad \left\langle {}_{q}^{p}\tau\right\rangle(\omega)=0. \tag{103}$$

Finally, we have

$$\left\langle {}_{q}^{p}\lambda_{\omega}^{U}\right\rangle_{n}=\left\langle {}_{q}^{p}L_{\omega}^{U}\right\rangle\left[\Lambda_{n}\right]=\left\langle {}_{q}^{p}\lambda_{n}\right\rangle+\eta\Lambda_{n}(\omega)\,z^{\omega}, \tag{104}$$

from which we obtain the exponential generating functions of $\left\langle {}_{q}^{p}\mu_{\omega}^{U}\right\rangle_{n}(z)$

$$\sum_{n=0}^{\infty}\left\langle {}_{q}^{p}\mu_{\omega}^{U}\right\rangle_{n}(z)\,\frac{t^{n}}{n!}=\left\langle {}_{q}^{p}\mu\right\rangle_{0}\left(ze^{t}\right)+\eta\left(ze^{t}\right)^{\omega},$$

and $\left\langle {}_{q}^{p}\nu_{\omega}^{U}\right\rangle_{n}(z)$

$$\sum_{n=0}^{\infty}\left\langle {}_{q}^{p}\nu_{\omega}^{U}\right\rangle_{n}(z)\,\frac{t^{n}}{n!}=\left\langle {}_{q}^{p}\nu\right\rangle_{0}(z+zt)+\eta\,(z+zt)^{\omega}.$$

3.4 Truncated linear functionals

Let $N\in\mathbb{N}_0$ and the *truncated functional* $\left\langle {}_{q}^{p}L^{T}\right\rangle$ be defined by

$$\left\langle {}_{q}^{p}L^{T}\right\rangle[u]=\sum_{x=0}^{N}u(x)\left\langle {}_{q}^{p}\rho\right\rangle(x)\,z^{x},\quad u\in\mathbb{K}[x], \tag{105}$$

as long as

$$\left\langle {}_{q}^{p}\lambda^{T}\right\rangle_{0}(z)=\sum_{x=0}^{N}\left\langle {}_{q}^{p}\rho\right\rangle(x)\,z^{x}\neq 0.$$

Remark 13 *If $\left\langle {}_{q}^{p}\tau \right\rangle (N) = 0$, then the functional (59) is already a truncated functional, since*

$$(-N)_x = 0, \quad x > N.$$

Therefore, we assume that $\left\langle {}_{q}^{p}\tau \right\rangle (N) \neq 0$.

Using the Pearson equation (18), we have

$$\begin{aligned} &\left[\left\langle {}_{q}^{p}\sigma \right\rangle (\vartheta) - z \left\langle {}_{q}^{p}\tau \right\rangle (\vartheta)\right] \left[\left\langle {}_{q}^{p}\lambda^{T} \right\rangle_0\right] \\ &= \sum_{x=0}^{N} \left[\left\langle {}_{q}^{p}\sigma \right\rangle (x) \left\langle {}_{q}^{p}\rho \right\rangle (x)\, z^{x} - \left\langle {}_{q}^{p}\tau \right\rangle (x) \left\langle {}_{q}^{p}\rho \right\rangle (x)\, z^{x+1}\right] \\ &= \sum_{x=0}^{N} \left\langle {}_{q}^{p}\sigma \right\rangle (x) \left\langle {}_{q}^{p}\rho \right\rangle (x)\, z^{x} - \sum_{x=1}^{N+1} \left\langle {}_{q}^{p}\tau \right\rangle (x-1) \left\langle {}_{q}^{p}\rho \right\rangle (x-1)\, z^{x} \\ &= -\left\langle {}_{q}^{p}\tau \right\rangle (N) \left\langle {}_{q}^{p}\rho \right\rangle (N) \frac{z^{N+1}}{N!}, \end{aligned}$$

and we conclude that $\lambda_0^T(z)$ satisfies the ODE

$$(\vartheta - N - 1) \left[\left\langle {}_{q}^{p}\sigma \right\rangle (\vartheta) - z \left\langle {}_{q}^{p}\tau \right\rangle (\vartheta)\right] \left[\left\langle {}_{q}^{p}\lambda^{T} \right\rangle_0\right] = 0. \tag{106}$$

Using (9) in (106), we obtain

$$(\vartheta - N - 1) \left\langle {}_{q}^{p}\sigma \right\rangle (\vartheta) \left[\left\langle {}_{q}^{p}\lambda^{T} \right\rangle_0\right] = z (\vartheta - N) \left\langle {}_{q}^{p}\tau \right\rangle (\vartheta) \left[\left\langle {}_{q}^{p}\lambda^{T} \right\rangle_0\right],$$

and therefore we have

$$\left\langle {}_{q}^{p}\lambda^{T} \right\rangle_0 = \Omega_{-N}^{-N} \left[\left\langle {}_{q}^{p}\lambda \right\rangle_0\right], \quad N \in \mathbb{N}_0. \tag{107}$$

Remark 14 *If we use the formula [89, 16.2.4]*

$$\sum_{k=0}^{N} \frac{(\mathbf{a})_k}{(\mathbf{b})_k} \frac{z^k}{k!} = \frac{z^N}{N!} \frac{(\mathbf{a})_N}{(\mathbf{b})_N} \, {}_{q+2}F_p \left(\begin{array}{c} -N, 1-\mathbf{b}-N, 1 \\ 1-\mathbf{a}-N \end{array} ; \frac{(-1)^{q+p+1}}{z} \right), \tag{108}$$

we obtain the hypergeometric representation

$$\left\langle {}_{q}^{p}\lambda^{T} \right\rangle_0 = \frac{z^N}{N!} \frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N} \, {}_{q+2}F_p \left(\begin{array}{c} -N, -\mathbf{b}-N, 1 \\ 1-\mathbf{a}-N \end{array} ; \frac{(-1)^{q+p+1}}{z} \right). \tag{109}$$

Using the integral representation [89, 16.5.3]

$$ {}_{p+1}F_q\left(\begin{matrix}\mathbf{a},\alpha\\ \mathbf{b}\end{matrix};\frac{x}{z}\right)=\frac{z^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}\,{}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix};xt\right)e^{-zt}dt \tag{110}$$

with $\alpha=1$, *we see that* $\left\langle {}_q^p\lambda^T\right\rangle_0$ *can be represented as a Laplace transform*

$$\left\langle {}_q^p\lambda^T\right\rangle_0(z)=\frac{z^{N+1}}{N!}\frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N}\int_0^{\infty}{}_{q+1}F_p\left(\begin{matrix}-N,-\mathbf{b}-N\\ 1-\mathbf{a}-N\end{matrix};(-1)^{q+p+1}t\right)e^{-zt}dt. \tag{111}$$

From (67) and (107), we get

$$\left\langle {}_q^p\Phi^T\right\rangle_n=(\mathfrak{S}_n-N-1)\left[\left\langle {}_q^p\Phi\right\rangle_n\right], \tag{112}$$

while (70) gives

$$\left\langle {}_q^p\Psi^T\right\rangle_n=(\mathfrak{S}_n+n-N)\left[\left\langle {}_q^p\Psi\right\rangle_n\right]. \tag{113}$$

Remark 15 *Note that since*

$$\sum_{x=0}^{N}\phi_n(x)\frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x}\frac{z^x}{x!}=\sum_{x=n}^{N}\frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x}\frac{z^x}{(x-n)!}=\sum_{x=0}^{N-n}\frac{(\mathbf{a})_{x+n}}{(\mathbf{b}+1)_{x+n}}\frac{z^{x+n}}{x!},$$

we have

$$\left\langle {}_q^p\nu^T\right\rangle_n(z)=z^n\frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n}\sum_{x=0}^{N-n}\frac{(\mathbf{a}+n)_x}{(\mathbf{b}+1+n)_x}\frac{z^x}{x!}. \tag{114}$$

Thus, we can use (108) and obtain

$$\left\langle {}_q^p\nu^T\right\rangle_n(z)=\frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N}\frac{z^N}{(N-n)!}\,{}_{q+2}F_p\left(\begin{matrix}n-N,-\mathbf{b}-N,1\\ 1-\mathbf{a}-N\end{matrix};\frac{(-1)^{q+p+1}}{z}\right). \tag{115}$$

In particular,

$$\left\langle {}_q^p\nu^T\right\rangle_N(z)=\frac{(\mathbf{a})_N}{(\mathbf{b}+\mathbf{1})_N}z^N,\quad \left\langle {}_q^p\nu^T\right\rangle_n(z)=0,\quad n>N.$$

Using (110) and (115), we get the integral representation

$$\left\langle {}_q^p\nu^T\right\rangle_n(z)=\frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N}\frac{z^{N+1}}{(N-n)!}\int_0^{\infty}{}_{q+1}F_p\left(\begin{matrix}n-N,-\mathbf{b}-N\\ 1-\mathbf{a}-N\end{matrix};(-1)^{q+p+1}t\right)e^{-zt}dt. \tag{116}$$

4 Examples

In this section, we will illustrate the application of the formulas that we have derived. We will consider all the polynomials of class $s \leq 2$ and also look at the subclasses obtained by applying one or more of the moment transformations from the previous section.

4.1 Polynomials of class 0 (discrete classical polynomials)

The discrete classical orthogonal polynomials (Charlier, Meixner, Krawtchouk) first appeared in the literature in the years 1905–1934, and were considered at the time as a generalization of the continuous classical polynomials (Hermite, Laguerre, Jacobi).

The last member of this class (Hahn polynomials) were introduced by Chebyshev (1875) and Hahn (1949), but we don't consider them by themselves since they are a special case ($z = 1$) of the Generalized Hahn polynomials (see Section 4.2.4).

We will use the notation $(p, q; N)$ to indicate that one of the upper parameters in the hypergeometric representation of the first moment is a negative integer $-N$, $N \in \mathbb{N}$.

For additional references, see [11], [22], [33], [34], [51], [88], [96].

4.1.1 Polynomials of type $(0, 0)$ (Charlier polynomials)

Linear functional

$$\left\langle {}_{0}^{0}L \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_{0}^{0}\lambda \right\rangle_0 (z) = {}_0F_0 \left(\begin{matrix} - \\ - \end{matrix} ; z \right) = e^z.$$

ODE satisfied by the first moment

$$(\vartheta - z) \left[\left\langle {}_{0}^{0}\lambda \right\rangle_0 \right] = 0. \tag{117}$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{0}\Phi \right\rangle_n [\mu] = \mu^{n+1} - z(\mu + 1)^n. \tag{118}$$

Recurrence of the standard moments

$$\left\langle {}_{0}^{0}\mu \right\rangle_{n+1} = z \sum_{k=0}^{n} \binom{n}{k} \left\langle {}_{0}^{0}\mu \right\rangle_{k}.$$

Representation of the standard moments in terms of the polynomials $Q_n(z)$

$$\left\langle {}_{0}^{0}\mu \right\rangle_{n} = \left\langle {}_{0}^{0}\mu \right\rangle_{0} \left\langle {}_{0}^{0}Q \right\rangle_{n}, \quad n \geq 0,$$

with

$$\left\langle {}_{0}^{0}Q \right\rangle_{n+1} = (\vartheta + z) \left\langle {}_{0}^{0}Q \right\rangle_{n}, \quad \left\langle {}_{0}^{0}Q \right\rangle_{0} = 1.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{0}^{0}\nu \right\rangle_{n}(z) = z^{n} \, {}_{0}F_{0}\left(\begin{matrix} - \\ - \end{matrix} ; z \right) = z^{n} e^{z}. \tag{119}$$

Remark 16 *Using (58) and (119), we have*

$$\left\langle {}_{0}^{0}\mu \right\rangle_{n}(z) = e^{z} \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} z^{k},$$

and therefore

$$\left\langle {}_{0}^{0}Q \right\rangle_{n}(z) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} z^{k}.$$

The polynomials $\left\langle {}_{0}^{0}Q \right\rangle_{n}$ *are known as the Touchard (or exponential, or Bell) polynomials [106].*

Modified moments recurrence operator

$$\left\langle {}_{0}^{0}\Psi \right\rangle_{n} = \Upsilon_{n+1}(\emptyset) - z\Upsilon_{n}(\emptyset) = \mathfrak{S}_{n} - z.$$

Remark 17 *The Charlier polynomials have the hypergeometric representation [89, 18.20.8]*

$$\left\langle {}_{0}^{0}P \right\rangle_{n}(x; z) = {}_{2}F_{0}\left(\begin{matrix} -n, -x \\ - \end{matrix} ; -z^{-1} \right). \tag{120}$$

The Charlier polynomials were introduced by Carl Vilhelm Ludwig Charlier (1862–1934) in his paper [23].

4.1.2 Polynomials of type $(1,0)$ (Meixner polynomials)

Linear functional

$$\left\langle {}_0^1 L \right\rangle [u] = \sum_{x=0}^{\infty} u(x)\ (a)_x \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_0^1 \lambda \right\rangle_0 (z) = {}_1F_0 \left(\begin{matrix} a \\ - \end{matrix} ; z \right) = (1-z)^{-a}.$$

The Meixner polynomials can be obtained from the Charlier polynomials by means of the transformation

$$\left\langle {}_0^1 \lambda \right\rangle_0 = \Omega^a \left[\left\langle {}_0^0 \lambda \right\rangle_0 \right].$$

Using (60) and (117), we obtain the ODE satisfied by the first moment

$$[\vartheta - z(\vartheta + a)] \left[\left\langle {}_0^1 \lambda \right\rangle_0 \right] = 0. \tag{121}$$

Standard moments recurrence operator

$$\left\langle {}_0^1 \Phi \right\rangle_n [\mu] = \mu^{n+1} - z(\mu + a)(\mu + 1)^n.$$

Standard moments recurrence

$$\left\langle {}_0^1 \mu \right\rangle_{n+1} = \frac{z}{1-z} \left[a \left\langle {}_0^1 \mu \right\rangle_n + \sum_{k=0}^{n-1} \binom{n}{k} \left(\left\langle {}_0^1 \mu \right\rangle_{k+1} + a \left\langle {}_0^1 \mu \right\rangle_k \right) \right].$$

Representation of the standard moments in terms of the polynomials $Q_n(z)$

$$\left\langle {}_0^1 \mu \right\rangle_n = (1-z)^{-a-n} \left\langle {}_0^1 Q \right\rangle_n (z), \quad n \geq 0,$$

with

$$\left\langle {}_0^1 Q \right\rangle_{n+1} = [(1-z)\vartheta + (n+a)z] \left\langle {}_0^1 Q \right\rangle_n, \quad \left\langle {}_0^1 Q \right\rangle_0 = 1.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_0^1 \nu \right\rangle_n (z) = z^n (a)_n \ {}_1F_0 \left[\begin{matrix} a+n \\ - \end{matrix} ; z \right],$$

and therefore

$$\left\langle {}_0^1 \nu \right\rangle_n (z) = z^n (a)_n (1-z)^{-a-n}. \tag{122}$$

Remark 18 *Using (58) and (122), we have*

$$\left\langle {}_{0}^{1}\mu \right\rangle_{n}(z) = (1-z)^{-a} \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (a)_{k} \left(\frac{z}{1-z} \right)^{k},$$

and therefore

$$\left\langle {}_{0}^{1}Q \right\rangle_{n}(z) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (a)_{k} \ z^{k} (1-z)^{n-k}.$$

Using (35), we get

$$\sum_{n=0}^{\infty} \left\langle {}_{0}^{1}\mu \right\rangle_{n} \frac{t^{n}}{n!} = \left(1 - ze^{t}\right)^{-a},$$

and hence

$$\sum_{n=0}^{\infty} \frac{\left\langle {}_{0}^{1}Q \right\rangle_{n}}{n!} \left(\frac{t}{1-z} \right)^{n} = \left(\frac{1-z}{1-ze^{t}} \right)^{a}.$$

or

$$\sum_{n=0}^{\infty} \left\langle {}_{0}^{1}Q \right\rangle_{n} \frac{t^{n}}{n!} = \left(\frac{1-z}{1-ze^{(1-z)t}} \right)^{a}.$$

This shows that the polynomials $\left\langle {}_{0}^{1}Q \right\rangle_{n}(z)$ *are related to the* Eulerian polynomials*, defined by the exponential generating function [21]*

$$\sum_{n=0}^{\infty} A_{n} \frac{t^{n}}{n!} = \frac{1-z}{e^{t}-z}.$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{1}\Psi \right\rangle_{n} = \Upsilon_{n+1}(\emptyset) - z\Upsilon_{n}(a) = (1-z)\,\mathfrak{S}_{n} - z(n+a).$$

Remark 19 *The Meixner polynomials have the hypergeometric representation [89, 18.20.7]*

$$\left\langle {}_{0}^{1}P \right\rangle_{n}(x;z) = {}_{2}F_{1}\left(\begin{matrix} -n, -x \\ a \end{matrix} ; 1 - z^{-1} \right).$$

The Meixner polynomials were introduced by Josef Meixner (1908 – 1994) in his paper [86], although Ladislav Truksa (1891–?) already considered them in his 1931 papers [107], [108], [109] (see [20]).

4.1.3 Polynomials of type $(1,0;N)$ (Krawtchouk polynomials)

These polynomials are a particular case of the Meixner polynomials, with $-a = N \in \mathbb{N}$.

Linear functional

$$\left\langle {}_{0}^{1;N}L \right\rangle [u] = \sum_{x=0}^{N} u(x)\ (-N)_x \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_{0}^{1;N}\lambda \right\rangle_0 (z) = (1-z)^N .$$

ODE satisfied by the first moment

$$[\vartheta - z(\vartheta - N)] \left[\left\langle {}_{0}^{1;N}\lambda \right\rangle_0 \right] = 0.$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{1;N}\Phi \right\rangle_n = \left\langle {}_{0}^{1;N}\mu \right\rangle^{n+1} - z\left(\left\langle {}_{0}^{1;N}\mu \right\rangle - N \right) \left(\left\langle {}_{0}^{1;N}\mu \right\rangle + 1 \right)^n .$$

Representation of the standard moments in terms of the polynomials $Q_n(z)$

$$\left\langle {}_{0}^{1;N}\mu \right\rangle_n = (1-z)^{N-n} \left\langle {}_{0}^{1}Q \right\rangle_n (z), \quad n \geq 0,$$

with

$$\left\langle {}_{0}^{1}Q \right\rangle_{n+1} = [(1-z)\vartheta + (n-N)z] \left\langle {}_{0}^{1}Q \right\rangle_n, \quad \left\langle {}_{0}^{1}Q \right\rangle_0 = 1.$$

Modified moments

$$\left\langle {}_{0}^{1;N}\nu \right\rangle_n (z) = z^n (-N)_n (1-z)^{N-n} .$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{1;N}\Psi \right\rangle_n = \Upsilon_{n+1}(\emptyset) - z\Upsilon_n(-N) = (1-z)\mathfrak{S}_n - z(n-N) .$$

The Krawtchouk polynomials were introduced by Mykhailo Pylypovych Kravchuk (1892 –1942) in his paper [66].

4.2 Polynomials of class 1

In [39], we classified the discrete semiclassical orthogonal polynomials of class $s = 1$. There are 4 main families and 8 subfamilies, obtained by applying rational spectral transformations to the Charlier and Meixner polynomials.

To help the reader with some of the results, we note the formula

$$\Upsilon_n(\mathbf{c}) = \mathfrak{S}_n^2 + [1 + e_1(\mathbf{c} + n)]\,\mathfrak{S}_n + e_2(\mathbf{c} + n), \quad \mathbf{c} = (c_1, c_2),$$

where the elementary symmetric polynomials were defined in (26).

For additional references, see [99], [98], [84], [15], [49].

4.2.1 Polynomials of type $(0,1)$ (Generalized Charlier polynomials)

Linear functional

$$\left\langle {}_{1}^{0}L \right\rangle [u] = \sum_{x=0}^{\infty} u(x)\ \frac{1}{(b+1)_x}\frac{z^x}{x!}.$$

First moment

$$\left\langle {}_{1}^{0}\lambda \right\rangle_0 (z) = {}_0F_1\left[\begin{matrix} - \\ b+1 \end{matrix}\ ; z\right] = \Gamma(b+1)\, z^{-\frac{b}{2}}\ \mathrm{I}_b\left(2\sqrt{z}\right),$$

where $\mathrm{I}_\nu(z)$ is the *modified Bessel function of the first kind [89, 10.25.2].*

The Generalized Charlier polynomials can be obtained from the Charlier polynomials by means of the transformation

$$\left\langle {}_{1}^{0}\lambda \right\rangle_0 = \Omega_{b+1}\left[\left\langle {}_{0}^{0}\lambda \right\rangle_0\right].$$

Using (62) and (117), we obtain the ODE satisfied by the first moment

$$[\vartheta(\vartheta + b) - z]\left[\left\langle {}_{1}^{0}\lambda \right\rangle_0\right] = 0. \tag{123}$$

Standard moments recurrence operator

$$\left\langle {}_{1}^{0}\Phi \right\rangle_n [\mu] = (\mu + b)\,\mu^{n+1} - z(\mu+1)^n.$$

Standard moments recurrence

$$\left\langle {}_{1}^{0}\mu \right\rangle_{n+2} = -b\left\langle {}_{1}^{0}\mu \right\rangle_{n+1} + z\sum_{k=0}^{n}\binom{n}{k}\left\langle {}_{1}^{0}\mu \right\rangle_k.$$

In particular,

$$\left\langle {}_{1}^{0}\mu\right\rangle_{2}=z\left\langle {}_{1}^{0}\mu\right\rangle_{0}-b\left\langle {}_{1}^{0}\mu\right\rangle_{1}.$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\left\langle {}_{1}^{0}\mu\right\rangle_{n}=\left\langle {}_{1}^{0}\overrightarrow{Q}\right\rangle_{n}\cdot\left\langle {}_{1}^{0}\overrightarrow{\mu}\right\rangle_{n},$$

with

$$\left\langle {}_{1}^{0}\overrightarrow{Q}\right\rangle_{n+1}=\left(\vartheta+\left\langle {}_{1}^{0}M\right\rangle\right)\left\langle {}_{1}^{0}\overrightarrow{Q}\right\rangle_{n},\quad \left\langle {}_{1}^{0}\overrightarrow{Q}\right\rangle_{0}=\overrightarrow{\varepsilon_{0}},$$

and

$$\left\langle {}_{1}^{0}M\right\rangle=\begin{bmatrix}0 & z\\ 1 & -b\end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{1}^{0}\nu\right\rangle_{n}(z)=\frac{z^{n}}{(b+1)_{n}}\,{}_{0}F_{1}\left[\begin{matrix}-\\ b+1+n\end{matrix};z\right].$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{0}\Psi\right\rangle_{n}=\Upsilon_{n+1}(b)-z\Upsilon_{n}(\emptyset)=\mathfrak{S}_{n}^{2}+(n+1+b)\,\mathfrak{S}_{n}-z.$$

or,

$$\left\langle {}_{1}^{0}\Psi\right\rangle_{n}=\mathfrak{S}_{n}^{2}+(n+1+b)\,\mathfrak{S}_{n}-z.$$

Remark 20 *If we write*

$$\left\langle {}_{1}^{0}\nu\right\rangle_{n}=A^{n}h_{n},$$

then the recurrence $\left\langle {}_{1}^{0}\Psi\right\rangle_{n}\left[\left\langle {}_{1}^{0}\nu\right\rangle\right]=0$ *becomes*

$$h_{n+1}+\frac{(n+b)}{A}h_{n}-\frac{z}{A^{2}}h_{n-1}=0.$$

Choosing

$$\frac{1}{A}=-2x,\quad -\frac{z}{A^{2}}=1,$$

we get

$$h_{n+1}-2\,(n+b)\,xh_{n}+h_{n-1}=0. \tag{124}$$

The orthogonal polynomials satisfying the 3-term recurrence relation (124) with initial conditions

$$h_{0}=1,\quad h_{1}=2bx$$

are the modified Lommel polynomials *having the hypergeometric representation*

$$h_n(x) = (b)_n (2x)^n \ {}_2F_3\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ b, -n, 1-b-n \end{matrix} ; -x^{-2} \right).$$

See [32], [46], [68], [73] .

Remark 21 *Another possibility is to define*

$$\left\langle {}^0_1\nu \right\rangle_n = (-1)^n r_n,$$

where the monic polynomials $r_n(b)$ satisfy the 3*-term recurrence relation*

$$br_n = r_{n+1} - nr_n + zr_{n-1}, \quad r_{-1} = 0, \quad r_0 = 1.$$

For additional references, see [27], [56], [102], [111].

4.2.2 Polynomials of type $(1,1)$ (Generalized Meixner polynomials)

Linear functional

$$\left\langle {}^1_1L \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}.$$

First moment

$$\left\langle {}^1_1\mu \right\rangle_0 (z) = \ {}_1F_1\left[\begin{matrix} a \\ b+1 \end{matrix} ; z \right].$$

The Generalized Meixner polynomials can be obtained from the Meixner polynomials by means of the transformation

$$\left\langle {}^1_1\lambda \right\rangle_0 = \Omega_{b+1}\left[\left\langle {}^1_0\lambda \right\rangle_0 \right].$$

Using (62) and (121), we obtain the ODE satisfied by the first moment

$$[\vartheta(\vartheta+b) - z(\vartheta+a)] \left[\left\langle {}^1_1\lambda \right\rangle_0 \right] = 0. \tag{125}$$

Standard moments recurrence operator

$$\left\langle {}^1_1\Phi \right\rangle_n [\mu] = (\mu+b)\mu^{n+1} - z(\mu+a)(\mu+1)^n.$$

Standard moments recurrence

$$\left\langle {}^1_1\mu \right\rangle_{n+2} = -b \left\langle {}^1_1\mu \right\rangle_{n+1} + z \sum_{k=0}^{n} \binom{n}{k} \left(\left\langle {}^1_1\mu \right\rangle_{k+1} + a \left\langle {}^1_1\mu \right\rangle_k \right).$$

In particular,

$$\left\langle {}_1^1\mu \right\rangle_2 = az \left\langle {}_1^1\mu \right\rangle_0 + (z-b) \left\langle {}_1^1\mu \right\rangle_1 .$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\left\langle {}_1^1\mu \right\rangle_n = \left\langle {}_1^1\overrightarrow{Q} \right\rangle_n \cdot \left\langle {}_1^1\overrightarrow{\mu} \right\rangle_n ,$$

with

$$\left\langle {}_1^1\overrightarrow{Q} \right\rangle_{n+1} = \left(\vartheta + \left\langle {}_1^1 M \right\rangle\right) \left\langle {}_1^1\overrightarrow{Q} \right\rangle_n , \quad \left\langle {}_1^1\overrightarrow{Q} \right\rangle_0 = \overrightarrow{\varepsilon_0},$$

and

$$\left\langle {}_1^1 M \right\rangle = \begin{bmatrix} 0 & az \\ 1 & z-b \end{bmatrix} .$$

Hypergeometric representation of the modified moments

$$\left\langle {}_1^1\nu \right\rangle_n (z) = z^n \frac{(a)_n}{(b+1)_n} \, {}_1F_1 \left[\begin{matrix} a+n \\ b+1+n \end{matrix} ; z \right] .$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_1^1\Psi \right\rangle_n &= \Upsilon_{n+1}(b) - z\Upsilon_n(a) \\ &= \mathfrak{S}_n^2 + (n+1+b-z)\,\mathfrak{S}_n - z(n+a) . \end{aligned}$$

Remark 22 *If we define*

$$\left\langle {}_1^1\nu \right\rangle_n = (-1)^n r_n,$$

then the monic polynomials $r_n(b)$ *satisfy the* 3*-term recurrence relation*

$$br_n = r_{n+1} - (n-z)\, r_n + z(n+a-1)\, r_{n-1}.$$

For additional references, see [16], [27], [47].

Christoffel-Charlier polynomials The Christoffel-Charlier polynomials [45] can be obtained from the Charlier polynomials by means of the transformation

$$\left\langle {}_0^0\lambda^C \right\rangle_0 = \Omega_{-\omega}^{-\omega+1} \left[\left\langle {}_0^0\lambda \right\rangle_0 \right] .$$

Using (79) and (119), we have

$$\left\langle {}_0^0\lambda^C \right\rangle_0 (z) = (z-\omega)\, e^z.$$

Linear functional

$$\left\langle {}_{0}^{0}L^{C}\right\rangle [u]=\sum_{x=0}^{\infty}(x-\omega)\,u(x)\ \frac{z^{x}}{x!}.$$

Using (73), we obtain

$$\left[\vartheta\left(\vartheta-\omega-1\right)-z\left(\vartheta-\omega+1\right)\right]\left[\left\langle {}_{0}^{0}\lambda^{C}\right\rangle_{0}\right]=0,$$

which is a special case of (125) with

$$a=-\omega+1,\quad b=-\omega-1.$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{0}\Phi^{C}\right\rangle_{n}[\mu]=\left(\mu-\omega-1\right)\mu^{n+1}-z\left(\mu-\omega+1\right)\left(\mu+1\right)^{n}.$$

From (78) and (119), we have

$$\left\langle {}_{0}^{0}\nu^{C}\right\rangle_{n}=\left\langle {}_{0}^{0}\nu\right\rangle_{n+1}+(n-\omega)\left\langle {}_{0}^{0}\nu\right\rangle_{n}=(z+n-\omega)\,z^{n}e^{z}. \tag{126}$$

Modified moments recurrence operator

$$\begin{aligned}\left\langle {}_{0}^{0}\Psi^{C}\right\rangle_{n}&=\Upsilon_{n+1}\left(-\omega-1\right)-z\Upsilon_{n}\left(-\omega+1\right)\\&=\mathfrak{S}_{n}^{2}+\left(n-\omega-z\right)\mathfrak{S}_{n}-z\left(n-\omega+1\right).\end{aligned}$$

Remark 23 *Using (126), we see that the modified moments satisfy the* first order *recurrence* $\left\langle {}_{0}^{0}\psi^{C}\right\rangle_{n}\left[\left\langle {}_{0}^{0}\nu^{C}\right\rangle\right]=0$*, where*

$$\left\langle {}_{0}^{0}\psi^{C}\right\rangle_{n}=\left(n-\omega+z\right)\mathfrak{S}_{n}-z\left(n-\omega+1+z\right).$$

This agrees with the recurrence $\left\langle {}_{0}^{0}\Psi^{C}\right\rangle_{n}\left[\left\langle {}_{0}^{0}\nu^{C}\right\rangle\right]=0$*, since*

$$\left(\mathfrak{S}_{n}+n+1-\omega\right)\left\langle {}_{0}^{0}\psi^{C}\right\rangle_{n}=\left(n-\omega+1+z\right)\left\langle {}_{0}^{0}\Psi^{C}\right\rangle_{n}.$$

Geronimus-Charlier polynomials The Geronimus-Charlier polynomials can be obtained from the Charlier polynomials by means of the transformation

$$\left\langle {}_{0}^{0}\lambda^{G}\right\rangle_{0}=\Omega_{1-\omega}^{-\omega}\left[\left\langle {}_{0}^{0}\lambda\right\rangle_{0}\right],\quad \omega\notin\mathbb{N}_{0}.$$

Linear functional

$$\left\langle {}_0^0 L^G \right\rangle [u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega} \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_0^0 \lambda^G \right\rangle_0 (z) = -\frac{1}{\omega} \, {}_1F_1 \left(\begin{array}{c} -\omega \\ -\omega+1 \end{array} ; z \right).$$

Using (81), we obtain

$$\left[\vartheta (\vartheta - \omega) - z (\vartheta - \omega) \right] \left[\left\langle {}_0^0 \lambda^G \right\rangle_0 \right] = 0,$$

which is a special case of (125) with

$$a = -\omega, \quad b = -\omega.$$

Standard moments recurrence operator

$$\left\langle {}_0^0 \Phi^G \right\rangle_n [\mu] = (\mu - \omega) \left\langle {}_0^0 \Phi \right\rangle_n [\mu].$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_0^0 \Psi^G \right\rangle_n &= \Upsilon_{n+1}(-\omega) - z \Upsilon_n(-\omega) = \left\langle {}_0^0 \Psi \right\rangle_n \circ \Upsilon_n(-\omega) \\ &= \mathfrak{S}_n^2 + (n+1-\omega-z)\, \mathfrak{S}_n - z(n-\omega). \end{aligned}$$

Using (85) and (119), we have

$$\left\langle {}_0^0 \nu^G \right\rangle_n (z) = \int_0^1 t^{-\omega-1} \left\langle {}_0^0 \nu \right\rangle_n (zt)\, dt$$

$$= z^n \int_0^1 t^{n-\omega-1} e^{zt} dt = (-1)^n (-z)^{\omega} \gamma (n-\omega, -z),$$

where $\gamma(a,z)$ is the *incomplete gamma function* defined by [89, 8.2.1]

$$\gamma(a,z) = z^a \int_0^1 t^{a-1} e^{-zt} dt.$$

Reduced Uvarov Charlier polynomials Since for the Charlier polynomials $\left\langle {}_{0}^{0}\sigma \right\rangle (\vartheta) = \vartheta$, we will have a reduced case for their Uvarov transformation if $\omega = 0$. The Reduced-Uvarov-Charlier polynomials can be obtained from the Charlier polynomials by means of the transformation

$$\left\langle {}_{0}^{0}\lambda_{0}^{U} \right\rangle_{0} = \left\langle {}_{0}^{0}\lambda \right\rangle_{0} + \eta = e^{z} + \eta, \quad \eta \in \mathbb{K}.$$

Linear functional

$$\left\langle {}_{0}^{0}L_{0}^{U} \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ \frac{z^{x}}{x!} + \eta u(0).$$

Using (96), we obtain

$$[\vartheta(\vartheta - 1) - z\vartheta] \left[\left\langle {}_{0}^{0}\lambda_{0}^{U} \right\rangle_{0} \right] = 0,$$

which is a special case of (125) with

$$a = 0, \quad b = -1.$$

Standard moments

$$\left\langle {}_{0}^{0}\mu_{0}^{U} \right\rangle_{n} = \left\langle {}_{0}^{0}\mu \right\rangle_{n} + \eta \times 0^{n} = \left\langle {}_{0}^{0}\mu \right\rangle_{n} + \eta \delta_{n,0}.$$

Standard moments recurrence operator

$$\left\langle \Phi_{0}^{U} \right\rangle_{n} = (\mathfrak{S}_{n} - 1) \left[\left\langle \Phi_{0} \right\rangle_{n} \right].$$

Modified moments

$$\left\langle {}_{0}^{0}\nu_{0}^{U} \right\rangle_{n} = \left\langle {}_{0}^{0}\nu \right\rangle_{n} + \eta \phi_{n}(0) = z^{n} e^{z} + \eta \delta_{n,0}.$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_{0}^{0}\Psi_{0}^{U} \right\rangle_{n} &= \Upsilon_{n+1}(-1) - z \Upsilon_{n}(0) = \Upsilon_{n}(0) \circ \left\langle {}_{0}^{0}\Psi \right\rangle_{n} \\ &= \mathfrak{S}_{n}^{2} + (n - z) \mathfrak{S}_{n} - nz. \end{aligned}$$

For additional references, see [13], [44].

Truncated Charlier polynomials The truncated Charlier polynomials can be obtained from the Charlier polynomials by means of the transformation
$$\left\langle {}_0^0\lambda^T \right\rangle_0 = \Omega_{-N}^{-N}\left[\left\langle {}_0^0\lambda \right\rangle_0\right], \quad N \in \mathbb{N}.$$

Linear functional
$$\left\langle {}_0^0L^T \right\rangle [u] = \sum_{x=0}^{N} u(x)\ \frac{z^x}{x!}.$$

First moment
$$\left\langle {}_0^0\lambda^T \right\rangle_0 (z) = \sum_{x=0}^{N} \frac{z^x}{x!}.$$

From (106), we have
$$\left[\vartheta\left(\vartheta - N - 1\right) - z\left(\vartheta - N\right)\right]\left[\left\langle {}_0^0\lambda^T \right\rangle_0\right] = 0, \tag{127}$$
which is a special case of (125) with
$$a = -N, \quad b = -N - 1.$$

Standard moments recurrence operator
$$\left\langle {}_0^0\Phi^T \right\rangle_n = \left(\mathfrak{S}_n - N - 1\right)\left[\left\langle {}_0^0\Phi \right\rangle_n\right].$$

Modified moments recurrence operator
$$\begin{aligned}\left\langle {}_0^0\Psi^T \right\rangle_n &= \Upsilon_{n+1}\left(-N-1\right) - z\Upsilon_n\left(-N\right) = \Upsilon_n\left(-N\right) \circ \left\langle {}_0^0\Psi \right\rangle_n \\ &= \mathfrak{S}_n^2 + \left(n - N - z\right)\mathfrak{S}_n - \left(n - N\right) z.\end{aligned}$$

Using (116), we obtain
$$\left\langle {}_0^0\nu^T \right\rangle_n (z) = \frac{z^{N+1}}{(N-n)!}\int_0^\infty {}_1F_0\left(\begin{matrix} n-N \\ - \end{matrix} ; -t\right) e^{-zt} dt$$
$$= \frac{z^{N+1}}{(N-n)!}\int_0^\infty \left(1+t\right)^{N-n} e^{-zt} dt = z^n \frac{\Gamma\left(N-n+1, z\right)}{(N-n)!} e^z,$$
where $\Gamma(a, z)$ is the incomplete gamma function defined by [89, 8.6.5]
$$\Gamma\left(a, z\right) = z^a e^{-z}\int_0^\infty \left(1+t\right)^{a-1} e^{-zt} dt.$$

Comparing with (119), we conclude that

$$\left\langle {}_{0}^{0}\nu^{T}\right\rangle_{n}(z)=\frac{\Gamma(N-n+1,z)}{(N-n)!}\left\langle {}_{0}^{0}\nu\right\rangle_{n}(z), \quad 0\leq n\leq N.$$

For additional references, see [53].

4.2.3 Polynomials of type $(2,0;N)$ (Generalized Krawtchouk polynomials)

Linear functional

$$\left\langle {}_{0}^{2;N}L\right\rangle[u]=\sum_{x=0}^{N}u(x)\ (a)_{x}(-N)_{x}\frac{z^{x}}{x!}, \quad N\in\mathbb{N}.$$

First moment

$$\left\langle {}_{0}^{2;N}\lambda\right\rangle_{0}(z)={}_{2}F_{0}\left[\begin{array}{c}-N,a\\ -\end{array};z\right].$$

Remark 24 *If we use the hypergeometric representation (120), we can write the first moment in terms of the Charlier polynomials*

$$\left\langle {}_{0}^{2;N}\lambda\right\rangle_{0}(z)=\left\langle {}_{0}^{0}P\right\rangle_{N}\left(-a;-z^{-1}\right).$$

The Generalized Krawtchouk polynomials can be obtained from the Krawtchouk polynomials by means of the transformation

$$\left\langle {}_{0}^{2;N}\lambda\right\rangle_{0}=\Omega^{a}\left[\left\langle {}_{0}^{1;N}\lambda\right\rangle_{0}\right].$$

Using (60), we obtain the ODE satisfied by the first moment

$$\left[\vartheta-z(\vartheta+a)(\vartheta-N)\right]\left\langle {}_{0}^{2;N}\lambda\right\rangle_{0}=0. \tag{128}$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{2;N}\Phi\right\rangle_{n}[\mu]=\mu^{n+1}-z(\mu+a)(\mu-N)(\mu+1)^{n}.$$

Standard moments recurrence

$$\mu_{n+2}=\left(N-a+z^{-1}\right)\mu_{n+1}+aN\mu_{n}$$
$$-\sum_{k=0}^{n-1}\binom{n}{k}\left[\mu_{k+2}+(a-N)\mu_{k+1}-aN\mu_{k}\right].$$

In particular,

$$\left\langle {}_{0}^{2;N}\mu \right\rangle_{2}=aN\left\langle {}_{0}^{2;N}\mu \right\rangle_{0}+\left(N-a+z^{-1}\right)\left\langle {}_{0}^{2;N}\mu \right\rangle_{1}.$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\left\langle {}_{0}^{2;N}\mu \right\rangle_{n}=z^{-n}\left\langle {}_{0}^{2;N}\overrightarrow{Q} \right\rangle_{n}\cdot\left\langle {}_{0}^{2;N}\overrightarrow{\mu} \right\rangle,\quad n\geq 0.$$

From (33), we have

$$\left\langle {}_{0}^{2;N}\overrightarrow{Q} \right\rangle_{n+1}=z\left(\vartheta+\left\langle {}_{0}^{2;N}M \right\rangle-nI\right)\left\langle {}_{0}^{2;N}\overrightarrow{Q} \right\rangle_{n},\quad \left\langle {}_{0}^{2;N}\overrightarrow{Q} \right\rangle_{0}=\overrightarrow{\varepsilon}_{0},$$

with

$$\left\langle {}_{0}^{2;N}M \right\rangle=\begin{bmatrix}0 & aN\\ 1 & N-a+z^{-1}\end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{0}^{2;N}\nu \right\rangle_{n}(z)=z^{n}(a)_{n}(-N)_{n}\ {}_{2}F_{0}\left[\begin{array}{c}a+n,n-N\\ -\end{array};z\right].$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{2;N}\Psi \right\rangle_{n}=\Upsilon_{n+1}(\emptyset)-z\Upsilon_{n}(a,-N),$$

or,

$$-z^{-1}\left\langle {}_{0}^{2;N}\Psi \right\rangle_{n}=\mathfrak{S}_{n}^{2}+\left(1+a-N+2n-z^{-1}\right)\mathfrak{S}_{n}+(n+a)(n-N).$$

Remark 25 *If we set $z^{-1}=x$, we see that the modified moments are a family of monic orthogonal polynomials $r_n(x)$, satisfying the 3-term recurrence relation*

$$xr_{n}=r_{n+1}+(a-N+2n-1)r_{n}+(n+a-1)(n-N-1)r_{n-1}.$$

4.2.4 Polynomials of type $(2,1)$ (Generalized Hahn polynomials of type I)

Linear functional

$$\left\langle {}^2_1 L \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}.$$

First moment

$$\left\langle {}^2_1 \lambda \right\rangle_0 (z) = \ {}_2F_1 \left[\begin{array}{c} a_1, a_2 \\ b+1 \end{array} ; z \right].$$

The generalized Hahn polynomials of type I can be obtained from the Meixner polynomials by means of the double transformation

$$\left\langle {}^2_1 \lambda \right\rangle_0 = \Omega^a_{b+1} \left[\left\langle {}^1_0 \lambda \right\rangle_0 \right].$$

Using (64), we obtain the ODE satisfied by the first moment

$$\left[\vartheta (\vartheta + b) - z (\vartheta + a_1)(\vartheta + a_2) \right] \left[\left\langle {}^2_1 \lambda \right\rangle_0 \right] = 0. \tag{129}$$

Standard moments recurrence operator

$$\left\langle {}^2_1 \Phi \right\rangle_n [\mu] = (\mu + b) \mu^{n+1} - z (\mu + a_1)(\mu + a_2)(\mu + 1)^n .$$

Standard moments recurrence

$$\begin{aligned} (1-z) z^{-1} \mu_{n+2} &= \left(a_1 + a_2 - b z^{-1} \right) \mu_{n+1} + a_1 a_2 \mu_n \\ &+ \sum_{k=0}^{n-1} \binom{n}{k} \left[\mu_{k+2} + (a_1 + a_2) \mu_{k+1} + a_1 a_2 \mu_k \right]. \end{aligned}$$

In particular,

$$\left\langle {}^2_1 \mu \right\rangle_2 = \frac{z}{1-z} \left[a_1 a_2 \left\langle {}^2_1 \mu \right\rangle_0 + \left(a_1 + a_2 - b z^{-1} \right) \left\langle {}^2_1 \mu \right\rangle_1 \right].$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\left\langle {}^2_1 \mu \right\rangle_n = (1-z)^{-n} \left\langle {}^2_1 \overrightarrow{Q} \right\rangle_n \cdot \left\langle {}^2_1 \overrightarrow{\mu} \right\rangle, \quad n \geq 0.$$

From (34), we have

$$\left\langle {}_{1}^{2}\overrightarrow{Q}\right\rangle_{n+1} = \left[(1-z)\left(\vartheta + \left\langle {}_{1}^{2}M\right\rangle\right) + nzI\right]\left\langle {}_{1}^{2}\overrightarrow{Q}\right\rangle_{n}, \quad \left\langle {}_{1}^{2}\overrightarrow{Q}\right\rangle_{0} = \overrightarrow{\varepsilon}_{0},$$

with

$$(1-z)\left\langle {}_{1}^{2}M\right\rangle = \begin{bmatrix} 0 & a_1 a_2 z \\ 1-z & (a_1+a_2)\, z - b \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{1}^{2}\nu\right\rangle_{n}(z) = z^n \frac{(a_1)_n (a_2)_n}{(b+1)_n}\, {}_2F_1\left[\begin{array}{c} a_1+n, a_2+n \\ b+1+n \end{array}; z\right]. \tag{130}$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{2}\Psi\right\rangle_{n} = \Upsilon_{n+1}(b) - z\Upsilon_n(a_1, a_2)$$
$$= (1-z)\,\mathfrak{S}_n^2 + [n+1+b-z(1+2n+a_1+a_2)]\,\mathfrak{S}_n - z(n+a_1)(n+a_2).$$

Remark 26 *If we set $b=-x$, we see that the modified moments are a family of orthogonal polynomials $r_n(x)$, satisfying the 3-term recurrence relation*

$$\begin{aligned} x r_n &= (1-z)\, r_{n+1} + [n - z(2n-1+a_1+a_2)]\, r_n \\ &\quad + z(n-1+a_1)(n-1+a_2)\, r_{n-1}. \end{aligned}$$

Remark 27 *The special case $z=1$, corresponds to the Hahn polynomials [62]. Note that in this case (130) can be reduced using the identity [89, 15.4.20]*

$${}_2F_1\left[\begin{array}{c} a_1, a_2 \\ b \end{array}; 1\right] = \frac{(b-a_2)_{-a_1}}{(b)_{-a_1}}, \quad \operatorname{Re}(b-a_2) > \operatorname{Re}(a_1).$$

Choosing $a_1=-N$, $N\in\mathbb{N}$, $a_2=a$, we get

$$\left\langle {}_{1}^{2}\nu\right\rangle_{n}(1) = \frac{(-N)_n (a)_n}{(b+1)_n}\frac{(b+1-a)_{N-n}}{(b+1+n)_{N-n}},$$

and since

$$(b+1)_n (b+1+n)_{N-n} = (b+1)_N,$$
$$(b+1-a)_{N-n} = (-1)^{N-n}(a-b+n-N)_{N-n} = (-1)^n \frac{(b+1-a)_N}{(a-b-N)_n},$$

we obtain

$$\left\langle {}_1^2\nu \right\rangle_n (1) = \frac{(b+1-a)_N}{(b+1)_N} (-1)^n \frac{(-N)_n (a)_n}{(a-b-N)_n}.$$

This agrees with the recurrence

$$\left[\Upsilon_{n+1}(b) - \Upsilon_n(-N,a)\right] \left[\left\langle {}_1^2\nu \right\rangle_n (1)\right] = 0,$$

which becomes

$$\frac{\left\langle {}_1^2\nu \right\rangle_{n+1} (1)}{\left\langle {}_1^2\nu \right\rangle_n (1)} = -\frac{(n-N)(n+a)}{n+a-b-N}.$$

For additional references, see [36], [48].

Christoffel-Meixner polynomials The Christoffel-Meixner polynomials [45] can be obtained from the Meixner polynomials by means of the transformation

$$\left\langle {}_0^1\lambda^C \right\rangle_0 = \Omega_{-\omega}^{-\omega+1} \left[\left\langle {}_0^1\lambda \right\rangle_0\right].$$

Linear functional

$$\left\langle {}_0^1L^C \right\rangle [u] = \sum_{x=0}^{\infty} (x-\omega)\, u(x) \ (a)_x \frac{z^x}{x!}.$$

Using (79), we have

$$\left\langle {}_0^1\lambda^C \right\rangle_0 (z) = (z\omega + az - \omega)(1-z)^{-a-1},$$

and therefore we need

$$z(\omega + a) - \omega \neq 0.$$

From (73), we obtain

$$\left[\vartheta(\vartheta - \omega - 1) - z(\vartheta - \omega + 1)(\vartheta + a)\right] \left[\left\langle {}_0^1\lambda^C \right\rangle_0\right] = 0,$$

which is a special case of (129) with

$$a_1 = a, \quad a_2 = -\omega + 1, \quad b = -\omega - 1.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi^C \right\rangle_n [\mu] = (\mu - \omega - 1)\mu^{n+1} - z(\mu - \omega + 1)(\mu + a)(\mu + 1)^n.$$

From (78) and (122), we get

$$\left\langle {}_0^1\nu^C \right\rangle_n = (z\omega + az + n - \omega)\, z^n \,(1-z)^{-a-n-1}\,(a)_n\,. \tag{131}$$

Modified moments recurrence operator

$$\begin{gathered}\left\langle {}_0^1\Psi^C \right\rangle_n = \Upsilon_{n+1}(-\omega-1) - z\Upsilon_n(a, -\omega+1)\\ = (1-z)\,\mathfrak{S}_n^2 + [n - \omega - z(2 + a - \omega + 2n)]\,\mathfrak{S}_n - z(n-\omega+1)(n+a)\,.\end{gathered}$$

Remark 28 *Using (131), we see that the modified moments satisfy the* first order *recurrence* $\left\langle {}_0^1\psi^C \right\rangle_n \left[{}_0^1\nu^C\right] = 0$, *with*

$$\left\langle {}_0^1\psi^C \right\rangle_n = (1-z)(n - \omega + z\omega + az)\,\mathfrak{S}_n - z(n+a)(n + 1 - \omega + z\omega + az)\,.$$

This agrees with the second order recurrence $\left\langle {}_0^1\Psi^C \right\rangle_n \left[{}_0^1\nu^C\right] = 0$, *since*

$$(\mathfrak{S}_n + n + 1 - \omega)\left[\left\langle {}_0^1\psi^C \right\rangle_n\right] = (n + 1 - \omega + z\omega + az)\left\langle {}_0^1\Psi^C \right\rangle_n\,.$$

Geronimus Meixner polynomials These polynomials can be obtained from the Meixner polynomials by means of the transformation

$$\left\langle {}_0^1\lambda^G \right\rangle_0 = \Omega_{1-\omega}^{-\omega}\left[\left\langle {}_0^1\lambda \right\rangle_0\right], \quad \omega \notin \mathbb{N}_0.$$

Linear functional

$$\left\langle {}_0^1L^G \right\rangle[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega}\;(a)_x\;\frac{z^x}{x!}.$$

First moment

$$\left\langle {}_0^1\lambda^G \right\rangle_0(z) = -\frac{1}{\omega}\;{}_2F_1\left(\begin{matrix} a, -\omega \\ -\omega+1\end{matrix}\;;z\right).$$

Using (81), we obtain

$$[\vartheta(\vartheta-\omega) - z(\vartheta-\omega)(\vartheta+a)]\left[\left\langle {}_0^1\lambda^G \right\rangle_0\right] = 0,$$

which is a special case of (129) with

$$a_1 = a, \quad a_2 = -\omega, \quad b = -\omega.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi^G \right\rangle_n [\mu] = (\mu - \omega) \left\langle {}_0^1\Phi \right\rangle_n [\mu] .$$

Modified moments recurrence operator

$$\begin{gathered}\left\langle {}_0^1\Psi^G \right\rangle_n = \Upsilon_{n+1}(-\omega) - z\Upsilon_n(a, -\omega) = \left\langle {}_0^1\Psi \right\rangle_n \circ \Upsilon_n(-\omega) \\ = (1-z)\mathfrak{S}_n^2 + [n - \omega + 1 - z(1 + a - \omega + 2n)]\mathfrak{S}_n - z(n-\omega)(n+a) .\end{gathered}$$

Using (85) and (122), we have

$$\begin{gathered}\left\langle {}_0^1\nu^G \right\rangle_n (z) = \int_0^1 t^{-\omega-1} \left\langle {}_0^1\nu \right\rangle_n (zt)\, dt = (a)_n \ z^n \int_0^1 t^{n-\omega-1} (1-zt)^{-a-n}\, dt \\ = (a)_n \ z^{\omega} B_z(n-\omega, 1-a-n) ,\end{gathered}$$

where $B_z(a,b)$ is the *incomplete beta function* defined by [89, 8.17.1]

$$B_z(a,b) = z^a \int_0^1 t^{a-1} (1-zt)^{b-1}\, dt.$$

Reduced-Uvarov Meixner polynomials Since for the Meixner polynomials we have

$$\left\langle {}_0^1\sigma \right\rangle = \vartheta, \quad \left\langle {}_0^1\tau \right\rangle = \vartheta + a,$$

we will have reduced cases for their Uvarov transformation if $\omega = 0$ or $\omega = -a$.

i) $\omega = 0$ In this case, we have

$$\left\langle {}_0^1\lambda_0^U \right\rangle_0 = \left\langle {}_0^1\lambda \right\rangle_0 + \eta = (1-z)^{-a} + \eta.$$

Linear functional

$$\left\langle {}_0^1L_0^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ (a)_x \ \frac{z^x}{x!} + \eta u(0) .$$

Using (96), we obtain

$$[\vartheta(\vartheta-1) - z\vartheta(\vartheta+a)] \left[\left\langle {}_0^1\lambda_0^U \right\rangle_0 \right] = 0, \tag{132}$$

which is a special case of (129) with

$$a_1 = a, \quad a_2 = 0, \quad b = -1.$$

Standard moments

$$\left\langle {}_0^1\mu_0^U \right\rangle_n = \left\langle {}_0^1\mu \right\rangle_n + \eta \times 0^n = \left\langle {}_0^1\mu \right\rangle_n + \eta\delta_{n,0}.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_0^U \right\rangle_n = \left(\mathfrak{S}_n - 1\right) \left\langle {}_0^1\Phi \right\rangle_n.$$

Modified moments

$$\left\langle {}_0^1\nu_0^U \right\rangle_n = \left\langle {}_0^1\nu \right\rangle_n + \eta\phi_n\left(0\right) = z^n \left(a\right)_n \left(1-z\right)^{-a-n} + \eta\delta_{n,0}.$$

Modified moments recurrence operator

$$\begin{gathered} \left\langle {}_0^1\Psi_0^U \right\rangle_n = \Upsilon_{n+1}\left(-1\right) - z\Upsilon_n\left(a,0\right) = \Upsilon_n\left(0\right) \circ \left\langle {}_0^1\Psi \right\rangle_n \\ = \left(1-z\right)\mathfrak{S}_n^2 + \left[n - z\left(1+a+2n\right)\right]\mathfrak{S}_n - zn\left(n+a\right). \end{gathered}$$

For additional references, see [6], [14], [43].

ii) $\omega = -a$ In this case, we have

$$\left\langle {}_0^1\lambda_{-a}^U \right\rangle_0 = \left\langle {}_0^1\lambda \right\rangle_0 + \eta z^{-a} = \left(1-z\right)^{-a} + \eta z^{-a}.$$

Linear functional

$$\left\langle {}_0^1L_{-a}^U \right\rangle [u] = \sum_{x=0}^{\infty} u\left(x\right)\ \left(a\right)_x\ \frac{z^x}{x!} + \eta u\left(-a\right) z^{-a}.$$

Using (97), we obtain

$$\left[\vartheta\left(\vartheta+a\right) - z\left(\vartheta+a\right)\left(\vartheta+a+1\right)\right]\left[\left\langle {}_0^1\lambda_{-a}^U \right\rangle_0\right] = 0, \tag{133}$$

which is a special case of (129) with

$$a_1 = a, \quad a_2 = a+1, \quad b = a.$$

Standard moments

$$\left\langle {}_0^1\mu_{-a}^U \right\rangle_n = \left\langle {}_0^1\mu \right\rangle_n + \eta\left(-a\right)^n z^{-a}.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_{-a}^U \right\rangle_n = (\mathfrak{S}_n + a)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Modified moments

$$\left\langle {}_0^1\nu_{-a}^U \right\rangle_n = \left\langle {}_0^1\nu \right\rangle_n + \eta\phi_n(-a)\, z^{-a} = z^n (a)_n (1-z)^{-a-n} + \eta\phi_n(-a)\, z^{-a}.$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_0^1\Psi_{-a}^U \right\rangle_n &= \Upsilon_{n+1}(a) - z\Upsilon_n(a, a+1) = \Upsilon_n(a+1) \circ \left\langle {}_0^1\Psi \right\rangle_n \\ &= (1-z)\,\mathfrak{S}_n^2 + (1-2z)(n+a+1)\,\mathfrak{S}_n - z(n+a)(n+a+1). \end{aligned}$$

Truncated Meixner polynomials These polynomials can be obtained from the Meixner polynomials by means of the transformation

$$\left\langle {}_0^1\lambda^T \right\rangle_0 = \Omega_{-N}^{-N}\left[\left\langle {}_0^1\lambda \right\rangle_0\right], \quad N \in \mathbb{N}.$$

Linear functional

$$\left\langle {}_0^1L^T \right\rangle [u] = \sum_{x=0}^{N} u(x)\ (a)_x\ \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_0^1\lambda^T \right\rangle_0 (z) = \sum_{x=0}^{N} (a)_x\ \frac{z^x}{x!}.$$

From (106), we have

$$\left[\vartheta(\vartheta - N - 1) - z(\vartheta + a)(\vartheta - N)\right]\left[\left\langle {}_0^1\lambda^T \right\rangle_0\right] = 0, \tag{134}$$

which is a special case of (129) with

$$a_1 = a, \quad a_2 = -N, \quad b = -N-1.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi^T \right\rangle_n = (\mathfrak{S}_n - N - 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_0^1\Psi^T \right\rangle_n &= \Upsilon_{n+1}(-N-1) - z\Upsilon_n(a, -N) = \Upsilon_n(-N) \circ \left\langle {}_0^1\Psi \right\rangle_n \\ &= (1-z)\,\mathfrak{S}_n^2 + \left[n - N - z(2n + a - N + 1)\right]\mathfrak{S}_n - z(n-N)(n+a). \end{aligned}$$

From (115), we have

$$\left\langle {}_{0}^{1}\nu^{T}\right\rangle_{n}(z)=\frac{(a)_{N}\ \ z^{N}}{(N-n)!}\ {}_{2}F_{1}\left(\begin{array}{c}n-N,1\\1-a-N\end{array};z^{-1}\right).$$

Using the transformation 15.8.7

$${}_{2}F_{1}\left(\begin{array}{c}-N,a\\b\end{array};z\right)=\frac{(b-a)_{N}}{(b)_{N}}z^{N}\ {}_{2}F_{1}\left(\begin{array}{c}-N,1-b-N\\a-b-N+1\end{array};1-z^{-1}\right),$$

we obtain

$$\left\langle {}_{0}^{1}\nu^{T}\right\rangle_{n}(z)=\frac{(a)_{N+1}}{a+n}\frac{z^{n}}{(N-n)!}\ {}_{2}F_{1}\left(\begin{array}{c}n-N,a+n\\a+n+1\end{array};1-z\right).$$

Since the incomplete beta function has the hypergeometric representation [89, 8.17.7]

$$B_{z}(a,b)=\frac{z^{a}}{a}\ {}_{2}F_{1}\left(\begin{array}{c}1-b,a\\a+1\end{array};z\right),$$

we conclude that

$$\left\langle {}_{0}^{1}\nu^{T}\right\rangle_{n}=\frac{(a)_{N+1}}{(N-n)!}z^{n}(1-z)^{-a-n}B_{1-z}(a+n,N-n+1),$$

and comparing with (122), we see that

$$\left\langle {}_{0}^{1}\nu^{T}\right\rangle_{n}=\frac{(a+n)_{N-n+1}}{(N-n)!}B_{1-z}(a+n,N-n+1)\left\langle {}_{0}^{1}\nu\right\rangle_{n}\ .$$

4.3 Polynomials of class 2

In [40], we classified the discrete semiclassical orthogonal polynomials of class $s=2$. There are 6 main families and 13 subfamilies, obtained by applying rational spectral transformations to the polynomials of class $s=1$.

To help the reader with some of the results, we note the formula

$$\begin{gathered}\Upsilon_{n}(\mathbf{c})=\mathfrak{S}_{n}^{3}+e_{1}(\mathbf{c}+n+1)\,\mathfrak{S}_{n}^{2}\\+\left[1+e_{1}(\mathbf{c}+n)+e_{2}(\mathbf{c}+n)\right]\mathfrak{S}_{n}+e_{3}(\mathbf{c}+n),\end{gathered}$$

$\mathbf{c}=(c_{1},c_{2},c_{3})$, where the elementary symmetric polynomials were defined in (26).

For additional references, see [77].

4.3.1 Polynomials of type (0,2)

Linear functional

$$\left\langle {}_{2}^{0}L \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ \frac{1}{(b_1+1)_x (b_2+1)_x} \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_{2}^{0}\lambda \right\rangle_0 (z) = {}_0F_2 \left[\begin{matrix} - \\ b_1+1, b_2+1 \end{matrix} ; z \right]$$

These polynomials can be obtained from the generalized Charlier polynomials by means of the transformation

$$\left\langle {}_{2}^{0}\lambda \right\rangle_0 = \Omega_{b+1} \left[\left\langle {}_{1}^{0}\lambda \right\rangle_0 \right].$$

Using (62), we obtain the ODE satisfied by the first moment

$$\left[\vartheta (\vartheta + b_1)(\vartheta + b_2) - z \right] \left[\left\langle {}_{2}^{0}\lambda \right\rangle_0 \right] = 0.$$

Standard moments recurrence operator

$$\left\langle {}_{2}^{0}\Phi \right\rangle_n [\mu] = (\mu + b_1)(\mu + b_2) \mu^{n+1} - z (\mu + 1)^n .$$

Using (28), we get

$$\left\langle {}_{2}^{0}\mu \right\rangle_3 = z \left\langle {}_{2}^{0}\mu \right\rangle_0 - b_1 b_2 \left\langle {}_{2}^{0}\mu \right\rangle_1 - (b_1 + b_2) \left\langle {}_{2}^{0}\mu \right\rangle_2 .$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\begin{gathered} \left\langle {}_{2}^{0}\mu \right\rangle_n = \left\langle {}_{2}^{0}\overrightarrow{Q} \right\rangle_n \cdot \left\langle {}_{2}^{0}\overrightarrow{\mu} \right\rangle, \quad n \geq 0, \\ \left\langle {}_{2}^{0}\overrightarrow{Q} \right\rangle_{n+1} = \left(\vartheta + \left\langle {}_{2}^{0}M \right\rangle \right) \left\langle {}_{2}^{0}\overrightarrow{Q} \right\rangle_n, \quad \left\langle {}_{2}^{0}\overrightarrow{Q} \right\rangle_0 = \overrightarrow{\varepsilon}_0, \end{gathered}$$

with

$$\left\langle {}_{2}^{0}M \right\rangle = \begin{bmatrix} 0 & 0 & z \\ 1 & 0 & -b_1 b_2 \\ 0 & 1 & -(b_1 + b_2) \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}^0_2\nu \right\rangle_n (z) = \frac{z^n}{(b_1+1)_n (b_2+1)_n} \, {}_0F_2 \left[\begin{matrix} - \\ b_1+1+n, b_2+1+n \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\begin{gathered} \left\langle {}^0_2\Psi \right\rangle_n = \Upsilon_{n+1}(b_1, b_2) - z\Upsilon_n(\emptyset) \\ = \mathfrak{S}_n^3 + [1 + e_1(\mathbf{b}+n+1)]\,\mathfrak{S}_n^2 + e_2(\mathbf{b}+n+1)\,\mathfrak{S}_n - z. \end{gathered}$$

4.3.2 Polynomials of type (1,2)

Linear functional

$$\left\langle {}^1_2L \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \, \frac{(a)_x}{(b_1+1)_x (b_2+1)_x} \frac{z^x}{x!}.$$

First moment

$$\left\langle {}^1_2\lambda \right\rangle_0 (z) = {}_1F_2 \left[\begin{matrix} a \\ b_1+1, b_2+1 \end{matrix} ; z \right].$$

These polynomials can be obtained from the generalized Meixner polynomials by means of the transformation

$$\left\langle {}^1_2\lambda \right\rangle_0 = \Omega_{b+1} \left[\left\langle {}^1_1\lambda \right\rangle_0 \right].$$

Using (62), we obtain the ODE satisfied by the first moment

$$[\vartheta(\vartheta+b_1)(\vartheta+b_2) - z(\vartheta+a)] \left[\left\langle {}^1_2\lambda \right\rangle_0 \right] = 0. \tag{135}$$

Standard moments recurrence operator

$$\left\langle {}^1_2\Phi \right\rangle_n [\mu] = (\mu+b_1)(\mu+b_2)\,\mu^{n+1} - z(\mu+a)(\mu+1)^n.$$

Using (28), we get

$$\left\langle {}^1_2\mu \right\rangle_3 = az \left\langle {}^1_2\mu \right\rangle_0 + (z - b_1 b_2) \left\langle {}^1_2\mu \right\rangle_1 - (b_1+b_2) \left\langle {}^1_2\mu \right\rangle_2.$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\begin{gathered} \left\langle {}^1_2\mu \right\rangle_n = \left\langle {}^1_2\overrightarrow{Q} \right\rangle_n \cdot \left\langle {}^1_2\overrightarrow{\mu} \right\rangle, \quad n \geq 0, \\ \left\langle {}^1_2\overrightarrow{Q} \right\rangle_{n+1} = \left(\vartheta + \left\langle {}^1_2M \right\rangle\right) \left\langle {}^1_2\overrightarrow{Q} \right\rangle_n, \quad \left\langle {}^1_2\overrightarrow{Q} \right\rangle_0 = \overrightarrow{\varepsilon}_0, \end{gathered}$$

with

$$\left\langle {}_{2}^{1}M \right\rangle = \begin{bmatrix} 0 & 0 & az \\ 1 & 0 & z - b_1 b_2 \\ 0 & 1 & -(b_1 + b_2) \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{2}^{1}\nu \right\rangle_n (z) = \frac{z^n (a)_n}{(b_1+1)_n (b_2+1)_n} \, {}_1F_2 \left[\begin{matrix} a+n \\ b_1+1+n, b_2+1+n \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\left\langle {}_{2}^{1}\Psi \right\rangle_n = \Upsilon_{n+1}(b_1, b_2) - z \Upsilon_n (a)$$
$$= \mathfrak{S}_n^3 + [1 + e_1 (\mathbf{b} + n + 1)] \mathfrak{S}_n^2 + [e_2 (\mathbf{b} + n + 1) - z] \mathfrak{S}_n - z (n + a).$$

Christoffel Generalized Charlier polynomials These polynomials can be obtained from the Generalized Charlier polynomials by means of the transformation

$$\left\langle {}_{1}^{0}\lambda^C \right\rangle_0 = \Omega_{-\omega}^{-\omega+1} \left[\left\langle {}_{1}^{0}\lambda \right\rangle_0 \right].$$

Linear functional

$$\left\langle {}_{1}^{0}L^C \right\rangle [u] = \sum_{x=0}^{\infty} (x - \omega) u(x) \ \frac{1}{(b+1)_x} \frac{z^x}{x!}.$$

Using (74), we have

$$\left\langle {}_{1}^{0}\lambda^C \right\rangle_0 (z) = -\omega \ {}_1F_2 \left[\begin{matrix} -\omega+1 \\ b+1, -\omega \end{matrix} ; z \right].$$

Using (73), we obtain

$$[\vartheta (\vartheta + b) (\vartheta - \omega - 1) - z (\vartheta - \omega + 1)] \left[\left\langle {}_{1}^{0}\lambda^C \right\rangle_0 \right] = 0,$$

which is a special case of (135) with

$$a = -\omega + 1, \quad b_1 = b, \quad b_2 = -\omega - 1.$$

Standard moments recurrence operator

$$\left\langle {}_{1}^{0}\Phi^C \right\rangle_n [\mu] = (\mu + b)(\mu - \omega - 1) \mu^{n+1} - z (\mu - \omega + 1)(\mu + 1)^n.$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{0}\Psi^C \right\rangle_n = \mathfrak{S}_n^3 + (b - \omega + 2n + 2) \mathfrak{S}_n^2$$
$$+ [(n + b + 1)(n - \omega) - z] \mathfrak{S}_n - z (n - \omega + 1).$$

Geronimus Generalized Charlier polynomials These polynomials can be obtained from the Generalized Charlier polynomials by means of the transformation

$$\left\langle {}_{1}^{0}\lambda^{G}\right\rangle_{0}=\Omega_{1-\omega}^{-\omega}\left[\left\langle {}_{1}^{0}\lambda\right\rangle_{0}\right],\quad \omega\notin\mathbb{N}_{0}.$$

Linear functional

$$\left\langle {}_{1}^{0}L^{G}\right\rangle[u]=\sum_{x=0}^{\infty}\frac{u(x)}{x-\omega}\,\frac{1}{(b+1)_{x}}\,\frac{z^{x}}{x!}.$$

First moment

$$\left\langle {}_{1}^{0}\lambda^{G}\right\rangle_{0}(z)=-\frac{1}{\omega}\,{}_{1}F_{2}\left(\begin{array}{c}-\omega\\ b+1,-\omega+1\end{array};z\right).$$

Using (81), we obtain

$$\left[\vartheta\left(\vartheta+b\right)\left(\vartheta-\omega\right)-z\left(\vartheta-\omega\right)\right]\left[\left\langle {}_{1}^{0}\lambda^{G}\right\rangle_{0}\right]=0,$$

which is a special case of (135) with

$$a=-\omega,\quad b_{1}=b,\quad b_{2}=-\omega.$$

Standard moments recurrence operator

$$\left\langle {}_{1}^{0}\Phi^{G}\right\rangle_{n}[\mu]=(\mu-\omega)\left\langle {}_{1}^{0}\Phi\right\rangle_{n}[\mu].$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{0}\Psi^{G}\right\rangle_{n}=\Upsilon_{n+1}(b,-\omega)-z\Upsilon_{n}(-\omega)=\left\langle {}_{1}^{0}\Psi\right\rangle_{n}\circ\Upsilon_{n}(-\omega).$$

Reduced Uvarov Generalized Charlier polynomials Since for the Generalized Charlier polynomials we have

$$\left\langle {}_{1}^{0}\sigma\right\rangle=\vartheta\left(\vartheta+b\right),\quad\left\langle {}_{1}^{0}\tau\right\rangle=\vartheta,$$

we will have reduced cases for their Uvarov transformation if $\omega=0,-b$.

i) $\omega = 0$ In this case, we have

$$\left\langle {}_1^0\lambda_0^U \right\rangle_0 = \left\langle {}_1^0\lambda \right\rangle_0 + \eta.$$

Linear functional

$$\left\langle {}_1^0 L_0^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \, \frac{1}{(b+1)_x} \, \frac{z^x}{x!} + \eta u(0).$$

Using (96) and (123), we obtain

$$\left[\vartheta (\vartheta + b) (\vartheta - 1) - z\vartheta\right] \left[\left\langle {}_1^0\lambda_0^U \right\rangle_0\right] = 0,$$

which is a special case of (135) with

$$a = 0, \quad b_1 = b, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\left\langle {}_1^0\Phi_0^U \right\rangle_n = (\mathfrak{S}_n - 1) \left\langle {}_1^0\Phi \right\rangle_n.$$

Standard moments recurrence

$$\mu \left[(\mu + b)(\mu - 1)\mu^n - z(\mu + 1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_1^0\Psi_{-b}^U \right\rangle_n = \Upsilon_{n+1}(b, -1) - z\Upsilon_n(0) = \Upsilon_n(0) \circ \left\langle {}_1^0\Psi \right\rangle_n.$$

ii) $\omega = -b$ In this case, we have

$$\left\langle {}_1^0\lambda_{-b}^U \right\rangle_0 = \left\langle {}_1^0\lambda \right\rangle_0 + \eta z^{-b}.$$

Linear functional

$$\left\langle {}_1^0 L_{-b}^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \, \frac{1}{(b+1)_x} \, \frac{z^x}{x!} + \eta u(-b).$$

Using (96) and (123), we obtain

$$\left[\vartheta (\vartheta + b)(\vartheta + b - 1) - z(\vartheta + b)\right] \left[\left\langle {}_1^0\lambda_{-b}^U \right\rangle_0\right] = 0,$$

which is a special case of (135) with

$$a = b, \quad b_1 = b, \quad b_2 = b - 1.$$

Standard moments recurrence operator

$$\left\langle {}_1^0\Phi_{-b}^U \right\rangle_n = (\mathfrak{S}_n + b - 1) \left[\left\langle {}_1^0\Phi \right\rangle_n \right].$$

Standard moments recurrence

$$(\mu + b) \left[(\mu + b - 1) \mu^{n+1} - z (\mu + 1)^n \right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_1^0\Psi_{-b}^U \right\rangle_n = \Upsilon_{n+1} (b, b-1) - z \Upsilon_n (b) = \Upsilon_n (b) \circ \left\langle {}_1^0\Psi \right\rangle_n.$$

Truncated Generalized Charlier polynomials These polynomials can be obtained from the Generalized Charlier polynomials by means of the transformation

$$\left\langle {}_1^0\lambda^T \right\rangle_0 = \Omega_{-N}^{-N} \left[\left\langle {}_1^0\lambda \right\rangle_0 \right], \quad N \in \mathbb{N}.$$

Linear functional

$$\left\langle {}_1^0L^T \right\rangle [u] = \sum_{x=0}^{N} u (x) \ \frac{1}{(b+1)_x} \ \frac{z^x}{x!}.$$

First moment

$$\left\langle {}_1^0\lambda^T \right\rangle_0 (z) = \sum_{x=0}^{N} \frac{1}{(b+1)_x} \ \frac{z^x}{x!},$$

and using (108), we get

$$\left\langle {}_1^0\lambda^T \right\rangle_0 = \frac{1}{(b+1)_N} \frac{z^N}{N!} \ {}_3F_0 \left(\begin{array}{c} -N, 1-b-N, 1 \\ - \end{array} ; z^{-1} \right).$$

From (106), we have

$$[\vartheta (\vartheta + b) (\vartheta - N - 1) - z (\vartheta - N)] \left[\left\langle {}_1^0\lambda^T \right\rangle_0 \right] = 0,$$

which is a special case of (135) with

$$a = -N, \quad b_1 = b, \quad b_2 = -N - 1.$$

Standard moments recurrence operator

$$\left\langle {}^{0}_{1}\Phi^{T}\right\rangle_{n} = \left(\mathfrak{S}_{n} - N - 1\right)\left[\left\langle {}^{0}_{1}\Phi\right\rangle_{n}\right].$$

Standard moments recurrence

$$\left(\mu - N - 1\right)\left(\mu + b\right)\mu^{n+1} - z\left(\mu - N\right)\left(\mu + 1\right)^{n} = 0.$$

Modified moments recurrence operator

$$\left\langle {}^{0}_{1}\Psi^{T}\right\rangle_{n} = \Upsilon_{n+1}\left(b, -N-1\right) - z\Upsilon_{n}\left(-N\right) = \Upsilon_{n}\left(-N\right)\circ\left\langle {}^{0}_{1}\Psi\right\rangle_{n}.$$

From (115), we see that

$$\left\langle {}^{0}_{1}\nu^{T}\right\rangle_{n}(z) = \frac{1}{\left(b+1\right)_{N}}\frac{z^{N}}{\left(N-n\right)!}\ {}_{3}F_{0}\left(\begin{array}{c} n-N, -b-N, 1 \\ - \end{array} ; z^{-1}\right).$$

4.3.3 Polynomials of type (2,2)

Linear functional

$$\left\langle {}^{2}_{2}L\right\rangle[u] = \sum_{x=0}^{\infty} u\left(x\right)\ \frac{\left(a_{1}\right)_{x}\left(a_{2}\right)_{x}}{\left(b_{1}+1\right)_{x}\left(b_{2}+1\right)_{x}}\frac{z^{x}}{x!}.$$

First moment

$$\left\langle {}^{2}_{2}\lambda\right\rangle_{0}(z) = {}_{2}F_{2}\left[\begin{array}{c} a_{1}, a_{2} \\ b_{1}+1, b_{2}+1 \end{array} ; z\right].$$

These polynomials can be obtained from the generalized Hahn polynomials by means of the transformation

$$\left\langle {}^{2}_{2}\lambda\right\rangle_{0} = \Omega_{b+1}\left[\left\langle {}^{2}_{1}\lambda\right\rangle_{0}\right].$$

Using (62), we obtain the ODE satisfied by the first moment

$$\left[\vartheta\left(\vartheta + b_{1}\right)\left(\vartheta + b_{2}\right) - z\left(\vartheta + a_{1}\right)\left(\vartheta + a_{2}\right)\right]\left[\left\langle {}^{2}_{2}\lambda\right\rangle_{0}\right] = 0. \tag{136}$$

From (136), we obtain a recurrence for the standard moments

$$\left(\left\langle {}^{2}_{2}\mu\right\rangle + b_{1}\right)\left(\left\langle {}^{2}_{2}\mu\right\rangle + b_{2}\right)\left\langle {}^{2}_{2}\mu\right\rangle^{n+1} = z\left(\left\langle {}^{2}_{2}\mu\right\rangle + a_{1}\right)\left(\left\langle {}^{2}_{2}\mu\right\rangle + a_{2}\right)\left(\left\langle {}^{2}_{2}\mu\right\rangle + 1\right)^{n}.$$

From (28), we have

$$\sum_{k=0}^{2} e_{2-k}(\mathbf{b}) \left\langle {}_{2}^{2}\mu \right\rangle_{k+1} = z \sum_{j=0}^{2} e_{2-j}(\mathbf{a}) \left\langle {}_{2}^{2}\mu \right\rangle_{j},$$

and therefore

$$\left\langle {}_{2}^{2}\mu \right\rangle_{3} = a_1 a_2 z \left\langle {}_{2}^{2}\mu \right\rangle_{0} + (z - b_1 - b_2) \left\langle {}_{2}^{2}\mu \right\rangle_{1} + [(a_1 + a_2) z - b_1 b_2] \left\langle {}_{2}^{2}\mu \right\rangle_{2}.$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_n}(z)$

$$\left\langle {}_{2}^{2}\mu \right\rangle_{n} = \left\langle {}_{2}^{2}\overrightarrow{Q} \right\rangle_{n} \cdot \left\langle {}_{2}^{2}\overrightarrow{\mu} \right\rangle, \quad n \geq 0,$$
$$\left\langle {}_{2}^{2}\overrightarrow{Q} \right\rangle_{n+1} = \left(\vartheta + \left\langle {}_{2}^{2}M \right\rangle\right) \left\langle {}_{2}^{2}\overrightarrow{Q} \right\rangle_{n}, \quad \left\langle {}_{2}^{2}\overrightarrow{Q} \right\rangle_{0} = \overrightarrow{\varepsilon}_{0},$$

with

$$\left\langle {}_{2}^{2}M \right\rangle = \begin{bmatrix} 0 & 0 & a_1 a_2 z \\ 1 & 0 & z - b_1 - b_2 \\ 0 & 1 & (a_1 + a_2) z - b_1 b_2 \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{2}^{2}\nu \right\rangle_{n}(z) = \frac{z^n (a_1)_n (a_2)_n}{(b_1 + 1)_n (b_2 + 1)_n} \, {}_{2}F_{2}\left[\begin{matrix} a_1 + n, a_2 + n \\ b_1 + 1 + n, b_2 + 1 + n \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_{2}^{2}\Psi \right\rangle_{n} &= \Upsilon_{n+1}(b_1, b_2) - z \Upsilon_{n}(a_1, a_2) \\ &= \mathfrak{S}_n^3 + [1 + e_1(\mathbf{b} + n + 1) - z] \mathfrak{S}_n^2 \\ &+ [e_2(\mathbf{b} + n + 1) - z - z e_1(\mathbf{a} + n)] \mathfrak{S}_n - z e_2(\mathbf{a} + n). \end{aligned}$$

Uvarov Charlier polynomials Suppose that $\omega \neq 0$. The Uvarov Charlier polynomials can be obtained from the Charlier polynomials by means of the transformation

$$\left\langle {}_{0}^{0}\lambda_{\omega}^{U} \right\rangle_{0} = \left\langle {}_{0}^{0}\lambda \right\rangle_{0} + \eta z^{\omega} = e^{z} + \eta z^{\omega}.$$

Linear functional

$$\left\langle {}_{0}^{0}L_{\omega}^{U} \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \frac{z^x}{x!} + \eta u(\omega) z^{\omega}.$$

Using (91), we obtain

$$\left[\vartheta\left(\vartheta-\omega\right)\left(\vartheta-\omega-1\right)-z\left(\vartheta-\omega+1\right)\left(\vartheta-\omega\right)\right]\left[\left\langle {}_0^0\lambda_\omega^U\right\rangle_0\right]=0,$$

which is a special case of (136) with

$$a_1=-\omega,\quad a_2=-\omega+1,\quad b_1=-\omega-1,\quad b_2=-\omega.$$

Standard moments recurrence operator

$$\left\langle {}_0^0\Phi_\omega^U\right\rangle_n=\left(\mathfrak{S}_n-\omega\right)\left(\mathfrak{S}_n-\omega-1\right)\left[\left\langle {}_0^0\Phi\right\rangle_n\right].$$

Standard moments recurrence

$$\left(\mu-\omega\right)\left[\left(\mu-\omega-1\right)\mu^{n+1}-z\left(\mu-\omega+1\right)\left(\mu+1\right)^n\right]=0$$

Modified moments recurrence operator

$$\begin{aligned}\left\langle {}_0^0\Psi_\omega^U\right\rangle_n&=\Upsilon_{n+1}\left(-\omega-1,-\omega\right)-z\Upsilon_n\left(-\omega,-\omega+1\right)\\&=\Upsilon_n\left(-\omega,-\omega+1\right)\circ\left\langle {}_0^0\Psi\right\rangle_n.\end{aligned}$$

Double Uvarov Charlier polynomials Since for the Reduced Uvarov Charlier polynomials we have

$$\left\langle {}_0^0\sigma_0^U\right\rangle\left(\vartheta\right)=\vartheta\left(\vartheta-1\right),$$

we will have a reduced case for their Uvarov transformation if we add an extra mass point at $\omega=1$. The Double Uvarov Charlier polynomials can be obtained from the Charlier polynomials by means of the double transformation

$$\left\langle {}_0^0\lambda_{0,1}^U\right\rangle_0=\left\langle {}_0^0\lambda\right\rangle_0+\eta_1+\eta_2 z=e^z+\eta_1+\eta_2 z.$$

Linear functional

$$\left\langle {}_0^0L_{0,1}^U\right\rangle[u]=\sum_{x=0}^{\infty}u\left(x\right)\ \frac{z^x}{x!}+\eta_1 u\left(0\right)+\eta_2 u\left(1\right)z.$$

Using (96), we obtain

$$\left[\vartheta\left(\vartheta-1\right)\left(\vartheta-2\right)-z\vartheta\left(\vartheta-1\right)\right]\left[\left\langle {}_0^0\lambda_{0,1}^U\right\rangle_0\right]=0,$$

which is a special case of (136) with

$$a_1 = 0, \quad a_2 = -1, \quad b_1 = -1, \quad b_2 = -2.$$

Standard moments recurrence operator

$$\left\langle {}_0^0\Phi_{0,1}^U \right\rangle_n = (\mathfrak{S}_n - 1)(\mathfrak{S}_n - 2)\left[\left\langle {}_0^0\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$\mu(\mu-1)\left[(\mu-2)\mu^n - z(\mu+1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_0^0\Psi_{0,1}^U \right\rangle_n = \Upsilon_{n+1}(-1,-2) - z\Upsilon_n(0,-1) = \Upsilon_n(0,-1) \circ \left\langle {}_0^0\Psi \right\rangle_n.$$

Reduced Uvarov Truncated Charlier polynomials Since for the Truncated Charlier polynomials we have

$$\left\langle {}_0^0\sigma^T \right\rangle(\vartheta) = \vartheta(\vartheta - N - 1), \quad \left\langle {}_0^0\tau^T \right\rangle(\vartheta) = \vartheta - N,$$

we will have reduced cases for their Uvarov transformation if $\omega = 0, N, N+1$.

i) $\omega = 0$

In this case, the polynomials can be obtained from the Truncated Charlier polynomials by means of the transformation

$$\left\langle {}_0^0\lambda_0^{T,U} \right\rangle_0 = \left\langle {}_0^0\lambda^T \right\rangle_0 + \eta.$$

Linear functional

$$\left\langle {}_0^0L_0^{T,U} \right\rangle[u] = \sum_{x=0}^{N} u(x)\ \frac{z^x}{x!} + \eta u(0).$$

Using (96) and (127), we obtain

$$\left[\vartheta(\vartheta-1)(\vartheta-N-1) - z\vartheta(\vartheta-N)\right]\left[\left\langle {}_0^0\lambda_0^{T,U} \right\rangle_0\right] = 0, \tag{137}$$

which is a special case of (136) with

$$a_1 = 0, \quad a_2 = -N, \quad b_1 = -1, \quad b_2 = -N-1.$$

Standard moments recurrence operator

$$\left\langle {}_0^0\Phi_0^{T,U} \right\rangle_n [\mu] = \mu \left[(\mu-1)(\mu-N-1)\mu^n - z(\mu-N)(\mu+1)^n \right]$$
$$= (\mathfrak{S}_n - 1)(\mathfrak{S}_n - N - 1) \left[\left\langle {}_0^0\Phi \right\rangle_n \right] [\mu].$$

Modified moments recurrence operator

$$\left\langle {}_0^0\Psi_0^{T,U} \right\rangle_n = \Upsilon_{n+1}(-1,-N-1) - z\Upsilon_n(0,-N) = \Upsilon_n(0,-N) \left[\left\langle {}_0^0\Psi \right\rangle_n \right].$$

Remark 29 *We could have also go the other way: start with the Reduced Uvarov Charlier polynomials and apply a truncation transformation*

$$\left\langle {}_0^0\lambda_0^{T,U} \right\rangle_0 = \Omega_{-N}^{-N} \left[\left\langle {}_0^0\lambda_0^{U} \right\rangle_0 \right], \quad N \in \mathbb{N}.$$

In either case, we obtain the same ODE (137).

ii) $\omega = N$

In this case, the polynomials can be obtained from the Truncated Charlier polynomials by means of the transformation

$$\left\langle {}_0^0\lambda_N^{T,U} \right\rangle_0 = \left\langle {}_0^0\lambda^{T} \right\rangle_0 + \eta z^N.$$

Linear functional

$$\left\langle {}_0^0L_N^{T,U} \right\rangle [u] = \sum_{x=0}^{N} u(x) \ \frac{z^x}{x!} + \eta u(N).$$

Using (97) and (127), we obtain

$$\left[\vartheta(\vartheta-N)(\vartheta-N-1) - z(\vartheta-N+1)(\vartheta-N) \right] \left[\left\langle {}_0^0\lambda_N^{T,U} \right\rangle_0 \right] = 0,$$

which is a special case of (136) with

$$a_1 = -N, \quad a_2 = -N+1, \quad b_1 = -N-1, \quad b_2 = -N.$$

Standard moments recurrence operator

$$\left\langle {}_0^0\Phi_N^{T,U} \right\rangle_n [\mu] = (\mu-N) \left[(\mu-N-1)\mu^{n+1} - z(\mu-N+1)(\mu+1)^n \right]$$
$$= (\mathfrak{S}_n - N)(\mathfrak{S}_n - N - 1) \left[\left\langle {}_0^0\Phi \right\rangle_n \right] [\mu].$$

Modified moments recurrence operator

$$\left\langle {}_0^0\Psi_N^{T,U} \right\rangle_n = \Upsilon_{n+1}(-N-1,-N) - z\Upsilon_n(-N,-N+1)$$
$$= \Upsilon_n(-N,-N+1) \circ \left\langle {}_0^0\Psi \right\rangle_n .$$

ii) $\omega = N+1$

In this case, the polynomials can be obtained from the Truncated Charlier polynomials by means of the transformation

$$\left\langle {}_0^0\lambda_{N+1}^{T,U} \right\rangle_0 = \left\langle {}_0^0\lambda^T \right\rangle_0 + \eta z^{N+1}.$$

Linear functional

$$\left\langle {}_0^0L_{N+1}^{T,U} \right\rangle [u] = \sum_{x=0}^{N} u(x) \ \frac{z^x}{x!} + \eta u(N+1).$$

Using (96) and (127), we obtain

$$[\vartheta(\vartheta-N-1)(\vartheta-N-2) - z(\vartheta-N)(\vartheta-N-1)] \left[\left\langle {}_0^0\lambda_{N+1}^{T,U} \right\rangle_0 \right] = 0,$$

which is a special case of (136) with

$$a_1 = -N, \quad a_2 = -N-1, \quad b_1 = -N-1, \quad b_2 = -N-2.$$

Standard moments recurrence operator

$$\left\langle {}_0^0\Phi_N^{T,U} \right\rangle_n [\mu] = (\mu-N-1)\left[(\mu-N-2)\mu^{n+1} - z(\mu-N)(\mu+1)^n\right]$$
$$= (\mathfrak{S}_n - N-1)(\mathfrak{S}_n - N-2)\left[\left\langle {}_0^0\Phi \right\rangle_n\right][\mu].$$

Modified moments recurrence operator

$$\left\langle {}_0^0\Psi_N^{T,U} \right\rangle_n = \Upsilon_{n+1}(-N-1,-N-2) - z\Upsilon_n(-N,-N-1)$$
$$= \Upsilon_n(-N,-N-1) \circ \left\langle {}_0^0\Psi \right\rangle_n .$$

Reduced Uvarov Generalized Meixner Polynomials Since for the Generalized Meixner Polynomials we have

$$\left\langle {}_{1}^{1}\sigma \right\rangle (\vartheta) = \vartheta(\vartheta + b), \quad \left\langle {}_{1}^{1}\tau \right\rangle (\vartheta) = \vartheta + a,$$

we will have reduced cases for their Uvarov transformation if $\omega = 0, -a, -b$.

i) $\omega = 0$

In this case, the polynomials can be obtained from the Generalized Meixner Polynomials by means of the transformation

$$\left\langle {}_{1}^{1}\lambda_{0}^{U} \right\rangle_{0} = \left\langle {}_{1}^{1}\lambda \right\rangle_{0} + \eta.$$

Linear functional

$$\left\langle {}_{1}^{1}L_{0}^{U} \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \, \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!} + \eta u(0).$$

Using (96) and (125), we obtain

$$[\vartheta(\vartheta - 1)(\vartheta + b) - z\vartheta(\vartheta + a)] \left[\left\langle {}_{1}^{1}\lambda_{0}^{U} \right\rangle_{0} \right] = 0,$$

which is a special case of (136) with

$$a_1 = a, \quad a_2 = 0, \quad b_1 = b, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\begin{aligned} \left\langle {}_{1}^{1}\Phi_{0}^{U} \right\rangle_{n} [\mu] &= \mu \left[(\mu - 1)(\mu + b)\mu^{n} - z(\mu + a)(\mu + 1)^{n} \right] \\ &= (\mathfrak{S}_{n} - 1) \left[\left\langle {}_{1}^{1}\Phi \right\rangle_{n} \right] [\mu]. \end{aligned}$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{1}\Psi_{0}^{U} \right\rangle_{n} = \Upsilon_{n+1}(b, -1) - z\Upsilon_{n}(a, 0) = \Upsilon_{n}(0) \circ \left\langle {}_{1}^{1}\Psi \right\rangle_{n}.$$

ii) $\omega = -a$

In this case, the polynomials can be obtained from the Generalized Meixner Polynomials by means of the transformation

$$\left\langle {}_{1}^{1}\lambda_{-a}^{U} \right\rangle_{0} = \left\langle {}_{1}^{1}\lambda \right\rangle_{0} + \eta z^{-a}.$$

Linear functional

$$\left\langle {}_1^1L_{-a}^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!} + \eta u(-a) z^{-a}.$$

Using (97) and (125), we obtain

$$\left[\vartheta(\vartheta+a)(\vartheta+b) - z(\vartheta+a)(\vartheta+a+1)\right]\left[\left\langle {}_1^1\lambda_{-a}^U \right\rangle_0\right] = 0,$$

which is a special case of (136) with

$$a_1 = a, \quad a_2 = a+1, \quad b_1 = b, \quad b_2 = a.$$

Standard moments recurrence operator

$$\left\langle {}_1^1\Phi_{-a}^U \right\rangle_n = (\mathfrak{S}_n + a)\left[\left\langle {}_1^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$(\mu+a)\left[(\mu+b)\mu^{n+1} - z(\mu+a+1)(\mu+1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_1^1\Psi_{-a}^U \right\rangle_n = \Upsilon_{n+1}(b,a) - z\Upsilon_n(a,a+1) = \Upsilon_n(a+1) \circ \left\langle {}_1^1\Psi \right\rangle_n.$$

iii) $\omega = -b$

In this case, the polynomials can be obtained from the Generalized Meixner Polynomials by means of the transformation

$$\left\langle {}_1^1\lambda_{-b}^U \right\rangle_0 = \left\langle {}_1^1\lambda \right\rangle_0 + \eta z^{-b}.$$

Linear functional

$$\left\langle {}_1^1L_{-b}^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!} + \eta u(-b) z^{-b}.$$

Using (96) and (125), we obtain

$$\left[\vartheta(\vartheta+b)(\vartheta+b-1) - z(\vartheta+a)(\vartheta+b)\right]\left[\left\langle {}_1^1\lambda_{-b}^U \right\rangle_0\right] = 0,$$

which is a special case of (136) with

$$a_1 = a, \quad a_2 = b, \quad b_1 = b, \quad b_2 = b-1.$$

Standard moments recurrence operator

$$\left\langle {}_1^1\Phi_{-b}^U \right\rangle_n [\mu] = (\mathfrak{S}_n + b - 1)\left[\left\langle {}_1^1\Phi \right\rangle_n\right][\mu].$$

Standard moments recurrence

$$(\mu + b)\left[(\mu + b - 1)\,\mu^{n+1} - z\,(\mu + a)\,(\mu + 1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_1^1\Psi_{-b}^U \right\rangle_n = \Upsilon_{n+1}(b, b-1) - z\Upsilon_n(a, b) = \Upsilon_n(b) \circ \left\langle {}_1^1\Psi \right\rangle_n.$$

Truncated Generalized Meixner Polynomials These polynomials can be obtained from the Generalized Meixner polynomials by means of the transformation

$$\left\langle {}_1^1\lambda^T \right\rangle_0 = \Omega_{-N}^{-N}\left[\left\langle {}_1^1\lambda \right\rangle_0\right], \quad N \in \mathbb{N}.$$

Linear functional

$$\left\langle {}_1^1L^T \right\rangle [u] = \sum_{x=0}^{N} u(x)\, \frac{(a)_x}{(b+1)_x}\,\frac{z^x}{x!}.$$

First moment

$$\left\langle {}_1^1\lambda^T \right\rangle_0 (z) = \sum_{x=0}^{N} \frac{(a)_x}{(b+1)_x}\,\frac{z^x}{x!}.$$

From (108), we get

$$\left\langle {}_1^1\lambda^T \right\rangle_0 = \frac{(a)_N}{(b+1)_N}\frac{z^N}{N!}\ {}_3F_1\left(\begin{array}{c} -N, -b-N, 1 \\ 1-a-N \end{array};-z^{-1}\right).$$

From (106), we have

$$\left[\vartheta\,(\vartheta + b)\,(\vartheta - N - 1) - z\,(\vartheta + a)\,(\vartheta - N)\right]\left[\left\langle {}_1^1\lambda^T \right\rangle_0\right] = 0,$$

which is a special case of (136) with

$$a_1 = a, \quad a_2 = -N, \quad b_1 = b, \quad b_2 = -N - 1.$$

Standard moments recurrence operator

$$\left\langle {}_1^1\Phi^T \right\rangle_n = (\mathfrak{S}_n - N - 1)\left[\left\langle {}_1^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$\left(\mu-N-1\right)\left(\mu+b\right)\mu^{n+1}-z\left(\mu-N\right)\left(\mu+a\right)\left(\mu+1\right)^{n}=0.$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{1}\Psi^{T}\right\rangle_{n}=\Upsilon_{n+1}\left(b,-N-1\right)-z\Upsilon_{n}\left(a,-N\right)=\Upsilon_{n}\left(-N\right)\circ\left\langle {}_{1}^{1}\Psi\right\rangle_{n}.$$

From (115), we see that

$$\left\langle {}_{1}^{1}\nu^{T}\right\rangle_{n}(z)=\frac{(a)_{N}}{(b+1)_{N}}\frac{z^{N}}{(N-n)!}\,{}_{3}F_{1}\left(\begin{array}{c} n-N,-b-N,1 \\ 1-a-N \end{array};-z^{-1}\right).$$

4.3.4 Polynomials of type (3,0;N)

Linear functional

$$\left\langle {}_{0}^{3;N}L\right\rangle[u]=\sum_{x=0}^{N}u(x)\ (-N)_{x}\left(a_{1}\right)_{x}\left(a_{2}\right)_{x}\frac{z^{x}}{x!},\quad N\in\mathbb{N}.$$

First moment

$$\left\langle {}_{0}^{3;N}\lambda\right\rangle_{0}(z)=\ {}_{3}F_{0}\left[\begin{array}{c} -N,a_{1},a_{2} \\ - \end{array};z\right].$$

These polynomials can be obtained from the generalized Krawtchouk polynomials by means of the transformation

$$\left\langle {}_{0}^{3;N}\lambda\right\rangle_{0}=\Omega^{a}\left[\left\langle {}_{0}^{2;N}\lambda\right\rangle_{0}\right].$$

Using (60), we obtain the ODE satisfied by the first moment

$$\left[\vartheta-z\left(\vartheta+\mathbf{a}_{3,N}\right)\right]\left[\left\langle {}_{0}^{3;N}\lambda\right\rangle_{0}\right]=0,$$

with

$$\mathbf{a}_{3,N}=\left(-N,a_{1},a_{2}\right). \tag{138}$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{3;N}\Phi\right\rangle_{n}[\mu]=\mu^{n+1}-z\left(\mu+\mathbf{a}_{3,N}\right)\left(\mu+1\right)^{n}.$$

From (28), we have

$$\left\langle {}^{3;N}_{0}\mu \right\rangle_3 = -e_3\left(\mathbf{a}_{3,N}\right)\left\langle {}^{3;N}_{0}\mu \right\rangle_0 + \left[z^{-1} - e_2\left(\mathbf{a}_{3,N}\right)\right]\left\langle {}^{3;N}_{0}\mu \right\rangle_1 - e_1\left(\mathbf{a}_{3,N}\right)\left\langle {}^{3;N}_{0}\mu \right\rangle_2 .$$

Representation of the standard moments in terms of the polynomials $\overrightarrow{Q_n}(z)$

$$\left\langle {}^{3;N}_{0}\mu \right\rangle_n = z^{-n}\left\langle {}^{3;N}_{0}\overrightarrow{Q} \right\rangle_n \cdot \left\langle {}^{3;N}_{0}\overrightarrow{\mu} \right\rangle, \quad n \geq 0.$$

From (33), we have

$$\left\langle {}^{3;N}_{0}\overrightarrow{Q} \right\rangle_{n+1} = z\left(\vartheta + \left\langle {}^{3;N}_{0}M \right\rangle - nI\right)\left\langle {}^{3;N}_{0}\overrightarrow{Q} \right\rangle_n, \quad \left\langle {}^{3;N}_{0}\overrightarrow{Q} \right\rangle_0 = \overrightarrow{\varepsilon}_0,$$

with

$$\left\langle {}^{3;N}_{0}M \right\rangle = \begin{bmatrix} 0 & 0 & -e_3\left(\mathbf{a}_{3,N}\right) \\ 1 & 0 & z^{-1} - e_2\left(\mathbf{a}_{3,N}\right) \\ 0 & 1 & -e_1\left(\mathbf{a}_{3,N}\right) \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}^{3;N}_{0}\nu \right\rangle_n (z) = z^n (-N)_n (a_1)_n (a_2)_n \ {}_3F_0\left[\begin{matrix} n-N, a_1+n, a_2+n \\ - \end{matrix}; z\right].$$

Modified moments recurrence operator

$$\left\langle {}^{3;N}_{0}\Psi \right\rangle_n = \Upsilon_{n+1}(\emptyset) - z\Upsilon_n(-N, a_1, a_2),$$

and therefore,

$$\begin{aligned} & -z^{-1}\left\langle {}^{3;N}_{0}\Psi \right\rangle_n = \mathfrak{S}_n^3 + e_1\left(\mathbf{a}_{3,N}+n+1\right)\mathfrak{S}_n^2 \\ & + \left[1 + e_1\left(\mathbf{a}_{3,N}+n\right) + e_2\left(\mathbf{a}_{3,N}+n\right) - z^{-1}\right]\mathfrak{S}_n + e_3\left(\mathbf{a}_{3,N}+n\right). \end{aligned}$$

4.3.5 Polynomials of type (3,1;N)

Linear functional

$$\left\langle {}^{3;N}_{1}L \right\rangle [u] = \sum_{x=0}^{N} u(x)\frac{(-N)_x (a_1)_x (a_2)_x}{(b+1)_x}\frac{z^x}{x!}, \quad N \in \mathbb{N}.$$

First moment

$$\left\langle {}^{3;N}_{1}\lambda \right\rangle_0 (z) = {}_3F_1\left[\begin{matrix} -N, a_1, a_2 \\ b+1 \end{matrix}; z\right].$$

These polynomials can be obtained from the generalized Hahn polynomials by means of the transformation

$$\left\langle {}_{1}^{3;N}\lambda \right\rangle_{0} = \Omega^{-N}\left[\left\langle {}_{1}^{2}\lambda \right\rangle_{0} \right].$$

Using (60), we obtain the ODE satisfied by the first moment

$$\left[\vartheta \left(\vartheta + b \right) - z \left(\vartheta + \mathbf{a}_{3,N} \right) \right] \left[\left\langle {}_{1}^{3;N}\lambda \right\rangle_{0} \right] = 0, \tag{139}$$

where $\mathbf{a}_{3,N}$ was defined in (138).

Standard moments recurrence operator

$$\left\langle {}_{0}^{3;N}\Phi \right\rangle_{n}[\mu] = \left(\mu + b \right) \mu^{n+1} - z \left(\mu + \mathbf{a}_{3,N} \right) \left(\mu + 1 \right)^{n}.$$

From (28), we have

$$\begin{aligned} \left\langle {}_{1}^{3;N}\mu \right\rangle_{3} = &-e_{3}\left(\mathbf{a}_{3,N} \right) \left\langle {}_{1}^{3;N}\mu \right\rangle_{0} + \left[bz^{-1} - e_{2}\left(\mathbf{a}_{3,N} \right) \right] \left\langle {}_{1}^{3;N}\mu \right\rangle_{1} \\ &+ \left[z^{-1} - e_{1}\left(\mathbf{a}_{3,N} \right) \right] \left\langle {}_{1}^{3;N}\mu \right\rangle_{2}. \end{aligned}$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_{n}}(z)$

$$\left\langle {}_{1}^{3;N}\mu \right\rangle_{n} = z^{-n} \left\langle {}_{1}^{3;N}\overrightarrow{Q} \right\rangle_{n} \cdot \left\langle {}_{1}^{3;N}\overrightarrow{\mu} \right\rangle, \quad n \geq 0.$$

From (33), we have

$$\left\langle {}_{1}^{3;N}\overrightarrow{Q} \right\rangle_{n+1} = z \left(\vartheta + \left\langle {}_{1}^{3;N}M \right\rangle - nI \right) \left\langle {}_{1}^{3;N}\overrightarrow{Q} \right\rangle_{n}, \quad \left\langle {}_{1}^{3;N}\overrightarrow{Q} \right\rangle_{0} = \overrightarrow{\varepsilon}_{0},$$

with

$$\left\langle {}_{1}^{3;N}M \right\rangle = \begin{bmatrix} 0 & 0 & -e_{3}\left(\mathbf{a}_{3,N} \right) \\ 1 & 0 & bz^{-1} - e_{2}\left(\mathbf{a}_{3,N} \right) \\ 0 & 1 & z^{-1} - e_{1}\left(\mathbf{a}_{3,N} \right) \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{1}^{3;N}\nu \right\rangle_{n}(z) = z^{n} \frac{(-N)_{n} \left(a_{1} \right)_{n} \left(a_{2} \right)_{n}}{(b+1)_{n}} \, {}_{3}F_{1}\left[\begin{matrix} n-N, a_{1}+n, a_{2}+n \\ b+1+n \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\left\langle {}_{1}^{3;N}\Psi \right\rangle_n = \Upsilon_{n+1}(b) - z\Upsilon_n(-N, a_1, a_2),$$

and therefore,

$$-z^{-1}\left\langle {}_{1}^{3;N}\Psi \right\rangle_n = \mathfrak{S}_n^3 + \left[e_1(\mathbf{a}_{3,N}+n+1) - z^{-1}\right]\mathfrak{S}_n^2$$
$$+\left[1 + e_1(\mathbf{a}_{3,N}+n) + e_2(\mathbf{a}_{3,N}+n) - (b+n+1)z^{-1}\right]\mathfrak{S}_n + e_3(\mathbf{a}_{3,N}+n).$$

Reduced Uvarov Generalized Krawtchouk Polynomials Since for the Generalized Krawtchouk Polynomials we have

$$\left\langle {}_{0}^{2,N}\sigma \right\rangle(\vartheta) = \vartheta, \quad \left\langle {}_{0}^{2,N}\tau \right\rangle(\vartheta) = (\vartheta + a)(\vartheta - N),$$

we will have reduced cases for their Uvarov transformation if $\omega = 0, -a, N$.

i) $\omega = 0$

In this case, the polynomials can be obtained from the Generalized Krawtchouk Polynomials by means of the transformation

$$\left\langle {}_{0}^{2,N}\lambda_0^U \right\rangle_0 = \left\langle {}_{0}^{2,N}\lambda \right\rangle_0 + \eta.$$

Linear functional

$$\left\langle {}_{0}^{2,N}L_0^U \right\rangle[u] = \sum_{x=0}^{N} u(x)\ (-N)_x (a)_x \frac{z^x}{x!} + \eta u(0).$$

Using (96) and (128), we obtain

$$[\vartheta(\vartheta-1) - z\vartheta(\vartheta+a)(\vartheta-N)]\left[\left\langle {}_{0}^{2,N}\lambda_0^U \right\rangle_0\right] = 0,$$

which is a special case of (139) with

$$a_1 = a, \quad a_2 = 0, \quad a_3 = -N, \quad b = -1.$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{2,N}\Phi_0^U \right\rangle_n = (\mathfrak{S}_n - 1)\left[\left\langle {}_{0}^{2,N}\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$\mu\left[(\mu-1)\,\mu^{n}-z\,(\mu+a)\,(\mu-N)\,(\mu+1)^{n}\right]=0.$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{2,N}\Psi_{0}^{U}\right\rangle_{n}=\Upsilon_{n+1}\left(-1\right)-z\Upsilon_{n}\left(a,-N,0\right)=\Upsilon_{n}\left(0\right)\circ\left\langle {}_{0}^{2,N}\Psi\right\rangle_{n}.$$

ii) $\omega=-a$

In this case, the polynomials can be obtained from the Generalized Krawtchouk Polynomials by means of the transformation

$$\left\langle {}_{0}^{2,N}\lambda_{-a}^{U}\right\rangle_{0}=\left\langle {}_{0}^{2,N}\lambda\right\rangle_{0}+\eta z^{-a}.$$

Linear functional

$$\left\langle {}_{0}^{2,N}L_{-a}^{U}\right\rangle[u]=\sum_{x=0}^{N}u\left(x\right)\ \left(-N\right)_{x}\left(a\right)_{x}\frac{z^{x}}{x!}+\eta u\left(-a\right)z^{-a}.$$

Using (97) and (128), we obtain

$$\left[\vartheta\left(\vartheta+a\right)-z\left(\vartheta+a\right)\left(\vartheta+a+1\right)\left(\vartheta-N\right)\right]\left[\left\langle {}_{0}^{2,N}\lambda_{-a}^{U}\right\rangle_{0}\right]=0,$$

which is a special case of (139) with

$$a_{1}=a,\quad a_{2}=a+1,\quad a_{3}=-N,\quad b=a.$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{2,N}\Phi_{-a}^{U}\right\rangle_{n}=\left(\mathfrak{S}_{n}-1\right)\left[\left\langle {}_{0}^{2,N}\Phi\right\rangle_{n}\right].$$

Standard moments recurrence

$$\mu\left[(\mu-1)\,\mu^{n}-z\,(\mu-N)\,(\mu+a)\,(\mu+1)^{n}\right]=0.$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{2,N}\Psi_{-a}^{U}\right\rangle_{n}=\Upsilon_{n+1}\left(a\right)-z\Upsilon_{n}\left(a,-N,a+1\right)=\Upsilon_{n}\left(a+1\right)\circ\left\langle {}_{0}^{2,N}\Psi\right\rangle_{n}.$$

iii) $\omega=N$

In this case, the polynomials can be obtained from the Generalized Krawtchouk Polynomials by means of the transformation

$$\left\langle {}_{0}^{2,N}\lambda_{N}^{U}\right\rangle_{0} = \left\langle {}_{0}^{2,N}\lambda\right\rangle_{0} + \eta z^{N}.$$

Linear functional

$$\left\langle {}_{0}^{2,N}L_{N}^{U}\right\rangle [u] = \sum_{x=0}^{N} u(x)\ (-N)_{x}(a)_{x}\frac{z^{x}}{x!} + \eta u(N)\, z^{N}.$$

Using (97) and (128), we obtain

$$[\vartheta(\vartheta - N) - z(\vartheta + a)(\vartheta - N + 1)(\vartheta - N)]\left[\left\langle {}_{0}^{2,N}\lambda_{N}^{U}\right\rangle_{0}\right] = 0,$$

which is a special case of (139) with

$$a_1 = a, \quad a_2 = -N + 1, \quad a_3 = -N, \quad b = -N.$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{2,N}\Phi_{N}^{U}\right\rangle_{n} = (\mathfrak{S}_{n} - N)\left[\left\langle {}_{0}^{2,N}\Phi\right\rangle_{n}\right].$$

Standard moments recurrence

$$(\mu - N)\left[\mu^{n+1} - z(\mu + a)(\mu - N + 1)(\mu + 1)^{n}\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{2,N}\Psi_{N}^{U}\right\rangle_{n} = \Upsilon_{n+1}(-N) - z\Upsilon_{n}(a, -N, -N + 1) = \Upsilon_{n}(-N + 1)\circ\left\langle {}_{0}^{2,N}\Psi\right\rangle_{n}.$$

4.3.6 Polynomials of type (3,2)

Linear functional

$$\left\langle {}_{2}^{3}L\right\rangle [u] = \sum_{x=0}^{\infty} u(x)\frac{(a_1)_x(a_2)_x(a_3)_x}{(b_1 + 1)_x(b_2 + 1)_x}\frac{z^x}{x!}.$$

First moment

$$\left\langle {}_{2}^{3}\lambda\right\rangle_{0}(z) = {}_{3}F_{2}\left[\begin{matrix} a_1, a_2, a_3 \\ b_1 + 1, b_2 + 1\end{matrix}; z\right].$$

These polynomials can be obtained from the polynomials of type $(2,2)$ by means of the transformation

$$\left\langle {}_{2}^{3}\lambda \right\rangle_{0} = \Omega^{a}\left[\left\langle {}_{2}^{2}\lambda \right\rangle_{0}\right].$$

Using (60), we obtain the ODE satisfied by the first moment

$$\left[\vartheta\left(\vartheta+b_{1}\right)\left(\vartheta+b_{2}\right)-z\left(\vartheta+a_{1}\right)\left(\vartheta+a_{2}\right)\left(\vartheta+a_{3}\right)\right]\left[\left\langle {}_{2}^{3}\lambda \right\rangle_{0}\right]=0. \tag{140}$$

From (140), we see that the standard moments $\left\langle {}_{2}^{3}\mu \right\rangle_{n}$ satisfy the recurrence

$$\left(\mu+b_{1}\right)\left(\mu+b_{2}\right)\mu^{n+1}-z\left(\mu+a_{1}\right)\left(\mu+a_{2}\right)\left(\mu+a_{3}\right)\left(\mu+1\right)^{n}=0.$$

From (28), we have

$$\sum_{k=0}^{2} e_{2-k}\left(\mathbf{b}\right)\left\langle {}_{2}^{3}\mu \right\rangle_{k+1} = z\sum_{j=0}^{3} e_{3-j}\left(\mathbf{a}\right)\left\langle {}_{2}^{3}\mu \right\rangle_{j},$$

and therefore

$$(1-z)\left\langle {}_{2}^{3}\mu \right\rangle_{3} = ze_{3}\left(\mathbf{a}\right)\left\langle {}_{2}^{3}\mu \right\rangle_{0}+\left[ze_{2}\left(\mathbf{a}\right)-e_{2}\left(\mathbf{b}\right)\right]\left\langle {}_{2}^{3}\mu \right\rangle_{1}+\left[ze_{1}\left(\mathbf{a}\right)-e_{1}\left(\mathbf{b}\right)\right]\left\langle {}_{2}^{3}\mu \right\rangle_{2}.$$

Representation of the standard moments in terms of the vector polynomials $\overrightarrow{Q_{n}}\left(z\right)$

$$\left\langle {}_{2}^{3}\mu \right\rangle_{n} = (1-z)^{-n}\left\langle {}_{2}^{3}\overrightarrow{Q}\right\rangle_{n}\cdot\left\langle {}_{2}^{3}\overrightarrow{\mu}\right\rangle, \quad \overrightarrow{Q_{0}}=\overrightarrow{\varepsilon_{0}}.$$

From (34), we have

$$\left\langle {}_{2}^{3}\overrightarrow{Q}\right\rangle_{n+1} = \left[(1-z)\left(\vartheta+\left\langle {}_{2}^{3}M\right\rangle\right)+nzI\right]\left\langle {}_{2}^{3}\overrightarrow{Q}\right\rangle_{n}, \quad \left\langle {}_{2}^{3}\overrightarrow{Q}\right\rangle_{0}=\overrightarrow{\varepsilon}_{0},$$

with

$$(1-z)\left\langle {}_{2}^{3}M\right\rangle = \begin{bmatrix} 0 & 0 & ze_{3}\left(\mathbf{a}\right) \\ 1-z & 0 & ze_{2}\left(\mathbf{a}\right)-e_{2}\left(\mathbf{b}\right) \\ 0 & 1-z & ze_{1}\left(\mathbf{a}\right)-e_{1}\left(\mathbf{b}\right) \end{bmatrix}.$$

Hypergeometric representation of the modified moments

$$\left\langle {}_{2}^{3}\nu \right\rangle_{n}(z) = z^{n}\frac{\left(a_{1}\right)_{n}\left(a_{2}\right)_{n}\left(a_{3}\right)_{n}}{\left(b+1\right)_{n}}\,{}_{3}F_{2}\left[\begin{matrix} a_{1}+n, a_{2}+n, a_{3}+n \\ b_{1}+1+n, b_{2}+1+n \end{matrix}; z\right].$$

Modified moments recurrence operator

$$\begin{aligned}\left\langle {}_{2}^{3}\Psi \right\rangle_{n} &= \Upsilon_{n+1}\left(b_{1},b_{2}\right)-z\Upsilon_{n}\left(a_{1},a_{2},a_{3}\right) \\ &= (1-z)\mathfrak{S}_{n}^{3}+\left[1+e_{1}\left(\mathbf{b}+n+1\right)-ze_{1}\left(\mathbf{a}+n+1\right)\right]\mathfrak{S}_{n}^{2} \\ &\quad+\left\{e_{2}\left(\mathbf{b}+n+1\right)-z\left[1+e_{1}\left(\mathbf{a}+n\right)+e_{2}\left(\mathbf{a}+n\right)\right]\right\}\mathfrak{S}_{n}-ze_{3}\left(\mathbf{a}+n\right).\end{aligned}$$

Uvarov Meixner polynomials Suppose that $\omega \neq 0, -a$. The Uvarov Meixner polynomials can be obtained from the Meixner polynomials by means of the transformation

$$\left\langle {}_0^1\lambda_\omega^U \right\rangle_0 = \left\langle {}_0^1\lambda \right\rangle_0 + \eta z^\omega = (1-z)^{-a} + \eta z^\omega.$$

Linear functional

$$\left\langle {}_0^1L_\omega^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x)\ (a)_x\ \frac{z^x}{x!} + \eta u(\omega) z^\omega.$$

Using (91) and (121), we obtain

$$\left[\vartheta(\vartheta-\omega)(\vartheta-\omega-1) - z(\vartheta+a)(\vartheta-\omega+1)(\vartheta-\omega)\right]\left[\left\langle {}_0^1\lambda_\omega^U \right\rangle_0\right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = -\omega, \quad a_3 = -\omega+1, \quad b_1 = -\omega-1, \quad b_2 = -\omega.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_\omega^U \right\rangle_n = (\mathfrak{S}_n - \omega)(\mathfrak{S}_n - \omega - 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$(\mu-\omega)\left[(\mu-\omega-1)\mu^{n+1} - z(\mu-\omega+1)(\mu+a)(\mu+1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_0^1\Psi_\omega^U \right\rangle_n = \Upsilon_{n+1}(-\omega-1,-\omega) - z\Upsilon_n(a,-\omega,-\omega+1) = \Upsilon_n(-\omega,-\omega+1) \circ \left\langle {}_0^1\Psi \right\rangle_n.$$

Double Uvarov Meixner polynomials Since for the Reduced Uvarov Meixner polynomials we have

$$\begin{aligned} \left\langle {}_0^1\sigma_0^U \right\rangle &= \vartheta(\vartheta-1), \quad \left\langle {}_0^1\tau_0^U \right\rangle = \vartheta(\vartheta+a) \\ \left\langle {}_0^1\sigma_{-a}^U \right\rangle &= \vartheta(\vartheta+a), \quad \left\langle {}_0^1\tau_{-a}^U \right\rangle = (\vartheta+a)(\vartheta+a+1) \end{aligned}$$

we will have a reduced case for their Uvarov transformations if we add an extra mass point at $\omega = 1, -a$, or $\omega = 0, -a-1$.

Case 1: $\omega_1 = 0, \omega_2 = 1$

First moment

$$\left\langle {}_0^1\lambda_{0,1}^U \right\rangle_0 = \left\langle {}_0^1\lambda \right\rangle_0 + \eta_1 + \eta_2 z = (1-z)^{-a} + \eta_1 + \eta_2 z.$$

Linear functional

$$\left\langle {}_0^1 L_{0,1}^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ (a)_x \ \frac{z^x}{x!} + \eta_1 u(0) + \eta_2 u(1) z.$$

Using (96) and (132), we obtain

$$\left[\vartheta(\vartheta-1)(\vartheta-2) - z(\vartheta+a)\vartheta(\vartheta-1)\right] \left[\left\langle {}_0^1\lambda_{0,1}^U \right\rangle_0\right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = 0, \quad a_3 = -1, \quad b_1 = -1, \quad b_2 = -2.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_{0,1}^U \right\rangle_n = \mathfrak{S}_n(\mathfrak{S}_n - 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$\mu(\mu-1)\left[(\mu-2)\mu^n - z(\mu+a)(\mu+1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_0^1\Psi_{0,1}^U \right\rangle_n = \Upsilon_{n+1}(-1,-2) - z\Upsilon_n(a,0,-1) = \Upsilon_n(0,-1) \circ \left\langle {}_0^1\Psi \right\rangle_n.$$

Case 2: $\omega_1 = 0, \omega_2 = -a$

First moment

$$\left\langle {}_0^1\lambda_{0,-a}^U \right\rangle_0 = \left\langle {}_0^1\lambda \right\rangle_0 + \eta_1 + \eta_2 z^{-a} = (1-z)^{-a} + \eta_1 + \eta_2 z^{-a}.$$

Linear functional

$$\left\langle {}_0^1 L_{0,-a}^U \right\rangle [u] = \sum_{x=0}^{\infty} u(x) \ (a)_x \ \frac{z^x}{x!} + \eta_1 u(0) + \eta_2 u(-a) z^{-a}.$$

Using (97) and (132), we obtain

$$\left[\vartheta(\vartheta-1)(\vartheta+a) - z(\vartheta+a)(\vartheta+a+1)\vartheta\right] \left[\left\langle {}_0^1\lambda_{0,-a}^U \right\rangle_0\right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = a+1, \quad a_3 = 0, \quad b_1 = a, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_{0,-a}^U \right\rangle_n = (\mathfrak{S}_n + a)(\mathfrak{S}_n - 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$\mu(\mu+a)\left[(\mu-1)\mu^n - z(\mu+a+1)(\mu+1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_0^1\Psi_{0,-a}^U \right\rangle_n = \Upsilon_{n+1}(a,-1) - z\Upsilon_n(a,a+1,0) = \Upsilon_n(a+1,0) \circ \left\langle {}_0^1\Psi \right\rangle_n.$$

Remark 30 *We omit the case $\omega_1 = -a, \omega_2 = 0$, since it's identical to case 2.*

Case 3: $\omega_1 = -a, \omega_2 = -a-1$
First moment

$$\left\langle {}_0^1\lambda_{-a,-a-1}^U \right\rangle_0 = \left\langle {}_0^1\lambda \right\rangle_0 + \eta_1 z^{-a} + \eta_2 z^{-a-1} = (1-z)^{-a} + \eta_1 z^{-a} + \eta_2 z^{-a-1}.$$

Linear functional

$$\left\langle {}_0^1L_{-a,-a-1}^U \right\rangle[u] = \sum_{x=0}^{\infty} u(x)\ (a)_x\ \frac{z^x}{x!} + \eta_1 u(-a) z^{-a} + \eta_2 u(-a-1) z^{-a-1}.$$

Using (97) and (133), we obtain

$$\left[\vartheta(\vartheta+a)(\vartheta+a+1) - z(\vartheta+a)(\vartheta+a+1)(\vartheta+a+2)\right]\left[\left\langle {}_0^1\lambda_{-a,-a-1}^U \right\rangle_0\right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = a+1, \quad a_3 = a+2, \quad b_1 = a, \quad b_2 = a+1.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_{-a,-a-1}^U \right\rangle_n = (\mathfrak{S}_n + a)(\mathfrak{S}_n + a + 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$(\mu+a)(\mu+a+1)\left[\mu^{n+1} - z(\mu+a+2)(\mu+1)^n\right] = 0.$$

Modified moments recurrence operator

$$\left\langle {}_0^1\Psi_{-a,-a-1}^U \right\rangle_n = \Upsilon_{n+1}(a,a+1) - z\Upsilon_n(a,a+1,a+2) = \Upsilon_n(a+1,a+2) \circ \left\langle {}_0^1\Psi \right\rangle_n.$$

Reduced Uvarov Truncated Meixner polynomials Since for the Truncated Meixner polynomials we have

$$\left\langle {}_{0}^{1}\sigma^{T}\right\rangle =\vartheta\left(\vartheta-N-1\right), \quad \left\langle {}_{0}^{1}\tau^{T}\right\rangle =\left(\vartheta+a\right)\left(\vartheta-N\right),$$

we will have reduced cases for their Uvarov transformation if

$$\omega=0,N+1,-a,N.$$

i) $\omega=0$

First moment

$$\left\langle {}_{0}^{1}\lambda_{0}^{T,U}\right\rangle_{0}=\left\langle {}_{0}^{1}\lambda^{T}\right\rangle_{0}+\eta.$$

Linear functional

$$\left\langle {}_{0}^{1}L_{0}^{T,U}\right\rangle [u]=\sum_{x=0}^{N}u\left(x\right)\ \left(a\right)_{x}\frac{z^{x}}{x!}+\eta u\left(0\right).$$

Using (96) and (134), we obtain

$$\left[\vartheta\left(\vartheta-1\right)\left(\vartheta-N-1\right)-z\left(\vartheta+a\right)\vartheta\left(\vartheta-N\right)\right]\left[\left\langle {}_{0}^{1}\lambda_{0}^{T,U}\right\rangle_{0}\right]=0,$$

which is a special case of (140) with

$$a_{1}=a,\quad a_{2}=0,\quad a_{3}=-N,\quad b_{1}=-1,\quad b_{2}=-N-1.$$

Standard moments recurrence operator

$$\left\langle {}_{0}^{1}\Phi_{0}^{T,U}\right\rangle_{n}=\left(\mathfrak{S}_{n}-N-1\right)\left(\mathfrak{S}_{n}-1\right)\left[\left\langle {}_{0}^{1}\Phi\right\rangle_{n}\right].$$

Standard moments recurrence

$$\mu\left[\left(\mu-1\right)\left(\mu-N-1\right)\mu^{n}-z\left(\mu-N\right)\left(\mu+a\right)\left(\mu+1\right)^{n}\right]=0.$$

Modified moments recurrence operator

$$\left\langle {}_{0}^{1}\Psi_{0}^{T,U}\right\rangle_{n}=\Upsilon_{n+1}\left(-1,-N-1\right)-z\Upsilon_{n}\left(a,0,-N\right)=\Upsilon_{n}\left(0,-N\right)\circ\left\langle {}_{0}^{1}\Psi\right\rangle_{n}.$$

ii) $\omega=N+1$

First moment

$$\left\langle {}_0^1\lambda_{N+1}^{T,U} \right\rangle_0 = \left\langle {}_0^1\lambda^T \right\rangle_0 + \eta z^{N+1}.$$

Linear functional

$$\left\langle {}_0^1L_{N+1}^{T,U} \right\rangle [u] = \sum_{x=0}^{N} u(x)\ (a)_x \frac{z^x}{x!} + \eta u(N+1)\, z^{N+1}.$$

Using (96) and (134), we obtain

$$[\vartheta(\vartheta - N - 1)(\vartheta - N - 2) - z(\vartheta + a)(\vartheta - N)(\vartheta - N - 1)] \left[\left\langle {}_0^1\lambda_{N+1}^{T,U} \right\rangle_0 \right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = -N, \quad a_3 = -N - 1, \quad b_1 = -N - 1, \quad b_2 = -N - 2.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_{N+1}^{T,U} \right\rangle_n = (\mathfrak{S}_n - N - 1)(\mathfrak{S}_n - N - 2) \left[\left\langle {}_0^1\Phi \right\rangle_n \right].$$

Standard moments recurrence

$$(\mu - N - 1)\left[(\mu - N - 2)\mu^{n+1} - z(\mu - N)(\mu + a)(\mu + 1)^n \right] = 0$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_0^1\Psi_{N+1}^{T,U} \right\rangle_n &= \Upsilon_{n+1}(-N - 1, -N - 2) - z\Upsilon_n(a, -N, -N - 1) \\ &= \Upsilon_n(-N, -N - 1) \circ \left\langle {}_0^1\Psi \right\rangle_n. \end{aligned}$$

iii) $\omega = -a$

First moment

$$\left\langle {}_0^1\lambda_{-a}^{T,U} \right\rangle_0 = \left\langle {}_0^1\lambda^T \right\rangle_0 + \eta z^{-a}.$$

Linear functional

$$\left\langle {}_0^1L_{-a}^{T,U} \right\rangle [u] = \sum_{x=0}^{N} u(x)\ (a)_x \frac{z^x}{x!} + \eta u(-a)\, z^{-a}.$$

Using (97) and (134), we obtain

$$[\vartheta(\vartheta + a)(\vartheta - N - 1) - z(\vartheta + a)(\vartheta + a + 1)(\vartheta - N)] \left[\left\langle {}_0^1\lambda_{-a}^{T,U} \right\rangle_0 \right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = a+1, \quad a_3 = -N, \quad b_1 = a, \quad b_2 = -N-1.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_{-a}^{T,U} \right\rangle_n = (\mathfrak{S}_n + a)(\mathfrak{S}_n - N - 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$(\mu + a)\left[(\mu - N - 1)\mu^{n+1} - z(\mu - N)(\mu + a + 1)(\mu + 1)^n\right] = 0.$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_0^1\Psi_{-a}^{T,U} \right\rangle_n &= \Upsilon_{n+1}(a, -N-1) - z\Upsilon_n(a, a+1, -N) \\ &= \Upsilon_n(a+1, -N) \circ \left\langle {}_0^1\Psi \right\rangle_n. \end{aligned}$$

iv) $\omega = N$

First moment

$$\left\langle {}_0^1\lambda_N^{T,U} \right\rangle_0 = \left\langle {}_0^1\lambda^T \right\rangle_0 + \eta z^N.$$

Linear functional

$$\left\langle {}_0^1L_N^{T,U} \right\rangle[u] = \sum_{x=0}^{N} u(x)\ (a)_x \frac{z^x}{x!} + \eta u(N) z^N.$$

Using (97) and (134), we obtain

$$[\vartheta(\vartheta - N)(\vartheta - N - 1) - z(\vartheta + a)(\vartheta - N + 1)(\vartheta - N)]\left[\left\langle {}_0^1\lambda_N^{T,U} \right\rangle_0\right] = 0,$$

which is a special case of (140) with

$$a_1 = a, \quad a_2 = -N+1, \quad a_3 = -N, \quad b_1 = -N, \quad b_2 = -N-1.$$

Standard moments recurrence operator

$$\left\langle {}_0^1\Phi_N^{T,U} \right\rangle_n = (\mathfrak{S}_n - N)(\mathfrak{S}_n - N - 1)\left[\left\langle {}_0^1\Phi \right\rangle_n\right].$$

Standard moments recurrence

$$(\mu - N)\left[(\mu - N - 1)\mu^{n+1} - z(\mu - N + 1)(\mu + a)(\mu + 1)^n\right] = 0.$$

Modified moments recurrence operator

$$\begin{aligned} \left\langle {}_0^1\Psi_N^{T,U} \right\rangle_n &= \Upsilon_{n+1}(-N, -N-1) - z\Upsilon_n(a, -N+1, -N) \\ &= \Upsilon_n(-N+1, -N) \circ \left\langle {}_0^1\Psi \right\rangle_n. \end{aligned}$$

5 Conclusion

We have studied the families of orthogonal polynomials characterized by the hypergeometric differential equation satisfied by the first moment $\lambda_0(z)$

$$[\vartheta q(\vartheta) - zp(\vartheta)][\lambda_0] = 0, \quad p, q \in \mathbb{K}[x].$$

We obtained recurrence relations for the moments on the monomial and falling factorial polynomial bases, and gave examples for all polynomials of class $s \leq 2$, where $s = \max\{\deg(q), \deg(p) - 1\}$.

We note that one could use the generating function (35) and the ODE it satisfies (36), as a different way of analyzing the standard moments $\mu_n(z)$. Similarly, one could study the modified moments $\nu_n(z)$ using (52) and (53).

We are currently working on further applications of our results to study some properties of the orthogonal polynomials themselves (representations, recurrence-relation coefficients, generating functions, etc).

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