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Comparative asymptotics for discrete semiclassical orthogonal polynomials

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Abstract

We study the ratio $\frac{P_n(x;z)}{\phi_n(x)}$ asymptotically as $n \rightarrow \infty$, where the polynomials $P_n(x; z)$ are orthogonal with respect to a discrete linear functional and $\phi_n(x)$ denote the falling factorial polynomials.

We give recurrences that allow the computation of high order asymptotic expansions of $P_n(x; z)$ and give examples for most discrete semiclassical polynomials of class $s \leq 2$.

We show several plots illustrating the accuracy of our results.

Keywords: Semiclassical orthogonal polynomials, asymptotic expansions, ordinary differential equations.

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1 Introduction

Let \mathbb{K} be a commutative ring (for our purposes, we mostly think of \mathbb{K} as the set of complex numbers \mathbb{C}) and \mathbb{N}_0 be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}.$$

We will denote by $\delta_{k,n}$ the *Kronecker delta*, defined by

$$\delta_{k,n} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}, \quad k, n \in \mathbb{N}_0,$$

and let \mathbb{F} be the ring of *formal power series* in the variable z

$$\mathbb{F} = \mathbb{K}[[z]] = \left\{ \sum_{n=0}^{\infty} c_n z^n : c_n \in \mathbb{K} \right\}.$$

We consider the differential operator $\vartheta : \mathbb{F} \rightarrow \mathbb{F}$ defined by [37, 16.8.2]

$$\vartheta = z\partial_z, \tag{1}$$

where ∂_z is the *derivative operator*

$$\partial_z = \frac{\partial}{\partial z}.$$

The action of ϑ on the monomials is given by

$$\vartheta^k [z^x] = x^k z^x, \tag{2}$$

where we always assume that x and z are **independent variables**.

Suppose that $L : \mathbb{F}[x] \rightarrow \mathbb{F}$ is a *linear functional* (acting on the variable x), and $\{\Lambda_n(x)\}_{n \geq 0} \subset \mathbb{K}[x]$ is a sequence of **monic polynomials** with $\deg(\Lambda_n) = n$. If the system of linear equations

$$L[\Lambda_k \Lambda_n] + \sum_{i=0}^{n-1} L[\Lambda_k \Lambda_i] \xi_{n,i} = 0, \quad 0 \leq k \leq n-1, \tag{3}$$

has a **unique solution** $\{\xi_{n,i}(z)\}_{0 \leq i \leq n-1} \subset \mathbb{F}$, we can define **monic polynomials** $P_n(x; z)$ by $P_0(x; z) = 1$ and

$$P_n(x; z) = \Lambda_n(x) + \sum_{i=0}^{n-1} \xi_{n,i}(z) \Lambda_i(x), \quad n \geq 1. \quad (4)$$

We say that $\{P_n(x; z)\}_{n \geq 0}$ is a sequence of (monic) *orthogonal polynomials* with respect to the functional L , [2], [4], [21], [22], [27], [28], [46].

In this paper, we focus on linear functionals of the form

$$L[u] = \sum_{x=0}^{\infty} u(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad u \in \mathbb{F}[x], \quad (5)$$

and we use the notation

$$\begin{aligned} (\mathbf{a})_n &= \prod_{i=1}^p (a_i)_n, & (\mathbf{b})_n &= \prod_{i=1}^q (b_i)_n, & n &\in \mathbb{N}_0, \\ \mathbf{c} + r &= (c_1 + r, c_2 + r, \dots, c_m + r) \in \mathbb{K}^m, & r &\in \mathbb{K}, \mathbf{c} \in \mathbb{K}^m, \end{aligned}$$

where

$$\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{K}^p, \quad \mathbf{b} = (b_1, \dots, b_q) \in \mathbb{K}^q, \quad p, q \in \mathbb{N}_0, \quad (6)$$

and the *Pochhammer polynomial* $(x)_n$ is defined by $(x)_0 = 1$ and [37, 18:12]

$$(x)_n = \prod_{j=0}^{n-1} (x + j), \quad n \in \mathbb{N}. \quad (7)$$

If $\mu_n(z) \in \mathbb{F}$ denote the *standard moments* of L on the monomial basis

$$\mu_n(z) = L[x^n], \quad n \in \mathbb{N}_0, \quad (8)$$

it follows from (2) and (5) that

$$\mu_{n+1} = \vartheta[\mu_n] = \vartheta^n[\mu_0], \quad n \in \mathbb{N}_0. \quad (9)$$

Moreover, using (5) we can see that [15]

$$L[\sigma(x)u(x)] = L[z\tau(x)u(x+1)], \quad u \in \mathbb{K}[x], \quad (10)$$

where

$$\sigma(x) = x(x + \mathbf{b})_1, \quad \tau(x) = (x + \mathbf{a})_1.$$

Because of (9), we say that the functional L is of *Toda-type* [3], [14], [38], [47], and because of (10) we also call L *discrete semiclassical* [1], [16], [18], [33], [36], [49]. The *class* of the functional L is defined by

$$s = \max \{ \deg(\sigma) - 1, \deg(\tau) - 1 \} = \max \{ p - 1, q \},$$

and semiclassical functional of class $s = 0$ are called *classical*.

Our objective is to obtain *comparative asymptotics* (also called *relative asymptotics*) [5], [23], [24], [25], [29], [30], [31], [32], [34], [39], [40], [41], [42], [43], [44], for the polynomials $P_n(x; z)$ with respect to the basis of *falling factorial polynomials* defined by $\phi_0(x) = 1$ and

$$\phi_n(x) = \prod_{k=0}^{n-1} (x - k), \quad n \in \mathbb{N}. \quad (11)$$

In other words, we want to study the limit

$$\lim_{n \rightarrow \infty} \frac{P_n(x; z)}{\phi_n(x)}, \quad x = O(1), \quad x \notin \mathbb{N}_0,$$

where z is a fixed number, and x belongs to a compact subset of the complex plane containing the origin. We already considered this type of limits in [10], [12] (Charlier and Meixner polynomials), and in [13] (Krawtchouk polynomials).

The organization of the paper is as follows: in Section 2 we review some of our results from [14]. The polynomials $P_n(x; z)$ have different asymptotic approximations depending on the relation between the parameters p and q defined in (6). Thus, we will consider the cases $p = q$ (Section 4.1), $p = q - 1$ (Section 4.2), $p < q - 1$ (Section 4.3), and $p = q + 1$ (Section 4.4). Finally, in the conclusions' section we will summarize the results and discuss future directions.

2 Previous results

In [14], we studied families of polynomials (that we said to be of *Toda type*), orthogonal with respect to a linear functional $L : \mathbb{F}[x] \rightarrow \mathbb{F}$ satisfying

$$D_z L[u] = L[xu], \quad u \in \mathbb{F}[x],$$

where $D_z : \mathbb{F} \rightarrow \mathbb{F}$ is a fixed *derivation* (on the variable z) associated to L .

In this section, we review some of the results that we obtained, and apply them to the particular cases:

- (i) $D_z = \vartheta$, where the operator ϑ was defined in (1).
- (ii) The variable transformation

$$D_w = w(1-w)\partial_w, \quad w = \frac{z}{z-1}.$$

2.1 Toda-type orthogonal polynomials

The linear system (3) can be written as

$$L[\Lambda_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,$$

and we see that the sequence $\{P_n(x; z)\}_{n \geq 0}$ satisfies the *orthogonality conditions*

$$L[P_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n, \quad (12)$$

where $h_n(z) \in \mathbb{F} \setminus \{0\}$ is the *norm* of $P_n(x; z)$.

From (12), we see that

$$L[xP_k P_n] = 0, \quad k \neq n, n \pm 1,$$

and therefore the polynomials $P_n(x; z)$ satisfy the *three term recurrence relation*

$$xP_n(x; z) = P_{n+1}(x; z) + \beta_n(z)P_n(x; z) + \gamma_n(z)P_{n-1}(x; z) \quad (13)$$

with $P_{-1} = 0$, $P_0 = 1$. The coefficients $\beta_n(z), \gamma_n(z) \in \mathbb{F}$ are given by [8]

$$\beta_0 = \frac{L[x]}{L[1]}, \quad \gamma_0 = 0, \quad (14)$$

and

$$\beta_n = \frac{L[xP_n^2]}{h_n}, \quad \gamma_n = \frac{L[xP_n P_{n-1}]}{h_{n-1}}, \quad n \in \mathbb{N}. \quad (15)$$

If we define $\sigma_n(z) \in \mathbb{F}$ by

$$P_n(x; z) = x^n - \sigma_n(z)x^{n-1} + u_n(x; z), \quad \deg(u_n) \leq n-2, \quad (16)$$

we have $\sigma_0 = 0$, and using (13) we get

$$x^{n+1} - \sigma_n x^n + x u_n = x^{n+1} - \sigma_{n+1} x^n + u_{n+1} + \beta_n (x^n - \sigma_n x^{n-1} + u_n) + \gamma_n P_{n-1}.$$

Comparing coefficients of x^n , we obtain $-\sigma_n = -\sigma_{n+1} + \beta_n$, or

$$\beta_n = \sigma_{n+1} - \sigma_n. \quad (17)$$

Our next result relates σ_n, h_n, β_n and γ_n .

Proposition 1 *Let ϑ be defined by (1), h_n be defined by (12), β_n, γ_n be defined by (15), and σ_n be defined by (16). Then, we have*

$$\vartheta [\sigma_n] = \gamma_n \quad (18)$$

and

$$\vartheta [\ln h_n] = \beta_n. \quad (19)$$

Proof. See [14]. ■

As a direct consequence, we see that (β_n, γ_n) are solutions of the *Toda equations* [47].

Corollary 2 *The coefficients of the 3-term recurrence relation (13) are solutions of the differential-difference equations*

$$\vartheta [\beta_n] = \Delta \gamma_n, \quad \vartheta [\ln \gamma_n] = \nabla \beta_n, \quad (20)$$

with initial conditions (14), where

$$\Delta f(n) = f(n+1) - f(n), \quad \nabla f(n) = f(n) - f(n-1). \quad (21)$$

Essential for our work in this paper is the following theorem.

Theorem 3 *The polynomials $P_n(x; z)$ defined by (12) satisfy the recurrence*

$$\vartheta [P_n] = -\gamma_n P_{n-1}, \quad (22)$$

and the ODE

$$[\vartheta^2 + (x - \beta_n) \vartheta + \gamma_n] [P_n] = 0. \quad (23)$$

Proof. See [14]. ■

Since $\vartheta = z\partial_z$, we have

$$z\partial_z P_n = -\gamma_n P_{n-1},$$

and

$$z(z\partial_z^2 P_n + \partial_z P_n) + (x - \beta_n) z\partial_z P_n + \gamma_n P_n = 0. \quad (24)$$

If we define $g_n(z) \in \mathbb{F}$ by

$$\gamma_n(z) = z g_n(z), \quad (25)$$

then

$$P'_n = -g_n P_{n-1}, \quad (26)$$

and (24) becomes

$$zP''_n + (x + 1 - \beta_n) P'_n + g_n P_n = 0, \quad (27)$$

where we will **always** use the notation

$$P'_n = \partial_z P_n.$$

2.2 The function $\sigma_n(z)$

A fundamental quantity in our studies is $\sigma_n(z)$ defined in (16).

Theorem 4 *The coefficients in the power series expansion*

$$\sigma_n(z) = \sum_{k=0}^{\infty} s_k(n) z^k, \quad (28)$$

are given by

$$s_0(n) = \frac{n(n-1)}{2}, \quad s_1(n) = n \frac{(n-1+\mathbf{a})_1}{(n+\mathbf{b})_1}, \quad (29)$$

and

$$s_k(n) = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) s_{k-j}(n) \Delta \nabla [s_j(n)], \quad k \geq 2, \quad (30)$$

Δ, ∇ are the finite difference operators (acting on n) defined in (21).

Proof. See [14]. ■

Using (17) and (18), we obtain the following result.

Corollary 5 *The coefficients of the 3-term recurrence relation (13) admit the formal power series*

$$\beta_n(z) = \sum_{k=0}^{\infty} \Delta s_k(n) z^k, \quad \gamma_n(z) = \sum_{k=1}^{\infty} k s_k(n) z^k, \quad (31)$$

where the coefficients $s_k(n)$ are defined by (28). In particular,

$$\beta_n(0) = n, \quad \gamma_n(0) = 0. \quad (32)$$

Remark 6 *From (25) and (31), we have*

$$g_n(z) = \sum_{k=0}^{\infty} (k+1) s_{k+1}(n) z^k. \quad (33)$$

From (29), we see that

$$s_1(n) = n^{\theta} \frac{(1 - n^{-1} + n^{-1}\mathbf{a})_1}{(1 + n^{-1}\mathbf{b})_1},$$

where

$$\theta = p + 1 - q. \quad (34)$$

If we write

$$s_1(n) = n^{\theta} \sum_{k=0}^{\infty} r_k n^{-k}, \quad (35)$$

we get

$$\sum_{j=0}^k e_{k-j}(\mathbf{b}) r_j = e_k(\mathbf{a} - 1),$$

where the *elementary symmetric polynomials* $e_n(\mathbf{c})$ are defined by the generating function [37, 19.19.4]

$$\sum_{n=0}^{\infty} e_n(\mathbf{c}) t^n = \prod_{i=1}^m (1 + t c_i), \quad \mathbf{c} \in \mathbb{K}^m. \quad (36)$$

Since $e_0 = 1$, we obtain the recurrence

$$r_k = e_k (\mathbf{a} - 1) - \sum_{j=0}^{k-1} e_{k-j} (\mathbf{b}) r_j, \quad r_0 = 1. \quad (37)$$

The first two coefficients r_k are

$$\begin{aligned} r_1 &= e_1 (\mathbf{a} - 1) - e_1 (\mathbf{b}), \\ r_2 &= e_2 (\mathbf{a} - 1) - e_2 (\mathbf{b}) - e_1 (\mathbf{a} - 1) e_1 (\mathbf{b}) + e_1^2 (\mathbf{b}). \end{aligned}$$

Theorem 7 *Let*

$$\Theta_k = (\theta - 2) k + \eta(\theta),$$

with

$$\eta(\theta) = \begin{cases} 0, & \theta = 1 \\ 1, & \theta = 0 \\ 2, & \theta \neq 0, 1 \end{cases}.$$

We have:

(i) *If $\theta < 0$, then*

$$s_k(n) \sim A_k(\theta) n^{\Theta_k}, \quad n \rightarrow \infty, \quad (38)$$

where $A_1 = 1$ and for $k \geq 2$

$$A_k = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) \Theta_j (\Theta_j - 1) A_j A_{k-j}. \quad (39)$$

(ii) *If $\theta = 0$, then as $n \rightarrow \infty$,*

$$s_1(n) \sim 1, \quad s_k(n) \sim r_1 C(k-1) n^{-2k+1}, \quad k \geq 2,$$

where $C(k)$ is the k^{th} Catalan number [37, 26.5(i)]

$$C(k) = \frac{1}{k+1} \binom{2k}{k}.$$

(iii) *If $\theta = 1$, then as $n \rightarrow \infty$,*

$$s_1(n) \sim n, \quad s_k(n) \sim r_2 n^{-k}, \quad k \geq 2.$$

Proof. See [14]. ■

Remark 8 *Using induction, we can see that the solution of (39) is given by*

$$A_k(\theta) = -\theta \frac{(1-\theta)^k}{(k-1)!} (1+k-\theta k)_{k-3}.$$

As a direct application of (30), we can illustrate the results of Theorem 7 for some particular cases.

Example 9 *Let $\theta = 1$. As $n \rightarrow \infty$, we have*

$$\begin{aligned} s_2 &= r_2 n^{-2} + (r_1 r_2 + 3r_3) n^{-3} + O(n^{-4}), \\ s_3 &= r_2 n^{-3} + 3(r_1 r_2 + 2r_3) n^{-4} + O(n^{-5}), \end{aligned}$$

and we see that $s_k(n) \sim r_2 n^{-k}$, $n \geq 2$, as expected. Also,

$$\begin{aligned} \sigma_n(z) &= \frac{n^2}{2} + \left(z - \frac{1}{2}\right) n + r_1 z + r_2 z n^{-1} + (r_3 + r_2 z) z n^{-2} \\ &\quad + [r_4 + (r_1 r_2 + 3r_3) z + r_2 z^2] z n^{-3} + O(n^{-4}), \end{aligned}$$

$$\beta_n(z) = n + z - r_2 z n^{-2} + [(1 - 2z) r_2 - 2r_3] z n^{-3} + O(n^{-4}), \quad (40)$$

and

$$g_n(z) = n + r_1 + r_2 n^{-1} + (2z r_2 + r_3) n^{-2} + O(n^{-3}). \quad (41)$$

Example 10 *Let $\theta = 0$. As $n \rightarrow \infty$, we have*

$$\begin{aligned} s_2 &= r_1 n^{-3} + (r_1^2 + 3r_2) n^{-4} + O(n^{-5}), \\ s_3 &= 2r_1 n^{-5} + 2(3r_1^2 + 5r_2) n^{-6} + O(n^{-7}), \end{aligned}$$

and we see that $s_k(n) \sim C(k-1) r_1 n^{-2k+1}$, $n \geq 2$, as expected. Also,

$$\sigma_n(z) = \frac{n^2}{2} - \frac{1}{2} n + z + r_1 z n^{-1} + r_2 z n^{-2} + (r_1 z + r_3) z n^{-3} + O(n^{-4}),$$

$$\begin{aligned} \beta_n(z) &= n - r_1 z n^{-2} + (r_1 - 2r_2) z n^{-3} \\ &\quad - [r_1(3z + 1) - 3(r_2 - r_3)] z n^{-4} + O(n^{-5}), \end{aligned} \quad (42)$$

and

$$g_n(z) = 1 + r_1 n^{-1} + r_2 n^{-2} + (2z r_1 + r_3) n^{-3} + O(n^{-4}). \quad (43)$$

Example 11 Let $\theta = -1$. As $n \rightarrow \infty$, we have

$$\begin{aligned} s_2 &= n^{-4} + 4r_1n^{-5} + (1 + 3r_1^2 + 7r_2)n^{-6} + O(n^{-7}), \\ s_3 &= 4n^{-7} + 28r_1n^{-8} + (20 + 51r_1^2 + 61r_2)n^{-9} + O(n^{-10}), \end{aligned}$$

and we see that $s_k(n) \sim A(k) r_1n^{-3k+2}$, $n \geq 2$, as expected. Also,

$$\sigma_n(z) = \frac{n^2}{2} - \frac{1}{2}n + zn^{-1} + r_1zn^{-2} + r_2zn^{-3} + (z + r_3)zn^{-4} + O(n^{-5}),$$

$$\beta_n(z) = n - zn^{-2} + (1 - 2r_1)zn^{-3} - [1 + 3(r_2 - r_1)]zn^{-4} + O(n^{-5}), \quad (44)$$

and

$$g_n(z) = n^{-1} + r_1n^{-2} + r_2n^{-3} + (2z + r_3)n^{-4} + O(n^{-5}). \quad (45)$$

2.3 The function $\Phi_n(z; x)$

Sometimes, the falling factorial polynomials $\phi_n(x)$ defined in (11), are called *binomial polynomials*, since we have

$$\frac{\phi_n(x)}{n!} = \binom{x}{n}, \quad n \in \mathbb{N}_0. \quad (46)$$

From the definition (11), we see that

$$\phi_{n+1}(x) = (x - n)\phi_n(x) = x\phi_n(x - 1), \quad n \geq 0, \quad (47)$$

and from (7) it follows that the falling factorial polynomials and the Pochhammer polynomials are related by

$$\phi_n(x) = (-1)^n (-x)_n = (x + 1 - n)_n.$$

Using (32) in (13), we obtain

$$P_{n+1}(x; 0) = (x - n)P_n(x; 0), \quad P_0(x; 0) = 1,$$

and comparing with the recurrence satisfied by the falling factorial polynomials (47), we conclude that

$$P_n(x; 0) = \phi_n(x). \quad (48)$$

Note that from (26) and (48), we see that

$$P'_n(x; 0) = -g_n(0) \phi_{n-1}(x). \quad (49)$$

If we define $\Phi_n(z; x)$ by

$$P_n(x; z) = \phi_n(x) \Phi_n(z; x), \quad (50)$$

then (47) and (49) give the recurrence

$$\Phi'_n(z; x) = -\frac{g_n(z)}{x+1-n} \Phi_{n-1}(z; x). \quad (51)$$

It also follows from (27) and (48) that $\Phi_n(z; x)$ is the solution of the ODE

$$z\Phi''_n + (x+1-\beta_n)\Phi'_n + g_n\Phi_n = 0, \quad (52)$$

with initial condition

$$\Phi_n(0; x) = 1. \quad (53)$$

Note that setting $z = 0$ in (52) and using (32) gives

$$\Phi'_n(0; x) = -\frac{g_n(0)}{x+1-n}$$

in agreement with (51).

Proposition 12 *Suppose that*

$$\Phi_n(z; x) = \sum_{k=0}^{\infty} \frac{\alpha_k(n)}{(x+1-n)_k} \frac{z^k}{k!}, \quad \alpha_0(n) = 1. \quad (54)$$

Then, the coefficients $\alpha_k(n)$ satisfy the recurrence

$$\alpha_{k+1}(n) = -\sum_{j=0}^k s_{j+1}(n) \alpha_{k-j}(n-1) (x+2-n+k-j)_j. \quad (55)$$

In particular,

$$\alpha_1(n) = -s_1(n). \quad (56)$$

Proof. Taking a derivative in (54), we have

$$\Phi'_n(z; x) = \sum_{k=0}^{\infty} \frac{k\alpha_k(n)}{(x+1-n)_k} \frac{z^{k-1}}{k!} = \frac{1}{x+1-n} \sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n)}{(x+2-n)_k} \frac{z^k}{k!},$$

since from (7) we see that

$$(x)_{k+1} = x(x+1)_k.$$

From (51), we conclude that

$$\sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n)}{(x+2-n)_k} \frac{z^k}{k!} = -g_n(z) \sum_{k=0}^{\infty} \frac{\alpha_k(n-1)}{(x+2-n)_k} \frac{z^k}{k!},$$

and using (33), we get

$$\frac{\alpha_{k+1}(n)}{(x+2-n)_k} = -\sum_{j=0}^k s_{j+1}(n) \frac{\alpha_{k-j}(n-1)}{(x+2-n)_{k-j}}. \quad (57)$$

The result follows after using the identity

$$\frac{(x)_n}{(x)_m} = (x+m)_{n-m}, \quad m \leq n.$$

■

Remark 13 Suppose that $\theta < 2$. It follows from (57) that to find the leading term in the asymptotic expansion of $\alpha_k(n)$ as $n \rightarrow \infty$, one needs to consider only the term with $j = 0$. Thus,

$$\alpha_{k+1}(n) \sim -s_1(n) \alpha_k(n-1), \quad n \rightarrow \infty$$

and we conclude that

$$\alpha_k(n) \sim (-1)^k \prod_{j=0}^{k-1} s_1(n-j), \quad n \rightarrow \infty.$$

Using (35), we get

$$\alpha_k(n) = (-1)^k n^{k\theta} \left[1 + k \left(r_1 - \frac{k-1}{2} \theta \right) n^{-1} + O(n^{-2}) \right], \quad n \rightarrow \infty.$$

Example 14 Let $\theta = 1$. As $n \rightarrow \infty$, we have

$$\frac{\alpha_k(n)}{(x+1-n)_k} = 1 + \frac{x+1+r_1}{n}k + O(n^{-2}),$$

and therefore

$$\Phi_n(z; x) = e^z \left[1 + \frac{x+1+r_1}{n}z + O(n^{-2}) \right], \quad n \rightarrow \infty. \quad (58)$$

2.4 The variable w

If we use (30) with $\theta = 2$, we get

$$\begin{aligned} s_1 &= n^2 + r_1n + r_2 + r_3n^{-1} + O(n^{-2}), \\ s_2 &= n^2 + r_1n + r_2 + 2r_3n^{-1} + O(n^{-2}), \\ s_3 &= n^2 + r_1n + r_2 + 3r_3n^{-1} + O(n^{-2}), \end{aligned}$$

and this is clearly **not** an asymptotic sequence. As we showed in [14], what we need is to change variables from z to

$$w = \frac{z}{z-1}. \quad (59)$$

Theorem 15 Let $\sigma_n(z)$ defined by (16). If we write

$$\sigma_n(w) = \sum_{k=0}^{\infty} \xi_k(n) w^k,$$

we have

$$\xi_0(n) = \frac{n(n-1)}{2}, \quad \xi_1(n) = -n \frac{(n-1+\mathbf{a})_1}{(n+\mathbf{b})_1}, \quad (60)$$

and

$$\xi_k = \xi_{k-1} + \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) \xi_{k-j} \nabla \Delta \xi_j, \quad k \geq 2. \quad (61)$$

Proof. See [14]. ■

Remark 16 *If we use (35) in (60), we get*

$$\xi_1(n) = -n^2 \sum_{k=0}^{\infty} r_k n^{-k}, \quad (62)$$

where the coefficients r_k can be computed using (37).

The asymptotic behavior of the coefficients $\xi_k(n)$ is given in the following result.

Theorem 17 *For all $k \geq 2$, we have*

$$\xi_k(n) = O(n^{-k+1}), \quad n \rightarrow \infty. \quad (63)$$

Proof. See [14]. ■

Remark 18 *For the first few $\xi_k(n)$, we can use (61) and (62), and obtain*

$$\begin{aligned} \xi_2(n) &= \frac{r_3}{n} + \frac{r_1 r_3 + 3r_4}{n^2} + O(n^{-3}), \\ \xi_3(n) &= -\frac{r_1 r_3 + 2r_4}{n^2} + O(n^{-3}), \\ \xi_4(n) &= \frac{(1 + r_1^2 + r_2)r_3 + 5(r_1 r_4 + r_5)}{n^3} + O(n^{-4}), \end{aligned} \quad (64)$$

as $n \rightarrow \infty$, in agreement with (63).

Note that we have

$$\gamma_n = z\sigma'_n(z) = w(1-w)\dot{\sigma}_n(w),$$

where we will **always** use the notation

$$\dot{\Phi}_n = \partial_w \Phi_n.$$

Therefore, in this case we define

$$\gamma_n(w) = w(1-w)\mathfrak{g}_n(w), \quad (65)$$

with

$$\mathfrak{g}_n(w) = \sum_{k=0}^{\infty} (k+1)\xi_{n,k+1}w^k.$$

Example 19 Using (62) and (64), we can compute the first terms in the asymptotic expansions of $\sigma_n(w)$, $\beta_n(w)$, and $\mathfrak{g}_n(w)$:

$$\sigma_n(w) = \left(\frac{1}{2} - w\right) n^2 - \left(\frac{1}{2} + r_1 w\right) n - r_2 w + r_3 (w - 1) w n^{-1} + O(n^{-2}),$$

$$\beta_n(w) = (1 - 2w) n - (1 + r_1) w - r_3 (w - 1) w n^{-2} + O(n^{-3}), \quad (66)$$

and

$$\mathfrak{g}_n(w) = -n^2 - r_1 n - r_2 + r_3 (2w - 1) n^{-1} + O(n^{-2}), \quad (67)$$

as $n \rightarrow \infty$.

3 Numerical results

Since we can write the falling factorial polynomials in terms of factorials (46), we can use the reflection formula for the Gamma function [37, 5.5.3]

$$\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)},$$

and obtain

$$\phi_n(x) = \frac{x!}{\Gamma(x + 1 - n)} = \frac{x! \sin[\pi(n - x)]}{\pi} \Gamma(n - x).$$

But

$$\sin(\pi(n - x)) = -\cos(\pi n) \sin(\pi x) = (-1)^{n+1} \sin(\pi x),$$

and therefore

$$\phi_n(x) = (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi} \Gamma(n - x).$$

Thus, in order to plot the different asymptotic approximations for $P_n(x; z)$, we will consider two cases:

i) On the negative real axis, we shall graph

$$\frac{P_n(x; z)}{\Gamma(n - x)} \quad \text{and} \quad (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi} \Phi_n(z; x), \quad (68)$$

since both functions are analytic, nonzero, and bounded in this region.

ii) On the positive real axis (with $x < n$), we shall graph

$$\frac{P_n(x; z)}{x! \Gamma(n-x)} \quad \text{and} \quad (-1)^{n+1} \frac{\sin(\pi x)}{\pi} \Phi_n(z; x), \quad (69)$$

since both functions are analytic and bounded in this region.

To compute the polynomials $P_n(x; z)$, we first compute the moments of L on the monomial basis (8) to a **very** high order of accuracy (with error less than $\varepsilon = 10^{-100}$), solve the system of equations (3)

$$\mu_{n+k} + \sum_{i=0}^{n-1} \mu_{k+i} \xi_{n,i} = 0, \quad 0 \leq k \leq n-1,$$

and construct the polynomials using (4),

$$P_n(x; z) = x^n + \sum_{i=0}^{n-1} \xi_{n,i}(z) x^i.$$

After that, we double-check that

$$|L[x^k P_n]| < \varepsilon, \quad 0 \leq k \leq n-1, \quad |L[x^n P_n]| > \varepsilon.$$

We have tried other methods (using Hankel determinants, recurrences, or the Toda equations and the 3-term recurrence relation), but found them unsatisfactory from a numerical point of view.

4 Asymptotic analysis

In this section, we will obtain asymptotic approximations for $P_n(x; z)$ as $n \rightarrow \infty$, with $x = O(1)$ and all other parameters fixed. Because of the moments' recurrence (9), the analyticity of **all** the moments $\mu_n(z)$ (and in consequence the polynomials P_n themselves) as functions of z will agree with that of the first moment $\mu_0(z)$.

But since $\mu_0(z)$ is a hypergeometric function,

$$\mu_0(z) = {}_pF_q \left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}; z \right) = \sum_{x=0}^{\infty} \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad \mathbf{a} \in \mathbb{K}^p, \mathbf{b} \in \mathbb{K}^q,$$

its domain of analyticity depends on the parameters p, q . We have three cases to consider:

(i) If $p < q + 1$, then $\mu_0(z)$ is an entire function of z . From (34), we see that this corresponds to the case $\theta < 2$.

(ii) If $p = q + 1$ ($\theta = 2$), then $\mu_0(z)$ is analytic inside the unit circle, $|z| < 1$, and can be extended by analytic continuation to the cut plane $\mathbb{C} \setminus [1, \infty)$.

(iii) If $p > q + 1$ ($\theta > 2$), then $\mu_0(z)$ diverges for all $z \neq 0$, except when one of the numerator parameters is a negative integer, and $\mu_0(z)$ becomes a polynomial (in z) of degree N . We will not study this situation in this paper, since in this case we need to scale n in terms of N and consider the limit as $N \rightarrow \infty$ (see [13] for the Krawtchouk polynomials).

We will divide the first case (i) in 3 subcases:

(a) When $p = q$ ($\theta = 1$), $\mu_0(z)$ is entire (but barely!) and the asymptotic expansion of $P_n(x; z)$ will contain an exponential multiple e^z .

(b) When $p = q - 1$ ($\theta = 0$), $P_n(x; z)$ will have a regular asymptotic expansion.

(c) When $p < q - 1$ ($\theta < 0$), some of the first terms in the asymptotic expansion of $P_n(x; z)$ will be missing.

If $p = q + 1$ ($\theta = 2$), then $\mu_0(z)$ will have a logarithmic singularity at $z = 1$. Thus, we expect that the asymptotic expansion of $P_n(x; z)$ will have a factor of the form $(1 - z)^s$, where the power could depend on n (and x). In this case, it is better to perform a change of variables and work with w defined in (59).

Notation 20 *We say that a family of polynomials is of type (p, q) , if it's orthogonal with respect to the functional (5) with $\mathbf{a} \in \mathbb{K}^p$ and $\mathbf{b} \in \mathbb{K}^q$.*

4.1 Case $p = q$ ($\theta = 1$)

From (58), we see that in this case we should "peel off" an exponential term from $\Phi_n(z; x)$. Thus, if

$$\Phi_n(z; x) = e^z \Lambda_n(z; x), \quad (70)$$

we have

$$\Phi'_n = e^z (\Lambda_n + \Lambda'_n), \quad \Phi''_n = e^z (\Lambda_n + 2\Lambda'_n + \Lambda''_n),$$

and (52) becomes

$$z\Lambda_n'' + (2z + x + 1 - \beta_n) \Lambda_n' + (z + x + 1 - \beta_n + g_n) \Lambda_n = 0. \quad (71)$$

From (40) and (41), we see that

$$\beta_n = n + \tilde{\beta}_n, \quad g_n = n + \tilde{g}_n, \quad \tilde{\beta}_n = O(1), \quad \tilde{g}_n = O(1), \quad n \rightarrow \infty,$$

and hence

$$z\Lambda_n'' + (2z + x + 1 - n - \tilde{\beta}_n) \Lambda_n' + (z + x + 1 + \tilde{g}_n - \tilde{\beta}_n) \Lambda_n = 0. \quad (72)$$

Thus, we shall have $\Lambda_n = O(1)$, $n \rightarrow \infty$. Replacing

$$\tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad \tilde{g}_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k},$$

and

$$\Lambda_n(z; x) = \sum_{k=0}^{\infty} \lambda_k(z; x) n^{-k},$$

in (72) and comparing coefficients of n^{-k} , we obtain the recurrence

$$\lambda'_{k+1} = z\lambda_k'' + (2z + x + 1) \lambda_k' + (z + x + 1) \lambda_k + \sum_{j=0}^k [(u_{k-j} - v_{k-j}) \lambda_j - v_{k-j} \lambda_j']. \quad (73)$$

From (53) and (70) we have $\Lambda_n(0; x) = \Phi_n(0; x) = 1$, and therefore

$$\lambda_k(0; x) = \delta_{0,k}, \quad k \geq 0. \quad (74)$$

Note that from (40) and (41) we see that

$$\begin{aligned} u_0 &= r_1, & u_1 &= r_2, & u_2 &= 2zr_2 + r_3, \\ v_0 &= z, & v_1 &= 0, & v_2 &= -r_2z. \end{aligned}$$

When $k = -1$, (73) and (74) give

$$\lambda'_0 = 0, \quad \lambda_0(0; x) = 1,$$

and thus

$$\lambda_0(z; x) = 1. \quad (75)$$

Using (75) in (73), we get

$$\lambda'_1 = z + x + 1 + u_0 - v_0 = x + 1 + r_1,$$

and since $\lambda_1(0; x) = 0$, we obtain

$$\lambda_1(z; x) = (x + 1 + r_1)z. \quad (76)$$

Similarly, using (75) and (76) in (73), we get after some simplification

$$\lambda'_2 = \lambda'_1(x + 1 + z) + \lambda_1\lambda'_1 + r_2,$$

and since $\lambda_2(0; x) = 0$, we conclude that

$$\lambda_2 = \lambda'_1 \left(x + \frac{z}{2} + 1 \right) z + \frac{1}{2} (\lambda_1)^2 + r_2 z,$$

or

$$\lambda_2(z; x) = [(x + 1)(x + 1 + r_1) + r_2]z + (x + 1 + r_1)(x + 2 + r_1) \frac{z^2}{2}. \quad (77)$$

4.1.1 Polynomials of type (0, 0) (Charlier polynomials).

The Charlier polynomials were introduced by Carl Vilhelm Ludwig Charlier (1862–1934) in his paper [7] and have the hypergeometric representation

$$P_n(x; z) = (-z)^n {}_2F_0 \left[\begin{matrix} -n, -x \\ - \\ -\frac{1}{z} \end{matrix} \right].$$

For this family, we have $r_k = 0$, $k \geq 1$, and therefore

$$\beta_n = n + z, \quad g_n = n.$$

Replacing in (71), we get

$$z\Lambda''_n + (z + x + 1 - n)\Lambda'_n + (x + 1)\Lambda_n = 0. \quad (78)$$

Therefore, the recurrence (73) becomes

$$\lambda'_{k+1} = z\lambda''_k + (z + x + 1)\lambda'_k + (x + 1)\lambda_k,$$

or

$$\lambda_{k+1}(z) = z(\lambda'_k + \lambda_k) + x[\lambda_k(z) - \lambda_k(0)] + x \int_0^z \lambda_k(t) dt.$$

Starting with $\lambda_0(z) = 1$, we obtain

$$\begin{aligned} \lambda_1(z) &= (x+1)z, & \lambda_2(z) &= (x+1)^2 z + (x+1)_2 \frac{z^2}{2}, \\ \lambda_3(z) &= (x+1)^3 z + (x+1)_2 (2x+3) \frac{z^2}{2} + (x+1)_3 \frac{z^3}{6}. \end{aligned} \quad (79)$$

However, in this case the ODE satisfied by $\Lambda_n(z; x)$ (78) has the exact solution [12]

$$\Lambda_n(z; x) = {}_1F_1 \left(\begin{matrix} x+1 \\ x+1-n \end{matrix}; -z \right),$$

where we have used the initial value $\Lambda_n(0; x) = 1$. Therefore,

$$\Lambda_n(z; x) = \sum_{k=0}^{\infty} \frac{(x+1)_k}{(x+1-n)_k} \frac{(-z)^k}{k!} \quad (80)$$

and using the first few terms we obtain

$$\begin{aligned} \sum_{k=0}^3 \frac{(x+1)_k}{(x+1-n)_k} \frac{(-z)^k}{k!} &= 1 + \frac{(x+1)z}{n} + \left[(x+1)^2 z + (x+1)_2 \frac{z^2}{2} \right] n^{-2} \\ &+ \left[(x+1)^3 z + (x+1)_2 (2x+3) \frac{z^2}{2} + (x+1)_3 \frac{z^3}{6} \right] n^{-3} + O(n^{-4}) \end{aligned}$$

as $n \rightarrow \infty$, in agreement with (79).

4.1.2 Polynomials of type (1, 1) (generalized Meixner)

For this family, we have

$$\frac{s_1(n)}{n} = \frac{n+a-1}{n+b} = 1 + \frac{a-b-1}{n+b} = 1 + (a-b-1) \sum_{k=1}^{\infty} \frac{(-b)^{k-1}}{n^k},$$

and therefore

$$r_k = (a-b-1)(-b)^{k-1}, \quad k \geq 1. \quad (81)$$

Using (81) in (75)–(77), we get $\lambda_0(z; x) = 1$,

$$\lambda_1(z; x) = (x+a-b)z,$$

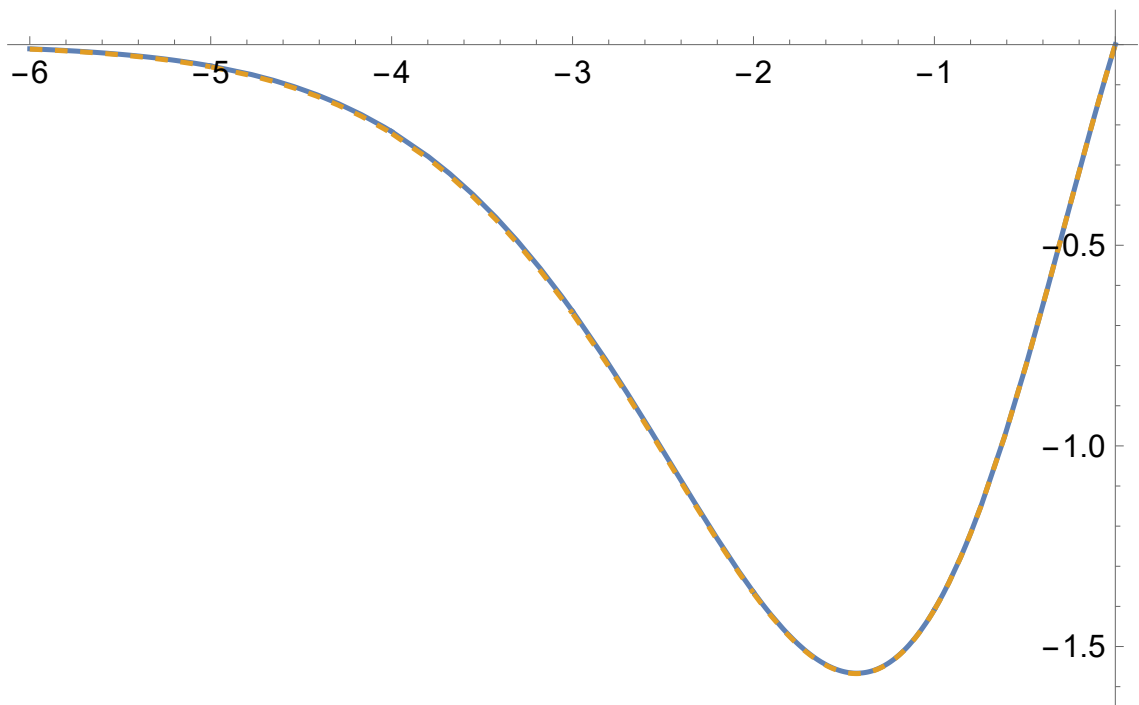


Figure 1: A plot of the scaled generalized Meixner polynomial $P_{10}(x; z)$ and its approximation.

and

$$\lambda_2(z; x) = [(x + a)(x + 1 - b) + b^2]z + (x + a - b + 1)(x + a - b)\frac{z^2}{2}.$$

In Figures 1 and 2, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = e^z \left[1 + \frac{\lambda_1(z; x)}{n} + \frac{\lambda_2(z; x)}{n^2} \right],$$

$n = 10$, $a = 0.2479357$, $b = 0.7146983$, and $z = 0.3974126$.

For additional information on these polynomials, see [6], [9], [15], [16], [17], [19].

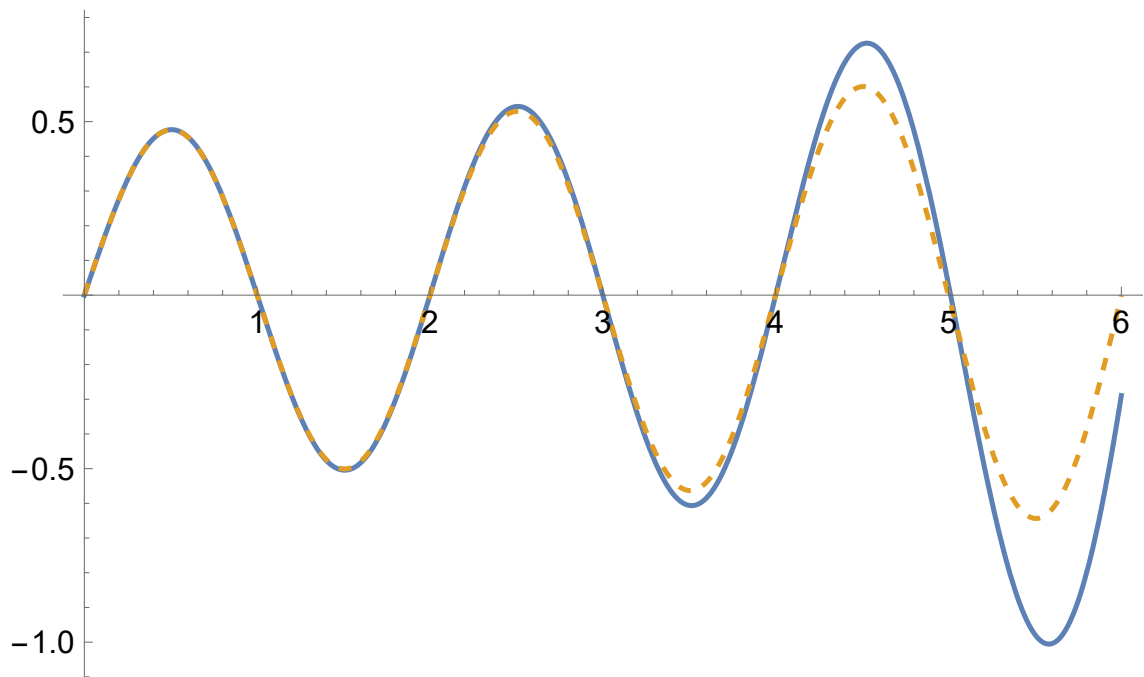


Figure 2: A plot of the scaled generalized Meixner polynomial $P_{10}(x; z)$ and its approximation.

4.1.3 Polynomials of type (2, 2)

For this family, we have

$$\frac{s_1(n)}{n} = \frac{(n+a_1-1)(n+a_2-1)}{(n+b_1)(n+b_2)} = 1 + \frac{(a_1-b_2-1)(a_2-b_2-1)}{(b_1-b_2)(n+b_2)} - \frac{(a_1-b_1-1)(a_2-b_1-1)}{(b_1-b_2)(n+b_1)}$$

and therefore

$$r_k = \frac{\tau_k^{(1)}(b_2) - \tau_k^{(1)}(b_1)}{b_1 - b_2}, \quad k \geq 1, \quad (82)$$

with

$$\tau_k^{(1)}(b) = (b-a_1+1)(b-a_2+1)(-b)^{k-1}.$$

In particular,

$$\begin{aligned} r_1 &= a_1 + a_2 - b_1 - b_2 - 2, \\ r_2 &= 1 - a_1 - a_2 - (a_1 + a_2 - 2)(b_1 + b_2) + b_1^2 + b_2^2 + b_1 b_2 + a_1 a_2. \end{aligned}$$

Using (82) in (75)–(77), we get $\lambda_0(z; x) = 1$,

$$\lambda_1(z; x) = (x + a_1 + a_2 - b_1 - b_2 - 1)z,$$

and

$$\begin{aligned} \lambda_2(z; x) &= [(x+1)(x+a_1+a_2-b_1-b_2-1) + r_2]z \\ &+ (x+a_1+a_2-b_1-b_2-1)(x+a_1+a_2-b_1-b_2)\frac{z^2}{2}. \end{aligned}$$

In Figures 3 and 4, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = e^z \left[1 + \frac{\lambda_1(z; x)}{n} + \frac{\lambda_2(z; x)}{n^2} \right],$$

$n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.

For additional information on these polynomials, see [15] and [17].

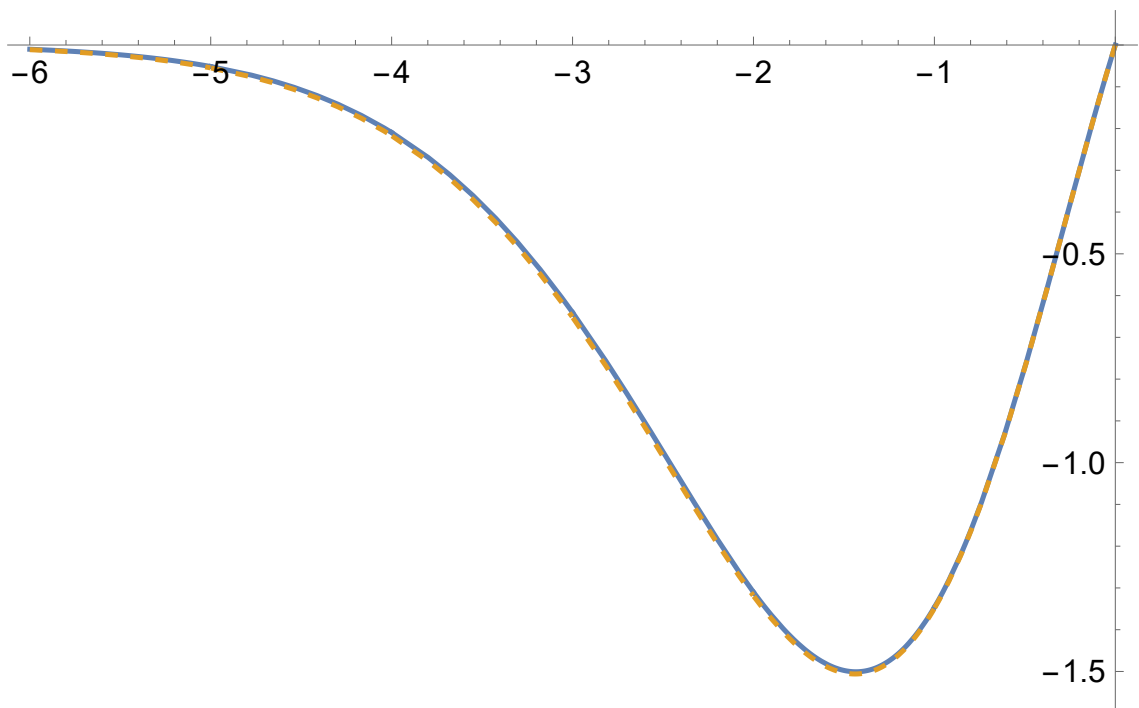


Figure 3: A plot of the scaled polynomial of type (2,2) $P_{10}(x; z)$ and its approximation.

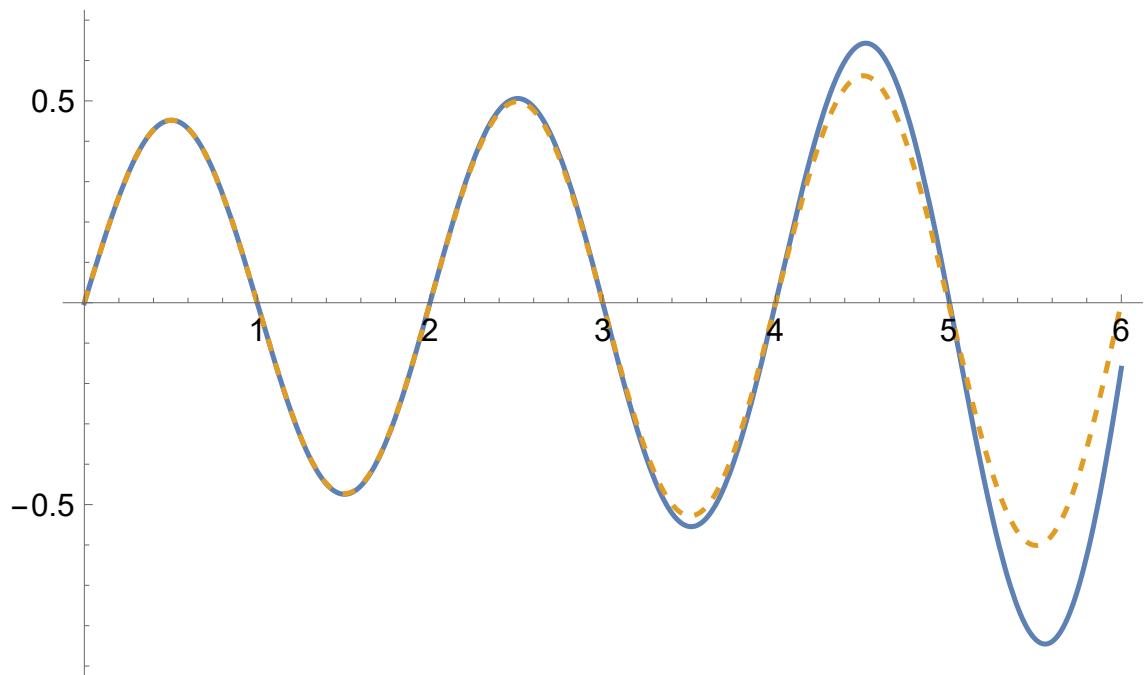


Figure 4: A plot of the scaled polynomial of type (2,2) $P_{10}(x; z)$ and its approximation.

4.2 Case $p = q - 1$ ($\theta = 0$)

From (42) and (43), we see that

$$\beta_n = n + n^{-2}\tilde{\beta}_n, \quad \tilde{\beta}_n = O(1), \quad g_n = O(1), \quad n \rightarrow \infty,$$

and replacing in (52), we get

$$z\Phi_n'' + \left(x + 1 - n - n^{-2}\tilde{\beta}_n\right)\Phi_n' + g_n\Phi_n = 0. \quad (83)$$

Thus, we shall have $\Phi_n = O(1)$, $n \rightarrow \infty$ with $\Phi_n(0; x) = 1$. Replacing

$$\tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k},$$

and

$$\Phi_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x) n^{-k}, \quad \varphi_k(0; x) = \delta_{0,k}, \quad k \geq 0,$$

in (83) and comparing coefficients of n^{-k} , we obtain the recurrence

$$\varphi'_{k+1} = z\varphi''_k + (x+1)\varphi'_k + \sum_{j=0}^k \varphi_j u_{k-j} - \sum_{j=0}^{k-2} \varphi'_j v_{k-2-j}. \quad (84)$$

Replacing $\varphi_0 = 1$ in (84) with $k = 0$, we have

$$\varphi'_1 = u_0 = 1,$$

and therefore

$$\varphi_1(z; x) = z. \quad (85)$$

Using $\varphi_0 = 1, \varphi_1 = z$ in (84) with $k = 1$, we get

$$\varphi'_2 = x + 1 + u_1 + zu_0 = x + 1 + r_1 + z,$$

and hence

$$\varphi_2(z; x) = (x + 1 + r_1)z + \frac{z^2}{2}. \quad (86)$$

Similarly, we have

$$\begin{aligned} \varphi'_3 &= z + (x+1)\varphi'_2 + \varphi_0 u_2 + \varphi_1 u_1 + \varphi_2 u_0 - \varphi'_0 v_0 \\ &= z + (x+1)\varphi'_2 + r_2 + r_1 z + \varphi_2, \end{aligned}$$

and we conclude that

$$\varphi_3(z; x) = [(x+1)(x+1+r_1) + r_2]z + [2(x+1+r_1) + 1]\frac{z^2}{2} + \frac{z^3}{6}. \quad (87)$$

4.2.1 Polynomials of type (0, 1) (generalized Charlier)

For this family, we have

$$s_1(n) = \frac{n}{n+b} = \sum_{k=0}^{\infty} \frac{(-b)^k}{n^k},$$

and therefore

$$r_k = (-b)^k, \quad k \geq 0. \quad (88)$$

Using (88) in (85)–(87), we get

$$\begin{aligned} \Phi_n(z; x) &\sim 1 + \frac{z}{n} + \frac{(x+1-b)z + \frac{z^2}{2}}{n^2} \\ &+ \frac{[(x+1)(x+1-b) + b^2]z + [2(x+1-b) + 1]\frac{z^2}{2} + \frac{z^3}{6}}{n^3} \end{aligned}$$

as $n \rightarrow \infty$.

In Figures 5 and 6, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = 1 + \frac{\varphi_1(z; x)}{n} + \frac{\varphi_2(z; x)}{n^2},$$

$n = 10$, $b = 0.7146983$, and $z = 0.3974126$.

For additional information on these polynomials, see [9], [15], [16], [17], [26], [45], [48].

4.2.2 Polynomials of type (1, 2)

For this family, we have

$$s_1(n) = \frac{n(n+a-1)}{(n+b_1)(n+b_2)} = 1 + \frac{(a-1-b_1)b_1}{(b_1-b_2)(n+b_1)} - \frac{(a-1-b_2)b_2}{(b_1-b_2)(n+b_2)},$$

and therefore

$$r_k = \frac{(b_1+1-a)(-b_1)^k + (a-1-b_2)(-b_2)^k}{b_1-b_2}, \quad k \geq 0.$$

In particular,

$$\begin{aligned} r_0 &= 1, \quad r_1 = a - b_1 - b_2 - 1, \\ r_2 &= (1-a)(b_1+b_2) + b_1^2 + b_2^2 + b_1b_2. \end{aligned} \quad (89)$$

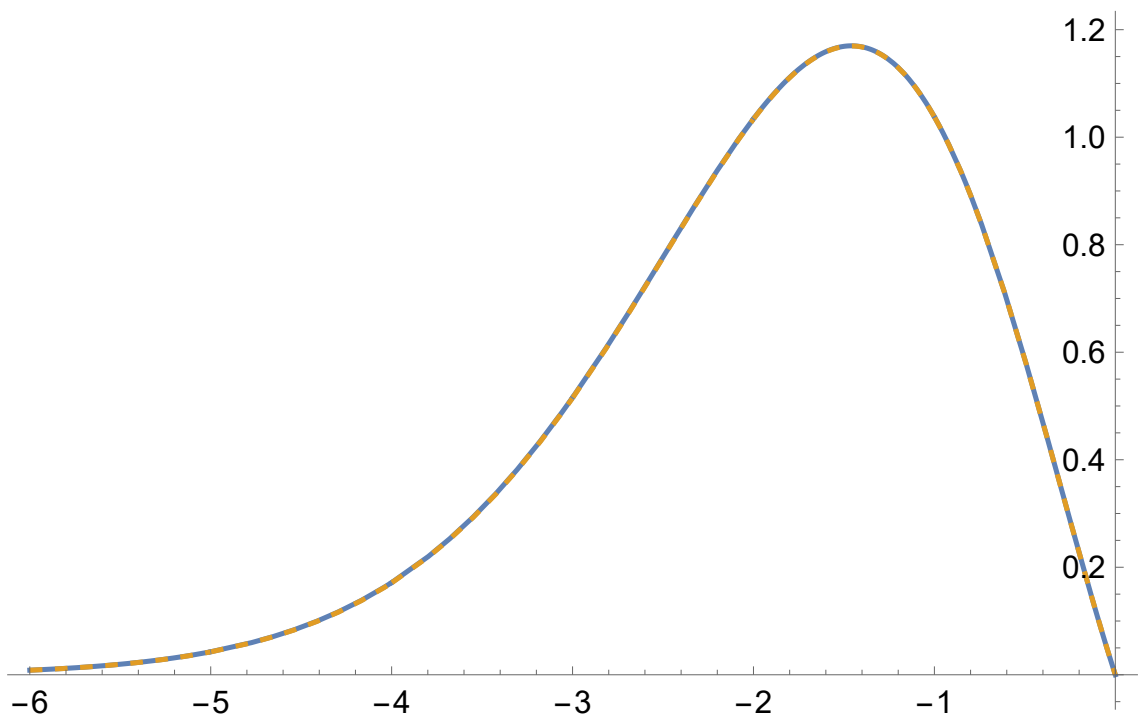


Figure 5: A plot of the scaled generalized Charlier polynomial $P_{10}(x; z)$ and its approximation.

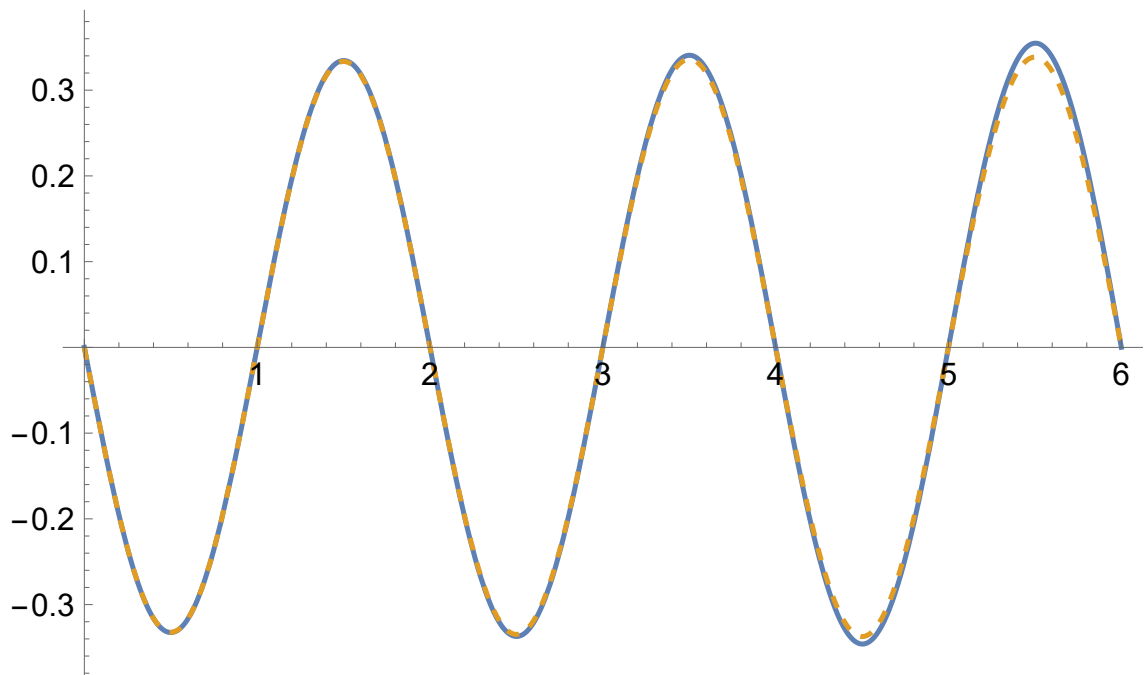


Figure 6: A plot of the scaled generalized Charlier polynomial $P_{10}(x; z)$ and its approximation.

Using (89) in (85)–(87), we get

$$\begin{aligned}\Phi_n(z; x) &= 1 + zn^{-1} + \left[(x + a - b_1 - b_2)z + \frac{z^2}{2} \right] n^{-2} \\ &\quad + [(x + 1)(x + a - b_1 - b_2) + r_2] zn^{-3} \\ &\quad + \left[\left(x + a - b_1 - b_2 + \frac{1}{2} \right) z^2 + \frac{z^3}{6} \right] n^{-3} + O(n^{-4})\end{aligned}$$

as $n \rightarrow \infty$.

In Figures 7 and 8, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = 1 + \frac{\varphi_1(z; x)}{n} + \frac{\varphi_2(z; x)}{n^2},$$

$n = 10$, $a = 0.2479357$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.

For additional information on these polynomials, see [15] and [17].

4.3 Case $p < q - 1$ ($\theta < 0$)

Looking at (44) and (45), suggests that as $n \rightarrow \infty$,

$$\beta_n = n + n^{\theta-1}\tilde{\beta}_n, \quad \tilde{\beta}_n = O(1), \quad g_n = n^\theta\tilde{g}_n, \quad \tilde{g}_n = O(1),$$

and replacing in (52), we get

$$z\Phi_n'' + \left(x + 1 - n - n^{\theta-1}\tilde{\beta}_n \right) \Phi_n' + n^\theta\tilde{g}_n\Phi_n = 0. \quad (90)$$

Thus, we expect that

$$\Phi_n(z; x) = 1 + n^{\theta-1}\tilde{\Phi}_n(z; x), \quad \tilde{\Phi}_n = O(1), \quad n \rightarrow \infty$$

with $\tilde{\Phi}_n(0; x) = 0$, and therefore the ODE (90) becomes

$$zn^{\theta-1}\tilde{\Phi}_n'' + \left(x + 1 - n - n^{\theta-1}\tilde{\beta}_n \right) n^{\theta-1}\tilde{\Phi}_n' + n^\theta\tilde{g}_n + n^{2\theta-1}\tilde{g}_n\tilde{\Phi}_n = 0,$$

or

$$z\tilde{\Phi}_n'' + \left(x + 1 - n - n^{\theta-1}\tilde{\beta}_n \right) \tilde{\Phi}_n' + n\tilde{g}_n + n^\theta\tilde{g}_n\tilde{\Phi}_n = 0. \quad (91)$$

Replacing

$$\tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k},$$

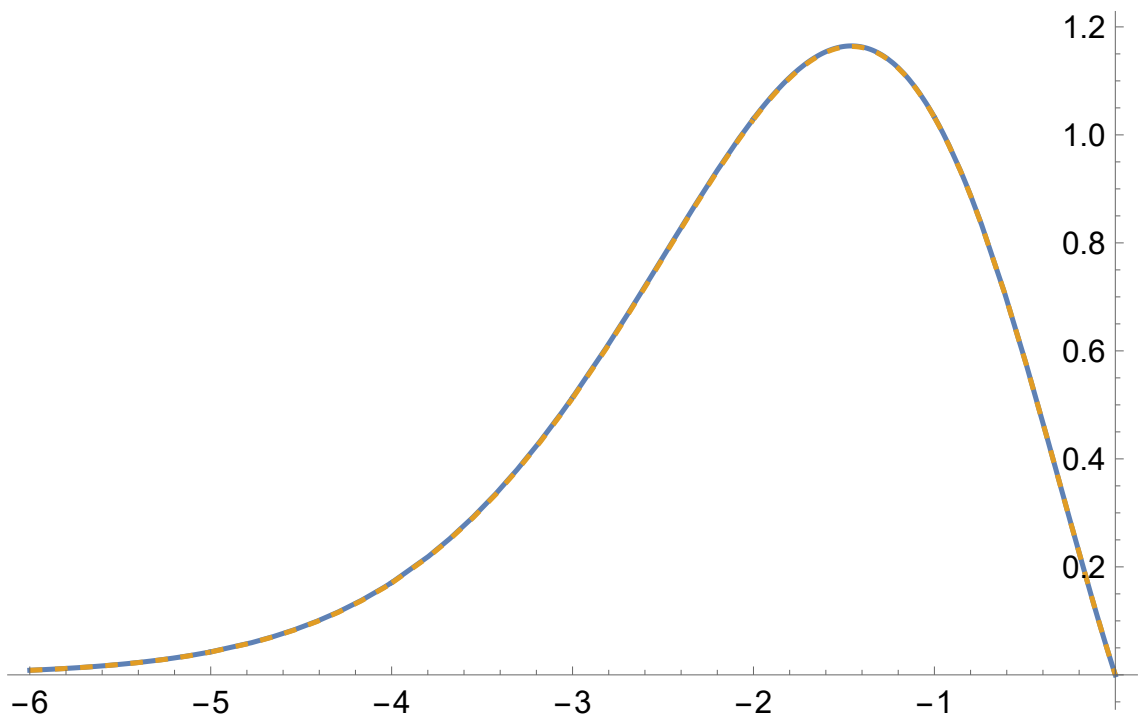


Figure 7: A plot of the scaled polynomial of type (1,2) $P_{10}(x; z)$ and its approximation.

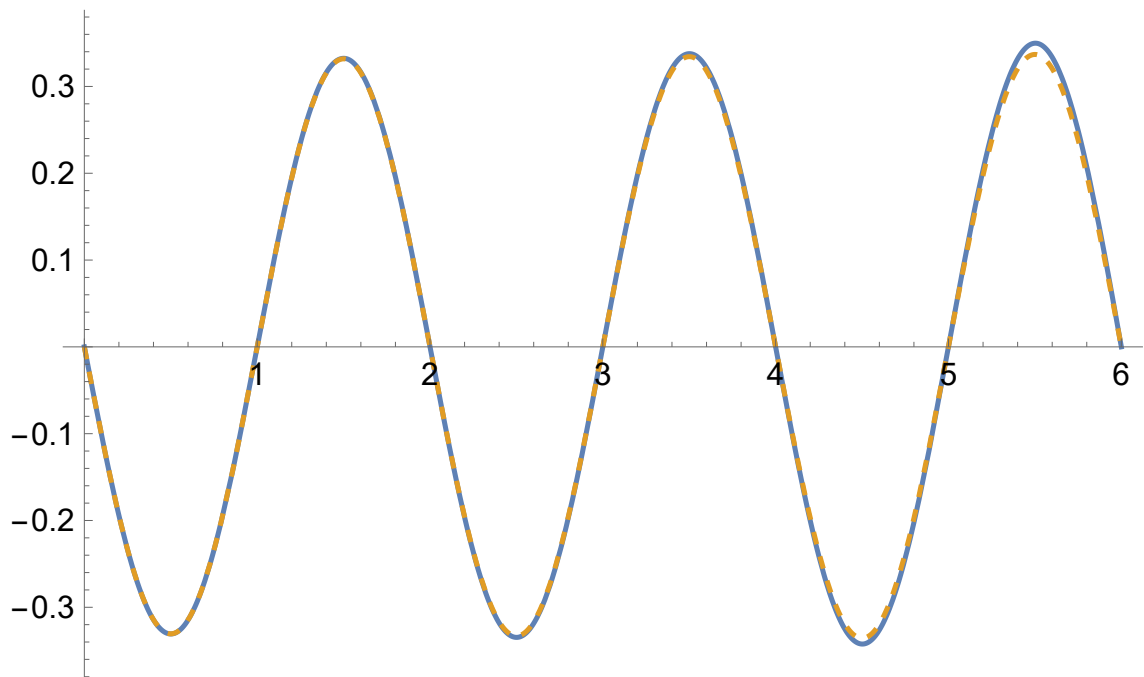


Figure 8: A plot of the scaled polynomial of type (1,2) $P_{10}(x; z)$ and its approximation.

and

$$\tilde{\Phi}_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x) n^{-k}, \quad \varphi_k(0; x) = 0, \quad k \geq 0$$

in (91) and comparing coefficients of n^{-k} , we obtain the recurrence

$$\varphi'_k = u_k + z\varphi''_{k-1} + (x+1)\varphi'_{k-1} + \sum_{j=0}^{k-1+\theta} \varphi_j u_{k-1+\theta-j} - \sum_{j=0}^{k+\theta-2} \varphi'_j v_{k+\theta-2-j}. \quad (92)$$

Setting $k = 0$ in (92), we get

$$\varphi'_0 = u_0 = 1,$$

and therefore

$$\varphi_0(z; x) = z. \quad (93)$$

For $k = 1$, we have

$$\varphi'_1 = u_1 + z\varphi''_0 + (x+1)\varphi'_0 + \sum_{j=0}^{\theta} \varphi_j u_{\theta-j} - \sum_{j=0}^{\theta-1} \varphi'_j v_{\theta-1-j},$$

but since $\theta < 0$ and $\varphi_0 = z$,

$$\varphi'_1 = u_1 + x + 1$$

and hence

$$\varphi_1(z; x) = (x+1+r_1)z. \quad (94)$$

Continuing this way, we see that

$$\varphi'_k = u_k + z\varphi''_{k-1} + (x+1)\varphi'_{k-1}, \quad 1 \leq k < 1 - \theta,$$

and for $k = 1 - \theta$

$$\varphi'_{1-\theta} = u_{1-\theta} + z\varphi''_{-\theta} + (x+1)\varphi'_{-\theta} + \varphi_0 u_0.$$

Thus,

$$\varphi_k(z; x) = \int_0^z u_k(t) dt + z\varphi'_{k-1}(z; x) + x\varphi_{k-1}(z; x), \quad 1 \leq k < 1 - \theta, \quad (95)$$

and

$$\varphi_{1-\theta}(z; x) = \int_0^z u_{1-\theta}(t) dt + z\varphi'_{-\theta}(z; x) + x\varphi_{-\theta}(z; x) + \frac{z^2}{2}. \quad (96)$$

4.3.1 Polynomials of type (0, 2)

For this family, we have

$$\frac{s_1(n)}{n^{-1}} = \frac{n^2}{(n+b_1)(n+b_2)} = 1 + \frac{b_2^2}{(b_1-b_2)(n+b_2)} - \frac{b_1^2}{(b_1-b_2)(n+b_1)},$$

and therefore

$$r_k = \frac{(-b_2)^{k+1} - (-b_1)^{k+1}}{b_1 - b_2}, \quad k \geq 0.$$

In particular,

$$r_0 = 1, \quad r_1 = -(b_1 + b_2), \quad r_2 = b_1 b_2 + b_1^2 + b_2^2. \quad (97)$$

Using (97) in (94) and (96), we get

$$\varphi_1(z; x) = (x + 1 - b_1 - b_2)z,$$

$$\varphi_2 = \int_0^z u_2(t) dt + z\varphi_1' + x\varphi_1 + \frac{z^2}{2} = \int_0^z r_2 dt + (x+1)(x+1-b_1-b_2)z + \frac{z^2}{2},$$

and hence

$$\varphi_2(z; x) = (b_1 b_2 + b_1^2 + b_2^2)z + (x+1)(x+1-b_1-b_2)z + \frac{z^2}{2}.$$

Combining the results above and recalling that $\varphi_0 = z$, we obtain

$$\begin{aligned} \Phi_n(z; x) &= 1 + \frac{z}{n^2} + (x+1-b_1-b_2)zn^{-3} \\ &+ \left[(b_1 b_2 + b_1^2 + b_2^2)z + (x+1)(x+1-b_1-b_2)z + \frac{z^2}{2} \right] n^{-4} + O(n^{-5}). \end{aligned}$$

In Figures 9 and 10, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = 1 + n^{-2} \left[\varphi_0(z; x) + \frac{\varphi_1(z; x)}{n} \right],$$

$n = 10$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.

For additional information on these polynomials, see [15] and [17].

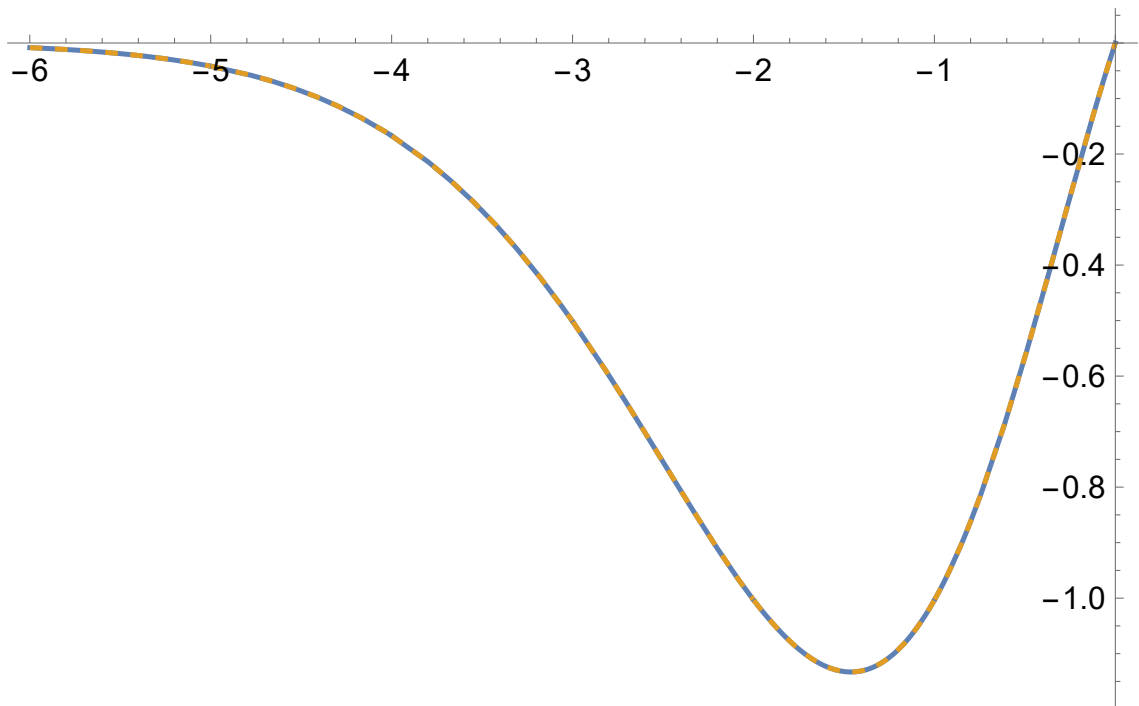


Figure 9: A plot of the scaled polynomial of type (0,2) $P_{10}(x; z)$ and its approximation.

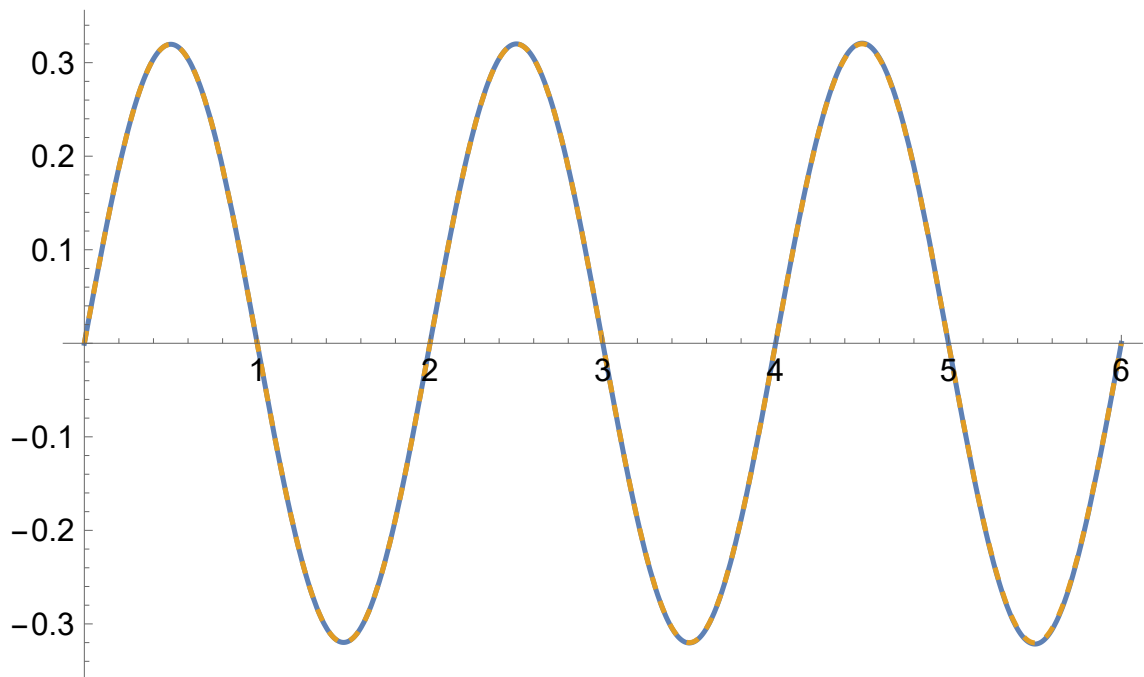


Figure 10: A plot of the scaled polynomial of type (0,2) $P_{10}(x; z)$ and its approximation.

4.4 Case $p = q + 1$ ($\theta = 2$)

Let w be defined by (59). Using

$$\partial_z = -(w-1)^2 \partial_w, \quad \partial_z^2 = (w-1)^4 \partial_w^2 + 2(w-1)^3 \partial_w,$$

in (24), we get

$$w^2(1-w)^2 \partial_w^2 \Phi_n + (x+1-\beta_n-2w)w(1-w) \partial_w \Phi_n + \gamma_n \Phi_n = 0,$$

and from (65) we have

$$w(1-w) \ddot{\Phi}_n + (x+1-\beta_n-2w) \dot{\Phi}_n + \mathfrak{g}_n \Phi_n = 0. \quad (98)$$

Based on the case $\theta = 1$ (Section 4.1), we expect that $\Phi_n(w; x)$ will contain an exponential term. Replacing

$$\Phi_n(w; x) = \exp[\Upsilon_n(w; x)], \quad \Upsilon_n(0; x) = 0,$$

in (98), we obtain

$$w(1-w) \left[\ddot{\Upsilon}_n + \left(\dot{\Upsilon}_n \right)^2 \right] + (x+1-\beta_n-2w) \dot{\Upsilon}_n + \mathfrak{g}_n = 0. \quad (99)$$

From (66)–(67), we have

$$\begin{aligned} \beta_n &= (1-2w)n - (1+r_1)w + \tilde{\beta}_n, & \tilde{\beta}_n &= O(n^{-2}), & n &\rightarrow \infty, \\ \mathfrak{g}_n &= -n^2 - r_1 n + \tilde{\mathfrak{g}}_n, & \tilde{\mathfrak{g}}_n &= O(1), & n &\rightarrow \infty, \end{aligned} \quad (100)$$

and replacing in (99) gives, to leading order,

$$w(1-w) \left(\dot{\Upsilon}_n \right)^2 \sim (1-2w)n \dot{\Upsilon}_n + n^2, \quad n \rightarrow \infty$$

and therefore

$$\dot{\Upsilon}_n \sim \frac{n}{w}, \quad \text{or} \quad \dot{\Upsilon}_n \sim \frac{n}{w-1}, \quad n \rightarrow \infty.$$

Since we want $\Upsilon_n(w; x)$ to be analytic in a neighborhood of $w = 0$, we choose

$$\Upsilon_n(w; x) \sim \ln(1-w)n, \quad n \rightarrow \infty,$$

and set

$$\Upsilon_n(w; x) = \ln(1-w)n + \sum_{k=0}^{\infty} \epsilon_k(w; x) n^{-k}, \quad \epsilon_k(0; x) = 0, \quad k \geq 0, \quad (101)$$

$$\tilde{\beta}_n(w) = \sum_{k=2}^{\infty} v_k(w; x) n^{-k}, \quad \tilde{\mathfrak{g}}_n(w) = \sum_{k=0}^{\infty} u_k(w; x) n^{-k}, \quad (102)$$

where from (66)–(67) we see that

$$v_2 = r_3(1-w)w, \quad u_0 = -r_2, \quad u_1 = r_3(2w-1). \quad (103)$$

Using (101)–(102) in (99) and comparing powers of n , we get

$$\dot{\epsilon}_0 = \frac{x+1+r_1}{w-1}.$$

Thus, since $\epsilon_0(0; x) = 0$,

$$\epsilon_0(w; x) = (x+1+r_1) \ln(1-w).$$

We could proceed in this manner, but instead we consider $\Psi_n(w; x)$ defined by

$$\Phi_n(w; x) = (1-w)^{n+x+1+r_1} \Psi_n(w; x), \quad (104)$$

so that

$$\Psi_n(w; x) = \exp \left[\sum_{k=1}^{\infty} \epsilon_k(w; x) n^{-k} \right] = O(1), \quad n \rightarrow \infty.$$

Using (100) and (104) in (98), we get

$$\begin{aligned} & w(1-w)^2 \ddot{\Psi}_n + (1-w) \left[x+1-w(r_1+2x+3) - \tilde{\beta}_n - n \right] \dot{\Psi}_n \\ & + \left[(n+x+1+r_1)\tilde{\beta}_n + (1-w)(\tilde{\mathfrak{g}}_n - (x+1)(x+1+r_1)) \right] \Psi_n = 0. \end{aligned} \quad (105)$$

Replacing (102) and

$$\Psi_n(w; x) = \sum_{k=0}^{\infty} \psi_k(w; x) n^{-k}, \quad \psi_k(0; x) = \delta_{0,k}, \quad k \geq 0$$

in (105), we obtain the recurrence

$$\begin{aligned} (1-w) \dot{\psi}_{k+1} &= w(1-w)^2 \ddot{\psi}_k + (1-w) [x+1 - (r_1+2x+3)w] \dot{\psi}_k \\ &+ (x+1)(x+1+r_1)(w-1) \psi_k + (1-w) \sum_{j=0}^k \psi_j u_{k-j} \\ &+ \sum_{j=0}^{k-1} \psi_j v_{k+1-j} + \sum_{j=0}^{k-2} \left[(x+1+r_1) \psi_j - \dot{\psi}_j \right] v_{k-j} = 0. \end{aligned} \quad (106)$$

Setting $k = 0$ and $\psi_0 = 1$ in (106), we obtain

$$\dot{\psi}_1 = -(x+1)(x+1+r_1) + u_0,$$

and since $u_0 = -r_2$ and $\psi_1(0; x) = 0$, we conclude that

$$\psi_1(w; x) = -[(x+1)(x+1+r_1) + r_2]w. \quad (107)$$

Replacing $k = 1$ and $\psi_0 = 1$ in (106), we have

$$(1-w)\dot{\psi}_2 = (1-w)[x+1 - (r_1 + 2x + 3)w]\dot{\psi}_1 \\ + (x+1)(x+1+r_1)(w-1)\psi_1 + (1-w)(u_1 + \psi_1 u_0) + v_2,$$

and using (103) and $\psi_1 = w\dot{\psi}_1$, we get

$$(1-w)\dot{\psi}_2 = (1-w)(x+1 - (r_1 + 2x + 3)w)\dot{\psi}_1 \\ + (x+1)(x+1+r_1)(w-1)w\dot{\psi}_1 \\ + (1-w)\left(r_3(2w-1) - r_2w\dot{\psi}_1\right) + r_3(1-w)w,$$

or

$$\dot{\psi}_2 = [x+1 - ((x+2)(x+2+r_1) + r_2)w]\dot{\psi}_1 + r_3(3w-1).$$

Since $\psi_2(0; x) = 0$, we conclude that

$$\psi_2(w; x) = \left[(x+1)w - ((x+2)(x+2+r_1) + r_2)\frac{w^2}{2} \right] \dot{\psi}_1 + \frac{r_3}{2}w(3w-2),$$

and noting from (107) that

$$-[(x+2)(x+2+r_1) + r_2]w = \psi_1(w; x+1),$$

we can write

$$\psi_2(w; x) = \left[x+1 + \frac{1}{2}\psi_1(w; x+1) \right] \psi_1(w; x) + \frac{r_3}{2}w(3w-2). \quad (108)$$

4.4.1 Polynomials of type $(1, 0)$ (Meixner polynomials)

The Meixner polynomials were introduced by Josef Meixner (1908 – 1994) in his paper [35] and have the representation

$$P_n(x; z) = (a)_n \left(1 - \frac{1}{z}\right)^{-n} {}_2F_1 \left[\begin{matrix} -n, -x \\ a \end{matrix}; 1 - \frac{1}{z} \right], \quad z \in \mathbb{C} \setminus [1, \infty).$$

For this family, we have

$$-\frac{\xi_1(n)}{n^2} = \frac{n+a-1}{n},$$

and therefore

$$r_0 = 1, \quad r_1 = a-1, \quad r_k = 0, \quad k \geq 2, \quad (109)$$

and

$$\beta_n(w) = (1-2w)n - aw, \quad \mathfrak{g}_n(w) = -n^2 - (a-1)n. \quad (110)$$

Thus, in this case $\tilde{\beta}_n = \tilde{\mathfrak{g}}_n = 0$, and using (109) in (105), we obtain

$$\begin{aligned} w(1-w)\ddot{\Psi}_n + [x+1 - (2x+2+a)w - n]\dot{\Psi}_n \\ - (x+1)(x+a)\Psi_n = 0, \end{aligned} \quad (111)$$

while the recurrence (106) becomes

$$\dot{\psi}_{k+1} = w(1-w)\ddot{\psi}_k + [x+1 - (2x+2+a)w]\dot{\psi}_k - (x+1)(x+a)\psi_k.$$

It follows that, as $n \rightarrow \infty$,

$$\begin{aligned} \Psi_n(w; x) &\sim 1 - (x+1)(x+a)wn^{-1} \\ &- \left[x+1 - \frac{1}{2}(x+2)(x+1+a)w \right] (x+1)(x+a)wn^{-2}. \end{aligned} \quad (112)$$

However, the ODE (111) can be solved exactly, and we have [12]

$$\Psi_n(w; x) = {}_2F_1 \left(\begin{matrix} x+1, x+a \\ x+1-n \end{matrix}; w \right),$$

and using the first couple of terms, we get

$$\begin{aligned} \Psi_n(w; x) &\sim \sum_{k=0}^2 \frac{(x+1)_k (x+a)_k w^k}{(x+1-n)_k k!} \sim - (x+1)(x+a)wn^{-1} \\ &- (x+1)(x+a)w \left[x+1 - \frac{1}{2}(x+2)(x+1+a)w \right] n^{-2}, \quad n \rightarrow \infty, \end{aligned}$$

in agreement with (112).

4.4.2 Polynomials of type (2, 1) (generalized Hahn polynomials of type I)

For this family, we have

$$\begin{aligned} -\frac{\xi_1(n)}{n^2} &= \frac{(n+a_1-1)(n+a_2-1)}{n(n+b)} \\ &= 1 + \frac{(a_1-1)(a_2-1)}{bn} - \frac{(b+1-a_1)(b+1-a_2)}{b(n+b)}, \end{aligned}$$

and therefore

$$\begin{aligned} r_0 &= 1, \quad r_1 = a_1 + a_2 - 2 - b, \\ r_k &= (b+1-a_1)(b+1-a_2)(-b)^{k-2}, \quad k \geq 2. \end{aligned} \tag{113}$$

Using (113) in (107)–(108), we get

$$\psi_1(w; x) = -[(x+1)(x+a_1+a_2-1-b) + (b-a_1+1)(b-a_2+1)]w$$

and

$$\begin{aligned} \psi_2(w; x) &= \left[x+1 + \frac{1}{2}\psi_1(w; x+1) \right] \psi_1(w; x) \\ &\quad - \frac{1}{2}(b-a_1+1)(b-a_2+1)bw(3w-2). \end{aligned}$$

In Figures 11 and 12, we plot the functions (68)–(69) with

$$\Phi_n(w; x) = (1-w)^{n+x+1+r_1} \left[1 + \frac{\psi_1(w; x)}{n} + \frac{\psi_2(w; x)}{n^2} \right],$$

$n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b = 0.7146983$, $z = -0.01574126$, and $w = 0.0154973$.

For additional information on these polynomials, see [11], [15], [16], [17], [20].

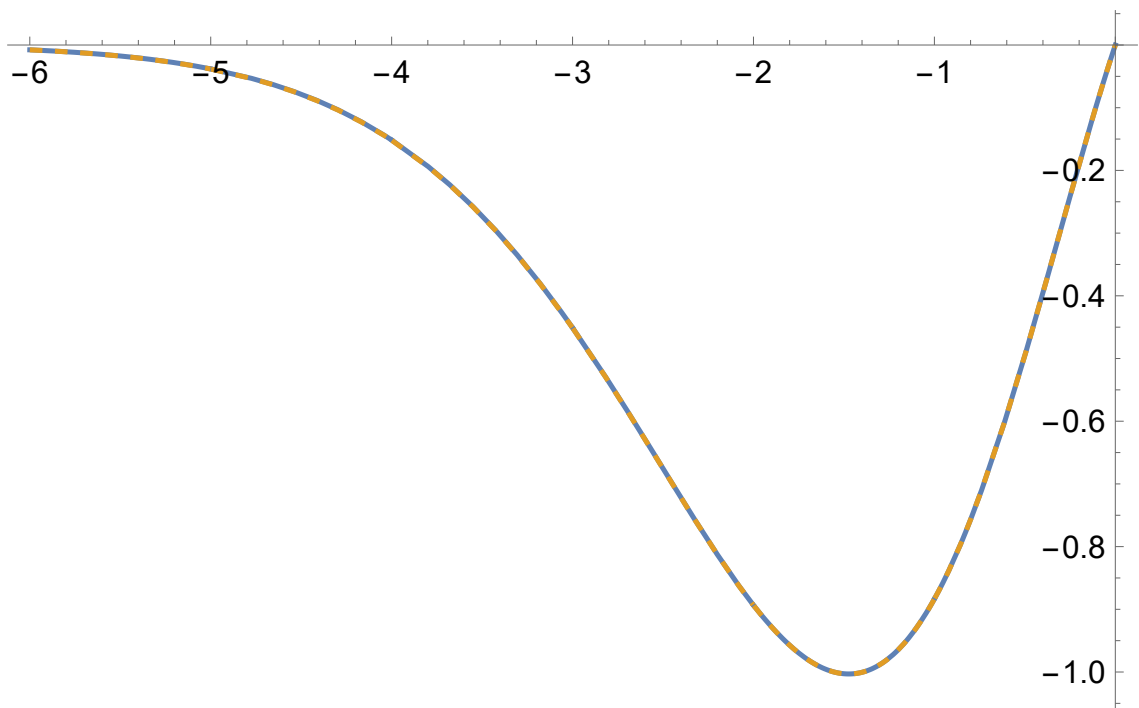


Figure 11: A plot of the scaled generalized Hahn polynomial $P_{10}(x; z)$ and its approximation.

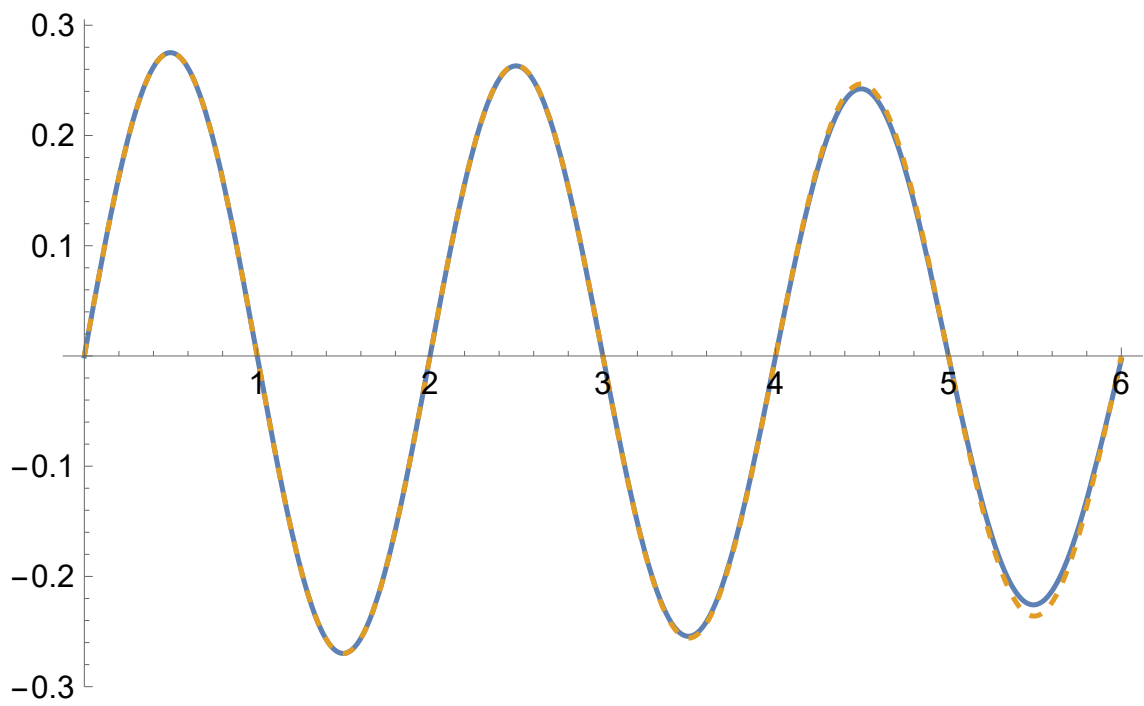


Figure 12: A plot of the scaled generalized Hahn polynomial $P_{10}(x; z)$ and its approximation.

4.4.3 Polynomials of type (3, 2)

For this family, we have

$$\begin{aligned} & -\frac{\xi_1(n)}{n^2} = \frac{(n+a_1-1)(n+a_2-1)(n+a_3-1)}{n(n+b_1)(n+b_2)} \\ = 1 + & \frac{(a_1-1)(a_2-1)(a_3-1)}{b_1 b_2 n} + \frac{(a_1-b_1-1)(a_2-b_1-1)(a_3-b_1-1)}{(b_1-b_2)b_1(n+b_1)} \\ & - \frac{(a_1-b_2-1)(a_2-b_2-1)(a_3-b_2-1)}{(b_1-b_2)b_2(n+b_2)}, \end{aligned}$$

and therefore

$$\begin{aligned} r_0 &= 1, \quad r_1 = a_1 + a_2 + a_3 - 3 - b_1 - b_2, \\ r_k &= \frac{\tau_k^{(2)}(b_1) - \tau_k^{(2)}(b_2)}{b_1 - b_2}, \quad k \geq 2, \end{aligned} \tag{114}$$

where

$$\tau_k^{(2)}(b) = (b - a_1 + 1)(b - a_2 + 1)(b - a_3 + 1)(-b)^{k-2}.$$

At this point, we truly reach the limit of being able to type expressions in a compact way. For the first terms in the asymptotic expansion of these polynomials, we refer to the general formulas (107)–(108) with r_1, r_2 given by (114).

In Figures 13 and 14, we plot the functions (68)–(69) with

$$\Phi_n(w; x) = (1-w)^{n+x+1+r_1} \left[1 + \frac{\psi_1(w; x)}{n} + \frac{\psi_2(w; x)}{n^2} \right],$$

$n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $a_3 = 0.3614782$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, $z = -0.01574126$, and $w = 0.0154973$.

For additional information on these polynomials, see [15] and [17].

5 Conclusions

We have given asymptotic expansions for the ratio

$$\frac{P_n(x; z)}{\phi_n(x)}, \quad x = O(1), \quad x \notin \mathbb{N}_0,$$

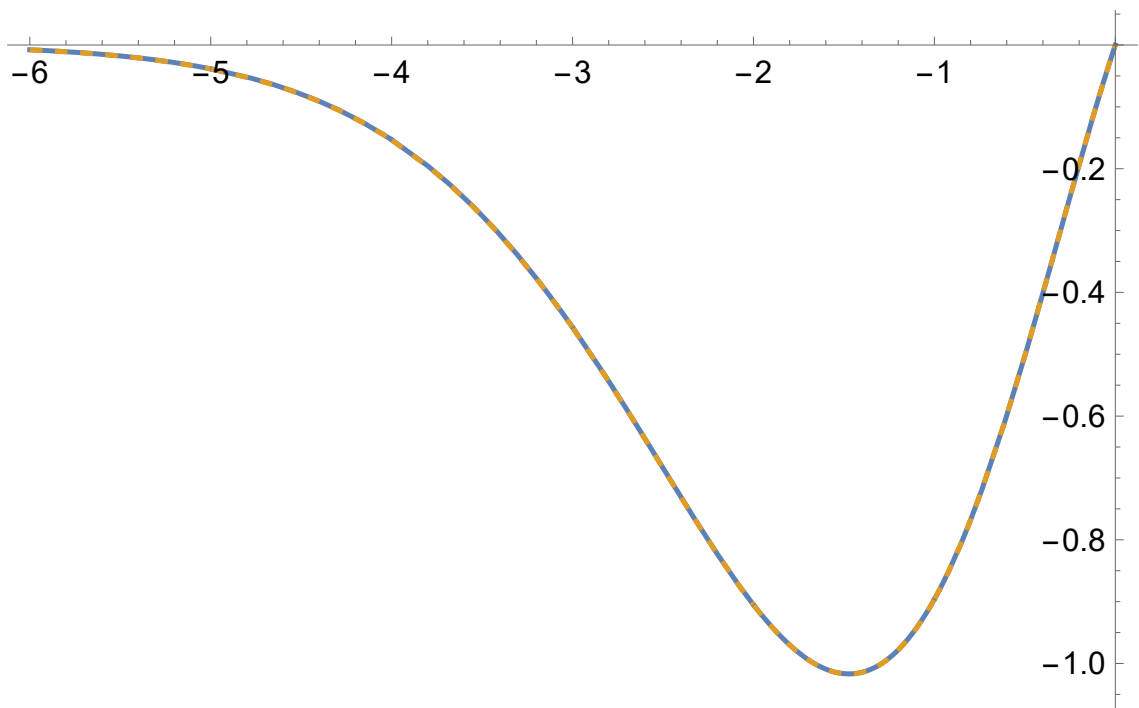


Figure 13: A plot of the scaled polynomial of type (3,2) $P_{10}(x; z)$ and its approximation.

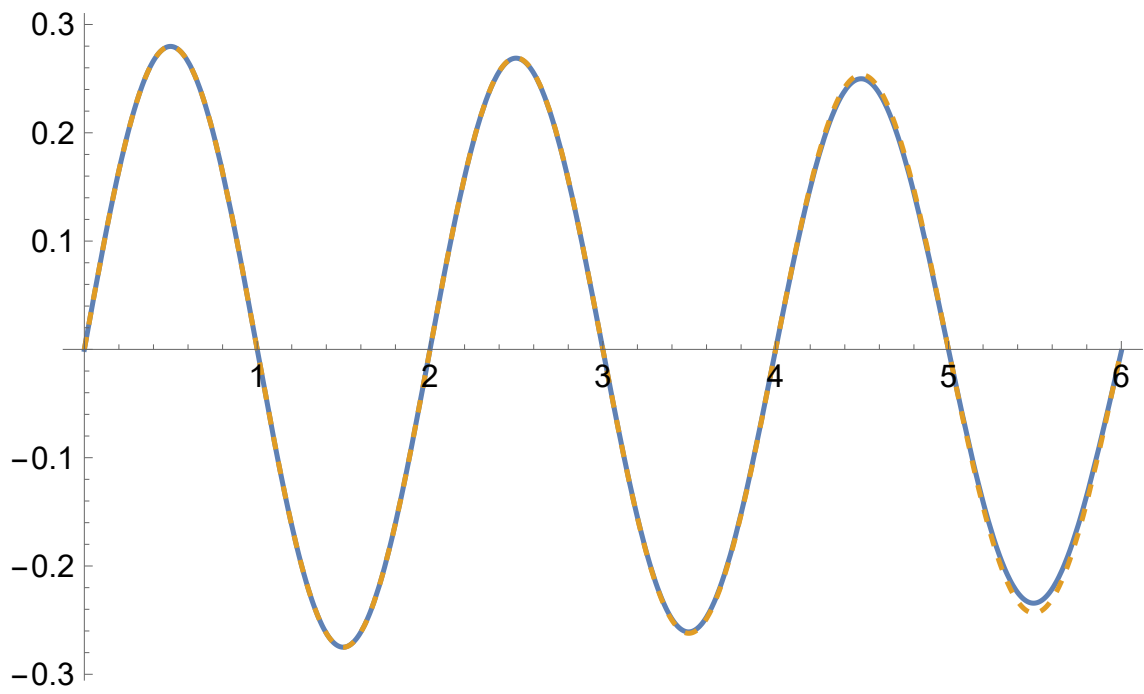


Figure 14: A plot of the scaled polynomial of type (3,2) $P_{10}(x; z)$ and its approximation.

as $n \rightarrow \infty$, where z (and any other parameters) is fixed. The polynomials $P_n(x; z)$ are orthogonal with respect to the linear functional

$$L[u] = \sum_{x=0}^{\infty} u(x) \frac{(\mathbf{a})_x}{(\mathbf{b} + 1)_x} \frac{z^x}{x!}, \quad \mathbf{a} \in \mathbb{K}^p, \mathbf{b} \in \mathbb{K}^q,$$

and depending on the value of the parameter $\theta = p + 1 - q$, we have the following cases:

(i) If $\theta < 1$, then

$$\frac{P_n(x; z)}{\phi_n(x)} = 1 + zn^{\theta-1} \left[1 + \frac{x+1+r_1}{n} + O(n^{-2}) \right], \quad n \rightarrow \infty,$$

where

$$\frac{(1 - n^{-1} + \mathbf{a}n^{-1})_1}{(1 + \mathbf{b}n^{-1})_1} = \sum_{k=0}^{\infty} r_k n^{-k}.$$

(ii) If $\theta = 1$, then as $n \rightarrow \infty$

$$\frac{P_n(x; z)}{\phi_n(x)} = e^z \left[1 + \frac{x+1+r_1}{n} z + O(n^{-2}) \right].$$

This result extends our previous work on the Charlier polynomials, [10], [12].

(iii) If $\theta = 2$, then as $n \rightarrow \infty$

$$\frac{P_n(x; w)}{\phi_n(x)} = (1 - w)^{n+x+1+r_1} \left[1 - \frac{(x+1)(x+1+r_1) + r_2}{n} w + O(n^{-2}) \right],$$

where $w = \frac{z}{z-1}$. This result extends our previous work on the Meixner polynomials, [10], [12].

(iv) If $\theta > 2$, then the polynomials $P_n(x; w)$ depend on a parameter N , with $-N \in \mathbb{N}$. We have not analyzed this case, since it will require scaling N in terms of n . For some related work on the Krawtchouk polynomials, see [13]. We plan to study this case in a forthcoming paper.

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