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Abstract

We define the family of truncated Hermite polynomials $P_n(x; z)$, orthogonal with respect to the linear functional

$$L[p] = \int_{-z}^{z} p(x) e^{-x^2} dx, \quad p \in \mathbb{R}[x], \quad z > 0.$$

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The connection of $P_n(x;z)$ with the Hermite and Rys polynomials is stated. The semiclassical character of $P_n(x;z)$ as polynomials of class 2 is emphasized.

As a consequence, several properties of $P_n(x;z)$ concerning the coefficients $\gamma_n(z)$ in the three-term recurrence relation they satisfy as well as the moments and the Stieltjes function of L are given. Ladder operators associated with the linear functional L, a holonomic differential equation (in x) for the polynomials $P_n(x;z)$, and a nonlinear ODE for the functions $\gamma_n(z)$ are deduced.

MSC class: 33C47 (primary), 33C45, 60E05 (secondary).

1 Introduction

The classical *Hermite polynomials* (Laplace 1810, Chebyshev 1859, Hermite 1864), are orthogonal with respect to the linear functional L defined by [5]

$$L_H[p] = \int_{-\infty}^{\infty} p(x)\, e^{-x^2} dx, \quad p \in \mathbb{R}[x]. \tag{1}$$

They have multiple applications in several areas of mathematics, including Brownian motion, Gaussian quadrature, random matrices, and wavelet series. They also appear in the framework of Sturm-Liouville equations associated with the Schrödinger equation of the harmonic oscillator in $\mathbb{R}$ [18].

Related to them are the *Rys polynomials* (named after John Rys, graduate student of Harry F. King), orthogonal with respect to the linear functional

$$L_R[p] = \int_I p(x)\, e^{-ax^2} dx, \quad p \in \mathbb{R}[x], \quad a > 0. \tag{2}$$

They are usually denoted by $R_n(x;a)$ if $I = [0,1]$ and $J_n(x;a)$ if $I = [-1,1]$. The polynomials $R_n(x;a)$ were introduced in [8] to compute integrals related to electron repulsion in molecular quantum mechanics [2], [16], [17], [28],[30]. [32].

The Rys polynomials have been studied by several authors, mostly from a computational point of view, and mainly related to the implementation of quadrature rules [1], [27],[29]. Their zeros and associated quadrature weights (Christoffel numbers) have been extensively analyzed [9] [14].

A basic point is the study of the moment sequence for the linear functional (2). Indeed, the moments of L_R can be expressed in terms of the incomplete gamma function, but for small values of a they are readily evaluated by a polynomial approximation to a non-alternating power series expansion in a that is valid over a specified range of a. As a next step, the practical evaluation of the zeros is done in terms of a low-order approximation that is valid on finite intervals of a, and by asymptotic expansions for large a. Another interesting point is the analytic relationships between zeros and weights as well as their variation in terms of the parameter a [15].

Quadrature formulas for more general linear functionals

$$\int_{-1}^{1} p(x)\left(1-x^2\right)^{\lambda-\frac{1}{2}} e^{-zx^2}dx, \quad \lambda > -\frac{1}{2}, \quad p \in \mathbb{R}[x],$$

were considered in [6], [11], [24], [25]. By using a transformation of quadrature rules from the interval $(-1,1)$ with N nodes to the interval $(0,1)$ with $\frac{N+1}{2}$ nodes, the method of modified moments [12] allows one to get the coefficients in the corresponding three-term recurrence relation

In this paper, we will study the *truncated Hermite polynomials* $P_n(x;z)$, orthogonal with respect to the linear functional

$$L[p] = \int_{-z}^{z} p(x)\, e^{-x^2}dx, \quad p \in \mathbb{R}[x], \quad z > 0, \tag{3}$$

and satisfying a three-term recurrence relation

$$xP_n(x;z) = P_{n+1}(x;z) + \gamma_n(z)P_{n-1}(x;z), \quad n \geq 0, \tag{4}$$

with $P_{-1} = 0$.

Since the Rys polynomials $J_n(x;z^2)$ satisfy

$$\frac{1}{z}\int_{-z}^{z} J_n\left(\frac{x}{z};z^2\right) J_m\left(\frac{x}{z};z^2\right) e^{-x^2}dx = \|J_n\|^2\, \delta_{n,m},$$

we can see that the polynomials $P_n(x;z)$ are related to the Rys polynomials by

$$P_n(x;z) = z^n J_n\left(\frac{x}{z};z^2\right), \quad h_n(z) = \|P_n\|^2 = z^{2n+1}\|J_n\|^2.$$

Using the relation $\gamma_n(z) = \frac{h_n}{h_{n-1}}$, it follows that the coefficients in the three-term recurrence relations are connected by

$$\gamma_n(z) = z^2 \frac{\|J_n\|^2}{\|J_{n-1}\|^2} = z^2 \gamma_n^{(J)}(z).$$

Nevertheless, our choice of (3) is based on the fact that when $z \to \infty$, we recover the standard Hermite linear functional (1) in a direct way. Our approach has an analytic (rather than numerical) flavor based on the $D-$semiclassical character of the linear functional L. Thus, we can analyze the structure relation (ladder operator) and the second order linear differential (holonomic) equation associated with the corresponding sequence of orthogonal polynomials, and this ODE provides an essential tool for an electrostatic interpretation of their zeros.

The structure of the manuscript is as follows. In Section 2, we present a basic background concerning linear functionals and orthogonal polynomials, with a special emphasis on the symmetric and $D-$semiclassical cases. In Section 3, we study some properties (especially the Pearson equation) that the linear functional (3) satisfies. The behavior of moments and the associated Stieltjes function follows in a natural way. Section 4 is focused on the nonlinear Laguerre-Freud equation that the coefficients of the three-term recurrence relation $\gamma_n(z)$ satisfy. In Section 5, the ladder operators associated with such a linear functional yield a second order linear differential (holonomic) equation for the polynomials $P_n(x;z)$. An electrostatic interpretation of their zeros in terms of an external potential in a such a way they are in an equilibrium state is given in Section 6. Finally, in Section 7, a Toda interpretation of parameters of the three-term recurrence relation as well as of the orthogonal polynomials in terms of the parameter z is discussed. We also find a nonlinear ODE (perhaps related to the Painlevé equations) satisfied by $\gamma_n(z)$

2 Basic background

Let $\mathfrak{L} : \mathbb{R}[x] \to \mathbb{R}$ be a linear functional and let μ_n denote the *moments* of L on the monomial basis

$$\mathfrak{L}[x^n] = \mu_n. \tag{5}$$

A sequence $\{p_n\}_{n\geq 0}$, $\deg(p_n) = n$, is called an *orthogonal polynomial sequence* with respect to $\mathfrak{L}$ if

$$\mathfrak{L}[p_k p_n] = h_n \delta_{k,n}, \quad k,n \in \mathbb{N}_0, \quad h_n \neq 0, \tag{6}$$

where $\delta_{k,n}$ denotes the Kronecker delta. If $h_n = 1$, then $\{p_n\}_{n\geq 0}$ is said to be an *orthonormal polynomial sequence*. Notice that it is unique with the convention that the leading coefficient is a positive real number. In such a case, the linear functional is said to be *quasi-definite*, If $h_n > 0$ for every $n \geq 0$, then the linear functional is said to be *positive definite*.

Let's denote by $\{P_n\}_{n\geq 0}$ the sequence of **monic** polynomials, orthogonal with respect to $\mathfrak{L}$. From (6), we see that

$$\mathfrak{L}[xP_k P_n] = 0, \quad k \neq n, n \pm 1,$$

and therefore the polynomials $P_n(x)$ satisfy the *three-term recurrence relation*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), n \geq 1, \tag{7}$$

with initial values $P_0(x) = 1, \ \ P_1(x) = x - \beta_0$. The coefficients β_n, γ_n are given by [5]

$$\beta_n = \frac{\mathfrak{L}[xP_n^2]}{h_n}, \quad \gamma_n = \frac{\mathfrak{L}[xP_n P_{n-1}]}{h_{n-1}}, \quad n \geq 1, \tag{8}$$

with initial values

$$\beta_0 = \frac{\mu_1}{\mu_0}. \tag{9}$$

Note that using (6) we have

$$h_n = \mathfrak{L}[x^n P_n] = \mathfrak{L}[xP_n P_{n-1}] = \gamma_n h_{n-1}, n \geq 1,$$

and hence

$$\gamma_n = \frac{h_n}{h_{n-1}}, \quad n \geq 1. \tag{10}$$

Definition 1 *A linear functional* $\mathfrak{L}$ *is called* symmetric *if* $\mu_{2n-1} = 0$, *for all* $n \in \mathbb{N}$.

Symmetric functionals can be characterized as follows.

Theorem 2 *Let $\{P_n\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to $\mathfrak{L}$. Then, the following statements are equivalent:*

1. *$\mathfrak{L}$ is symmetric.*

2. *$\beta_n = 0, \quad n \in \mathbb{N}$.*

3. *For all $n \in \mathbb{N}$*
$$P_n(-x) = (-1)^n P_n(x). \tag{11}$$

Proof. See [5, Theorem 4.3]. ■

Note that if $\mathfrak{L}$ is symmetric there exist two sequences of polynomials $\{P_n^e(x)\}_{n\geq 0}$ and $\{P_n^o(x)\}_{n\geq 0}$ such that

$$P_{2n}(x) = P_n^e(x^2), \quad P_{2n+1}(x) = xP_n^o(x^2).$$

The polynomials $P_n^e(x), P_n^o(x)$ are orthogonal with respect to the linear functionals $\mathfrak{U}$, $\mathfrak{V}$ satisfying $\mathfrak{U}[x^n] = \mathfrak{L}[x^{2n}]$ and $\mathfrak{V}[x^n] = \mathfrak{U}[x^{n+1}]$, respectively (see [5, Theorem 8.1]) .

It is clear from (3) that $\mathfrak{L}$ is symmetric and therefore the polynomials $P_n(x;z)$ satisfy the recurrence relation (4) with initial conditions $P_0(x;z) = 1, P_1(x;z) = x$. If we write

$$P_n(x;z) = x^n - c_n(z)\, x^{n-2} + d_n(z)\, x^{n-4} + O\left(x^{n-6}\right), \tag{12}$$

with $c_0 = c_1 = 0$, $d_0 = d_1 = d_2 = d_3 = 0$, then (4) gives

$$\begin{aligned} & x^{n+1} - c_n x^{n-1} + d_n x^{n-3} + O\left(x^{n-5}\right) \\ &= x^{n+1} - c_{n+1} x^{n-1} + d_{n+1} x^{n-3} + O\left(x^{n-5}\right) \\ &+\gamma_n \left(x^{n-1} - c_{n-1} x^{n-3} + d_{n+1} x^{n-5} + O\left(x^{n-7}\right)\right), \end{aligned}$$

and comparing powers of x we get

$$c_n = c_{n+1} - \gamma_n, \quad d_n = d_{n+1} - \gamma_n c_{n-1}. \tag{13}$$

We conclude that

$$c_n(z) = \sum_{k=1}^{n-1} \gamma_k(z), \quad n \geq 2,$$

and

$$d_n(z) = \sum_{k=3}^{n-1} \gamma_k(z)\, c_{k-1}(z), \quad n \geq 4.$$

Reversing (12), we obtain

$$\begin{aligned} x^n &= P_n(x;z) + c_n(z)\, P_{n-2}(x;z) \\ &- [d_n(z) - c_n(z)\, c_{n-2}(z)]\, P_{n-4}(x;z) + O\left(x^{n-6}\right). \end{aligned} \tag{14}$$

Taking the derivative with respect to x in (12), we have

$$\partial_x P_n = n x^{n-1} - (n-2)\, c_n x^{n-3} + (n-4)\, d_n x^{n-5} + O\left(x^{n-7}\right),$$

and using (14) we see that

$$\partial_x P_n = n\left(P_{n-1} + c_{n-1} P_{n-3}\right) - (n-2)\, c_n P_{n-3} + O\left(x^{n-5}\right).$$

Since $c_{n-1} - c_n = -\gamma_{n-1}$, we conclude that

$$\partial_x P_n(x;z) = n P_{n-1}(x;z) + [2c_n(z) - n\gamma_{n-1}]\, P_{n-3}(x;z) + O\left(x^{n-5}\right). \tag{15}$$

An interesting family of linear functionals is the so called $D-$*semiclassical* (with respect to the derivative operator). In such a case, $\mathfrak{L}$ satisfies a first order linear differential equation (*Pearson equation*)

$$\partial_x^* (\phi \mathfrak{L}) + \psi \mathfrak{L} = 0,$$

where $\phi(x)$ is a monic polynomial, $\psi(x)$ is a polynomial of degree at least 1, and the *adjoints* of the derivative and multiplication operators are defined by [10]

$$(\partial_x^* \mathfrak{L})[p] = -\mathfrak{L}[\partial_x p], \quad (x\mathfrak{L})[p] = \mathfrak{L}[xp]. \tag{16}$$

Notice that a semiclassical linear functional satisfies many Pearson equations taking into account the choices of the polynomials ϕ, ψ. The **minimal degree** choice of ϕ, ψ yields the definition of the *class* of a semiclassical linear functional $\mathfrak{L}$ as

$$s = \max\{\deg(\phi) - 2, \deg(\psi) - 1\}.$$

Notice that $D-$*classical* linear functionals (Hermite, Laguerre, Jacobi, Bessel) are semiclassical of class $s = 0$. The description of $D-$semiclassical linear functional of class $s = 1$ has been done in [3]. Characterizations of the $D-$semiclassical orthogonal polynomial sequences were pioneered by Pascal Maroni [20], [21], [22], [23].

3 Truncated Hermite linear functional

In this section we study the truncated Hermite linear functional L defined by (3), which is a $D-$semiclassical functional of class $s = 2$. In order to prove it, we will first find the corresponding Pearson equation. Next, we will deduce a second order linear recurrence equation satisfied by its moments. By using the z-transform of the sequence of moments (Stieltjes function), we will get a first order linear differential equation that the Stieltjes function satisfies.

3.1 Pearson equation

Proposition 3 *Let $p \in \mathbb{R}[x]$ and $\phi(x;z)$, $\psi(x;z)$ be defined by*

$$\phi(x;z) = x^2 - z^2, \quad \psi(x;z) = 2x\phi(x;z). \tag{17}$$

The functional L defined by (3) satisfies the Pearson equation

$$L[\partial_x(\phi p)] = L[\psi p], \tag{18}$$

or equivalently

$$L[\phi \partial_x p] = L[2x(\phi - 1)p]. \tag{19}$$

Proof. Let $p \in \mathbb{R}[x]$. We have

$$L[\partial_x(\phi p)] = \int_{-z}^{z} \partial_x(\phi p) e^{-x^2} dx$$

$$= \left[\phi(x) p(x) e^{-x^2}\right]_{-z}^{z} - \int_{-z}^{z} -2x\phi(x) p(x) e^{-x^2} dx = L[2x\phi p],$$

and we obtain (18). Using the product rule, we see that equation (19) follows immediately from (18). ■

Using the adjoint operators defined by (16), we can write $\phi\partial_x^* L[p] = -L[\partial_x(\phi p)]$, and therefore the Pearson equation (18) has the form

$$(\phi\partial_x^* + \psi) L = 0. \tag{20}$$

Remark 4 *For the linear functional (1) associated with the Hermite polynomials, we have the Pearson equation*

$$\partial_x^* L_H + 2xL_H = 0. \tag{21}$$

Multiplying (21) by $\phi(x,z)$, *we get*

$$0 = \phi\partial_x^* L_H + 2x\phi L_H = (\phi\partial_x^* + \psi) L_H, \tag{22}$$

which yields another Pearson equation satisfied by L_H *but that has no minimal degree. Note that (22) is equivalent to (20).*

3.2 Moments

It follows from the definition of L that the odd moments are zero. Setting $s = x^2$ in (5), we have

$$\mu_{2n}(z) = 2\int_0^z x^{2n} e^{-x^2} dx = \int_0^{z^2} s^{n-\frac{1}{2}} e^{-s} ds = \widehat{\gamma}\left(n + \frac{1}{2}, z^2\right), \tag{23}$$

where the *incomplete gamma function* $\widehat{\gamma}(a,z)$ is defined by [26, 8.2.1]

$$\widehat{\gamma}(a,z) = \int_0^z t^{a-1} e^{-t} dt.$$

Note that we use the nonstandard notation $\widehat{\gamma}(a,z)$ to distinguish the incomplete gamma function from the coefficient $\gamma_n(z)$ in the recurrence relation (4).

The function $\widehat{\gamma}(a,z)$ has the hypergeometric representation [26, 8.5.1]

$$\widehat{\gamma}(a,z) = a^{-1} z^a e^{-z} \; {}_1F_1\left(\begin{array}{c} 1 \\ a+1 \end{array}; z\right), \tag{24}$$

where the (generalized) *hypergeometric function* ${}_pF_q$ is defined by [26, 16.2.1]

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; z\right) = \sum_{x=0}^{\infty} \frac{(a_1)_x \cdots (a_p)_x}{(b_1)_x \cdots (b_q)_x} \frac{z^x}{x!},$$

and the *Pochhammer symbol* is defined by [26, 5.2.4]

$$(c)_n = \prod_{j=0}^{n-1} (c+j), \quad n \in \mathbb{N}, \quad (c)_0 = 1. \tag{25}$$

Using (24) in (23), we get

$$\mu_{2n}(z) = \frac{2}{2n+1} z^{2n+1} e^{-z^2} \, {}_1F_1\left(\begin{matrix} 1 \\ n+\frac{3}{2} \end{matrix} ; z^2 \right), \tag{26}$$

and for $n = 0$ we have

$$\mu_0(z) = 2\int_0^z e^{-x^2} dx = \sqrt{\pi}\, \mathrm{erf}(z), \tag{27}$$

where the $\mathrm{erf}(z)$ denotes the *error function* [26, 7.2.1]. Using the recurrence relation [26, 8.8.1]

$$\widehat{\gamma}(a+1, z) = a\widehat{\gamma}(a, z) - z^a e^{-z},$$

we get

$$\mu_{2n+2}(z) = \left(n + \frac{1}{2}\right) \mu_{2n}(z) - z^{2n+1} e^{-z^2}. \tag{28}$$

In particular,

$$\mu_2(z) = \frac{1}{2}\mu_0(z) - \sqrt{z} e^{-z^2} = \frac{\sqrt{\pi}}{2} \mathrm{erf}(z) - z e^{-z^2}.$$

To obtain a second order homogeneous recurrence equation that the moments satisfy, we can use the Pearson equation (19).

Proposition 5 *Let $u_n(z)$ be defined by*

$$u_n(z) = \mu_{2n}(z). \tag{29}$$

Then, $u_n(z)$ satisfies the recurrence equation

$$2u_{n+2} - \left(2n+3+2z^2\right) u_{n+1} + (2n+1) z^2 u_n = 0, \tag{30}$$

with initial conditions

$$u_0 = \sqrt{\pi}\, \mathrm{erf}(z), \quad u_1 = \frac{\sqrt{\pi}}{2} \mathrm{erf}(z) - z e^{-z^2}.$$

Proof. Using (19) with $p(x) = x^{2n+1}$, we have

$$L\left[(2n+1)\left(x^2 - z^2\right)x^{2n}\right] = L\left[2x\left(x^2 - z^2 - 1\right)x^{2n+1}\right],$$

or

$$(2n+1)\left(\mu_{2n+2} - z^2\mu_{2n}\right) = 2\left[\mu_{2n+4} - \left(z^2+1\right)\mu_{2n+2}\right].$$

■

Remark 6 *From the asymptotic expansion [26, 8.11.2]*

$$\widehat{\gamma}(a,z) \sim \Gamma(a)\left[1 - z^{a-1}e^{-z}\sum_{k\geq 0}\frac{z^{-k}}{\Gamma(a-k)}\right], \quad z \to \infty,$$

in (23), we see that for fixed n

$$u_n(z) \sim \Gamma\left(n+\frac{1}{2}\right)\left[1 - e^{-z^2}\sum_{k\geq 0}\frac{z^{2(n-k)-1}}{\Gamma\left(n+\frac{1}{2}-k\right)}\right], \quad z \to \infty. \tag{31}$$

The even moments of the Hermite polynomials are [26, 5.9.1]

$$\mu_{2n}^{H} = \int_0^\infty s^{n-\frac{1}{2}}e^{-s}ds = \Gamma\left(n+\frac{1}{2}\right),$$

and therefore we can rewrite (31) as the ratio asymptotics

$$\frac{\mu_{2n}(z)}{\mu_{2n}^{H}} \sim 1 - e^{-z^2}\sum_{k\geq 0}\frac{z^{2(n-k)-1}}{\Gamma\left(n+\frac{1}{2}-k\right)}, \quad z \to \infty.$$

3.3 Stieltjes function

The *Stieltjes function* associated with a linear functional $\mathfrak{L}$ is defined by [10], [10], [21]

$$S(t) = \mathfrak{L}\left[\frac{1}{t-x}\right] = \sum_{n\geq 0}\frac{\mu_n}{t^{n+1}}, \tag{32}$$

where the sum is a formal power series.

Proposition 7 *The Stieltjes function $S(t;z)$ associated with the linear functional L satisfies the first order **nonhomogeneous** ODE*

$$\phi(t;z)\,\partial_t S+\psi(t;z)\,S=[2\phi(t;z)-1]\,u_0(z)+2u_1(z)\,, \tag{33}$$

where $\phi(x;z)$ and $\psi(x;z)$ were defined in (17).

Proof. For the linear functional L, the Stieltjes function reads

$$S(t;z)=\sum_{n\geq 0}\frac{u_n(z)}{t^{2n+1}}, \tag{34}$$

and therefore

$$\sum_{n\geq 0}(2n+1)\,\frac{u_n(z)}{t^{2n+1}}=-t\partial_t S, \tag{35}$$

i. e.,

$$\sum_{n\geq 0}\frac{u_{n+k}(z)}{t^{2n+1}}=\sum_{n\geq k}\frac{u_n(z)}{t^{2n-2k+1}}=t^{2k}\left(S-\sum_{n=0}^{k-1}\frac{u_n(z)}{t^{2n+1}}\right). \tag{36}$$

Using (35) and (36), we have

$$\begin{gathered}2t^4\left[S-\left(\frac{u_0(z)}{t}+\frac{u_1(z)}{t^3}\right)\right]+t^2\left(t\partial_t S+\frac{u_0(z)}{t}\right)\\-2z^2t^2\left(S-\frac{u_0(z)}{t}\right)-z^2t\partial_t S=0,\end{gathered}$$

since

$$\sum_{n\geq 0}(2n+3)\,\frac{u_{n+1}(z)}{t^{2n+1}}=t^2\sum_{n\geq 1}(2n+1)\,\frac{u_n(z)}{t^{2n+1}}=t^2\left(-t\partial_t S-\frac{u_0(z)}{t}\right).$$

After simplification, we obtain

$$\left(t^2-z^2\right)(\partial_t S+2tS)=\left(2t^2-2z^2-1\right)u_0(z)+2u_1(z).$$

■

Note that differentiating (33) with respect to t, we get

$$\partial_t\left[\left(t^2-z^2\right)(\partial_t S+2tS)\right]=4tu_0(z),$$

and, therefore,

$$\partial_t \left(\frac{\partial_t \left[\left(t^2 - z^2\right) \left(\partial_t S + 2tS\right) \right]}{t} \right) = 0.$$

Thus, the function $S(t;z)$ satisfies the third order **homogeneous** linear ODE with polynomial coefficients

$$\begin{gathered} t\left(t^2 - z^2\right) \partial_t^3 S + \left(2t^4 - 2t^2 z^2 + 3t^2 + z^2\right) \partial_t^2 S \\ +2t(5t^2 - z^2)\partial_t S + 2(3t^2 + z^2) S = 0. \end{gathered}$$

Remark 8 *For the Hermite polynomials, we have [26, 7.7.2]*

$$S_H(t) = \int_{-\infty}^{\infty} \frac{e^{-x^2}}{t - x} dx = -\mathrm{i}\pi\omega(t), \quad \operatorname{Im}(t) > 0,$$

where $\mathrm{i}^2 = -1$, *and the function* $\omega(t)$ *is defined by [26, 7.2.3]*

$$\omega(t) = e^{-t^2} \left[1 - \operatorname{erf}(-\mathrm{i}t)\right].$$

The function $\omega(t)$ *satisfies [26, 7.10.2]*

$$\omega'(t) = -2t\omega(t) + \frac{2\mathrm{i}}{\sqrt{\pi}},$$

and therefore

$$S_H'(t) = -2tS_H(t) + 2\sqrt{\pi} = -2tS_H(t) + 2\mu_0^H. \tag{37}$$

Comparing (37) with (33), we see that the Stieltjes functions of the functionals L *and* L_H *are related by*

$$0 = \phi(t;z)\left(\partial_t S_H + 2tS_H - 2\mu_0^H\right) = \phi(t;z)\left(\partial_t S + 2tS - 2u_0\right) + u_0 - 2u_1.$$

4 Laguerre-Freud equations

The (generally nonlinear) equations satisfied by the coefficients of the three-term recurrence relation (7) are known in the literature as *Laguerre-Freud equations* (see [4]). They can be consider discrete analogues of the Painlevé equations [19].

First of all, we will find a second order nonlinear difference equation that the parameters of the three term recurrence relation satisfy.

Theorem 9 *The coefficients $\gamma_n(z)$ satisfy the Laguerre-Freud equation*

$$\frac{z^2}{2} = \gamma_n \left(\gamma_{n-1} + \gamma_n - z^2 + \frac{1}{2} - n\right) - \gamma_{n+1} \left(\gamma_{n+1} + \gamma_{n+2} - z^2 - n - \frac{3}{2}\right). \tag{38}$$

Proof. Taking $p(x) = P_n P_{n+1}$ in (18), we get

$$L\left[\partial_x \left(\phi P_n P_{n+1}\right)\right] = L\left[\psi P_n P_{n+1}\right]. \tag{39}$$

The left hand side gives

$$L\left[\partial_x \left(\phi P_n P_{n+1}\right)\right] = 2L\left[x P_n P_{n+1}\right] + L\left[\left(x^2 - z^2\right)\left(P_{n+1}\partial_x P_n + P_n \partial_x P_{n+1}\right)\right]$$

and using (6), we have

$$2L\left[x P_n P_{n+1}\right] = 2h_{n+1}, \quad L\left[\left(x^2 - z^2\right) P_{n+1}\partial_x P_n\right] = n h_{n+1},$$
$$L\left[-z^2 P_n \partial_x P_{n+1}\right] = -z^2 \left(n+1\right) h_n.$$

From (4), we see that

$$x^2 P_n = P_{n+2} + \left(\gamma_{n+1} + \gamma_n\right) P_n + \gamma_{n-1}\gamma_n P_{n-2}, \tag{40}$$

and using (15) we get

$$\partial_x P_{n+1} = \left(n+1\right) P_n + \left[2c_{n+1} - \left(n+1\right)\gamma_n\right] P_{n-2} + O\left(x^{n-4}\right).$$

Hence,

$$L\left[x^2 P_n \partial_x P_{n+1}\right] = \left(n+1\right)\left(\gamma_{n+1} + \gamma_n\right) h_n + \left[2c_{n+1} - \left(n+1\right)\gamma_n\right]\gamma_{n-1}\gamma_n h_{n-2}.$$

Using (10), we conclude that

$$L\left[x^2 P_n \partial_x P_{n+1}\right] = \left(n+1\right)\left(\gamma_{n+1} + \gamma_n\right) h_n + \left[2c_{n+1} - \left(n+1\right)\gamma_n\right] h_n,$$

and therefore

$$L\left[\partial_x \left(\phi P_n P_{n+1}\right)\right] = \left[\left(2n+3\right)\gamma_{n+1} + 2c_{n+1} - \left(n+1\right) z^2\right] h_n.$$

The right hand side in (39) reads

$$L\left[\psi P_n P_{n+1}\right] = L\left[2x(x^2 - z^2) P_n P_{n+1}\right],$$

and since (4) gives

$$\begin{aligned} x^3 P_n &= P_{n+3} + (\gamma_{n+2} + \gamma_{n+1} + \gamma_n) P_{n+1} \\ &+ \gamma_n (\gamma_{n+1} + \gamma_n + \gamma_{n-1}) P_{n-1} + \gamma_n \gamma_{n-1} \gamma_{n-2} P_{n-3}, \end{aligned} \tag{41}$$

we have

$$L[\psi P_n P_{n+1}] = 2(\gamma_{n+2} + \gamma_{n+1} + \gamma_n) h_{n+1} - 2z^2 h_{n+1}$$

or, using (10),

$$L[\psi P_n P_{n+1}] = 2\gamma_{n+1} \left(\gamma_{n+2} + \gamma_{n+1} + \gamma_n - z^2\right) h_n.$$

Thus,

$$(2n+3)\gamma_{n+1} + 2c_{n+1} - (n+1) z^2 = 2\gamma_{n+1} \left(\gamma_{n+2} + \gamma_{n+1} + \gamma_n - z^2\right), \tag{42}$$

and shifting $n \to n-1$,

$$(2n+1)\gamma_n + 2c_n - nz^2 = 2\gamma_n \left(\gamma_{n+1} + \gamma_n + \gamma_{n-1} - z^2\right). \tag{43}$$

Subtracting (43) from (42) and using (13), we obtain

$$(2n+3)\gamma_{n+1} - (2n-1)\gamma_n - z^2 = 2\gamma_{n+1} \left(\gamma_{n+2} + \gamma_{n+1} - z^2\right) - 2\gamma_n \left(\gamma_n + \gamma_{n-1} - z^2\right)$$

and the result follows. ■

Remark 10 *Notice that the nonlinear equation of order* 2 *(38) involves* 4 *consecutive terms of the sequence of parameters* $\gamma_n(z)$.

As an alternative to (38) we will next deduce a third order nonlinear difference equation involving 3 consecutive terms of the sequence of parameters $\gamma_n(z)$. Thus, the computation of them is more accurate.

Theorem 11 *The coefficients* $\gamma_n(z)$ *satisfy the equation*

$$\gamma_n \left(n + \frac{1}{2} - \gamma_n - \gamma_{n+1}\right)\left(n - \frac{1}{2} - \gamma_n - \gamma_{n-1}\right) = z^2 \left(\frac{n}{2} - \gamma_n\right)^2. \tag{44}$$

Proof. From (8), we have

$$h_{n-1}(z)\,\gamma_n(z) = L[xP_nP_{n-1}] = -\frac{1}{2}\left[P_nP_{n-1}e^{-x^2}\right]_{-z}^{z} + \frac{1}{2}L[\partial_x(P_nP_{n-1})],$$

while (11) gives

$$\begin{aligned}\left[P_nP_{n-1}e^{-x^2}\right]_{-z}^{z} &= P_n(z;z)\,P_{n-1}(z;z)\,e^{-z^2} - (-1)^{2n-1}P_n(z;z)\,P_{n-1}(z;z)\,e^{-z^2}\\ &= 2P_n(z;z)\,P_{n-1}(z;z)\,e^{-z^2}.\end{aligned}$$

But using (6), we see that $L[P_n\partial_xP_{n-1}] = 0$, and therefore

$$h_{n-1}(z)\,\gamma_n(z) = -P_n(z;z)\,P_{n-1}(z;z)\,e^{-z^2} + \frac{1}{2}L[\partial_xP_nP_{n-1}]. \tag{45}$$

Since $P_n(x;z) = x^n + O(x^{n-1})$, we have

$$\partial_xP_n(x;z) = nx^{n-1} + O\left(x^{n-2}\right) = nP_{n-1}(x;z) + O\left(x^{n-2}\right), \tag{46}$$

and hence

$$L[P_{n-1}\partial_xP_n] = nh_{n-1}(z). \tag{47}$$

From (45) and (47), we conclude that

$$P_n(z;z)\,P_{n-1}(z;z)\,e^{-z^2} = \left[\frac{n}{2} - \gamma_n(z)\right]h_{n-1}(z). \tag{48}$$

On the other hand, if follows from (4) that

$$x^2P_n^2(x;z) = (P_{n+1}(x;z) + \gamma_nP_{n-1}(x;z))^2 = P_{n+1}^2(x;z) + 2\gamma_nP_{n-1}(x;z)P_{n+1}(x;z) + \gamma_n^2P_{n-1}^2(x;z).$$

Thus (6) and (10) give

$$L\left[x^2P_n^2\right] = h_{n+1}(z) + \gamma_n^2(z)\,h_{n-1}(z) = h_{n+1}(z) + \gamma_n(z)\,h_n(z). \tag{49}$$

But

$$L\left[\partial_x\left(xP_n^2\right)\right] = \left[xP_n^2e^{-x^2}\right]_{-z}^{z} + 2L\left[x^2P_n^2\right] = 2zP_n^2(z;z)\,e^{-z^2} + 2L\left[x^2P_n^2\right],$$

and, as a consequence, we obtain

$$L\left[\partial_x\left(xP_n^2\right)\right] = 2zP_n^2(z;z)\,e^{-z^2} + 2\left[h_{n+1}(z) + \gamma_n^2(z)\,h_{n-1}(z)\right]. \tag{50}$$

On the other hand, since

$$xP_n(x;z)\,\partial_x P_n(x;z) = P_n(x;z)\left[nx^n + O\left(x^{n-1}\right)\right],$$

and using (6) we get

$$L\left[\partial_x\left(xP_n^2\right)\right] = L\left[P_n^2\right] + L\left[2xP_n\partial_x P_n\right] = (2n+1)h_n(z). \tag{51}$$

From (50) and (51), we conclude that

$$P_n^2(z;z)\,e^{-z^2} = \frac{(2n+1)\,h_n(z) - 2\left[h_{n+1}(z) + \gamma_n^2(z)\,h_{n-1}(z)\right]}{2z}. \tag{52}$$

Squaring (48), we have

$$P_n^2(z;z)\,P_{n-1}^2(z;z)\,e^{-2z^2} = \left[\frac{n}{2} - \gamma_n(z)\right]^2 h_{n-1}^2(z),$$

and using (52), we obtain

$$\begin{aligned}&\frac{(2n+1)\,h_n - 2\left(h_{n+1} + \gamma_n^2 h_{n-1}\right)}{2z}\,\frac{(2n-1)\,h_{n-1} - 2\left(h_n + \gamma_{n-1}^2 h_{n-2}\right)}{2z}\\ &= \left(\frac{n}{2} - \gamma_n\right)^2 h_{n-1}^2.\end{aligned}$$

Dividing by $h_{n-1}^2(z)$ and using (10), we get

$$\frac{\gamma_n\left(2n - 2\gamma_n - 2\gamma_{n+1} + 1\right)\left(2n - 2\gamma_n - 2\gamma_{n-1} - 1\right)}{4z^2} = \left(\frac{n}{2} - \gamma_n\right)^2,$$

since

$$\frac{h_{n+1}(z)}{h_n(z)} = \gamma_{n+1}(z), n \geq 0.$$

■

Corollary 12 *Let $g_n(z)$ be defined by*

$$g_n(z) = \frac{n}{2} - \gamma_n(z). \tag{53}$$

Then, $g_n(z)$ satisfies the nonlinear recurrence

$$\left(\frac{n}{2} - g_n\right)\left(g_n + g_{n+1}\right)\left(g_n + g_{n-1}\right) = z^2 g_n^2. \tag{54}$$

Remark 13 *For the Hermite polynomials $H_n(x)$, we have*

$$xH_n(x) = H_{n+1}(x) + \frac{n}{2}H_{n-1}(x),$$

and therefore

$$\gamma_H(n) = \frac{n}{2}.$$

Thus, from (53) we see that $g_n \to 0$ as $z \to \infty$.

To obtain an asymptotic expansion of $g_n(z)$ as $n \to \infty$, we can use the nonlinear recurrence (54).

Theorem 14 *For $z = O(1)$, we have*

$$g_n(z) \sim \frac{n}{2} - \frac{z^2}{4} - \frac{z^2}{16}n^{-2} - \frac{z^4}{16}n^{-3} - \frac{z^2}{64}(1+3z^4)n^{-4} + O(n^{-5}) \tag{55}$$

as $n \to \infty$.

Proof. Replacing

$$g_n(z) = \sum_{k \geq -1} \frac{\xi_k(z)}{n^k}$$

in (54) and comparing coefficients of n, we get

$$2\xi_{-1}^2(1 - 2\xi_{-1}) = 0, \quad O\left(n^3\right),$$

and therefore $\xi_{-1} = 0$, or $\xi_{-1}(z) = \frac{1}{2}$. The solution $\xi_{-1} = 0$ leads to $\xi_k(z) = 0$ for all k, and hence we choose $\xi_{-1}(z) = \frac{1}{2}$. For $O(n^2)$, we get

$$-\left(\xi_0 + \frac{z^2}{4}\right) = 0 \to \xi_0(z) = -\frac{z^2}{4}.$$

The next term, for $O(n)$ gives $\xi_1(z) = 0$, and continuing this way we obtain (55). ■

Corollary 15 *For $z = O(1)$, the recurrence coefficient $\gamma_n(z)$, satisfying the Laguerre-Freud equation (44) has the asymptotic expansion*

$$\gamma_n(z) \sim \frac{z^2}{4} + \frac{z^2}{16}n^{-2} + \frac{z^4}{16}n^{-3} + \frac{z^2}{64}(1+3z^4)n^{-4} + O(n^{-5}), \quad n \to \infty. \tag{56}$$

With the previous result, we can get a first estimate for the asymptotic behavior of $P_n(x;z)$ as $n \to \infty$.

Proposition 16 *For $x, z = O(1)$, the polynomials $P_n(x;z)$ satisfy*

$$P_n(x;z) \sim \Phi_+^n(x;z) + \Phi_-^n(x;z), \quad n \to \infty, \tag{57}$$

where

$$\Phi_\pm(x;z) = \frac{x \pm \sqrt{x^2 - z^2}}{2}.$$

Proof. Using (56) in (4), we see that [31]

$$\lim_{n\to\infty} [P_n(x;z)]^{\frac{1}{n}} = \Phi(x;z),$$

where $\Phi(x;z)$ is a solution of the quadratic equation

$$x = \Phi + \frac{z^2}{4}\frac{1}{\Phi}. \tag{58}$$

Thus,

$$\Phi(x;z) = \frac{x \pm \sqrt{x^2 - z^2}}{2},$$

and the result follows. ■

Note that for $x \in (-z, z)$

$$\Phi_\pm(x;z) = \frac{x \pm \mathrm{i}\sqrt{z^2 - x^2}}{2},$$

and therefore setting $x = z\cos(\theta)$ we have

$$P_n(x;z) \sim 2\left(\frac{z}{2}\right)^n \cos(n\theta), \quad n \to \infty.$$

Since the *Chebyshev polynomials of the first kind* are defined by [26, 18.5.1]

$$T_n(\cos(\theta)) = \cos(n\theta),$$

we see that

$$P_n(x;z) \sim 2\left(\frac{z}{2}\right)^n T_n\left(\frac{x}{z}\right), \qquad n \to \infty.$$

5 Structure relation and differential equation

Semiclassical polynomials (with respect to the derivative operator) are *holonomic functions* [13], meaning that they are solutions of a linear ODE with polynomial coefficients. In this section, we shall find the differential equation (in x) satisfied by $P_n(x;z)$.

5.1 Structure relation

One of the basic properties of semiclassical polynomials with respect to the derivative operator, is a relation between $\partial_x P_n$ and P_n [10]. For the polynomials $P_n(x;z)$, we have the following result.

Theorem 17 *The polynomials $P_n(x;z)$ satisfy the differential-recurrence relation*

$$\phi(x;z)\,\partial_x P_{n+1} = (n+1)\,P_{n+2} + \lambda_n P_n + \tau_n P_{n-2}, \tag{59}$$

where

$$\lambda_n(z) = \left[2\left(\gamma_n + \gamma_{n+1} + \gamma_{n+2} - z^2 - 1\right) - n\right]\gamma_{n+1}, \tag{60}$$

and

$$\tau_n(z) = 2\gamma_{n+1}\gamma_n\gamma_{n-1}. \tag{61}$$

Proof. If

$$\phi(x;z)\,\partial_x P_{n+1} = \sum_{k=0}^{n+2} d_{n,k}(z)\,P_k, \tag{62}$$

then using (6) and (11), we have

$$h_k d_{n,k} = L\left[\phi\partial_x P_{n+1} P_k\right] = 0, \quad n+1 \equiv k \mod(2), \tag{63}$$

since L is symmetric and ϕ is an even polynomial of x. Using (19), we get

$$\begin{aligned} h_k d_{n,k} &= L\left[\phi\partial_x\left(P_{n+1}P_k\right)\right] - L\left[\phi P_{n+1}\partial_x P_k\right] \\ &= L\left[2x\left(\phi-1\right)P_{n+1}P_k\right] - L\left[\phi P_{n+1}\partial_x P_k\right]. \end{aligned}$$

Since the polynomials P_n are orthogonal, we conclude that $d_{n,k} = 0$ for $0 \le k < n-2$, and because we are working with monic polynomials, we see from (62) that $d_{n,n+2} = n+1$. Hence, we obtain (59).

Using (40) and (41), we get

$$\begin{aligned}
x(\phi-1)P_k = \left[x^3-\left(z^2+1\right)x\right]P_k &= P_{k+3}+\gamma_k\gamma_{k-1}\gamma_{k-2}P_{k-3} \\
&+ \left[\gamma_k+\gamma_{k+1}+\gamma_{k+2}-\left(z^2+1\right)\right]P_{k+1} \\
&+ \gamma_k\left[\gamma_k+\gamma_{k+1}+\gamma_{k-1}-\left(z^2+1\right)\right]P_{k-1}.
\end{aligned}$$

Using (6), it follows that

$$\begin{aligned}
L\left[2x(\phi-1)P_{n+1}P_n\right] &= \left[\gamma_n+\gamma_{n+1}+\gamma_{n+2}-\left(z^2+1\right)\right]h_{n+1}, \\
L\left[2x(\phi-1)P_{n+1}P_{n-2}\right] &= h_{n+1},
\end{aligned}$$

and since $\phi\partial_x P_k = kP_{k+1}(x)+O\left(x^k\right)$, we see that

$$L\left[\phi P_{n+1}\partial_x P_n\right] = nh_{n+1}, \quad L\left[\phi P_{n+1}\partial_x P_{n-2}\right] = 0.$$

Hence, we conclude that

$$h_n\lambda_n = 2\left[\gamma_n+\gamma_{n+1}+\gamma_{n+2}-\left(z^2+1\right)\right]h_{n+1}-nh_{n+1}, \quad h_{n-2}\tau_n = 2h_{n+1},$$

and using (10) we get

$$\begin{aligned}
\lambda_n &= 2\left[(\gamma_n+\gamma_{n+1}+\gamma_{n+2})-\left(z^2+1\right)-\frac{n}{2}\right]\gamma_{n+1}, \\
\tau_n &= 2\frac{h_{n+1}}{h_{n-2}} = 2\gamma_{n+1}\gamma_n\gamma_{n-1}.
\end{aligned}$$

■

5.2 Differential equation

We will now obtain a lowering operator acting on the variable x for $P_n(x,z)$.

Proposition 18 *Let the operator U_n be defined by*

$$U_n = A_n(x;z)\,\partial_x - B_n(x;z), \quad n\in\mathbb{N},$$

where

$$A_n(x;z) = \frac{\phi(x;z)}{2\gamma_n(z)\,C_n(x;z)}, \quad B_n(x;z) = \frac{n-2\gamma_n(z)}{2\gamma_n(z)\,C_n(x;z)}x, \tag{64}$$

and

$$C_n(x;z) = \phi(x;z)+\gamma_n(z)+\gamma_{n+1}(z)-n-\frac{1}{2}. \tag{65}$$

Then,

$$U_nP_n = P_{n-1}, \quad n\in\mathbb{N}. \tag{66}$$

Proof. If we use (40) in (59), then we have

$$\begin{aligned}\phi\partial_x P_{n+1} &= (n+1) P_{n+2} + \lambda_n P_n + \tau_n \frac{\left[x^2 - (\gamma_n + \gamma_{n+1})\right] P_n - P_{n+2}}{\gamma_n \gamma_{n-1}} \\ &= \left(n+1 - \frac{\tau_n}{\gamma_n\gamma_{n-1}}\right) P_{n+2} + \left[\lambda_n + \frac{\tau_n \left(x^2 - \gamma_n - \gamma_{n+1}\right)}{\gamma_n\gamma_{n-1}}\right] P_n.\end{aligned}$$

From (4), we see that

$$\left[\phi\partial_x - \left(n+1 - \frac{\tau_n}{\gamma_n\gamma_{n-1}}\right) x\right] P_{n+1} = \left[\lambda_n + \tau_n \frac{x^2 - \gamma_n}{\gamma_n\gamma_{n-1}} - (n+1)\gamma_{n+1}\right] P_n. \tag{67}$$

Using (60) and (61) in (67), we get

$$\left[\phi\partial_x - (n+1-2\gamma_{n+1}) x\right] P_{n+1} = 2\gamma_{n+1}\left(\phi + \gamma_{n+1} + \gamma_{n+2} - n - \frac{3}{2}\right) P_n,$$

and the result follows. ∎

Taking into account the lowering operator U_n, we can deduce a second order linear differential equation (in x) for $P_n(x,z)$.

Theorem 19 *Let the differential operator D_n be defined by*

$$\begin{aligned} D_n &= \phi^2 C_n \partial_x^2 - 2x\phi\left[(\phi-1) C_n + \phi\right]\partial_x \\ &+ (n-2\gamma_n)\left[2x^2\phi - \left(\phi - 2x^2\phi + nx^2 - 2x^2\gamma_n\right) C_n\right] + 4\gamma_n C_{n-1} C_n^2. \end{aligned} \tag{68}$$

Then, $D_n P_n = 0$ for all $n \in \mathbb{N}$.

Proof. Using (66) in (4), we get

$$xU_n P_n = P_n + \gamma_{n-1} U_{n-1} U_n P_n.$$

If y is a function of x, we have

$$\begin{aligned} U_{n-1}U_n y &= (A_{n-1}\partial_x - B_{n-1})(A_n y' - B_n y) \\ &= A_{n-1}(\partial_x A_n y' + A_n y'' - \partial_x B_n y - B_n y') - B_{n-1}(A_n y' - B_n y), \end{aligned}$$

and therefore

$$\begin{aligned} (\gamma_{n-1}U_{n-1}U_n - xU_n + 1)\, y &= \gamma_{n-1}A_{n-1}A_n y'' \\ &+ \gamma_{n-1}\left[A_{n-1}(\partial_x A_n - B_n) - A_n B_{n-1}\right] y' - xA_n y' \\ &+ \gamma_{n-1}(B_n B_{n-1} - A_{n-1}\partial_x B_n)\, y + xB_n y + y. \end{aligned}$$

From (64) and (65), we see that

$$\gamma_{n-1}A_{n-1}A_n = \frac{\phi^2}{4\gamma_n C_{n-1}C_n},$$

$$\gamma_{n-1}\left[A_{n-1}\left(\partial_x A_n - B_n\right) - A_n B_{n-1}\right] - xA_n = -\frac{\left(\phi-1\right)C_n + \phi}{2\gamma_n C_{n-1}C_n^2}x\phi,$$

and

$$\begin{aligned}&\gamma_{n-1}\left(B_n B_{n-1} - A_{n-1}\partial_x B_n\right) + xB_n + 1\\ &= \frac{\left(n-2\gamma_n\right)\left[2x^2\phi - \left(\phi + nx^2 - 2x^2\phi - 2x^2\gamma_n\right)C_n\right]}{4\gamma_n C_{n-1}C_n^2} + 1.\end{aligned}$$

Multiplying by $4\gamma_n C_{n-1}C_n^2$, the result follows. ■

Remark 20 *We can write the third term in the differential operator D_n as*

$$\left(n-2\gamma_n\right)\left[\left(2x^2-1\right)C_n + 2x^2\right]\phi + \left[4\gamma_n C_{n-1}C_n - \left(n-2\gamma_n\right)^2 x^2\right]C_n$$

and since from (65) we see that $C_n\left(x;z\right) = \phi\left(x;z\right) + l_n\left(z\right),$ we have

$$4\gamma_n C_{n-1}C_n - \left(n-2\gamma_n\right)^2 x^2 = 4\gamma_n\left(\phi + l_{n-1}\right)\left(\phi + l_n\right) - \left(n-2\gamma_n\right)^2 x^2,$$

where

$$l_n\left(z\right) = \gamma_n\left(z\right) + \gamma_{n+1}\left(z\right) - n - \frac{1}{2}. \tag{69}$$

The Laguerre-Freud equation (44) can be written as

$$4\gamma_n l_n l_{n-1} = z^2\left(n-2\gamma_n\right)^2,$$

and therefore

$$\begin{aligned}4\gamma_n\left(\phi + l_{n-1}\right)\left(\phi + l_n\right) - \left(n-2\gamma_n\right)^2 x^2 &= 4\gamma_n\left(\phi + l_n + l_{n-1} - \frac{l_n l_{n-1}}{z^2}\right)\phi\\ &= \left[4\gamma_n\left(C_n + l_{n-1}\right) - \left(n-2\gamma_n\right)^2\right]\phi.\end{aligned}$$

As a consequence, we can write the differential equation for $P_n\left(x;z\right)$ in the reduced form

$$\begin{aligned}&\phi C_n\partial_x^2 P_n - 2x\left[\left(\phi-1\right)C_n + \phi\right]\partial_x P_n + \left(n-2\gamma_n\right)\left[\left(2x^2-1\right)C_n + 2x^2\right]P_n\\ &+ \left[4\gamma_n\left(C_n + l_{n-1}\right) - \left(n-2\gamma_n\right)^2\right]C_n P_n = 0.\end{aligned} \tag{70}$$

Using (56) in (68), we get

$$D_n \sim -n\phi^2\partial_x^2 + 2nx\phi\left(\phi - 1\right)\partial_x + n^3\phi, \quad n \to \infty,$$

and therefore we see that if $P_n\left(x;z\right) \sim \Phi^n\left(x;z\right)$ as $n \to \infty$, then to leading order

$$1 - \phi\left(\frac{\partial_x \Phi}{\Phi}\right)^2 = 0.$$

The solutions of this Riccati equation are

$$\Phi_{\pm}\left(x;z\right) = \frac{x \pm \sqrt{\phi\left(x;z\right)}}{2},$$

in agreement with (57).

6 Electrostatic interpretation of the zeros

It is very well known that the zeros of orthogonal polynomials with respect to a positive definite linear functional are real, simple, and located in the interior of the convex hull of the support of the linear functional [5]. Thus, let denote by $\{x_{n,k}\left(z\right)\}_{1\leq k\leq n}$ the zeros of $P_n\left(x;z\right)$ in an increasing order, i.e.

$$P_n\left(x_{n,k};z\right) = 0, \quad 1 \leq k \leq n,$$

and $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$.

Evaluating the operator D_n at $x = x_{n,k}$, we see that

$$\left[\frac{\partial_x^2 P_n}{\partial_x P_n}\right]_{x=x_{n,k}} - 2x_{n,k}\frac{\left[\phi\left(x_{n,k};z\right) - 1\right]C_n\left(x_{n,k};z\right) + \phi\left(x_{n,k};z\right)}{\phi\left(x_{n,k};z\right)C_n\left(x_{n,k};z\right)} = 0,$$

or, using (65),

$$\left[\frac{\partial_x^2 P_n}{\partial_x P_n}\right]_{x=x_{n,k}} - 2x_{n,k} + \frac{1}{x_{n,k} - z} + \frac{1}{x_{n,k} + z} - \frac{1}{x_{n,k} - \zeta_n(z)} - \frac{1}{x_{n,k} + \zeta_n(z)} = 0, \tag{71}$$

where

$$\zeta_n^2\left(z\right) = z^2 + n + \frac{1}{2} - \gamma_n - \gamma_{n+1}. \tag{72}$$

Using (72) in (44), we get

$$\gamma_n \left(\zeta_n^2(z) - z^2\right)\left(\zeta_{n-1}^2(z) - z^2\right) = z^2 \left(\frac{n}{2} - \gamma_n\right)^2,$$

and since

$$\zeta_0^2(z) - z^2 = \frac{1}{2} - \frac{\mu_1}{\mu_0} = \frac{z e^{-z^2}}{\sqrt{\pi}\,\mathrm{erf}\,(z)} > 0, \quad z \in \mathbb{R},$$

it follows by induction that $\zeta_n^2(z) - z^2 > 0$ for all $n \in \mathbb{N}_0$. In fact, one can show that

$$\min_{z\in\mathbb{R}} \zeta_n^2\,(z) = \zeta_n^2\,(0) = n + \frac{1}{2}, \quad n \in \mathbb{N}_0.$$

Using (56) in (72), we obtain

$$\zeta_n^2\,(z) \sim n + \frac{z^2+1}{2} - \frac{z^2}{8} n^{-2} + \frac{z^2\left(1-z^2\right)}{8} n^{-3}, \quad n \to \infty,$$

and therefore

$$\zeta_n\,(z) - z = \sqrt{n} - z + O\left(n^{-\frac{1}{2}}\right), \quad n \to \infty.$$

Thus, for $z = O\,(1)$, the points $\pm\zeta_n\,(z)$ are outside the interval $[-z, z]$, and "moving" outwards as $n \to \infty$.

Using the previous results, we have shown the following theorem.

Theorem 21 *The zeros of $P_n\,(x; z)$ are located at the equilibrium points of n unit charged particles located in the interval $(-z, z)$ under the influence of the potential*

$$V_n\,(x; z) = x^2 - \ln\left|x^2 - z^2\right| + \ln\left|x^2 - \zeta_n^2\,(z)\right|.$$

Proof. As it's well known, if we write

$$P_n\,(x; z) = \prod_{k=1}^{n} \left(x - x_{n,k}\right),$$

then [10, Chapter 10]

$$\left[\frac{\partial_x^2 P_n}{\partial_x P_n}\right]_{x=x_{n,k}} = \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2}{x_{n,k} - x_{n,j}},$$

and therefore (71) gives

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2}{x_{n,j}-x_{n,k}} + 2x_{n,k} - \frac{1}{x_{n,k}-z} - \frac{1}{x_{n,k}+z} + \frac{1}{x_{n,k}-\zeta_n} + \frac{1}{x_{n,k}+\zeta_n} = 0,$$

or equivalently

$$\frac{\partial E}{\partial x_{n,k}} = 0, \quad 1 \leq k \leq n,$$

where the total energy of the system is

$$E\left(x_{n,1},\ldots,x_{n,n}\right) = -2\sum_{1\leq j<k\leq n}^{n} \ln\left|x_{n,k}-x_{n,j}\right| + \sum_{k=1}^{n} x_{n,k}^2 - \ln\left|x_{n,k}^2 - z^2\right| + \ln\left|x_{n,k}^2 - \zeta_n^2\right|.$$

It follows that the external potential is

$$V_n\left(x;z\right) = x^2 - \ln\left|x^2 - z^2\right| + \ln\left|x^2 - \zeta_n^2\right|.$$

■

7 Toda-type behavior

Differentiating (3) with respect to z, we have

$$\partial_z L\left[p\left(x;z\right)\right] = e^{-z^2}\left[p\left(z;z\right) + p\left(-z;z\right)\right] + L\left[\partial_z p\left(x;z\right)\right], \tag{73}$$

and we note that

$$\partial_z L\left[\phi p\right] = L\left[\partial_z\left(\phi p\right)\right], \tag{74}$$

where $\phi\left(x;z\right)$ was defined in (17).

In particular,

$$u_n' = \partial_z L\left[x^{2n}\right] = 2z^{2n}e^{-z^2},$$

and using (28), we get

$$zu_n' = 2z^{2n+1}e^{-z^2} = \left(2n+1\right)u_n - 2u_{n+1}. \tag{75}$$

On the other hand, using (74) we have

$$u_{n+1}' - 2zu_n - z^2u_n' = \partial_z\left(u_{n+1} - z^2u_n\right) = -2zu_n,$$

and therefore

$$u'_{n+1} = z^2 u'_n, \quad n \geq 0. \tag{76}$$

Using the differential-recurrence for the moments (76), we can obtain a first order ODE (in z) for the Stieltjes function $S(t;z)$.

Proposition 22 *Let the function $S(t;z)$ be defined by (34), and $\phi(x;z)$ be defined by (17). Then,*

$$\phi(t;z)\,\partial_z S = 2te^{-z^2}. \tag{77}$$

Proof. Using (36), we have

$$\sum_{n\geq 0} \frac{u_{n+1}}{t^{2n+1}} = t^2 S - tu_0,$$

and therefore (76) gives

$$t^2 \partial_z S - tu'_0 = z^2 \partial_z S.$$

Using (27), the result follows. ■

Remark 23 *Clearly, (77) and the initial condition $S(t;0) = 0$ yield*

$$S(t;z) = 2t \int_0^z \frac{e^{-x^2}}{t^2 - x^2} dx,$$

which is just the definition (32), since

$$\sum_{n\geq 0} \frac{x^{2n}}{t^{2n+1}} = \frac{t}{t^2 - x^2}.$$

Next, we will obtain some equations relating h'_n, γ'_n with h_n, γ_n.

Theorem 24 *The functions $h_n(z), \gamma_n(z)$ satisfy the Toda-type equations*

$$\vartheta \ln(h_n) = 2n + 1 - 2(\gamma_{n+1} + \gamma_n), \tag{78}$$

and

$$\vartheta \ln(\gamma_n) = 2(\gamma_{n-1} - \gamma_{n+1} + 1), \tag{79}$$

where ϑ is the operator defined by

$$\vartheta = z\partial_z. \tag{80}$$

Proof. Using (73), we have

$$h_n' = \partial_z L\left[P_n^2\right] = 2e^{-z^2} P_n^2(z;z) + L\left[2P_n \partial_z P_n\right],$$

but since $\deg(\partial_z P_n) \leq n-2$ we can use (6) and (52) and obtain

$$zh_n' = (2n+1)\, h_n - 2\left(h_{n+1} + \gamma_n^2 h_{n-1}\right),$$

or, using (10)

$$z\frac{h_n'}{h_n} = 2n + 1 - 2\left(\gamma_{n+1} + \gamma_n\right).$$

Note that (10) gives

$$\frac{\gamma_n'}{\gamma_n} = \frac{h_n'}{h_n} - \frac{h_{n-1}'}{h_{n-1}},$$

and hence we can write (78) in terms of γ_n

$$z\frac{\gamma_n'}{\gamma_n} = 2\left(\gamma_{n-1} - \gamma_{n+1} + 1\right).$$

■

If we combine the Laguerre-Freud equation for γ_n (44) and the Toda-type equation for $h_n(z)$, we obtain the following result.

Proposition 25 *The function $h_n(z)$ satisfies*

$$h_n' h_{n-1}' = \left(nh_{n-1} - 2h_n\right)^2, \quad n \geq 1.$$

Proof. From (78), we see that

$$\frac{1}{2}\vartheta \ln(h_n) = n + \frac{1}{2} - \left(\gamma_{n+1} + \gamma_n\right),$$

and using this in (44), we get

$$\frac{1}{2}\vartheta \ln(h_n)\, \frac{1}{2}\vartheta \ln(h_{n-1}) = \frac{z^2}{\gamma_n}\left(\frac{n}{2} - \gamma_n\right)^2,$$

or using (10) and (80)

$$h_n \partial_z \ln(h_n)\, h_{n-1} \partial_z \ln(h_{n-1}) = \left(nh_{n-1} - 2h_n\right)^2,$$

and the result follows. ■

7.1 Nonlinear ODE

Using the Laguerre-Freud equations (38), (44) and the differential-recurrence relation (79), we can derive a nonlinear second order ODE for $\gamma_n(z)$.

Theorem 26 *The function $\gamma_n(z)$ satisfies*

$$z^2\left[\gamma_n'' + 2(6\gamma_n - n)(2\gamma_n - n)\right]^2 = 4\left(z^2 + 2\gamma_n - n\right)^2\left[\left(\gamma_n'\right)^2 + 4\gamma_n(2\gamma_n - n)^2\right]. \tag{81}$$

Proof. Setting $n \to n-1$ in (44), we have

$$\gamma_{n-1}\left(n - \frac{1}{2} - \gamma_{n-1} - \gamma_n\right)\left(n - \frac{3}{2} - \gamma_{n-1} - \gamma_{n-2}\right) = z^2\left(\frac{n-1}{2} - \gamma_{n-1}\right)^2,$$

and solving for γ_{n-2}, we obtain

$$\gamma_{n-2} = \frac{z^2\left(\frac{1}{2} - \frac{1}{2}n + \gamma_{n-1}\right)^2}{\gamma_{n-1}\left(\frac{1}{2} - n + \gamma_n + \gamma_{n-1}\right)} + n - \frac{3}{2} - \gamma_{n-1}. \tag{82}$$

Solving for γ_{n+2} in (38), we get

$$\gamma_{n+2} = \frac{\gamma_n\left(\gamma_n + \gamma_{n-1} - z^2 - n + \frac{1}{2}\right) + \gamma_{n+1}\left(n + \frac{3}{2} - \gamma_{n+1} + z^2\right) - \frac{1}{2}z^2}{\gamma_{n+1}}. \tag{83}$$

From (82) and (83) we conclude that

$$\begin{aligned}&\gamma_{n-1}\gamma_{n-2} + \gamma_{n+1}\gamma_{n+2} = \frac{z^2\left(\frac{1}{2} - \frac{1}{2}n + \gamma_{n-1}\right)^2}{\frac{1}{2} - n + \gamma_n + \gamma_{n-1}} + \left(n - \frac{3}{2} - \gamma_{n-1}\right)\gamma_{n-1} \\ &+\gamma_n\left(\gamma_n + \gamma_{n-1} - z^2 - n + \frac{1}{2}\right) + \gamma_{n+1}\left(n + \frac{3}{2} - \gamma_{n+1} + z^2\right) - \frac{1}{2}z^2.\end{aligned} \tag{84}$$

Differentiating (79) with respect to z, we see that

$$\frac{\left(z\gamma_n'\right)'}{2} = \gamma_n'\left(\gamma_{n-1} - \gamma_{n+1} + 1\right) + \gamma_n\left(\gamma_{n-1}' - \gamma_{n+1}'\right)$$

and using (79) again we have

$$\left(z\gamma_n'\right)' = \frac{z\left(\gamma_n'\right)^2}{\gamma_n} + \frac{4}{z}\gamma_n\left[\gamma_{n-1}\left(\gamma_{n-2} - \gamma_n + 1\right) - \gamma_{n+1}\left(\gamma_n - \gamma_{n+2} + 1\right)\right]. \tag{85}$$

Using (84) in (85), we get

$$\begin{aligned}\frac{z}{4\gamma_n}\left(z\gamma_n'\right)' = \left(\frac{z\gamma_n'}{2\gamma_n}\right)^2 + \frac{z^2\left(\frac{1}{2}-\frac{1}{2}n+\gamma_{n-1}\right)^2}{\frac{1}{2}-n+\gamma_n+\gamma_{n-1}} \\ +\gamma_n\left(\frac{1}{2}-n-z^2+\gamma_n-\gamma_{n+1}\right)-\frac{1}{2}z^2 \\ +\gamma_{n+1}\left(z^2+n+\frac{1}{2}-\gamma_{n+1}\right)+\gamma_{n-1}\left(n-\frac{1}{2}-\gamma_{n-1}\right).\end{aligned} \tag{86}$$

Note that (44), (79), and (86) are three equations relating γ_n'', γ_n', γ_n, γ_{n-1} and γ_{n+1}. Thus, $\gamma_{n+1}, \gamma_{n-1}$ can be eliminated from the system and we obtain (81). ■

7.2 Power series

Using the nonlinear recurrence (44), we can see that

$$\gamma_n(z) = \frac{n^2 z^2}{4n^2-1} + \frac{4n^3 z^4}{\left(4n^2-1\right)^2\left(4n^2-9\right)} + O\left(z^6\right), \quad z \to 0. \tag{87}$$

To obtain higher order terms, we can use (79).

Theorem 27 *The Maclaurin series of the function $\gamma_n(z)$ is*

$$\gamma_n(z) = \sum_{k=1}^{\infty} \eta_{n,k} z^{2k}, \tag{88}$$

where

$$\eta_{n,1} = \frac{n^2}{4n^2-1}, \tag{89}$$

and

$$\eta_{n,k} = \frac{1}{k-1}\sum_{j=1}^{k-1}\left(\eta_{n-1,j}-\eta_{n+1,j}\right)\eta_{n,k-j}, \quad k \geq 2. \tag{90}$$

Proof. The first term (89) follows immediately from (87). From (88) we have

$$\frac{z}{2}\gamma_n'(z) = \sum_{k=1}^{\infty} k\eta_{n,k} z^{2k}, \tag{91}$$

and

$$\gamma_{n-1}(z) - \gamma_{n+1}(z) = \sum_{k=1}^{\infty} (\eta_{n-1,k} - \eta_{n+1,k}) z^{2k}.$$

Using the Cauchy product, we get

$$\left[z^{2k}\right] (\gamma_{n-1} - \gamma_{n+1} + 1) \gamma_n = \eta_{n,k} + \sum_{j=1}^{k-1} (\eta_{n-1,j} - \eta_{n+1,j}) \eta_{n,k-j}, \tag{92}$$

where $[z^m]$ denotes the coefficient of z^m in the given expression.

Using (91) and (92) in (79), we obtain

$$k\eta_{n,k} = \eta_{n,k} + \sum_{j=1}^{k-1} (\eta_{n-1,j} - \eta_{n+1,j}) \eta_{n,k-j},$$

and (90) follows. ■

Remark 28 *If we introduce the forward and backward difference operators*

$$\Delta f(n) = f(n+1) - f(n), \quad \nabla f(n) = f(n) - f(n-1), \tag{93}$$

then we can write (90) as

$$\eta_{n,k} = -\frac{1}{k-1} \sum_{j=1}^{k-1} (\Delta + \nabla) \eta_{n,j} \eta_{n,k-j}, \quad k \geq 2,$$

and it follows that (88) is an asymptotic series as $n \to \infty$. Using the same methods that we introduced in [7], we can show that for $k \geq 2$

$$\eta_{n,k} = O\left(n^{-k-1}\right), \quad n \to \infty.$$

Using (4) and (88), we see that

$$P_n(x; z) = x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} z^2 + O\left(z^4\right).$$

We obtain higher order terms in the following theorem.

Theorem 29 *The Maclaurin series of the polynomials* $P_n(x;z)$ *is*

$$P_n(x;z) = x^n + \sum_{k=1}^{\infty} \alpha_{n,k}(x) z^{2k}, \tag{94}$$

where

$$\alpha_{0,k}(x) = \alpha_{1,k}(x) = 0, \quad k \geq 1,$$

and the coefficients $\alpha_{n,k}(x)$ *satisfy the recurrence*

$$\alpha_{n+1,k}(x) - x\alpha_{n,k}(x) + \eta_{n,k} x^{n-1} + \sum_{j=1}^{k-1} \alpha_{n-1,j}(x) \eta_{n,k-j} = 0. \tag{95}$$

In particular,

$$\alpha_{n,1}(x) = -\frac{n(n-1)}{2(2n-1)} x^{n-2}, \tag{96}$$

and

$$\alpha_{n,2}(x) = n(n-1) \frac{8n(n-1)x^2 + (2n+1)(2n-1)(n-2)(n-3)}{8(2n+1)(2n-1)^2(2n-3)} x^{n-4}. \tag{97}$$

Proof. Using (88) and (94) in (4), we have

$$x^{n+1} + \sum_{k=1}^{\infty} x\alpha_{n,k}(x) z^{2k} = x^{n+1} + \sum_{k=1}^{\infty} \alpha_{n+1,k}(x) z^{2k}$$
$$+x^{n-1}\gamma_n(z) + \gamma_n(z) \left(\sum_{k=1}^{\infty} \alpha_{n-1,k}(x) z^{2k} \right),$$

and therefore we obtain the recurrence

$$x\alpha_{n,k}(x) = \alpha_{n+1,k}(x) + \eta_{n,k} x^{n-1} + \sum_{j=1}^{k-1} \alpha_{n-1,j}(x) \eta_{n,k-j}.$$

If $k = 1$, then (95) becomes

$$\alpha_{n+1,1}(x) - x\alpha_{n,1}(x) = -\eta_{n,1} x^{n-1},$$

and the solution with initial condition $\alpha_{0,k}(x) = 0$ is

$$\alpha_{n,1}(x) = -x^{n-2}\sum_{i=0}^{n-1}\eta_{i,1}(x) = -\frac{n(n-1)}{2(2n-1)}x^{n-2}.$$

Setting $k = 2$ in (95) we get

$$\alpha_{n+1,2}(x) - x\alpha_{n,2}(x) = -\eta_{n,1}\alpha_{n-1,1}(x) - \eta_{n,2}x^{n-1},$$

and therefore

$$\alpha_{n,2}(x) = -x^{n-1}\sum_{i=0}^{n-1}\left[x^{-i}\eta_{i,1}\alpha_{i-1,1}(x) + \frac{\eta_{i,2}}{x}\right].$$

Using (90) and (96), we obtain (97). ■

8 Conclusions

We have defined the family of truncated Hermite polynomials $P_n(x;z)$, orthogonal with respect to the linear functional

$$L[p] = \int_{-z}^{z} p(x)\,e^{-x^2}dx, \quad p \in \mathbb{R}[x], \quad z > 0.$$

Such a linear functional satisfies the Pearson equation

$$L[\phi\partial_x p] = L[2x(\phi-1)p], \quad \phi(x;z) = x^2 - z^2.$$

We related $P_n(x;z)$ to the Hermite and Rys polynomials, and studied the sequence $P_n(x;z)$ as semiclassical polynomials of class 2. The expansion of $\phi\partial_x P_n$ in the $\{P_k\}_{k\geq 0}$ basis (structure relation), an asymptotic approximation (for large n), a lowering operator, second order ODE (in x), and power series (in z) for $P_n(x;z)$ are given.

We obtained a second order linear recurrence for the moments, as well as a differential-recurrence equation in terms of the variable z. An asymptotic approximation for the moments (as $z \to \infty$) is obtained. Differential equations (in t and z, respectively) for the Stieltjes function $S(t;z)$ of the moments associated with L are deduced.

We also got nonlinear recurrences (Laguerre-Freud equations) and a nonlinear ODE that the parameters $\gamma_n(z)$ satisfy. As a consequence, an asymptotic approximation (for large n), a differential-recurrence equation, and a power series for the coefficients $\gamma_n(z)$ in the recurrence relation of $P_n(x;z)$ are obtained.

We plan to continue our research on these polynomials in order to obtain asymptotic expansions for $P_n(x;z)$ as $n \to \infty$, $z \to \infty$ as well as when both $n, z \to \infty$ simultaneously. One should be able to obtain the well known asymptotic approximations for the Hermite polynomials in the last case.

Finally, we will deal with the analysis of truncated Laguerre polynomials, as well as other families of truncated semiclassical polynomials.

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