July 20, 2022

27th ICDEA, Paris–Saclay, France

Symbolic summation, linear difference equations and challenging applications

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Der Wissenschaftsfonds.

Outline

1. A warm-up example
2. The difference ring machinery for symbolic summation
3. Challenging applications

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Bigg)$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

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↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

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$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

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Summing the telescoping equation over j from 0 to a gives

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Summing the telescoping equation over j from 0 to a gives

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$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

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In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!}+\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!}+\frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

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In[3]:= $\mathbf{res} = \mathbf{SigmaReduce[mySum]}$

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= $\mathbf{SigmaLimit[res, \{n\}, a]}$

Out[4]= $$\frac{1}{n!}\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

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$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $k \geq 1$.

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no solution ☹

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $k \geq 1$.

no solution ☹

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + c_1(n)\,f(n+1,k)}$$

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for all $k \geq 1$.

Sigma *computes*: $c_0(n) = -n,\ c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

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FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + c_1(n)\,f(n+1,k)}$$

for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a}\Big[c_0(n)\,f(n,k) + c_1(n)\,f(n+1,k)\Big]}$$

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Zeilberger's creative telescoping paradigm

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Zeilberger's creative telescoping paradigm

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for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad\qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

In[5]:= $\mathbf{mySum = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};}$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

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Compute a recurrence

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Out[6]= $$-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)\left(S[1,a] + S[1,n] - S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $$-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n] + 1}{(n+1)^3}$$

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Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\,\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a] + S[1,n] - S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\,\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\left\{\left\{0, \frac{1}{n(n+1)}\right\}, \left\{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\right\}\right\}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

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Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

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Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j) &= \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}\\ &= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}\end{aligned}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Part 2: The difference ring machinery for symbolic summation

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1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

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n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \cdots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
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FIND a recurrence for $F(n)$

2. Recurrence solving

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Special cases:

$$S_{2,1}(n)=\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}$$ (harmonic sums)

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{j=1}^{n} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2}$$

(binomial sums)

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^{k} \frac{1}{j\binom{4j}{3j}^2}$$

$$
\begin{aligned}
&-2(1+n)^3(3+n)n!^2F(n)\\
&\qquad\qquad+(1+n)\big(8+9n+2n^2\big)n!F(n+1)-F(n+2)=0
\end{aligned}
$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\prod_{i=1}^{n}i!+c_2\Big(-2^n n!\prod_{i=1}^{n}i!+\tfrac{3}{2}\prod_{i=1}^{n}i!\sum_{i=1}^{n}2^i i!\Big)\mid c_1,c_2\in\mathbb{K}\Big\}$$

$$\begin{aligned}&\big(1+S_1(n)+nS_1(n)\big)^2\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2F(n)\\&-(1+n)(3+2n)S_1(n)\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2F(n+1)\\&\qquad+(1+n)^2(2+n)^3S_1(n)\big(1+S_1(n)+nS_1(n)\big)F(n+2)=0\end{aligned}$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\,S_1(n)\prod_{l=1}^{n}S_1(l)+c_2S_1(n)^2\prod_{l=1}^{n}S_1(l)\mid c_1,c_2\in\mathbb{K}\Big\}$$

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

F(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

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$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$||$$

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$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}]}$

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In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

This summation machinery is based on

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Part 2: The difference ring machinery for symbolic summation

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Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}}[s]}_{\text{polynomial ring}}$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{aligned} \mathrm{ev}' : \quad \mathbb{Q}(x) \times \mathbb{N} &\to \mathbb{Q} \\ \left(\tfrac{p(x)}{q(x)}, n\right) &\mapsto \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\mathrm{ev} : \quad \mathbb{Q}(x)[s] \times \mathbb{N} \quad \to \quad \mathbb{Q}$$

$$\mathbf{ev(s, n) = S_1(n)}$$

Simplify

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$$\begin{aligned} \mathrm{ev} : \quad & \mathbb{Q}(x)[s] \times \mathbb{N} && \to \quad \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, n\right) && \mapsto \quad \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) S_1(n)^i \qquad \mathbf{ev(s,n) = S_1(n)} \end{aligned}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau: & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \text{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

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Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0,1,2,3,\dots\rangle\langle 0,1,\frac{1}{2},\frac{1}{3},\dots\rangle \\ & & \Vert \\ & & \langle 0,1,1,1,\dots\rangle \\ & & \Vert \\ \tau(x\frac{1}{x}) = \tau(1) & = & \langle 1,1,1,1,\dots\rangle \end{array}$$

Simplify

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Consider the map

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \\ & & \| \\ & & \langle 0, 1, 1, 1, \dots \rangle \\ & & \| \\ \tau(x\,\frac{1}{x}) = \tau(1) & = & \langle 1, 1, 1, 1, \dots \rangle \end{array}$$

Simplify

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2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{rccc} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

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$$\sigma : \quad \mathbb{Q}(x)[s] \quad \to \quad \mathbb{Q}(x)[s]$$

$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\left(s + \frac{1}{x+1}\right)^i \end{array}$$

$$s \mapsto s + \frac{1}{x+1}$$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{llll} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\left(s + \frac{1}{x+1}\right)^i \end{array}$$

$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

In this example:

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

Simplify

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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τ is an injective difference ring homomorphism:

$$\begin{array}{ccc} \mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\ \big\downarrow \tau & = & \big\downarrow \tau \\ \mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S\quad} & \mathbb{K}^{\mathbb{N}}/\sim \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(n) \rangle_{n \geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}} / \sim, S)$$

$$\sum_{k=0}^{a} S_1(k) =?$$

$$\begin{array}{ccccc} (\mathbb{A},\sigma) & \simeq & (\tau(\mathbb{A}),S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim,S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}] & & \end{array}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

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$$\Updownarrow \quad \tau$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

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Output: $\bar{g} = x\,s - x$

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Output: $g(k) = k\,S_1(k) - k$

$\Updownarrow \quad \tau$

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$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0)$$

Given: $f(k) = S_1(k)$

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Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0) = (a+1)S_1(a+1) - (a+1)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

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Find: $\bar{g} \in \mathbb{A}$:

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Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

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- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

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Sk!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

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hypergeometric products $\quad\leftrightarrow\quad \sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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hypergeometric	$\leftrightarrow$	$\sigma(p_1) = a_1\, p_1$	$a_1 \in \mathbb{K}(x)^*$
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Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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hypergeometric products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)^* \end{aligned}$$

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$(-1)^k \leftrightarrow \quad \sigma(z) = -z \qquad z^2 = 1$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

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		$\sigma(p_e) = a_e p_e$	$a_e \in \mathbb{K}(x)^*$
α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}}$	$\leftrightarrow$	$\sigma(\mathbf{z}) = \alpha\,\mathbf{z}$	$\mathbf{z}^{\lambda} = \mathbf{1}$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)^* \end{aligned}$$

α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow \sigma(\mathbf{z}) = \alpha\, \mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$

$$\mathcal{S} S_1(k) = S_1(k) + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{x+1}$$

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(nested) sum	$\leftrightarrow$	$\sigma(s_1) = s_1 + f_1$	$f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z]$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

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$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \end{aligned}$$

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Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

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such that $\mathrm{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

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α is a primitive λth root of unity

(nested) su… $\dots, p_e^{-1}][z]$

GIVEN $f \in \mathbb{A}$;
FIND, in case of existence, a $g \in \mathbb{A}$ such that

$$\sigma(g) - g = f.$$

$$\begin{aligned} \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2]\dots[s_r]}$$

- with an automorphism as given in the previous slide.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

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- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\text{const}_\sigma \mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

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Theorem. The following statements are equivalent:

1. $\mathrm{const}_\sigma \mathbb{A} = \mathbb{K}$.

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2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

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3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

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CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

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3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

Remark 1: Related results have been worked out in the Galois theory of difference equations (van der Put/Singer, 1997)

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \dots [s_r]}$$

- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\text{const}_\sigma \mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

3. There is an embedding τ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

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Remark 2: Works more generally for (nested) mixed multibasic products

Example: algebraic independence of sequences

1. $(\mathbb{Q}(x)[s_1, s_2, \dots], \sigma)$ is an $R\Pi\Sigma$-ring with

$$\sigma(s_i) = s_i + \frac{1}{(x+1)^i} \qquad i = 1, 2, 3, \dots$$

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2. There is an embedding of the polynomial ring $\mathbb{Q}(x)[s_1, s_2, \dots]$ into $\mathbb{Q}^{\mathbb{N}} / \sim$ with

$$s_1 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n \geq 0}, \qquad s_2 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n \geq 0} \qquad \dots$$

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$\Rightarrow$ The generalized harmonic numbers

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}, \quad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}, \quad S_3(n) = \sum_{i=1}^{n} \frac{1}{i^3}, \quad \dots$$

are algebraically independent among each other over the rational sequences.

Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,n]`

$B(n)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

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(δ can be computed explicitly)

- and such that

the arising sums and products in $B(n)$ (except the alternating sign) are **algebraically independent**
(i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

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Application 1: the expression $B(n)$ is usually much smaller

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Application 3: we get canonical form representations

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \dots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

F(n)=combined solutions in terms of indefinite nested sums.

Part 3: Applications

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- combinatorics
- special functions
- number theory
- statistics
- numerics
- elementary particle physics (QCD)

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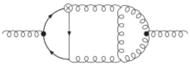
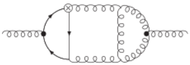
Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals



behavior of particles

$$\longrightarrow \quad \int \Phi(n, \epsilon, x)\, dx$$

Feynman integrals

Feynman integrals

$$\int_0^1 x^n \, dx$$

Feynman integrals

$$\int_0^1 x^n(1+x)^n \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^n(1+x)^n}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2$$

Feynman integrals

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Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1 + x_1)^n}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

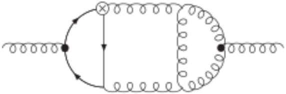
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^n(1+x_1)^{n-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals

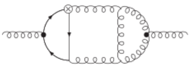


a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1}$$
$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$
$$\Bigg[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$
$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Bigg]$$
$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{n-3-j}$$
$$\times \left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{n-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

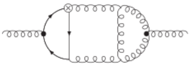
Evaluation of Feynman Integrals



behavior of particles ⟶ $\int \Phi(n, \epsilon, x)\,dx$

Feynman integrals

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n,\epsilon,x)\,dx$$

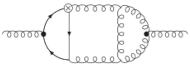
Feynman integrals

DESY

$$\sum f(n,\epsilon,k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



behavior of particles ⟶ $\int \Phi(n,\epsilon,x)\,dx$

Feynman integrals

Tactic 1:

symbolic summation

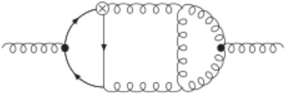
↓ **DESY**

$\sum f(n,\epsilon,k)$

complicated multi-sums

expression in special functions ⟵ **advanced difference ring theory** (Sigma-package)

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1} \qquad ||$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

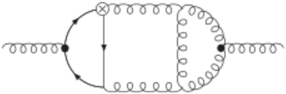
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

$$\Big[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

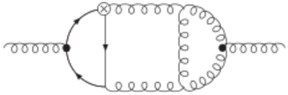
$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Big]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{n-3-j}$$

$$\times \left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{n-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\,\varepsilon^{0} + \dots$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

Simplify

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\ \frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-\left(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s)\right)$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+ \Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+ \big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
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$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

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$$+ \Big(2 \ldots 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
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$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\ldots} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

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$$+ \Big(2 \ldots - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big) \ldots S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(\ldots\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2 \ldots} - \frac{5}{\ldots}\Big) \ldots + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n \ldots}{n(n+\ldots} \ldots -1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)\ldots}{n\ldots} \ldots n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(- \ldots\Big) S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)\ldots}{n(n+1)} S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \Big(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\Big)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$S_{-2,1,1}(n) = \sum_{i=1}^{n} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Tactic 2: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$
- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

Strategy: uncouple and solve

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Strategy: uncouple and solve

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)),\ k > 1$$

Strategy: uncouple and solve

DE system	OreSys package (S. Gerhold) $\longrightarrow$ uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

Strategy: uncouple and solve

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

$$\textstyle\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\hat{I}_k(x) = \mathrm{expr}_k(\hat{I}_1(x)),\, k > 1$$

↓ holonomic closure prop. ↓

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

linear recurrence

$$\textstyle\sum_i a'_i(n) I_1(n+i) = r'(n)$$

Strategy: uncouple and solve

DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system
$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$
$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

↓ holonomic closure prop.

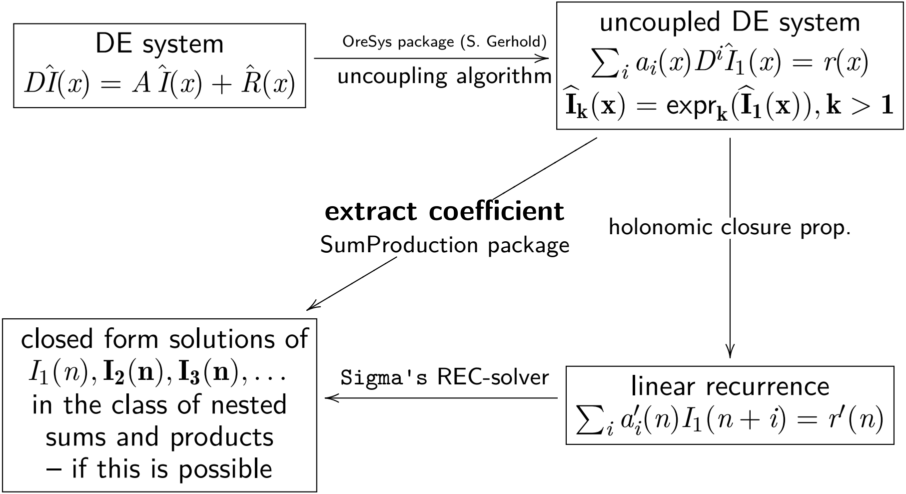
$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

linear recurrence
$\sum_i a'_i(n) I_1(n+i) = r'(n)$

← `Sigma's` REC-solver ←

closed form solutions of
$I_1(n)$
in the class of nested
sums and products
– if this is possible

Strategy: uncouple and solve



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$
- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

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- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

$$\downarrow \text{solver for } \hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$$

$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \dots + \varepsilon^{o_i} F_{o_i}(n) + \dots$$

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$
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$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \dots + \varepsilon^{o_i} F_{o_i}(n) + \dots$$

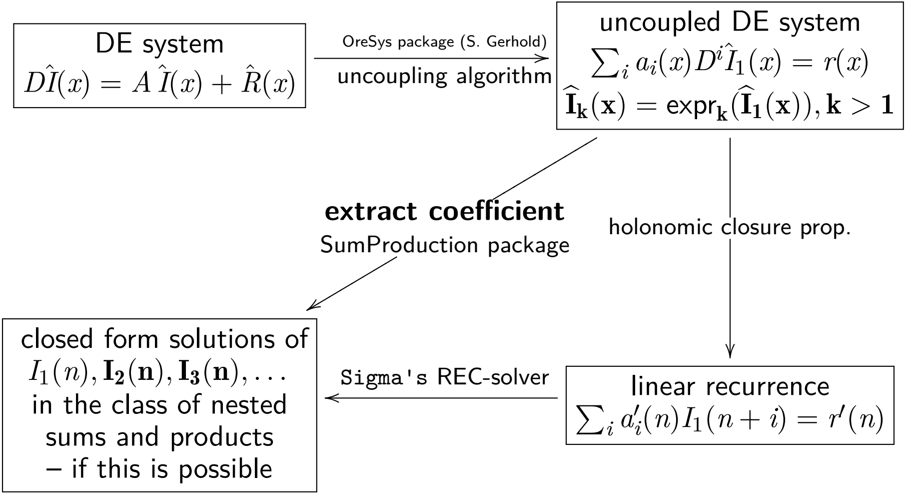
$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

$$P(n) = \varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n) + \varepsilon^{0} P_0(n) + \dots$$

Calculations based on “uncouple and solve”:

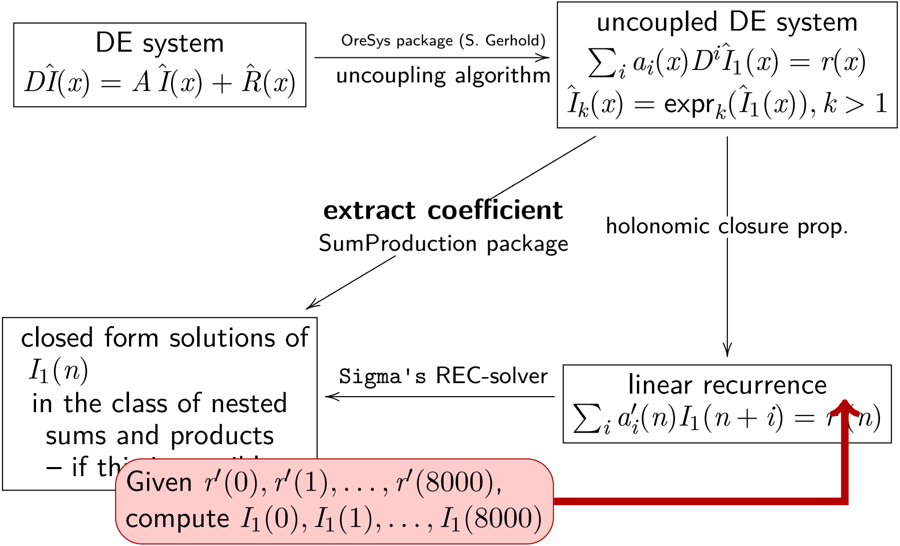
- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(n)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSN 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor Number Scheme at Next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.

Strategy: uncouple and solve



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Strategy: uncouple and compute numbers



J. Blümlein, CS, 2017, arXiv:1701.04614 [hep-ph].

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$
- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_{0}(n) + \dots}_{\text{only numbers}}$$

$n = 0, 1, \dots, 8000$

$\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{numbers}} + \dots$$

$n = 0, 1, \dots, 8000$

coupled systems for

$$\hat{P}(x) = \sum_{n=0}^{\infty} P(n)x^n$$

↓ `SolveCoupledSystem.m`

large no. of evaluations of $P(n)$

coupled systems for

$$\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

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large no. of evaluations of $P(n)$

↓ guessing (`ore_algebra` in Sage by M. Kauers)

recurrence for $P(n)$ with minimal order

coupled systems for

$$\hat{P}(x) = \sum_{n=0}^{\infty} P(n)x^n$$

↓ `SolveCoupledSystem.m`

large no. of evaluations of $P(n)$

↓ guessing (`ore_algebra` in Sage by M. Kauers)

recurrence for $P(n)$ with minimal order

↓ `Sigma.m`

expression in terms of indefinite nested sums over hypergeometric products

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial** $=<<$ **iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial =<< iFile16**

Out[2]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

52869784323778576751/80221899453696000,

49422862094045523994231/753773992230616156800,

33131879832907935920726113/509557943985299969760000,

5209274721836755168448777/80949984111854180459136,

56143711997344769021041145213/882589266383586456384353664,

453500433353845628194790025124807/7217228048879468556886950000000,

14061543374120479886110159898869387/226643167590350326435656036000000,

715586522666491903324905785178619936571168370307700222807811495895030000000,

16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,

1428729642632302467951426905844691837805299/239407595750341228278613159615736736000000,

498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000,

[illegible]}

In[3]:= **rec =<< rFile16**

Out[3]=
$$
\begin{aligned}
&(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + \\
&2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + \\
&873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + \\
&1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + \\
&575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + \\
&69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + \\
&901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + \\
&7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + \\
&32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + \\
&86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + \\
&124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + \\
&87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + \\
&25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + \\
&1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]
\end{aligned}
$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38}+161315106167193 6n^{37}+\\
&101748284195864576n^{36}+4135139115563745280n^{35}+121713599527855849472n^{34}+\\
&2765050919624810430464n^{33}+50453046277771391664128n^{32}+759760507477065230974976n^{31}+\\
&9628262076527899425374208n^{30}+104191253579306374131613696n^{29}+973595596739520084325171200n^{28}+\\
&7924537790312611436520013824n^{27}+56571687381518195331462463488n^{26}+\\
&356133102136059681954436399104n^{25}+1985507231916669869451824553984n^{24}+\\
&9836060321685410187563260035072n^{23}+43406506634905372676489415905280n^{22}+\\
&170945808151999530921656848106496n^{21}+601507760131008511164113355409920n^{20}+\\
&1892149418896523531194676203153920n^{19}+5321173806292333448534132495165440n^{18}+\\
&13370912745727662541153592039812160n^{17}+29987002021632029091547005084057760n^{16}+\\
&59921270253255984811455083696758912n^{15}+106434458966741189159011567116493072n^{14}+\\
&167533688453539238956436945725341004n^{13}+232781742346547554435545097479210510n^{12}+\\
&284125621128876904663642986868770746n^{11}+302806836393712159148051277734975424n^{10}+\\
&279679164311116651162116055961513301n^{9}+221781415386984655607595031093415136n^{8}+\\
&149214365004640710156345950062395186n^{7}+83882523964213110328265187672574356n^{6}+\\
&38609679702395410742361774562392789n^{5}+14149471988638475521561721269939086n^{4}+\\
&3963748138857399502678254252169734n^{3}+795659668131014454843348852372480n^{2}+\\
&101701393436276172443717692853400n+6204709909986751913151675960000\Big)\mathrm{P}[n+1]
\end{aligned}
$$

$$+2(n+3)^{2}(2n+5)^{3}(2n+9)(2n+11)\Big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+$$
$$274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+25180108291969215434326016n^{32}+$$
$$284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+$$
$$1178515602115604757944201871360n^{27}+6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+$$
$$722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+$$
$$28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+$$
$$424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+1499359936674956711935311062995422344n^{15}+$$
$$2378007025570977662661938772843220240n^{14}+3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+$$
$$4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+3761696365025837909581516781307249585n^{9}+$$
$$2726553473467254373993685951699145492n^{8}+1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+$$
$$371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+$$
$$6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\Big)P[n+2]$$

$$
\begin{aligned}
&-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40} + 3322784268681216n^{39} + 216160919414112256n^{38} + 9074528155284275200n^{37} + \\
&276348048819456311296n^{36} + 6506479077331107315712n^{35} + 123266585640616142569472n^{34} + 1931040885785102661976064n^{33} + 25510503383281445462081536n^{32} + \\
&288418124175428279391485952n^{31} + 2822442799033603081019326464n^{30} + 24120717233320712351821332480n^{29} + 181295944719289040999116701696n^{28} + \\
&1205246297785423925076555694080n^{27} + 7119049557560114436136213413888n^{26} + 37496933571993839665392189775872n^{25} + 176616172467048982234270428880896n^{24} + \\
&745539218875020737621728364206080n^{23} + 2824909633156578132652259733712896n^{22} + 9618101958268071244680677589035520n^{21} + \\
&29441860528446423517613263360742912n^{20} + 81033563306363873505877563416477312n^{19} + 200454769103641040142838133702338304n^{18} + \\
&445286624972461749049425309485328992n^{17} + 887028447418790661018847407251573152n^{16} + 1581538101499869694224895701784875304n^{15} + \\
&2517550244392724509968791166585362672n^{14} + 3566593026520465155504695877897282630n^{13} + 4479066125207404898722179511912639638n^{12} + \\
&4962006990874351800791769650243464872n^{11} + 4819992643914265990647887896664485209n^{10} + 4074895386694182240941538222230233221n^{9} + \\
&2970477229398746689186622534784613554n^{8} + 1845274131994015990683957902602775337n^{7} + 962091291302144537393228847830431614n^{6} + \\
&412595107814836563208757757032740146n^{5} + 141540723940232563767779647013785485n^{4} + 37292931812630561528276365992452010n^{3} + \\
&70748657772254167254528728953971 00n^{2} + 858794112392644074221312049837000n + 499973867382601126036151047800000\Big)P[n+3]
\end{aligned}
$$

$$+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} +$$
$$3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} +$$
$$38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} +$$
$$69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} +$$
$$32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} +$$
$$1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} +$$
$$20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} +$$
$$260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} +$$
$$2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} +$$
$$10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} +$$
$$34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} +$$
$$69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} +$$
$$82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} +$$
$$55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} +$$
$$19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} +$$
$$3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} +$$
$$168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n +$$
$$127412073235174465112560388640 0\Big)P[n+4]$$

$$
\begin{aligned}
&-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + \\
&1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + \\
&136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + \\
&83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + \\
&12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + \\
&510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + \\
&6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + \\
&18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + \\
&29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + \\
&18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + \\
&2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + \\
&19313175036907229252501700n + 1248723341516324359641600\Big)\mathrm{P}[n+5]==0;
\end{aligned}
$$

In[4]:= $\textbf{recSol} = \textbf{SolveRecurrence}[\textbf{rec}, \textbf{P}[\textbf{n}]]$

In[4]:= **recSol = SolveRecurrence[rec, P[n]]**

Out[4]=

$$\Big\{\Big\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\Big\}$$

$$\Big\{0, -\frac{(3+2n)\big(-8-9n+2n^2\big)}{(1+n)^2(1+2n)^2}\Big\}$$

$$\Big\{0, -\frac{(3+2n)\big(-5+8n^2\big)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big\}$$

$$\Big\{0, \frac{(3+2n)\big(-513-2184n-2416n^2+768n^4\big)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)\big(3+44n+48n^2\big)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)\big(3+44n+48n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big\}$$

$$\{0,\ \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2}+\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+\Big(-\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i}-\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-\frac{8(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{-1+2i}}{(1+n)(1+2n)}-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\},\ \{1,0\}\}$$

In[5]:= **sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]**

In[5]:= **sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]**

Out[5]=

$$
\begin{aligned}
&\frac{1}{3(1+n)^4(1+2n)^4}\left(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\right)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-\\
&\frac{(3+2n)\left(-3+101n+126n^2\right)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)\left(115+921n+1967n^2+1524n^3+340n^4\right)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+\frac{44(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\\
&\frac{(3+2n)\left(23+139n+130n^2\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3}{9(1+n)(1+2n)}+\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)\left(77+261n+190n^2\right)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\\
&\frac{16(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+\frac{2(3+2n)\left(13-153n-303n^2+12n^3+172n^4\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-\\
&\frac{4(3+2n)\left(-41-53n+2n^2\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+\frac{32(3+2n)\left(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\\
&\frac{4(3+2n)\left(23+139n+130n^2\right)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^2}{3(1+n)^2(1+2n)^2}+\frac{32(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^2}{3(1+n)(1+2n)}+\frac{64(3+2n)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^3}{9(1+n)(1+2n)}-\\
&\frac{16(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{i}}{3(1+n)(1+2n)}-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+\\
&\frac{128(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}\,3(1+n)(1+2n)}{+}\;\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{i}}{3(1+n)(1+2n)}+\frac{128(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{-1+2i}}{3(1+n)(1+2n)}
\end{aligned}
$$

In[6]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol,**

n, 2]//ToStandardForm, n]//CollectProdSum

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum**

Out[8]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\left(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\right)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\left(-\frac{2(3+2n)\left(147+985n+1871n^2+1268n^3+212n^4\right)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\right)S[1,2n]-\frac{4(3+2n)\left(23+123n+114n^2\right)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\left(53+229n+190n^2\right)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\left(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\right)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\left(1+8n+8n^2\right)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[6]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum**

In[9]:= **SExpansion[sol, n, 2]**

Out[9]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

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- A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, CS, K. Schönwald. The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements. Nuclear Physics B 948(114753), pp. 1-41. 2019.
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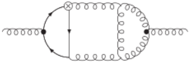
Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n,\epsilon,x)\,dx$$

Feynman integrals

DESY

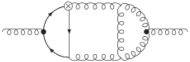
$$Dy = A\,y$$

coupled systems of linear DEs

difference ring machinery

expression in special functions

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n, \epsilon, x)\, dx$$

Feynman integrals

LHC at CERN

DESY

$$Dy = A\, y$$

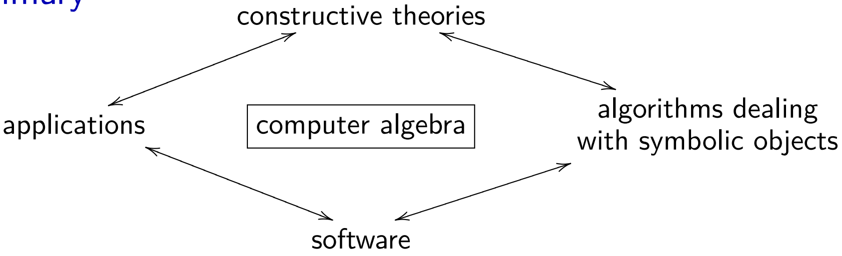
coupled systems of linear DEs

difference ring machinery

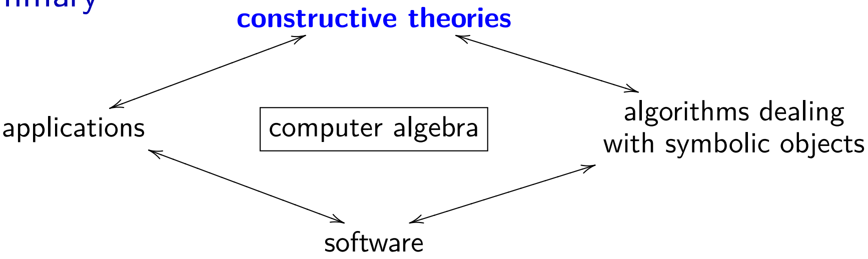
expression in special functions

applicable

Summary

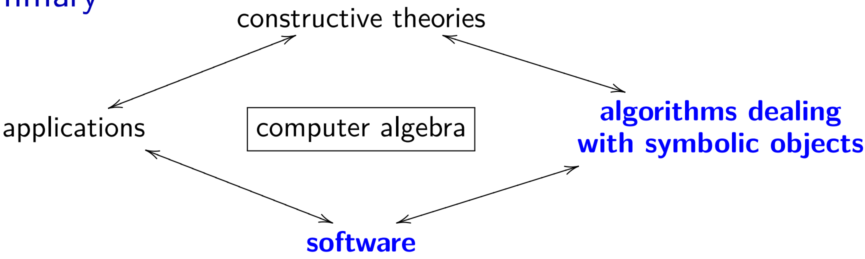

constructive theories
applications
computer algebra
algorithms dealing
with symbolic objects
software

Summary



- New constructive difference ring theory

Summary



- New constructive difference ring theory
- Efficient and stable algorithms/software packages
  - mass production of multiple sums (several millions)
  - solving large recurrences (up to order 50)
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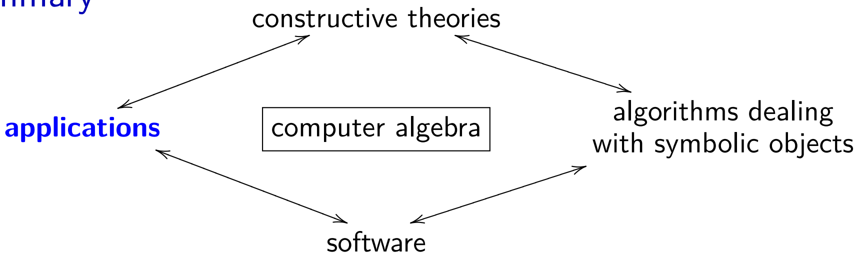
Summary



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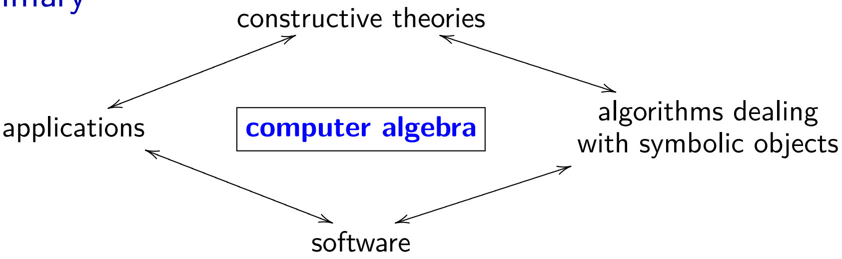
Summary



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