DESY–JKU/RISC Cooperation: 15 years, May 24, 2022

Symbolic methods for analytic solutions of differential & difference equations and related tools

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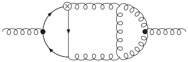
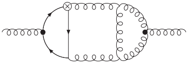
Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals

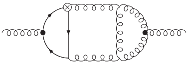


behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



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$$\int \Phi(N, \epsilon, x) dx$$

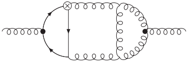
Feynman integrals

DESY

$$\sum f(N, \epsilon, k)$$

complicated multi-sums

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N, \epsilon, x)dx$$

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DESY

$$\sum f(N, \epsilon, k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions

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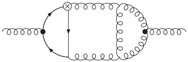

behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

LHC at CERN

DESY

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

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(Sigma-package)

expression in special functions

applicable

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||?$$

$$F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^0 + \ldots$$

$$F(\varepsilon, N) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\Big(-1-\frac{3\varepsilon}{2}\Big) \times$$
$$\times B\Big(2+k, \frac{\varepsilon}{2}\Big) B(-\varepsilon+k, -\varepsilon) B\Big(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\Big) \binom{N}{k}$$

where

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon,N)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

for general expansion methods see
J. Blümlein, CS, M. Saragnese, 2021. arXiv:2111.15501 [math-ph]

$$F(\varepsilon,N)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\ldots}$$

$$||$$

$$\Big(\sum_{k=1}^{N}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{N}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\sum_{k=1}^{N}f_{-1}(N,k)\Big)\varepsilon^{-1}+\ldots$$

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$$||$$

$$\Big(\sum_{k=1}^{N}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{N}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\boxed{\sum_{k=1}^{N}f_{-1}(N,k)}\Big)\varepsilon^{-1}+\dots$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

where

$$S_a(N) = \sum_{i=1}^{N} \frac{\mathsf{sign}(a)^i}{i^a} \text{ and } \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$\downarrow$ (summation package Sigma.m)

$$\begin{aligned}\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)&\\ -(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)&\\ +(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)&\\ =\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\tfrac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}&\end{aligned}$$

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$$\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\,\frac{1-4N}{N+1}+c_2\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

$$||$$ (recurrence finding and solving)

$$\Big(\frac{1}{12} - \frac{1}{8}\zeta_2\Big) \frac{1-4N}{N+1} + 1 \, \frac{-14N-13}{(N+1)^2}$$
$$+ \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+ \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \dots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

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Special cases:

$$S_{2,1}(n)=\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}$$ (harmonic sums)

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases:

$$\sum_{k=1}^{n}\frac{1}{(1+2k)^2}\sum_{j=1}^{k}\frac{1}{j^2}\sum_{i=1}^{j}\frac{1}{1+2i}$$

(cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases:

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}$$ (binomial sums)

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h\binom{2h}{h}\sum_{k=1}^{h}\frac{2^{2k}}{k^2\binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^{k} \frac{1}{j \binom{4j}{3j}^2}$$

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3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.

Sigma.m is based on difference ring/field theory

1. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
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21. CS. Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80(3), pp. 616-664. 2017.
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24. CS. Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation. 2021. Springer.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,-3\}]}$

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In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\rightarrow\{\varepsilon,-3,-3\}]}$

Out[5]= $\left\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)}\right\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **mySum =**

$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,-2\}]}$

Out[5]= $$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},$$
$$-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big\}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **mySum =**

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{N\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -1\}]}$

Out[5]=
$$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},$$
$$-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1},$$
$$\left(\frac{1}{12}-\frac{1}{8}\zeta(2)\right)\frac{1-4N}{N+1}+\frac{-14N-13}{(N+1)^2}+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+$$
$$\frac{(14N+13)S_1(N)}{3(N+1)^2}+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}\Big\}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Bigg(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Bigg)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left[\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)\right]$$

$$||$$

$$\left[\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right]$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

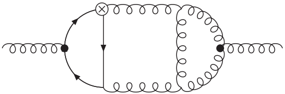
$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

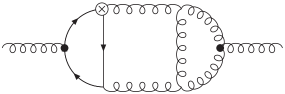
$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}$.



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+ \Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(- \quad \ldots \quad 1)^N(2N+1) \over N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \quad \ldots \quad 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big)S_1(N) + \big(\frac{3}{4} + (-1)^N\big)S_2(N)^2$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$\boxed{F_0(N)} =$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(\dots \frac{\dots 1)^N(2N+1)}{\dots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \dots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1\dots} \dots \big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \dots\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{\dots(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$F_0(N) =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(- [illegible] \frac{1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + [illegible] 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 [illegible]} [illegible])^N\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26 + 4 [illegible])^N [illegible] 1\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2([illegible]} - \frac{5}{[illegible]}\Big)S_2(N) + S_{-2}(N)\big(10S_1(N)^2 + \Big(\frac{[illegible](2N+1)}{N(N+1)}$$

$$+ \frac{4(3N - [illegible]}{N(N + [illegible]} [illegible] -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N [illegible]}{N([illegible]} [illegible] (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(- \frac{2(- [illegible]}{[illegible]} [illegible] \Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N [illegible]}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

The general tactic

Feynman integrals

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]
Blümlein, Hasselhuhn, CS, RADCOR'10, arXiv:1202.4303 [math-ph]
CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\big(-1-\frac{3\varepsilon}{2}\big) B\big(2+k, \frac{\varepsilon}{2}\big) B(-\varepsilon+k, -\varepsilon) B\big(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\big) \binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2 F(N) + \big(3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8\big) F(N+1)$$
$$-(2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - \big(2\zeta_2 - \frac{68}{3}\big)\varepsilon^0 + \dots$$

$$F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \big(\frac{\zeta_2}{4} + \frac{79}{24}\big)\varepsilon^{-1} + \dots$$
$$F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \big(\frac{\zeta_2}{3} + \frac{1415}{324}\big)\varepsilon^{-1} + \dots$$

$\downarrow$

$$\boxed{F(N) = F_{-3}(N)\,\varepsilon^{-3} + F_{-2}(N)\,\varepsilon^{-2} + F_{-1}(N)\,\varepsilon^{-1} + \dots}$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

↓ (summation package Sigma.m)

$$F(N)=\tfrac{4N}{3(N+1)}\varepsilon^{-3}-\Big(\tfrac{2(2N+1)}{3(N+1)}S_1(N)+\tfrac{2N(2N+3)}{3(N+1)^2}\Big)\varepsilon^{-2}$$
$$\Big(\tfrac{(1-4N)}{6(N+1)}S_1(N)^2-\tfrac{N\left(N^2-2\right)}{3(N+1)^3}+\tfrac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N)+\tfrac{(1-4N)}{6(N+1)}S_2(N)+\tfrac{N\zeta_2}{2(N+1)}\Big)\varepsilon^{-1}+\dots$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_x f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_y f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V-topologies

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$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

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The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

$$
\begin{aligned}
a_0(N,\varepsilon) = {} & (N+1)(N+2)\big(8\varepsilon^{10} + 104\varepsilon^9(N+3) + 4\varepsilon^8\big(96N^2 + 601N + 887\big) \\
& + 4\varepsilon^7\big(12N^3 + 414N^2 + 1583N + 1393\big) \\
& - 8\varepsilon^6\big(264N^4 + 2436N^3 + 8643N^2 + 14518N + 9947\big) \\
& - 16\varepsilon^5\big(156N^5 + 1690N^4 + 6847N^3 + 12661N^2 + 9537N + 717\big) \\
& + 32\varepsilon^4\big(68N^6 + 1158N^5 + 8155N^4 + 30114N^3 + 61712N^2 + 67616N + 31693\big) \\
& + 64\varepsilon^3\big(40N^7 + 560N^6 + 2755N^5 + 3729N^4 - 14194N^3 - 61920N^2 - 89140N - 46600\big) \\
& - 128\varepsilon^2(N+2)\big(12N^7 + 254N^6 + 2249N^5 + 10758N^4 + 30173N^3 + 50610N^2 \\
& + 49122N + 22706\big) \\
& + 256\varepsilon(N+2)^2(N+3)(N+4)\big(44N^4 + 501N^3 + 2044N^2 + 3455N + 1976\big) \\
& - 512(N+1)(N+2)^3(N+3)^2(N+4)\big(6N^2 + 47N + 95\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1(N,\varepsilon) = (N+2)\big(&-22\varepsilon^{11} - 2\varepsilon^{10}(157N+435) - \varepsilon^{9}\big(1500N^2+8611N+11745\big)\\
&- \varepsilon^{8}\big(2548N^3+22936N^2+63597N+54229\big)\\
&+ 4\varepsilon^{7}\big(266N^4+1857N^3+6065N^2+14351N+15987\big)\\
&+ 8\varepsilon^{6}\big(994N^5+12961N^4+67246N^3+174692N^2+226821N+116092\big)\\
&+ 16\varepsilon^{5}\big(336N^6+5348N^5+33569N^4+104918N^3+165290N^2+108259N+6100\big)\\
&- 16\varepsilon^{4}\big(404N^7+7578N^6+61778N^5+284762N^4+802660N^3+1382074N^2\\
&+ 1340455N+560287\big)\\
&- 64\varepsilon^{3}\big(94N^8+1823N^7+14305N^6+55870N^5+96299N^4-37256N^3\\
&- 447044N^2-704959N-379338\big)\\
&+ 128\varepsilon^{2}(N+3)\big(30N^8+715N^7+7667N^6+48253N^5+194086N^4+507439N^3\\
&+ 835393N^2+785327N+320382\big)\\
&- 256\varepsilon(N+2)(N+3)^2\big(107N^6+2070N^5+16342N^4+67226N^3+151557N^2\\
&+ 176932N+83196\big)\\
&+ 256(N+2)^3(N+3)^3(N+4)\big(30N^3+331N^2+1193N+1386\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_2(N,\varepsilon) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17N+45) + 2\varepsilon^{10}\big(620N^2+3553N+4795\big)\\
&+ 2\varepsilon^{9}\big(1504N^3+14190N^2+41901N+38907\big)\\
&+ 4\varepsilon^{8}\big(172N^4+4983N^3+30942N^2+69119N+50850\big)\\
&- 4\varepsilon^{7}\big(1996N^5+24056N^4+113313N^3+269119N^2+337198N+185290\big)\\
&- 16\varepsilon^{6}\big(450N^6+8210N^5+59749N^4+227386N^3+486841N^2+563176N+275664\big)\\
&+ 16\varepsilon^{5}\big(340N^7+4314N^6+19137N^5+25532N^4-55105N^3-206516N^2-191528N\\
&- 23458\big)\\
&+ 32\varepsilon^{4}\big(140N^8+2940N^7+26550N^6+139926N^5+493839N^4+1240186N^3\\
&+ 2161699N^2+2304248N+1100084\big)\\
&+ 64\varepsilon^{3}\big(4N^9+506N^8+8651N^7+63510N^6+236215N^5+395334N^4-105413N^3\\
&- 1551017N^2-2362944N-1217770\big)\\
&- 128\varepsilon^{2}(N+3)\big(12N^9+314N^8+3782N^7+29105N^6+160727N^5+640273N^4\\
&+ 1750874N^3+3052505N^2+3017094N+1276604\big)\\
&+ 256\varepsilon(N+2)(N+3)^2(N+4)\big(26N^6+825N^5+8967N^4+46529N^3+125411N^2\\
&+ 168628N+88652\big)\\
&- 512(N+1)(N+2)^2(N+3)^3(N+4)^2\big(6N^3+98N^2+459N+655\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3(N,\varepsilon) = \big(&-64\varepsilon^{12} - 8\varepsilon^{11}(113N+298) - 8\varepsilon^{10}\big(519N^2+2948N+3896\big)\\
&-4\varepsilon^9\big(1444N^3+13839N^2+39746N+34305\big)\\
&+4\varepsilon^8\big(1948N^4+17868N^3+63837N^2+112966N+84655\big)\\
&+16\varepsilon^7\big(1456N^5+20460N^4+112365N^3+304963N^2+412258N+221769\big)\\
&-8\varepsilon^6\big(320N^6+2050N^5+4192N^4+27408N^3+174901N^2+411759N+324872\big)\\
&-16\varepsilon^5\big(1756N^7+33154N^6+265889N^5+1186719N^4+3218059N^3+5349388N^2\\
&+5071913N+2113696\big)\\
&+32\varepsilon^4\big(188N^8+4802N^7+59527N^6+439922N^5+2025336N^4+5813984N^3\\
&+10076450N^2+9621283N+3878602\big)\\
&+64\varepsilon^3\big(140N^9+2768N^8+22500N^7+99545N^6+287700N^5+723136N^4\\
&+1854572N^3+3714620N^2+4272517N+2031600\big)\\
&-128\varepsilon^2\big(24N^{10}+830N^9+14362N^8+152630N^7+1053620N^6+4834279N^5\\
&+14824351N^4+29964399N^3+38244797N^2+27875896N+8824032\big)\\
&+256\varepsilon(N+2)(N+3)(N+4)\big(118N^7+2639N^6+24247N^5+118311N^4+329565N^3\\
&+520306N^2+426076N+136854\big)\\
&-512(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)\big(12N^3+97N^2+230N+144\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_4(N,\varepsilon) = \big(&64\varepsilon^{12} + 192\varepsilon^{11}(5N+14) + 16\varepsilon^{10}\big(297N^2 + 1769N + 2451\big)\\
&+ 16\varepsilon^9\big(453N^3 + 4462N^2 + 13094N + 11244\big)\\
&- 8\varepsilon^8\big(1084N^4 + 11117N^3 + 47258N^2 + 103981N + 94650\big)\\
&- 8\varepsilon^7\big(3304N^5 + 51138N^4 + 311957N^3 + 948722N^2 + 1440105N + 858544\big)\\
&+ 16\varepsilon^6\big(420N^6 + 5507N^5 + 36275N^4 + 169650N^3 + 536911N^2 + 952507N + 694370\big)\\
&+ 16\varepsilon^5\big(1828N^7 + 38868N^6 + 353301N^5 + 1801014N^4 + 5604391N^3 + 10664390N^2\\
&+ 11433064N + 5260048\big)\\
&- 32\varepsilon^4\big(316N^8 + 8356N^7 + 105800N^6 + 802421N^5 + 3836854N^4 + 11588223N^3\\
&+ 21401558N^2 + 22066744N + 9745752\big)\\
&- 64\varepsilon^3\big(116N^9 + 2424N^8 + 19923N^7 + 82966N^6 + 208191N^5 + 530980N^4 + 1847484N^3\\
&+ 4687014N^2 + 6120858N + 3111104\big)\\
&+ 128\varepsilon^2\big(24N^{10} + 826N^9 + 14897N^8 + 172000N^7 + 1314686N^6 + 6710299N^5\\
&+ 22873183N^4 + 51298261N^3 + 72551278N^2 + 58573022N + 20544948\big)\\
&- 256\varepsilon(N+2)(N+3)\big(106N^8 + 3278N^7 + 42903N^6 + 310942N^5 + 1366350N^4\\
&+ 3729418N^3 + 6173159N^2 + 5657732N + 2191212\big)\\
&+ 512(N+1)(N+2)^2(N+3)^2(N+4)(N+5)(N+6)\big(12N^3 + 121N^2 + 396N + 431\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_5(N,\varepsilon) = {} & (N+5)\big(-128\varepsilon^{11} - 128\varepsilon^{10}(11N+26) - 32\varepsilon^{9}\big(115N^2+592N+647\big) \\
& + 32\varepsilon^{8}\big(63N^3+430N^2+1665N+2384\big) \\
& + 16\varepsilon^{7}\big(714N^4+7881N^3+33802N^2+66225N+47654\big) \\
& - 16\varepsilon^{6}\big(234N^5+2444N^4+13989N^3+50862N^2+104083N+87848\big) \\
& - 16\varepsilon^{5}\big(580N^6+10181N^5+76586N^4+319207N^3+772120N^2+1012046N+547832\big) \\
& + 16\varepsilon^{4}\big(244N^7+5456N^6+61605N^5+401216N^4+1536277N^3+3408574N^2 \\
& + 4066436N+2026928\big) \\
& + 64\varepsilon^{3}\big(26N^8+357N^7+583N^6-11139N^5-65193N^4-120264N^3+11864N^2 \\
& + 272830N+222624\big) \\
& - 64\varepsilon^{2}(N+3)\big(12N^8+298N^7+4684N^6+49024N^5+306907N^4+1122441N^3 \\
& + 2350650N^2+2607576N+1185072\big) \\
& + 256\varepsilon(N+2)(N+3)\big(25N^7+743N^6+8856N^5+55358N^4+197497N^3+404131N^2 \\
& + 439902N+196128\big) \\
& - 256(N+1)(N+2)^2(N+3)^2(N+4)(N+6)(N+7)\big(6N^2+35N+54\big)\big).
\end{aligned}
$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

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$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

`Sigma.m` $\downarrow$ (2 hours)

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\cdots+F_4(n)\varepsilon^4+O(\varepsilon^5)$$

We get
$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$

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$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

$$\begin{aligned}F_{-1}(n) = (-1)^n &\left(\frac{2\left(9n^5+81n^4+295n^3+533n^2+500n+204\right)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)}\right)\\ &+ \frac{2\left(18n^5+150n^4+490n^3+755n^2+536n+132\right)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)}\\ &+ \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)}\right) S_2(n)\\ &+ \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)}\right) S_{-2}(n)\end{aligned}$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
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ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Find a recurrence for the integral

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

Holonomic/difference field approach (M. Round)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

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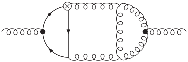
Evaluation of Feynman Integrals

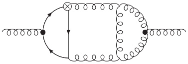


Behavior of particles

$$\longrightarrow \quad \int \Phi(N,\epsilon,x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x) dx$$

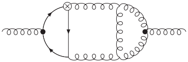
Feynman integrals

DESY

$$Dy = A\, y$$

coupled systems of linear DEs

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

DESY

$$Dy = A\,y$$

coupled systems of linear DEs

RISC

(new coupled system solver)

expression in special functions

Tactic 3: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\ldots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\ldots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \ldots\\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \ldots\\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \ldots\\ \hat{R}_\lambda(x)\end{pmatrix}$$

given

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$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinComb}(..., D^i\hat{R}_i(x), \dots)$$

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DE-solver

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

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DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

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Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$ (harmonic polylogarithms)

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

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Special cases of iterated integrals over hyperexponential functions:

$$H_{2,-2}(x) = \int_0^x \frac{1}{2-\tau_1} \int_0^{\tau_1} \frac{1}{2+\tau_2} d\tau_2 d\tau_1$$ (generalized polylogarithms)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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$$\int_0^x \frac{1}{1+\tau_1+\tau_1^2} \int_0^{\tau_1} \frac{1}{1+\tau_2^2} d\tau_2 d\tau_1 \qquad \text{(cyclotomic polylogarithms)}$$

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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$$\int_0^x \frac{1}{\sqrt{1+\tau_1}}\int_0^{\tau_1}\frac{1}{1+\tau_2}d\tau_2 d\tau_1 \qquad \text{(radical integrals)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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$$\int_0^x \frac{1}{1-\tau_1+\eta\tau_1} \int_0^{\tau_1} \sqrt{1-\tau_2}\sqrt{1-\tau_2+\eta\tau_2}\,d\tau_2 d\tau_1$$

(generalized radical integrals)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2} dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x\frac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 + c_4\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 \;\Big|\; c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}$$

Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

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$$\Updownarrow$$

There exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ with $$a_0(n) F(n) + \cdots + a_\delta(n) F(n+\delta) = 0.$$

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$\Updownarrow$ algorithmic

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Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x) \\ - \left(25x^2 - 208x\right) f''(x) - (15x - 8) f'(x) - f(x) = 0$$

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$$F(n) = \frac{1}{\binom{2n}{n}^3} = \frac{(1)_n (1)_n (1)_n (1)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!} \frac{1}{64^n}$$

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for

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Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x)$$
$$+ \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x)$$
$$+ 2\left(253x^2 - 786x + 72\right) f''(x) + 4(35x - 12) f'(x) + 4f(x) = 0$$

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$$(n+2)(n+1)^3 F(n) - 4(n+2)(2n+1)^2(2n+3) F(n+1) + 16(2n+1)^2(2n+3)^2 F(n+2) = 0$$

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$$\Big\downarrow \texttt{Sigma.m}$$

$$F(n)=\frac{1}{\binom{2n}{n}^2}\Big(c_1+c_2S_1(n)\Big)=\frac{(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!}\frac{1}{16^n}\Big(c_1+c_2S_1(n)\Big)$$

Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2\left[\begin{matrix}1,1,1\\ \frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{16}\right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

for

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DE-solver REC-solver

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

Tactic 3: the DE-REC approach

DE system

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→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

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$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

↓ holonomic closure prop. ↓

linear recurrence

$$\sum_i a'_i(N) I_1(N+i) = r'(N)$$

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uncoupled DE system
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↓ holonomic closure prop. ↓

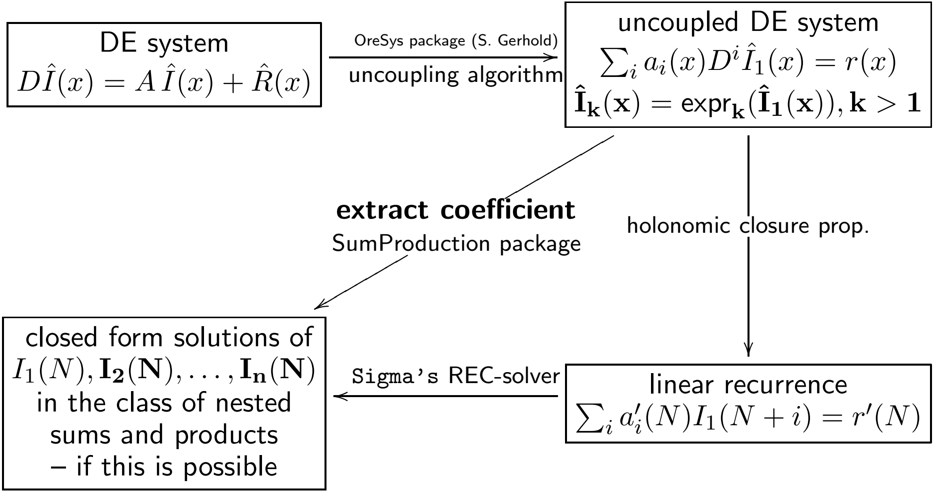
$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

linear recurrence
$$\sum_i a_i'(N) I_1(N+i) = r'(N)$$

← Sigma's REC-solver ←

closed form solutions of $I_1(N)$ in the class of nested sums and products – if this is possible

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

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$\downarrow$ IBP methods

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$\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_{0}(N) + \dots$$

General strategy: physical problem $\hat{P}(x)$

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- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_{0}(N) + \ldots$$

$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$$

$$P(N) = \varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N) + \varepsilon^{0} P_{0}(N) + \ldots$$

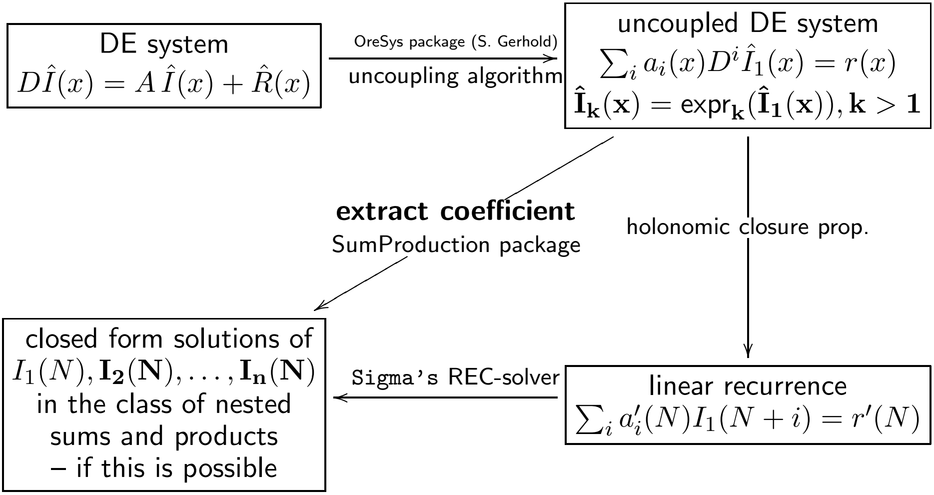
Calculations based on Tactic 3:

- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSN 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor Number Scheme at Next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.

Tactic 4: Compute large moments and guessing recurrences

[coming, e.g., from IBP methods]

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Tactic 4: compute large moments (SolveCoupledSystem package)

DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system
$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$
$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

extract coefficient
SumProduction package

holonomic closure prop.

linear recurrence
$\sum_i a'_i(N) I_1(N+i) = r'(N)$

← `Sigma's` REC-solver ←

closed form solutions of
$I_1(N)$
in the class of nested
sums and products
– if th

Given $r'(0), r'(1), \ldots, r'(5114)$,
compute $I_1(0), I_1(1), \ldots, I_1(5114)$

J. Blümlein, CS, 2017, arXiv:1701.04614 [hep-ph].

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \underbrace{\varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_0(N) + \dots}_{\text{only numbers}}$$

$N = 0, 1, \dots, 8000$

$\downarrow$ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$

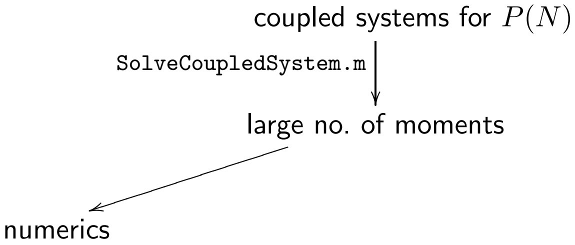
$$P(N) = \underbrace{\varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_0(N)}_{\text{numbers}} + \dots$$

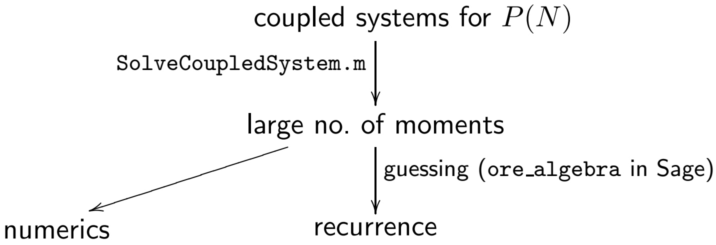
$N = 0, 1, \dots, 8000$

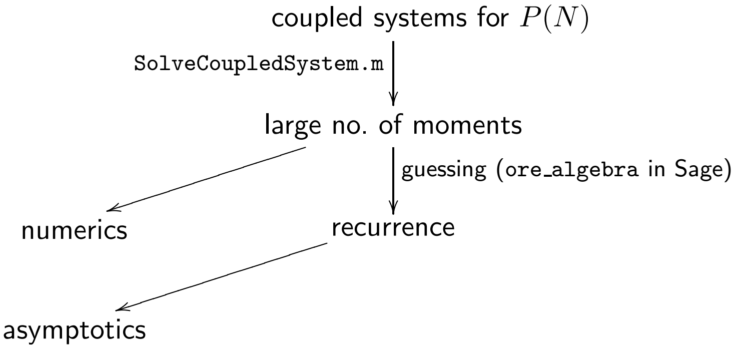
coupled systems for $P(N)$

`SolveCoupledSystem.m` ↓

large no. of moments


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
numerics


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence


coupled systems for $P(N)$
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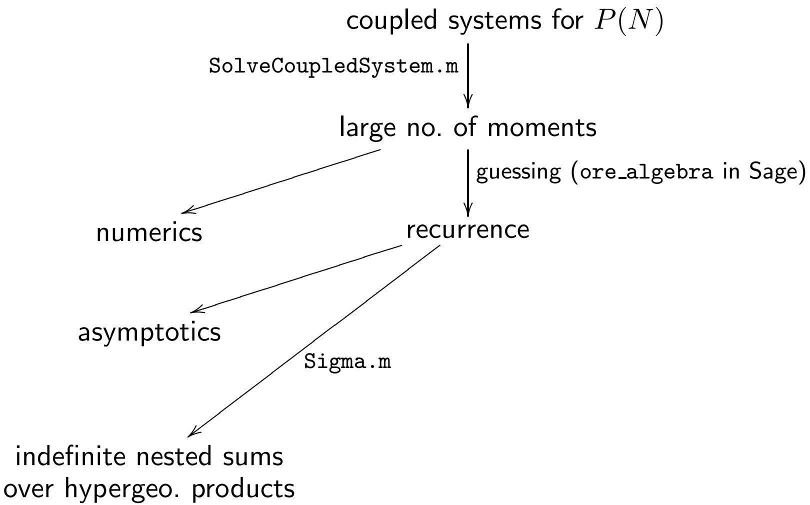
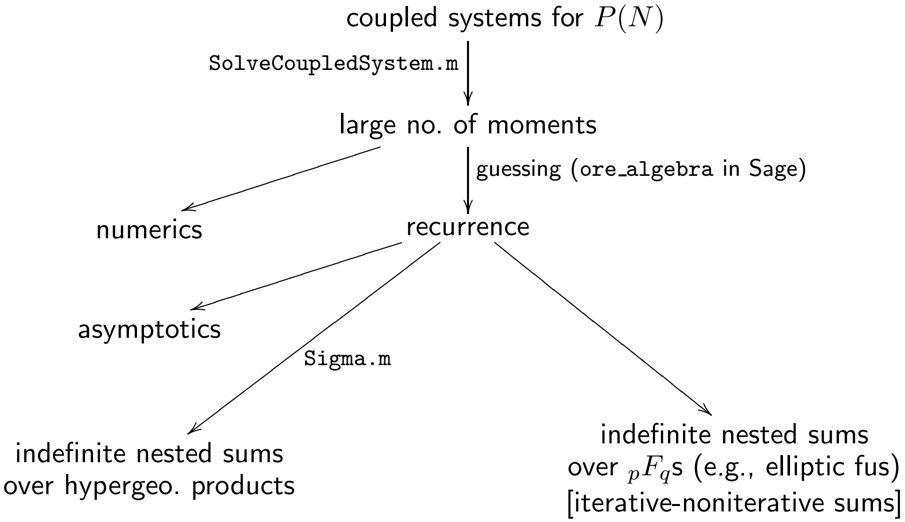
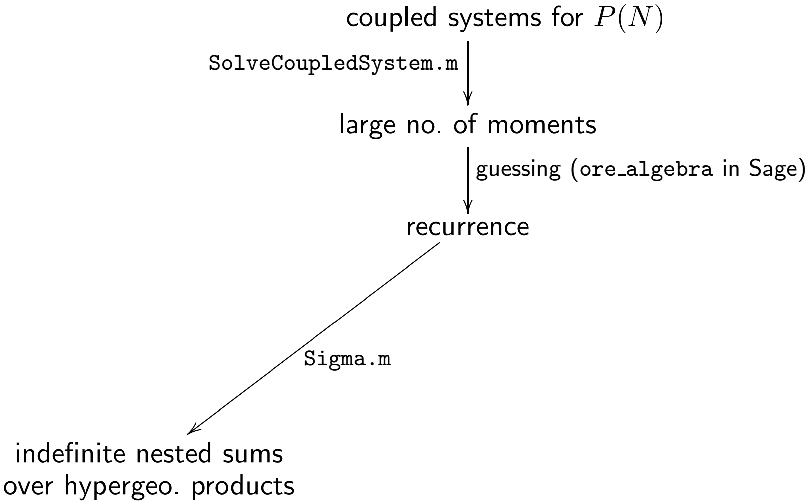
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics


coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products



coupled systems for $P(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums over hypergeo. products
indefinite nested sums over ${}_pF_q$s (e.g., elliptic fus) [iterative-noniterative sums]


coupled systems for $P(N)$
SolveCoupledSystem.m
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indefinite nested sums
over hypergeo. products

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$N = 0, 1, \ldots, 8000$

$\downarrow$ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$

$$P(N) = \underbrace{\varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(N)}_{\text{numbers}} + \ldots$$

$N = 0, 1, \ldots, 8000$

General strategy: physical problem $\hat{P}(x)$

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$\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \underbrace{\varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \varepsilon^{0} F_0(N) + \ldots}_{\text{only numbers}}$$

$\downarrow$ plug into $\hat{P}(x) = \sum_{N=0}^{\infty} P(N) x^N$

$$P(N) = \underbrace{\varepsilon^{-3} P_{-3}(N) + \varepsilon^{-2} P_{-2}(N) + \varepsilon^{-1} P_{-1}(N)}_{\text{nice}} + \underbrace{\varepsilon^{0} P_0(N)}_{\text{partially nice}} + \ldots$$

all N solution

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[8]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= **initial =<< iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[8]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= **initial =<< iFile16**

Out[9]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

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4942286209404552399423l/753773992230616156800,

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52092747218367551684487777/80949984111854180459136,

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14287296426323024679514269058446918378052 99/239407595750341228278613159615736736000 00,

49893869021959529450510280919915455078308076 7/846888366785297981317126230405400272000000 0, In[10]:= **rec =<< rFile16**

Out[10]=

$$(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + 2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + 873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + 1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + 575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + 69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + 901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + 7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + 32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + 86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + 124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + 87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + 25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + 1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 1613151061671936n^{37} + \\
&101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
&2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
&9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
&7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
&356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
&9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
&170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
&1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
&13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
&59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
&167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
&284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
&279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + \\
&149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + \\
&38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + \\
&3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + \\
&101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\,P[n+1]
\end{aligned}
$$

$$+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\Big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+$$

$$274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+$$

$$25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+$$

$$23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+$$

$$6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+$$

$$722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+$$

$$28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+$$

$$424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+$$

$$1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+$$

$$3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+$$

$$4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+$$

$$3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+$$

$$1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+$$

$$371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+$$

$$6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\Big)P[n+2]$$

$$-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+$$
$$276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+$$
$$25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+$$
$$24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+$$
$$7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+$$
$$745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+$$
$$29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+$$
$$445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+$$
$$1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+$$
$$3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+$$
$$4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+$$
$$4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+$$
$$1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+$$
$$412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+$$
$$7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)f[n+3]$$

$$+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} +$$
$$3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} +$$
$$38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} +$$
$$69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} +$$
$$32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} +$$
$$1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} +$$
$$20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} +$$
$$260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} +$$
$$2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} +$$
$$10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} +$$
$$34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} +$$
$$69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} +$$
$$82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} +$$
$$55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} +$$
$$19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} +$$
$$3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} +$$
$$168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n +$$
$$1274120732351744651125603886400\Big)P[n+4]$$

$$
\begin{aligned}
&-(\mathrm{n}+5)^2(\mathrm{n}+6)^4(2\mathrm{n}+5)(2\mathrm{n}+7)(2\mathrm{n}+9)^3(2\mathrm{n}+11)^4\Big(309237645312\mathrm{n}^{32}+28361279668224\mathrm{n}^{31}+\\
&1249518729297920\mathrm{n}^{30}+35220794552352768\mathrm{n}^{29}+713726163159089152\mathrm{n}^{28}+11076866026783113216\mathrm{n}^{27}+\\
&136959486138712588288\mathrm{n}^{26}+1385658801437173350400\mathrm{n}^{25}+11691772665924577918976\mathrm{n}^{24}+\\
&83438339505976242995200\mathrm{n}^{23}+508989054278115477684224\mathrm{n}^{22}+2675508113418826174332928\mathrm{n}^{21}+\\
&12193213796145039633072128\mathrm{n}^{20}+48399020537651722726242304\mathrm{n}^{19}+167881257973769248139515904\mathrm{n}^{18}+\\
&510012482113388176546187776\mathrm{n}^{17}+1358662126092561923541267968\mathrm{n}^{16}+3174925021159974655053814528\mathrm{n}^{15}+\\
&6504205668151125355938798848\mathrm{n}^{14}+11663792381020901870157176128\mathrm{n}^{13}+\\
&18263581057905911985340656960\mathrm{n}^{12}+24881010123632244515458585528\mathrm{n}^{11}+\\
&29346856353503020415409305704\mathrm{n}^{10}+29775859546803351930591002266\mathrm{n}^{9}+25770328899499991754425455738\mathrm{n}^{8}+\\
&18817114309842270306167785140\mathrm{n}^{7}+11424980760825630752861027739\mathrm{n}^{6}+5656051955667821083952617134\mathrm{n}^{5}+\\
&2221448212382554437709999491\mathrm{n}^{4}+664859653803075491350122060\mathrm{n}^{3}+142190920852333874895041748\mathrm{n}^{2}+\\
&19313175036907229252501700\mathrm{n}+1248723341516324359641600\Big)\,\mathrm{P}[\mathrm{n}+5]==0
\end{aligned}
$$

In[11]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

In[11]:= **recSol = SolveRecurrence[rec, P[n]]**

Out[11]=

$$\{\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\}$$

$$\{0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2}\}$$

$$\{0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\}$$

$$\{0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)(3+44n+48n^2)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^2}{(1+n)(1+2n)}\}$$

$$\{0, \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2}+$$

$$\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+\Big(-$$

$$\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+$$

$$\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+$$

$$\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i}-$$

$$\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-$$

$$\frac{8(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{-1+2i}}{(1+n)(1+2n)}$$

$$-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\}, \{1,0\}\}$$

In[12]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[12]:= **sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]**

Out[12]=

$$
\begin{aligned}
&\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-\\
&\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+\\
&\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+\\
&\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+\\
&\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-\\
&\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+\\
&\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+\\
&\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-\\
&\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+\\
&\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}\,3(1+n)(1+2n)}{+}\;\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+\\
&\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}
\end{aligned}
$$

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= $\mathbf{sol = TransformToSSums[sol];}$

In[15]:= $\mathbf{sol = ReduceToBasis[MultipleSumLimit[sol,}$

$\mathbf{n, 2]//ToStandardForm, n]//CollectProdSum;}$

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

$$\text{Out[15]=}\ \frac{1}{3(1+n)^4(1+2n)^4}\left(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\right)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\left(147+985n+1871n^2+1268n^3+212n^4\right)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Big)S[1,2n]-\frac{4(3+2n)\left(23+123n+114n^2\right)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\left(53+229n+190n^2\right)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\Big)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\left(1+8n+8n^2\right)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[16]:= **SExpansion[sol, n, 2]**

Out[16]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Calculations based on Tactic 4:

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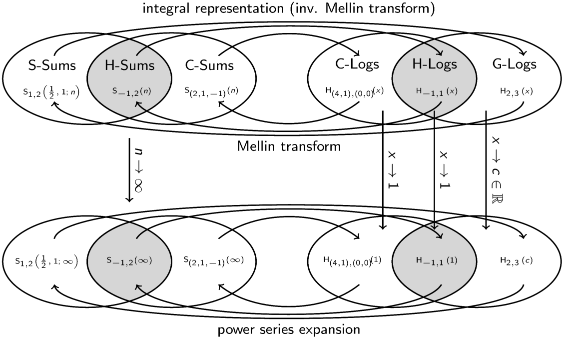
Symbolic tools for special functions

Nested sums	**Nested integrals**	**Special numbers**
Harmonic Sums $\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Harmonic Polylogarithms $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{1+z}$	multiple zeta values $\int_0^1 dx\frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2)+...$
gen. Harmonic Sums $\sum_{k=1}^{n}\frac{(1/2)^k}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	gen. Harmonic Polylogarithms $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{z-3}$	gen. multiple zeta values $\int_0^1 dx\frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3)+...$
Cycl. Harmonic Sums $\sum_{k=1}^{n}\frac{1}{(2k+1)}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Cycl. Harmonic Polylogarithms $\int_0^x \frac{dy}{1+y^2}\int_0^y \frac{dz}{1-z+z^2}$	cycl. multiple zeta values $\mathbf{C} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$
Binomial Sums $\sum_{k=1}^{n}\frac{1}{k^2}\binom{2k}{k}(-1)^k$	root-valued iterated integrals $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{z\sqrt{1+z}}$	associated numbers $\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$
	iterated integrals on ${}_2F_1$'s $\int_0^z \frac{\ln(x)}{1+x}\,{}_2F_1\left[\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]dx$	associated numbers $\int_0^1 {}_2F_1\left[\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]dx$

Nested sums	Nested integrals	Special numbers
Harmonic Sums $\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Harmonic Polylogarithms $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{1+z}$	multiple zeta values $\int_0^1 dx\frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2)+\ldots$
gen. Harmonic Sums $\sum_{k=1}^{n}\frac{(1/2)^k}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	gen. Harmonic Polylogarithms $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{z-3}$	gen. multiple zeta values $\int_0^1 dx\frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3)+\ldots$
Cycl. Harmonic Sums $\sum_{k=1}^{n}\frac{1}{(2k+1)}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Cycl. Harmonic Polylogarithms $\int_0^x \frac{dy}{1+y^2}\int_0^y \frac{dz}{1-z+z^2}$	cycl. multiple zeta values $\mathbf{C}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$
Binomial Sums $\sum_{k=1}^{n}\frac{1}{k^2}\binom{2k}{k}(-1)^k$	root-valued iterated integrals $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{z\sqrt{1+z}}$	associated numbers $\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$
	iterated integrals on ${}_2F_1$'s $\int_0^z \frac{\ln(x)}{1+x}\,{}_2F_1\left[\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]dx$	associated numbers $\int_0^1 {}_2F_1\left[\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]dx$



Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}}$$

Integral representation:

$$=\int_0^1 \frac{x^n-1}{1-x}\left(\int_0^x \frac{\int_0^y \frac{1}{1-z}dz}{y}dy-\zeta(2)\right)dx, \qquad \zeta(z):=\sum_{i=1}^{\infty}1/i^z$$

Asymptotic expansion:

$$=\left(\frac{1}{30n^5}-\frac{1}{6n^3}+\frac{1}{2n^2}-\frac{1}{n}\right)\ln(n)$$
$$-\tfrac{1}{100n^5}-\tfrac{1}{6n^4}+\tfrac{13}{36n^3}-\tfrac{1}{4n^2}-\tfrac{1}{n}+2\zeta(3)+O(\tfrac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{n}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{H_j}{j}}{i}}{k}=-\frac{21\zeta(2)^2}{20n}+\frac{1}{8n^2}+\frac{295}{216n^3}-\frac{1115}{96n^4}+O(n^{-5})$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)\zeta(2)$$
$$+2^n\Big(\frac{3}{2n}+\frac{3}{2n^2}+\frac{9}{2n^3}+\frac{39}{2n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{n}+\frac{3}{4n^2}-\frac{157}{36n^3}+\frac{19}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta(3)}{16}\Big)\zeta(2)-\frac{31\zeta(5)}{8}$$
$$+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})$$
$$+\log(2)\Big(6\zeta(2)-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n} \frac{4^j H_{j-1}}{\binom{2j}{j} j^2} = 7\zeta(3) + \sqrt{\pi}\sqrt{n}\Bigg\{\Bigg[-\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} + O(n^{-10})\Bigg] \ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} + O(n^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence and DE solver
3. coupled systems solver
4. the large moment method

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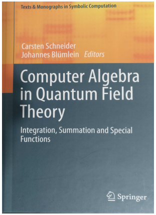

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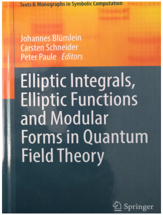

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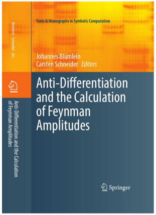

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Main CA-packages

In[17]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[18]:= << **MultiIntegrate.m**

MultiIntegrate by Jakob Ablinger © RISC-Linz

In[19]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[20]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[21]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[22]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

In[23]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

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Within the RISC-DESY cooperation we expect that we will discover and explore many

new algorithms in CA and results in QFT!