

Loops and Legs in QFT 2022

Ettal, Germany, March 27, 2021

Computer Algebra and Hypergeometric Structures for Feynman Integrals

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joint with Johannes Blümlein and Marco Saragnese (DESY, arXiv:2111.15501)

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Outline

1. Motivation: solving linear recurrences
and differential equations
2. Extracting HYP structures from D-equations
3. Conclusion

1. Motivation

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

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$$\underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

The integrand is

- hyperexponential in x, y, z :

$$\frac{D_x f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

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The integrand is

- hyperexponential in x, y, z :

$$\frac{D_y f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

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The integrand is

- ▶ hyperexponential in x, y, z :
- ▶ hypergeometric in n :

$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

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Ablinger's
MultiIntegrate.m

↓ (9 hours)

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

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$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

recurrence solver \downarrow

$F(\varepsilon, n) =$ expression in terms of special functions

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n

$h(n)$: expression in indefinite nested sums
defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n + \delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

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DECIDE constructively if $F(n)$ can be expressed in terms **indefinite nested sums** defined over **hypergeometric products**.

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Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j} \quad (\text{harmonic sums})$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];

J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases of **indefinite nested sums** over hypergeometric products:

$$\sum_{k=1}^n \frac{2^k}{k} \sum_{i=1}^k \frac{2^{-i}}{i} \sum_{j=1}^i \frac{S_1(j)}{j} \quad (\text{generalized harmonic sums})$$

- S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
 J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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$$\sum_{k=1}^n \frac{1}{(1+2k)^2} \sum_{j=1}^k \frac{1}{j^2} \sum_{i=1}^j \frac{1}{1+2i} \quad (\text{cyclotomic harmonic sums})$$

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases of **indefinite nested sums** over hypergeometric products:

$$\sum_{j=1}^n \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \quad (\text{binomial sums})$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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$$\sum_{h=1}^n 2^{-2h} (1 - \eta)^h \binom{2h}{h} \sum_{k=1}^h \frac{2^{2k}}{k^2 \binom{2k}{k}} \quad (\text{generalized binomial sums})$$

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\sum_{k=1}^n \left(\prod_{i=1}^k \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^k \frac{1}{j \binom{4j}{3j}^2}$$

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Ablinger's
MultiIntegrate.m

(9 hours)

↓

$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

recurrence solver

↓

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$$a_0(\varepsilon, n)F(\varepsilon, n) + a_1(\varepsilon, n)F(\varepsilon, n+1) + \dots + a_5(\varepsilon, n)F(\varepsilon, n+5) = 0$$

Sigma.m

↓

$$F(\varepsilon, n) = F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \dots$$

A refined recurrence solver (`Sigma.m`)

GIVEN a recurrence

$a_0(\varepsilon, n), \dots, a_\delta(\varepsilon, n)$: polynomials in ε, n
 $h_l(n), h_{l+1}(n), \dots, h_\lambda(n)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$\begin{aligned} a_0(\varepsilon, n)F(\varepsilon, n) + \dots + a_\delta(\varepsilon, n)F(\varepsilon, n + \delta) \\ = h_l(n)\varepsilon^l + h_{l+1}(n)\varepsilon^{l+1} + \dots h_\lambda(n)\varepsilon^r + O(\varepsilon^{r+1}); \end{aligned}$$

together with ε -expansions of $F(0), \dots, F(\delta - 1)$ up to a certain order.

A refined recurrence solver (`Sigma.m`)

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together with ε -expansions of $F(0), \dots, F(\delta - 1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(n)$ of

$$F(n) = F_l(n)\varepsilon^l + F_{l+1}(n)\varepsilon^{l+1} + \dots + F_\lambda(n)\varepsilon^r + O(\varepsilon^{r+1})$$

can be given in terms of **indefinite nested sums** defined over **hypergeometric products**.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

↓ (package `MultiIntegrate.m`)

$$a_0(\varepsilon, n)F(n) + a_1(\varepsilon, n)F(n+1) + \dots + a_5(\varepsilon, n)F(n+5) = 0$$

$$F(2) = \frac{20}{27\varepsilon^3} - \frac{40}{27\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{1393}{486} + \frac{5\zeta_2}{18} \right) + \dots$$

⋮

$$F(6) = \frac{22}{147\varepsilon^3} - \frac{535}{2058\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{630043}{1234800} + \frac{11\zeta_2}{196} \right) + \dots$$

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↓ (summation package Sigma.m)

$$F(n) = F_{-3}(n) \varepsilon^{-3} + F_{-2}(n) \varepsilon^{-2} + F_{-1}(n) \varepsilon^{-1} + \dots$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

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$$F_{-2}(n) = -\frac{4(-1)^n(3n^3+18n^2+31n+18)}{3(n+1)^3(n+2)^2} - \frac{4(6n^3+32n^2+51n+26)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

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$$\begin{aligned} F_{-1}(n) &= (-1)^n \left(\frac{2(9n^5 + 81n^4 + 295n^3 + 533n^2 + 500n + 204)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)} \right) \\ &+ \frac{2(18n^5 + 150n^4 + 490n^3 + 755n^2 + 536n + 132)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)} \\ &+ \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)} \right) S_2(n) \\ &+ \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)} \right) S_{-2}(n) \end{aligned}$$

We calculated it up to ε^4 in about 2 hours.

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
 Almquist/Zeilberger
 (Jakob Ablinger)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n + d) = h(\varepsilon, n)$$

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 ε -recurrence solver

multivariate
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(K. Wegschaider)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n + d) = h(\varepsilon, n)$$

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 ε -recurrence solver

multivariate
Almquist/Zeilberger
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
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Holonomic/difference ring approach

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n + d) = h(\varepsilon, n)$$

Motivation: solving recurrences and D-equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

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\downarrow
 uncoupling algorithms
 (Zürcher, Abramov/Zima, Gauss, ...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinCom}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinCom}(\dots, D^i\hat{R}_i(x), \dots)$$

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DE-solver

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uncoupling algorithms
 (Zürcher, Abramov/Zima, Gauss, ...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver

REC-solver

2. Extracting HYP structures from linear D-equations

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

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DECIDE constructively if $f(x)$ can be expressed in terms of **iterated integrals** defined over **hyperexponential functions**.

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Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \quad (\text{harmonic polylogarithm})$$

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

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$$H_{2,-2}(x) = \int_0^x \frac{1}{2 - \tau_1} \int_0^{\tau_1} \frac{1}{2 + \tau_2} d\tau_2 d\tau_1 \quad (\text{generalized polylogarithms})$$

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2} dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11 + 20x)f'(x) + (1 + x)(35 + 134x)f''(x) + 3(1 + x)^2(4 + 37x)f^{(3)}(x) + 18x(1 + x)^3f^{(4)}(x) = 0$$



$$\left\{ c_1 + c_2 \int_0^x \frac{1}{1 + \tau_1} d\tau_1 + c_3 \int_0^x \frac{1}{1 + \tau_1} \int_0^{\tau_1} \frac{\sqrt[3]{1 + \sqrt{1 + \tau_2}}}{1 + \tau_2} d\tau_2 d\tau_1 + c_4 \int_0^x \frac{1}{1 + \tau_1} \int_0^{\tau_1} \frac{\sqrt[3]{1 - \sqrt{1 + \tau_2}}}{1 + \tau_2} d\tau_2 d\tau_1 \mid c_1, c_2, c_3, c_4 \in \mathbb{K} \right\}$$

Connection: DE \longleftrightarrow REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

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\Updownarrow algorithmic

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Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\begin{aligned} & - (x^4 - 64x^3) f^{(4)}(x) - 2(5x^3 - 144x^2) f^{(3)}(x) \\ & - (25x^2 - 208x) f''(x) - (15x - 8) f'(x) - f(x) = 0 \end{aligned}$$

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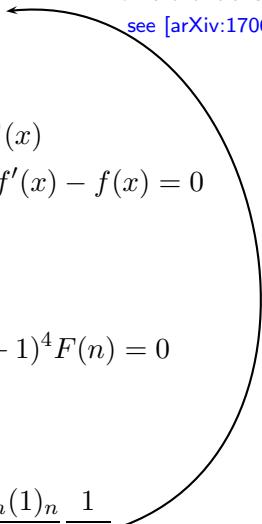
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for further transformations
see [arXiv:1706.01299]



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Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

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$$\begin{aligned} & (x^6 - 32x^5 + 256x^4) f^{(6)}(x) + (23x^5 - 528x^4 + 2560x^3) f^{(5)}(x) \\ & + (171x^4 - 2552x^3 + 6272x^2) f^{(4)}(x) + 2(245x^3 - 2002x^2 + 1728x) f^{(3)}(x) \\ & + 2(253x^2 - 786x + 72) f''(x) + 4(35x - 12) f'(x) + 4f(x) = 0 \end{aligned}$$

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↓ Sigma.m

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$$f(x) = c_1 \cdot {}_3F_2 \left[\begin{matrix} 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2} \end{matrix}; \frac{x}{16} \right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

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II. A linear DE-solver for hypergeometric structures

FIND

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such that

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$F(n)$ = expression in terms of indefinite nested sums
over hypergeometric products

III. A partial linear DE-solver

Find a power series solution

$$f(x_1, \dots, x_r) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} F(n_1, \dots, n_r) x_1^{n_1} \cdots x_r^{n_r}$$

for

$$\sum_{(s_1, \dots, s_r) \in T} \underbrace{b_{(s_1, \dots, s_r)}(x_1, \dots, x_r)}_{\in \mathbb{K}[x_1, \dots, x_r]} D_{x_1}^{s_1} \cdots D_{x_r}^{s_r} f(x_1, \dots, x_r) = 0 \quad \begin{array}{l} T \subset \mathbb{N}^r \\ \text{finite} \end{array}$$

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But: there are methods to hunt for solutions based on

M. Kauers, CS, *Partial denominator bounds for partial linear difference equations*, in: Proc. ISSAC'10 (2010)

M. Kauers, CS, *A refined denominator bounding algorithm for multivariate linear difference equations*, in: Proc. ISSAC'11 (2011)

J. Blümlein, M. Saragnese, CS, *Hypergeometric Structures in Feynman Integrals*, arXiv:2111.15501 [math-ph]

$$\begin{aligned} & (n+1)^2 (k + n^2 + 2) (3kn^2 - 4k^2 - 5kn - 12k + 2n^3 + 2n^2 - 8n - 8) F(n, k + 1) \\ & + (n+1)^2 (k + n^2 + 3) (2k^2 - 2kn^2 + 2kn + 6k - n^3 - n^2 + 4n + 4) F(n, k + 2) \\ & + (n+1)^2 (k + n + 1) (2k - n^2 + n + 4) (k + n^2 + 1) F(n, k) \\ & - (k + 1)n^2(n + 2)^2 (k + n^2 + 2n + 2) F(n + 1, k) \\ & + kn^2(n + 2)^2 (k + n^2 + 2n + 3) F(n + 1, k + 1) = 0 \end{aligned}$$

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& + (n+1)^2 (k+n+1) (2k - n^2 + n + 4) (k+n^2+1) F(n, k) \\
& - (k+1)n^2(n+2)^2 (k+n^2+2n+2) F(n+1, k) \\
& + kn^2(n+2)^2 (k+n^2+2n+3) F(n+1, k+1) = 0
\end{aligned}$$

$$\begin{array}{c}
\downarrow \\
W = \{S_1(k), S_1(n+k), S_{2,1}(n+k)\} \\
\text{degree bound 5}
\end{array}$$

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& + (n+1)^2 (k+n+1) (2k - n^2 + n + 4) (k+n^2+1) F(n, k) \\
& - (k+1)n^2(n+2)^2 (k+n^2+2n+2) F(n+1, k) \\
& + kn^2(n+2)^2 (k+n^2+2n+3) F(n+1, k+1) = 0
\end{aligned}$$

$$\begin{array}{c}
\downarrow \\
W = \{S_1(k), S_1(n+k), S_{2,1}(n+k)\} \\
\text{degree bound 5}
\end{array}$$

$$37 \text{ solutions } \frac{p}{(1+n)^2(1+k+n^2)} \text{ with}$$

$$\begin{aligned}
& (n+1)^2 (k+n^2+2) (3kn^2 - 4k^2 - 5kn - 12k + 2n^3 + 2n^2 - 8n - 8) F(n, k+1) \\
& + (n+1)^2 (k+n^2+3) (2k^2 - 2kn^2 + 2kn + 6k - n^3 - n^2 + 4n + 4) F(n, k+2) \\
& + (n+1)^2 (k+n+1) (2k - n^2 + n + 4) (k+n^2+1) F(n, k) \\
& - (k+1)n^2(n+2)^2 (k+n^2+2n+2) F(n+1, k) \\
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37 solutions $\frac{p}{(1+n)^2(1+k+n^2)}$ with

$$\begin{aligned}
p \in \left\{ 1 + \frac{1}{2}nS_1(k+n), k, n, kn, kn^2, kn^3, kn^4, kS_1(n), knS_1(n), kn^2S_1(n), kn^3S_1(n), kS_1(n)^2, \right. \\
knS_1(n)^2, kn^2S_1(n)^2, kS_1(n)^3, knS_1(n)^3, kS_1(n)^4, kS_{2,1}(n), knS_{2,1}(n), kn^2S_{2,1}(n), kn^3S_{2,1}(n), \\
kS_1(n)S_{2,1}(n), knS_1(n)S_{2,1}(n), kn^2S_1(n)S_{2,1}(n), kS_1(n)^2S_{2,1}(n), knS_1(n)^2S_{2,1}(n), \\
kS_1(n)^3S_{2,1}(n), kS_{2,1}(n)^2, knS_{2,1}(n)^2, kn^2S_{2,1}(n)^2, kS_1(n)S_{2,1}(n)^2, knS_1(n)S_{2,1}(n)^2, \\
\left. kS_1(n)^2S_{2,1}(n)^2, kS_{2,1}(n)^3, knS_{2,1}(n)^3, kS_1(n)S_{2,1}(n)^3, kS_{2,1}(n)^4 \right\}
\end{aligned}$$

IV. A partial linear coupled DE-solver (arXiv:2111.15501)

Find a power series solution

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F(n, m) x^n y^m$$

for

$$(x-1)yD_{xy}f(x, y) + (x(2\varepsilon + \frac{7}{2}) - \varepsilon + 1)D_x f(x, y) \\ + (x-1)xD_x^2 f(x, y) + y(2\varepsilon + 1)D_y f(x, y) + \frac{3}{2}(2\varepsilon + 1)f(x, y) = 0,$$

$$x(y-1)D_{xy}f(x, y) + x(4 - \varepsilon)D_x f(x, y) + (y-1)yD_y^2 f(x, y) \\ + (y(\frac{13}{2} - \varepsilon) - \varepsilon + 1)D_y f(x, y) + \frac{3(4-\varepsilon)}{2}f(x, y) = 0.$$

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$$\frac{3}{2}(2\varepsilon + 1)F(n, m) - n(\varepsilon - 1)F(n + 1, m) = 0, \\ -\frac{3}{2}(\varepsilon - 4)F(n, m) - m(\varepsilon - 1)F(n, m + 1) = 0.$$

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↓simplified Ore-Sato Theorem

$$F(n, m) = \left(\prod_{i=1}^n \frac{(1 + 2i)(3 + i - \varepsilon)}{2i(-2 + i + \varepsilon)} \right) \prod_{i=1}^m \frac{(1 + 2i + 2n)(i + 2\varepsilon)}{2i(-2 + i + n + \varepsilon)}$$

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Find a power series solution

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F(n, m) x^n y^m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{m+n} (4-\varepsilon)_n (1+2\varepsilon)_m}{m! n! (-1+\varepsilon)_{m+n}}$$

for

$$(x-1)yD_{xy}f(x, y) + (x(2\varepsilon + \frac{7}{2}) - \varepsilon + 1)D_x f(x, y) + (x-1)xD_x^2 f(x, y) + y(2\varepsilon + 1)D_y f(x, y) + \frac{3}{2}(2\varepsilon + 1)f(x, y) = 0,$$

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↓simplified Ore-Sato Theorem

$$F(n, m) = \frac{\left(\frac{3}{2}\right)_{m+n} (4-\varepsilon)_n (1+2\varepsilon)_m}{m! n! (-1+\varepsilon)_{m+n}}$$

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$F_{-1}(n, m)\varepsilon^{-1} + F_0(n, m)\varepsilon^0 + \dots$

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$$F_{-1}(n, m) = -\frac{1}{6} \frac{x^m y^n (3+n)! \left(\frac{3}{2}\right)_{m+n}}{n! (-2+m+n)!}$$

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 &= \varepsilon^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{-1}(n, m) + \varepsilon^0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_0(n, m) + \dots
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 \end{aligned}$$

|| Sigma.m

$$\begin{aligned}
 &\varepsilon^{-1} \left[-\frac{P_1(x, y)}{64(-1+x)^2(-1+y)^5(x-y)^3} - \frac{15x^6}{4(-1+x)^2(x-y)^4} \sum_{i=1}^{\infty} \frac{x^i \left(\frac{3}{2}\right)_i}{i!} \right. \\
 &\left. + \frac{P_2(x, y)}{64(-1+y)^5(x-y)^4} \sum_{i=1}^{\infty} \frac{y^i \left(\frac{3}{2}\right)_i}{i!} \right] + \varepsilon^0 [\dots] + \dots
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 f(x, y) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{m+n} (4-\varepsilon)_n (1+2\varepsilon)_m}{\underbrace{m!n!(-1+\varepsilon)_{m+n}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\dots}} \\
 &= \varepsilon^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{-1}(n, m) + \varepsilon^0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_0(n, m) + \dots
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 \end{aligned}$$

||

$$\varepsilon^{-1} \left[-\frac{15x^6}{4(x-y)^4(1-x)^{7/2}} - \frac{15y^3 Q(x, y)}{64(x-y)^4(1-y)^{13/2}} \right] + \varepsilon^0 [\dots] + \dots$$

V. Advanced expansion and summation

Consider the innocent DE:

$$(1 - x)x f''(x) - (4\varepsilon + 2x - 3)f'(x) - (\varepsilon + 1)f(x) = 0$$

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with

$$F_0 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^n (1 - i + i^2)}{n!(3)_n}$$

$$F_1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^n (1 - i + i^2)}{n!(3)_n} \left(-\frac{2n(3n+5)}{(n+1)(n+2)} + 4S_1(n) + \sum_{i=1}^n \frac{1}{1-i+i^2} \right)$$

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$$F_1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^n (1 - i + i^2)}{n!(3)_n} \left(-\frac{2n(3n+5)}{(n+1)(n+2)} + 4S_1(n) + \sum_{i=1}^n \frac{1}{1 - i + i^2} \right)$$

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$$\stackrel{\text{Sigma.m}}{=} -\frac{8}{3} + \frac{8C}{3} - \frac{20 \cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{9\pi} + \frac{2 \cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi} \sum_{i=1}^{\infty} \frac{1}{1 - i + i^2}$$

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with the extra constant

$$C = \sum_{k=1}^{\infty} \left(\frac{1}{\pi k} \cosh \left[\frac{\sqrt{3}\pi}{2} \right] - \frac{1}{(k!)^2} \prod_{l=1}^k (1 - l + l^2) \right)$$

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$$F_1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^n (1-i+i^2)}{n!(3)_n} \left(-\frac{2n(3n+5)}{(n+1)(n+2)} + 4S_1(n) + \sum_{i=1}^n \frac{1}{1-i+i^2} \right)$$

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$$= 1 + \frac{2 \cosh \left[\frac{\sqrt{3}\pi}{2} \right] \left\{ \Re \left[\psi \left(\frac{1}{2} + \frac{i\sqrt{3}}{2} \right) \right] + \gamma_E \right\}}{\pi}$$

with γ_E the Euler–Mascheroni constant and $\psi(x)$ the digamma function.

Conclusion

1. Up-to-date solver for linear recurrences and DEs (within `Sigma.m` and `HarmonicSums.m`)
2. Interplay: DE solver \longleftrightarrow RE solver
3. Finding (generalized) hypergeometric structures from DEs
4. A first prototype method to solve partial linear DE/RE equations in QCD
5. A tactic to find Apell-like structures of coupled DE-systems
6. Advance expansion and summation methods for the simplification of the found solutions