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Multi-Summation in Difference Rings and Applications

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Der Wissenschaftsfonds.

Outline

1. A warm-up example

2. The difference ring machinery for symbolic summation

3. Challenging applications

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad (=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= << **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= **SigmaLimit[res, {n}, a]**

Out[4]= $$\frac{1}{n!}\,\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Telescoping

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

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for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n, k)$

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n,\ c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) \Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\, f(n,k) + \sum_{k=1}^{a} c_1(n)\, f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n) f(n, k) + c_1(n)\, f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n) \sum_{k=1}^{a} f(n, k) + c_1(n) \sum_{k=1}^{a} f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n) f(n, k) + c_1(n)\, f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad = \quad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $$\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)\,(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, n][[1]]$

Out[6]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a] + S[1,n] - S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec} = \mathbf{LimitRec}[\mathbf{rec}, \mathbf{SUM}[n], \{n\}, a]$

Out[7]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n] + 1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[n]]$

Out[8]= $\left\{\left\{0, \frac{1}{n(n+1)}\right\}, \left\{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\right\}\right\}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-nSUM[n]+(1+n)(2+n)SUM[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-nSUM[n] + (1+n)(2+n)SUM[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)}\Big)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Part 2: The difference ring machinery for symbolic summation

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

$$-2(1+n)^3(3+n)n!^2A(n)$$
$$+(1+n)\big(8+9n+2n^2\big)n!A(1+n)-A(2+n)=0$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\prod_{i=1}^{n}i!+c_2\Big(-2^n n!\prod_{i=1}^{n}i!+\tfrac{3}{2}\prod_{i=1}^{n}i!\sum_{i=1}^{n}2^i i!\Big)\mid c_1,c_2\in\mathbb{K}\Big\}$$

$$\begin{aligned}&\big(1+S_1(n)+nS_1(n)\big)^2\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2A(n)\\-&(1+n)(3+2n)S_1(n)\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2A(n+1)\\&\qquad+(1+n)^2(2+n)^3S_1(n)\big(1+S_1(n)+nS_1(n)\big)A(n+2)=0\end{aligned}$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\,S_1(n)\prod_{l=1}^{n}S_1(l)+c_2S_1(n)^2\prod_{l=1}^{n}S_1(l)\mid c_1,c_2\in\mathbb{K}\Big\}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

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FIND all solutions expressible by indefinite nested products/sums
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3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

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$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

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$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

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In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}}\ [s]}_{\text{polynomial ring}}$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{aligned} \mathrm{ev}' : \quad \mathbb{Q}(x) \times \mathbb{N} \quad &\to \quad \mathbb{Q} \\ \left(\tfrac{p(x)}{q(x)}, n\right) \quad &\mapsto \quad \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\mathbf{ev(s,n) = S_1(n)}$$

Simplify

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$$\begin{array}{llll} \mathrm{ev} : & \mathbb{Q}(x)[s] \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, n\right) & \mapsto & \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) S_1(n)^i \end{array} \qquad \mathbf{ev(s, n) = S_1(n)}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \ldots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \rangle$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\tau(x)\tau(\tfrac{1}{x}) = \langle 0,1,2,3,\dots \rangle \langle 0,1,\tfrac{1}{2},\tfrac{1}{3},\dots \rangle$$

$$|| $$

$$\langle 0,1,1,1,\dots \rangle$$

$$||$$

$$\tau(x\,\tfrac{1}{x}) = \tau(1) = \langle 1,1,1,1,\dots \rangle$$

Simplify

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3. a ring automorphism

$$\begin{array}{rlll} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

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$$\sigma : \quad \mathbb{Q}(x)[s] \to \mathbb{Q}(x)[s]$$

$$s \mapsto s + \frac{1}{x+1}$$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Simplify

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$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\left(s + \frac{1}{x+1}\right)^i \end{array}$$

$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

In this example:

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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τ is an injective difference ring homomorphism:

$$\begin{array}{ccc}
\mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\
\downarrow \tau & = & \downarrow \tau \\
\mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S\quad} & \mathbb{K}^{\mathbb{N}}/\sim
\end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

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$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(n)\rangle_{n\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

$$\boxed{\begin{array}{ccccc} (\mathbb{A},\sigma) & \simeq & (\tau(\mathbb{A}),S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim,S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}] & & \end{array}}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k\geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\begin{array}{ccccc} (\mathbb{A}, \sigma) & \simeq & (\tau(\mathbb{A}), S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim, S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k\geq 0}] & & \end{array}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k)\rangle_{k\geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

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Output: $\bar{g} = x\,s - x$

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Find: $g = \langle g(k) \rangle_{k\geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\, S_1(k) - k$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\, s - x$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0) = (a+1)S_1(a+1) - (a+1)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\, S_1(k) - k$

$$\Updownarrow$$

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Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

Sk!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

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hypergeometric products $\leftrightarrow$ $\sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

$$\begin{array}{rcll} \text{hypergeometric} & \leftrightarrow & \sigma(p_1) = a_1\, p_1 & a_1 \in \mathbb{K}(x)^* \\ \text{products} & & \sigma(p_2) = a_2 p_2 & a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^* \end{array}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products $\leftrightarrow$

$$\begin{aligned}
\sigma(p_1) &= a_1\, p_1 && a_1 \in \mathbb{K}(x)^* \\
\sigma(p_2) &= a_2 p_2 && a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\
&\vdots \\
\sigma(p_e) &= a_e p_e && a_e \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^*
\end{aligned}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z]}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x)=x+1$$

$$\begin{array}{rcll}
\text{hypergeometric} & \leftrightarrow & \sigma(p_1)=a_1\,p_1 & a_1\in\mathbb{K}(x)^*\\
\text{products} & & \sigma(p_2)=a_2 p_2 & a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*\\
 & & \vdots & \\
 & & \sigma(p_e)=a_e p_e & a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*\\
(-\mathbf{1})^{\mathbf{k}} & \leftrightarrow & \sigma(\mathbf{z})=-\mathbf{z} & \mathbf{z}^{\mathbf{2}}=\mathbf{1}
\end{array}$$

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z]}$$

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hypergeometric products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^* \end{aligned}$$

α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow \quad \sigma(\mathbf{z}) = \alpha\, \mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1]}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x) = x+1$$

hypergeometric products	$\leftrightarrow$	$\sigma(p_1)=a_1\,p_1$	$a_1\in\mathbb{K}(x)^*$
		$\sigma(p_2)=a_2p_2$	$a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*$
		$\vdots$	
		$\sigma(p_e)=a_ep_e$	$a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*$
α is a primitive λth root of unity $\quad\alpha^{\mathbf{k}}$	$\leftrightarrow$	$\sigma(\mathbf{z})=\alpha\,\mathbf{z}$	$\mathbf{z}^{\lambda}=\mathbf{1}$
$\mathcal{S}H_k=H_k+\frac{1}{k+1}$	$\leftrightarrow$	$\sigma(s_1)=s_1+\frac{1}{x+1}$	

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1]}$$

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		$\vdots$	
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α is a primitive λth root of unity $\alpha^{\mathbf{k}}$	$\leftrightarrow$	$\sigma(\mathbf{z})=\alpha\,\mathbf{z}$	$\mathbf{z}^{\lambda}=\mathbf{1}$
(nested) sum	$\leftrightarrow$	$\sigma(s_1)=s_1+f_1$	$f_1\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

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hypergeometric products $\leftrightarrow$

$$\begin{aligned}
\sigma(p_1) &= a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\
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&\vdots\\
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\end{aligned}$$

α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow \quad \sigma(\mathbf{z}) = \alpha\,\mathbf{z} \qquad \mathbf{z}^{\lambda}=\mathbf{1}$

(nested) sum $\leftrightarrow$

$$\begin{aligned}
\sigma(s_1) &= s_1+f_1 && f_1\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]\\
\sigma(s_2) &= s_2+f_2 && f_2\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]
\end{aligned}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \dots}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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(nested) sum $\leftrightarrow$

$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \dots$$

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α is a primitive λth root of unity $\alpha^{\mathbf{k}}$ $\leftrightarrow$

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$$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z]$$

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$$\vdots$$

such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

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α is a primitive λth root of unity

(nested) su... $\dots, p_e^{-1}][z]$

$$\sigma(s_2)=s_2+f_2 \qquad f_2\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]$$
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$$\vdots$$

such that $\mathrm{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

GIVEN $f\in\mathbb{A}$;
FIND, in case of existence, a $g\in\mathbb{A}$ such that

$$\sigma(g)-g=f.$$

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \dots [s_r]}$$

- with an automorphism as given in the previous slide.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

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- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\mathrm{const}_\sigma \mathbb{A} = \mathbb{K}.$

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

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2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

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3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

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Remark: Related results have been worked out in the Galois theory of difference equations (van der Put/Singer, 1997)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

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$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

  $A(\lambda) = B(\lambda)$ for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
  (δ can be computed explicitly)

- and such that

  the arising sums and products in $B(k)$ (except the alternating sign) are **algebraically independent**
  (i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$$\downarrow \quad \texttt{SigmaReduce[A,k]}$$

$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Application 2: Canonical representations

A_1 A_2 expressions in a term algebra

Application 2: Canonical representations

$$\mathsf{ev}(A_1, n) \quad \stackrel{?}{=} \quad \mathsf{ev}(A_2, n)$$

expressions in a term algebra

Application 2: Canonical representations

$$\begin{array}{ccc} A_1 & & A_2 \\ \big\downarrow & \text{SigmaReduce} & \big\downarrow \\ B_1 & & B_2 \end{array}$$

expressions in a term algebra

$\forall n \geq 0 \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n)$ $\qquad$ $\mathsf{ev}(B_2, n) = \mathsf{ev}(A_2, n)$

Application 2: Canonical representations

A_1 $\qquad$ A_2 $\qquad$ expressions in a term algebra

$\downarrow$ SigmaReduce $\downarrow$

B_1 $\qquad$ B_2

$\forall n \geq 0\ \ \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \quad = \quad \mathsf{ev}(B_2, n) = \mathsf{ev}(A_2, n)$

$\Updownarrow$ canonical simplifier

$B_1 = B_2$

Application 2: Canonical representations

$$\mathsf{ev}(A_1,n) \overset{?}{=} \mathsf{ev}(A_2,n)$$

expressions in a term algebra

$$\downarrow \qquad\qquad \text{SigmaReduce} \qquad\qquad \downarrow$$

$$B_1 \qquad\qquad\qquad B_2$$

$$\forall n \geq 0 \quad \mathsf{ev}(A_1,n) = \mathsf{ev}(B_1,n) \quad = \quad \mathsf{ev}(B_2,n) = \mathsf{ev}(A_2,n)$$

$$\Updownarrow$$

canonical simplifier

$$B_1 = B_2$$

Application 2: Canonical representations

$\mathsf{ev}(A_1,n) \quad \stackrel{?}{=} \quad \mathsf{ev}(A_2,n)$ expressions in a term algebra

$\downarrow$ SigmaReduce $\downarrow$

$B_1 \qquad\qquad B_2$

$$\forall n \geq 0 \;\; \mathsf{ev}(A_1,n) = \mathsf{ev}(B_1,n) \quad = \quad \mathsf{ev}(B_2,n) = \mathsf{ev}(A_2,n)$$

$\Updownarrow$ canonical simplifier

$$B_1 = B_2$$

Application 3: We solve the zero-recognition problem.

$$A_1(k) \text{ evaluates to 0 from a certain point on} \quad \Leftrightarrow \quad B_1 = 0$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

Part 3: Challenging applications

Part 3: Challenging applications in number theory

Example: a challenging email

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider



Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
-Doron
```

[arose in the bounds on the run time of the simplex algorithm on a polytope]

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks
suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \boxed{\sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}}$$

with

$$S_1(j) := \sum_{i=1}^{j} \frac{1}{i}$$

The inner sum

$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

`Sigma.m`

Recurrence finder

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}$$

The inner sum

$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

Sigma.m

Recurrence solver

Sigma.m

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)} \in \left\{ c_1 \frac{S_1(k)}{k} + c_2 \frac{1}{k} + \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k)}{2k^2} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

where

$$S_2(k) = \sum_{i=1}^{k} \frac{1}{i^2}$$

The inner sum

$$k^2\mathbf{A}(k)-(k+1)(2k+1)\mathbf{A}(k+1)+(k+1)(k+2)\mathbf{A}(k+2)=\frac{1}{k+1}$$

Sigma.m **Recurrence solver** Sigma.m

$$\mathbf{A}(k)=\sum_{j=1}^{\infty}\frac{S_1(j)}{j(j+k)} = \boxed{0\frac{S_1(k)}{k}+\zeta(2)\frac{1}{k}+\frac{kS_1(k)^2-2S_1(k)+kS_2(k)}{2k^2}}$$

where

$$S_2(k)=\sum_{i=1}^{k}\frac{1}{i^2} \qquad \zeta(z)=\sum_{i=1}^{\infty}\frac{1}{i^z}$$

Simplify

$$\sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}.$$

$$||$$

$$\frac{kS_1(k)^2 - 2S_1(k) + kS_2(k) + 2k\zeta(2)}{2k^2}$$

Simplify

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}.$$

$$||$$

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \times \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k) + 2k\zeta(2)}{2k^2}$$

$||$telescoping + limit calculations

$$-4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5)$$

$$||$$

$$0.999222... \neq 1$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \, \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \, \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}\,.$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \, \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \, \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}.$$

Then

$$-\frac{1}{3} \sum_{\nu=n-m+1}^{\infty} \frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6} \sum_{\nu=1}^{\infty} \frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Theorem (CS, Sigma, Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}.$$

Then

$$-\frac{1}{3} \sum_{\nu=n-m+1}^{\infty} \frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6} \sum_{\nu=1}^{\infty} \frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

with

$$\alpha_0(n,m)=(2n-m)^5,$$

$$\alpha_1(n,m)=-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3$$
$$+30n^2m-14nm^2+2m^3+8n^2-10nm+2m^2-4n+m),$$

$$\alpha_2(n,m)=-(2n-m-1)^3(4n-m)(m+2).$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

$$\mathsf{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

$$
\begin{aligned}
S(n,m) = \sum_{j=0}^{n}\sum_{\nu=1}^{\infty} \Bigg(& \frac{\binom{n}{j}^2 \binom{j-m+2n}{n} (1+\nu)_{-m+2n} (1-j+\nu+n)_{-1+n}}{(1+\nu+n)_n (1+\nu+n)_{-m+2n} (\nu+n)^4 (\nu-m+2n)^3} \\
& \times \Bigg((\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) \\
& \quad + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
& \quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& - \nu(j-\nu-n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) \\
& \quad - S_1(\nu-m+3n) - S_1(-j+\nu+n) + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& + \nu(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) \\
& \quad - S_1(\nu-m+3n) - S_1(-j+\nu+n) + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& - (j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) \\
& \quad - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
& \quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& + \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{(j-\nu-2n)^2} - S_2(\nu) + 2S_2(\nu+n) \\
& \quad - S_2(\nu+2n) - S_2(\nu-m+3n) - S_2(-j+\nu+n) \\
& \quad + S_2(\nu-m+2n) + S_2(-j+\nu+2n)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+4(j+n)(\nu+n)-3(\nu+n)^2+n(-m+n)-j(m+2n)\Big)\\
&-2(\nu+n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)\\
&\quad+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&-3(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)\\
&\quad+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&-(\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}\\
&-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&\qquad\times\big(-S_1(\nu+n)+S_1(\nu+2n)\big)\\
&+(\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}\\
&\quad-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)
\end{aligned}
$$

$$
\begin{aligned}
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\times \big(- S_1(\nu) + S_1(\nu - m + 2n)\big) \\
&- (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} \\
&- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\times \big(- S_1(\nu+n) + S_1(\nu - m + 3n)\big) \\
&+ (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} \\
&- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 \\
&- (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\times \Big(- \frac{1}{-j+\nu+2n} - S_1(-j+\nu+n) + S_1(-j+\nu+2n)\Big)\Big)\Big)
\end{aligned}
$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \text{ Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$
$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$\downarrow$ Sigma.m with DR-creative telesoping

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$
$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$\downarrow$ Sigma.m with Holonomic-DR approach

$$(2n-m)^5S(n,m)$$
$$-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3+30n^2m-14nm^2$$
$$+2m^3+8n^2-10nm+2m^2-4n+m)S(n,m+1)$$
$$-(2n-m-1)^3(4n-m)(m+2)S(n,m+2)=R(n,m)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

SigmaReduce

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

Finally, check 2 initial values: another round of non-trivial summation...

Part 3: Challenging applications in combinatorics

Plane Partition

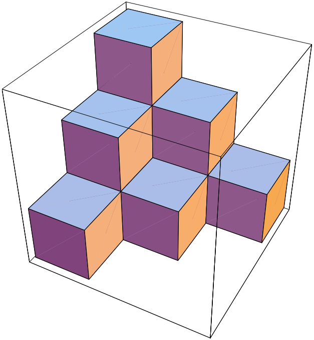
Example (n=3)

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

Plane Partition

Example (n=3)

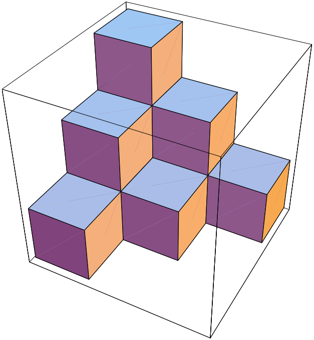
$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



Totally Symmetric Plane Partition (TSPP)

Example (n=3)

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



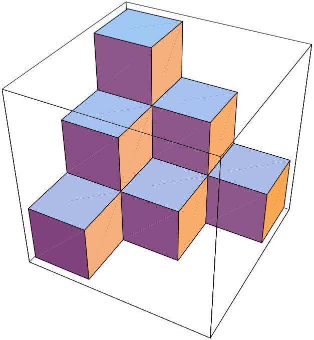
TSPP-conjecture (Andrews,Macdonald,Stanley; in 80ies). The number of totally symmetric plane partitions with largest part $\leq n$ is

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}, \qquad n \geq 1.$$

Totally **S**ymmetric **P**lane **P**artition (TSPP)

Example (n=3)

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \quad \longleftrightarrow$$



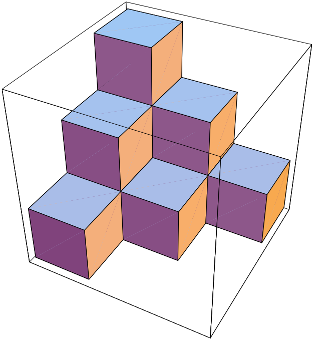
TSPP-Theorem (Stembridge, 1995). The number of totally symmetric plane partitions with largest part $\leq n$ is

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}, \qquad n \geq 1.$$

Totally Symmetric Plane Partition (TSPP)

Example (n=3)

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



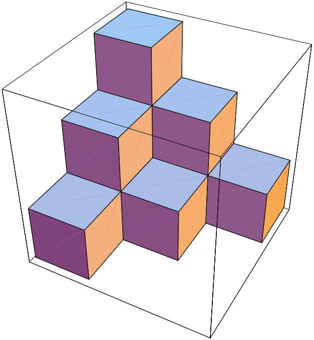
$\downarrow$ evaluation of a determinant (G.E. Andrews, ~ 1990)
(using an LU-decomposition)

multiple sum identities (proof?)

Totally **S**ymmetric **P**lane **P**artition (TSPP)

Example (n=3)

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



↓ evaluation of a determinant (G.E. Andrews, ~ 1990)
(using an LU-decomposition)

multiple sum identities (proof?)

↓ correctness proofs (Sigma, 2005)

TSPP-Theorem

A typical example (of these identities):

Define

$$h(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor - 1} \frac{k}{m-s} \binom{m-s}{2m-2s-k} \frac{(-1)^{s+k}}{2m\,4^s} \sum_{r=0}^{s} \frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}$$

$$A(i,m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} h(k,m),$$

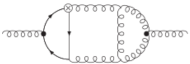
$$B(i,m) := \sum_{k=i}^{2m} (-1)^k h(k,m).$$

Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m+1$,

$$2h(i-2,m) - 5h(i-1,m) - A(i,m) + 6(-1)^i B(i,m) = 3(-1)^i \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2}.$$

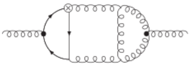
Part 3: Challenging applications in particle physics

Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals



behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x)\, dx$$

Feynman integrals

Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\, dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

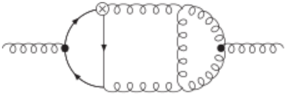
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\dots\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad ||$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

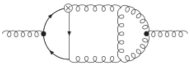
$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Evaluation of Feynman Integrals



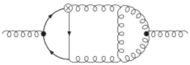
behavior of particles

$\longrightarrow$

$$\int \Phi(N, \epsilon, x)\,dx$$

Feynman integrals

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N, \epsilon, x)\,dx$$

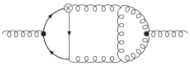
Feynman integrals

DESY

$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N, \epsilon, x)\, dx$$

Feynman integrals

DESY

$$\sum f(N, \epsilon, k)$$

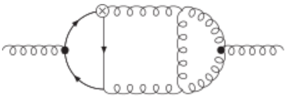
complicated multi-sums

advanced difference ring theory

(Sigma-package)

expression in special functions

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad \Big|\Big|$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

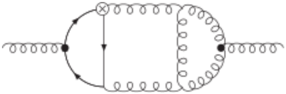
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

$$\Big[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

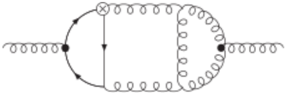
$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times\left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{N-3-j}dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}\varepsilon^{0} + \ldots$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}\varepsilon^0 + \dots$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$
$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$
$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$
$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+ \Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

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$$+ \Big(- \ldots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big) S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big) S_3(N)$$

$$+ \big(2 + \ldots 28 S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big) S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big) S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big) S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big) S_2(N) + S_{-2}(N)\Big(10 S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big) S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big) S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big) S_4(N) + \big(-6 + 5(-1)^N\big) S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big) S_{2,1}(N) + \big(20 + 2(-1)^N\big) S_{2,-2}(N) + \big(-17 + 13(-1)^N\big) S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big) S_{-3,1}(N) + \big(3 - 5(-1)^N\big) S_{2,1,1}(N)$$

$$+ 32 S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3 S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

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$$+ \big(- \ldots \frac{\ldots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ (2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big) \ldots {}^N\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(\ldots \Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

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$$+ \Big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big) \ldots 3 \ldots N)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(- \ldots N \ldots\Big)$$

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$$+ \frac{4(3N - \ldots}{N(N+ \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

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$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

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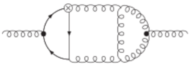
$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N, \epsilon, x)\,dx$$

Feynman integrals

DESY

$$\sum f(N, \epsilon, k)$$

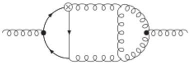
complicated multi-sums

advanced difference ring theory

(Sigma-package)

expression in special functions

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

DESY

LHC at CERN

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

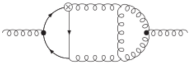
advanced difference ring theory

(Sigma-package)

expression in special functions

applicable

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

- What did the universe look like in the first second
- Do the 4 fundamental forces unite at high energies?
- Do the properties of the new particle agree with the predicted Higgs-Boson?

DESY

applicable

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

advanced difference ring theory

(Sigma-package)

expression in special functions

Conclusion

1. A warm-up example

2. The difference ring machinery for symbolic summation

$$\updownarrow$$

3. Challenging applications