Antidifferentiation and the Calculation of Feynman Amplitudes

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Summation Theory and Integration

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Der Wissenschaftsfonds.

Outline of the talk:

Part 1: A warm-up example

Part 2: The underlying framework: difference ring theory

Part 3: The simplification of Feynman integrals

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad(=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

anti-differenciation (discrete version of antidifferentiation)

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

$\uparrow$ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum}=\sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= **SigmaLimit[res, {n}, a]**

Out[4]= $$\frac{1}{n!}\frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}$$
$$\underbrace{+\frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\color{red}{\sum_{k=0}^{\infty}}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\color{red}{\sum_{k=1}^{\infty}}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\color{red}{\sum_{k=1}^{\infty}}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Telescoping

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

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for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k)+(-n-1)S_1(n)-kS_1(k+n)-2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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Zeilberger's creative telescoping paradigm

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for all $0 \leq k \leq n$ and all $n \geq 0$.

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$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

$$||\qquad\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n)+(2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $$\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-\mathtt{n}\text{SUM}[\mathtt{n}] + (1+\mathtt{n})(2+\mathtt{n})\text{SUM}[\mathtt{n}+1] == \frac{(\mathtt{n}+1)\mathtt{S}[1,\mathtt{n}]+1}{(\mathtt{n}+1)^3}$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, [arXiv:2005.04944])

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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$f(n,k)$: indefinite nested product-sum in k;
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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, [arXiv:2005.04944])

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\|$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+ \frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

In[1]:= $<< \mathbf{Sigma.m}$

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<< \mathbf{HarmonicSums.m}$

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In[3]:= $<< \mathbf{EvaluateMultiSums.m}$

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In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}]}$

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

Sigma.m is based on difference ring/field theory

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The underlying framework: difference ring theory

[a gentle introduction]

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

Simplify

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}}[s]}_{\text{polynomial ring}}$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{aligned} \mathrm{ev}' : \quad \mathbb{Q}(x) \times \mathbb{N} &\to \quad \mathbb{Q} \\ \left(\tfrac{p(x)}{q(x)}, k\right) &\mapsto \quad \begin{cases} \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\mathrm{ev} : \quad \mathbb{Q}(x)[s] \times \mathbb{N} \quad \to \quad \mathbb{Q}$$

$$\mathbf{ev(s,k) = S_1(k)}$$

Simplify
$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

$$\begin{array}{rlcl} \mathrm{ev}' : & \mathbb{Q}(x) \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\frac{p(x)}{q(x)}, k\right) & \mapsto & \begin{cases} \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

$$\begin{array}{rlcl} \mathrm{ev} : & \mathbb{Q}(x)[s] \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, k\right) & \mapsto & \sum_{i=0}^{d} \mathrm{ev}'(f_i, k) S_1(k)^i \end{array} \qquad \mathbf{ev(s,k) = S_1(k)}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau: & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \text{ev}(f,k) \rangle_{k \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \qquad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$$||$$

$$\langle 0,1,1,1,\dots \rangle$$

Simplify

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Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

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$$\begin{array}{rccc} \tau: & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}}/\sim \\ & f & \mapsto & \langle \mathrm{ev}(f,k) \rangle_{k \geq 0} \end{array}$$

$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0,1,2,3,\dots \rangle \langle 0,1,\frac{1}{2},\frac{1}{3},\dots \rangle \\ & & \| \\ & & \langle 0,1,1,1,\dots \rangle \\ & & \| \\ \tau(x\,\frac{1}{x}) = \tau(1) & = & \langle 1,1,1,1,\dots \rangle \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{llll} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

Simplify

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$$\sigma : \quad \mathbb{Q}(x)[s] \quad \to \quad \mathbb{Q}(x)[s] \qquad\qquad s \mapsto s + \tfrac{1}{x+1}$$

$$S_1(k+1) = S_1(k) + \tfrac{1}{k+1}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

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$$\begin{aligned} \sigma : \quad & \mathbb{Q}(x)[s] && \to && \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i && \mapsto && \sum_{i=0}^{d} \sigma'(f_i)\Big(s + \tfrac{1}{x+1}\Big)^i \end{aligned}$$

$s \mapsto s + \frac{1}{x+1}$

$S_1(k+1) = S_1(k) + \frac{1}{k+1}$

Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), k) = \mathrm{ev}(s + \tfrac{1}{x+1}, k) = S_1(k) + \tfrac{1}{k+1} = \mathrm{ev}(s, k+1)$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify
$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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τ is an injective difference ring homomorphism:

$$\begin{array}{ccc}
\mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\
\downarrow \tau & = & \downarrow \tau \\
\mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S\quad} & \mathbb{K}^{\mathbb{N}}/\sim
\end{array}$$

Simplify
$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
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$$\Updownarrow$$
$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(k)\rangle_{k\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

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Find: $G = \langle G(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

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CONSTRUCT a difference ring $(\mathbb{A}, \sigma)$ with ev for $F(k)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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		$\vdots$	
		$\sigma(p_e) = a_e p_e$	$a_e \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^*$
$\mathbf{(-1)^k}$	$\leftrightarrow$	$\sigma(\mathbf{z}) = -\mathbf{z}$	$\mathbf{z^2 = 1}$

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$\mathcal{S}S_1(k) = S_1(k) + \frac{1}{k+1}$	$\leftrightarrow$	$\sigma(s_1) = s_1 + \frac{1}{x+1}$	

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CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$ with ev for $F(k)$: (Karr81,CS16,CS17)

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Represent sums (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant set

$$\text{const}_\sigma \mathbb{A} := \{k \in \mathbb{A} | \sigma(k) = k\}.$$

Note 1: $\text{const}_\sigma \mathbb{A}$ is a ring that contains $\mathbb{Q}$

Note 2: We always take care that $\text{const}_\sigma \mathbb{A}$ is a field

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Such a difference ring extension $(\mathbb{A}[t], \sigma)$ of $(\mathbb{A}, \sigma)$ is called Σ^*-extension

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1. $\boxed{\nexists g \in \mathbb{A} : \ \sigma(g) = g + f}$: $(\mathbb{A}[t], \sigma)$ is a Σ^*-extension of $(\mathbb{A}, \sigma)$
2. $\boxed{\exists g \in \mathbb{A} : \sigma(g) = g + f}$: No need for a Σ^*-extension!

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The hypergeometric case

- Take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma|_{\mathbb{K}} = \text{id}$ and $\sigma(x) = x + 1$.
- Let $\alpha_1, \ldots, \alpha_r \in \mathbb{K}(x)^*$

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CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$ with ev for $F(k)$: (Karr81,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

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γ is a primitive λth root of unity $\quad \gamma^{\mathbf{k}} \leftrightarrow$

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(nested) su... $\dots{}_e,p_e^{-1}][z]$

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$$\sigma(s_3) = s_3 + f_3 \qquad f_3 \in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]$$

$$\vdots$$

GIVEN $f\in\mathbb{A}$ with $\mathrm{ev}(f,k)=F(k)$;
FIND, in case of existence, a $g\in\mathbb{A}$ such that

$$\sigma(g)-g=f.$$

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Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$A(\lambda) = B(\lambda)$ for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
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- and such that

the arising sums and products in $B(k)$ (except γ^n with $(\gamma^n)^\lambda = 1$) are **algebraically independent**
(i.e., they do not satisfy any polynomial relation)

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3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), k) = \mathrm{ev}(s + \tfrac{1}{x+1}, k) = S_1(k) + \tfrac{1}{k+1} = \mathrm{ev}(s, k+1)$$

$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(k) \rangle_{k \geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

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CS, A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS, Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

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Note 1: Similar results have been worked out in the Galois theory of difference equations (van der Put/Singer, 1997)

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Note: Quasi-shuffle relations produce such difference rings for cyclotomic sums; see [arXiv:1510.03692, Ablinger/CS] inspired by [arXiv:hep-ph/0311046, Blümlein].

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\sigma(s_3) &= s_3+f_3 && f_3\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]\\
&\vdots
\end{aligned}$$

CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$ with ev for $A(k)$: (Karr81,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x) = x+1$$

hypergeometric products	$\leftrightarrow$	$\sigma(p_1) = a_1\,p_1$	$a_1 \in \mathbb{K}(x)^*$
		$\sigma(p_2) = a_2 p_2$	$a_2 \in \mathbb{K}(x)[p_1,p_1^{-1}]^*$
		$\vdots$	
		$\sigma(p_e) = a_e p_e$	$a_e \in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*$
γ is a primitive λth root of unity $\gamma^{\mathbf{k}}$	$\leftrightarrow$	$\sigma(\mathbf{z}) = \gamma\,\mathbf{z}$	$\mathbf{z}^\lambda = \mathbf{1}$
(nested) sum	$\leftrightarrow$	$\sigma(s_1) = s_1 + f_1$	$f_1 \in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]$
		$\sigma(s_2) = s_2 + f_2$	$f_2 \in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]$
		$\sigma(s_3) = s_3 + f_3$	$f_3 \in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]$
		$\vdots$	

such that $\mathrm{const}_\sigma\mathbb{A} = \{c\in\mathbb{A}|\sigma(c)=c\} = \mathbb{K}$.

CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with ev for $A(k)$: (Karr81,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2][s_3]\dots$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

We get $a \in \mathbb{A}$ plus
an embedding τ from $(\mathbb{A}, \sigma)$ into the ring of sequences s.t.

$$\tau(a) = \langle A(\nu) \rangle_{\nu \geq 0}$$

$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2]$$

$\vdots$

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Reinterpreting a in terms of these nested sums and products yields $B(k)$

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$$\bigcap$$

$$\tau(\mathbb{A}) = \underbrace{\tau(\mathbb{K}(x))}_{\text{rational seq.}}\ [\langle\gamma^k\rangle_{k\geq 0}]\underbrace{[\tau(p_1),\tau(p_1^{-1})]\dots[\tau(p_e),\tau(p_e^{-1})]}_{\text{nested products}}\underbrace{[\tau(s_1)]\dots[\tau(s_r)]}_{\text{nested sums}}$$

$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]$

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$$\tau(\mathbb{A}) = \underbrace{\tau(\mathbb{K}(x))}_{\text{rational seq.}}\,[\langle\gamma^k\rangle_{k\geq 0}]\,\underbrace{\underbrace{[\tau(p_1),\tau(p_1^{-1})]\dots[\tau(p_e),\tau(p_e^{-1})]}_{\text{nested products}}\underbrace{[\tau(s_1)]\dots[\tau(s_r)]}_{\text{nested sums}}}_{\text{algebraic independent}}$$

$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2]$$

$$\vdots$$

such that $\mathrm{const}_\sigma\mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$A(\lambda) = B(\lambda)$ for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
(δ can be computed explicitly)

- and such that

the arising sums and products in $B(k)$ (except γ^n with $(\gamma^n)^\lambda = 1$) are **algebraically independent**
(i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Simplification of nested product-sum expressions

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$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Application 2: We solve the zero-recognition problem.

$$A(k) \text{ evaluates to 0 from a certain point on} \quad \Leftrightarrow \quad B(k) = 0$$

Application: The simplification of Feynman integrals

– a successful story of the RISC–DESY cooperation
(Johannes Blümlein and Peter Marquard)

Journal publications dealing with non-trivial calculations

- I. Bierenbaum, J. Blümlein, S. Klein, and C. Schneider. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
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- J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, CS, F. Wissbrock Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements. Nuclear Physics B. 864: 52-84, 2012.
- J. Blümlein, A. Hasselhuhn, S. Klein, CS. The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements. Nuclear Physics B: 866: 196-211, 2013.
- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
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- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
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- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.

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- J. Blümlein, CS. The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory. Physics Letters B 771, pp. 31-36. 2017.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, CS, F. Wißbrock. Three Loop Massive Operator Matrix Elements and Asymptotic Wilson Coefficients with Two Different Masses. Nucl. Phys. B. 921, pp. 585-688. 2017.
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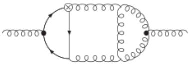
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS, K. Schönwald. The three-loop single mass polarized pure singlet operator matrix element. Nuclear Physics B 953(114945), pp. 1-25. 2020.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Polarized Operator Matrix Element $A_{gg,Q}^{(3)}$. Nuclear Physics B 955, pp. 1-70. 2020.

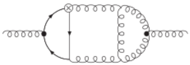
+ 38 proceedings publications within the particle physics community.

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles

$$\longrightarrow \int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Feynman integrals

$$\int_0^1 x^N \, dx = \frac{1}{N+1} \qquad \text{for } N = 0, 1, 2, 3, \ldots$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

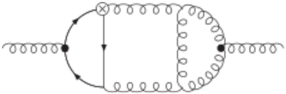
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad ||$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

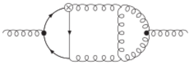
$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Evaluation of Feynman Integrals

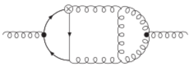


Behavior of particles

$$\longrightarrow \int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles → $\int \Phi(N,\epsilon,x)dx$

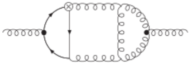
Feynman integrals

↓ **DESY** (J. Blümlein)

$\sum f(N,\epsilon,k)$

complicated multi-sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

DESY

(J. Blümlein)

$$\sum f(N, \epsilon, k)$$

complicated multi-sums

RISC

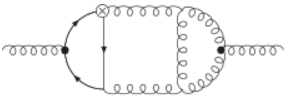
(Sigma-package)

expression in special functions

Example 1:

massive 3-loop ladder integrals

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$\Vert$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

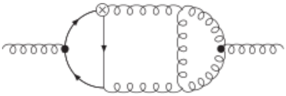
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

$$\Bigg[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

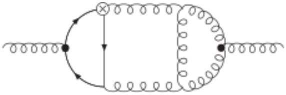
$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Bigg]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$\|$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$
$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$
$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$
$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$
\boxed{F_0(N)} =
$$

$$
\begin{aligned}
&\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2 \\
&+ \Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N) \\
&+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2 \\
&- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big) \\
&+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)} \\
&+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big) \\
&+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N) \\
&+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N) \\
&- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N) \\
&+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)
\end{aligned}
$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(-\cdots\frac{\cdots 1)^N(2N+1)}{\cdots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\Big(2+\cdots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

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$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

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$$+\Big(- \ldots (-1)^N(2N+1) \ldots N(N+1) - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1) \ldots} \ldots (-1)^N\big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \ldots (-1)^N \ldots + 1\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

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$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26 + 4 \ldots \frac{\ldots)^N}{\ldots 1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(\ldots} - \frac{5}{\ldots}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{\ldots(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N - \ldots}{N(N + \ldots} \ldots -1)^N\Big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N \ldots}{N(\ldots} \ldots (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-\ldots}{\ldots} \ldots\Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

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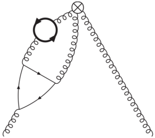
$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Example 2:

2-mass 3-loop Feynman integrals

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

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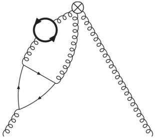

Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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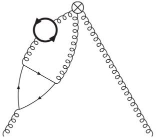

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Typical triple sum:

$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^{k}\times$$
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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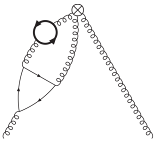

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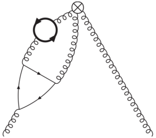
6 hours for this sum

$\sim$ 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

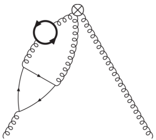
expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



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expression (377 MB) consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m

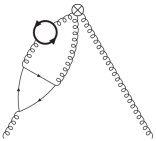
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation		number of indef. sums
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s	(2.1 days)	1188
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s	(11.4 days)	747
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s	(6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s	(8.0 days)	44
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s	(3.5 days)	1933
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s	(8.2 days)	621
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s	(5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s	(2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

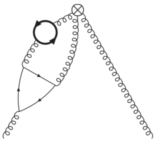
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left(\sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies ⟶

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

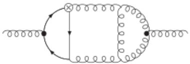
↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

⟵ Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

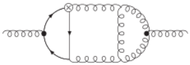
Evaluation of Feynman Integrals



Behavior of particles $\longrightarrow$ $\int \Phi(N, \epsilon, x)dx$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x)dx$$

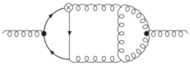
Feynman integrals

DESY
(J. Blümlein,
P. Marquard)

$$Dy = A\,y$$

coupled systems of
linear DE

Evaluation of Feynman Integrals



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$$Dy = A\,y$$

coupled systems of
linear DE

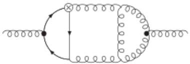
advanced difference ring theory
(new coupled system solver)

expression in
special functions

Taylored algorithms/packages for coupled systems coming from IBPs

- J. Blümlein, S. Klein, CS, F. Stan. A Symbolic Summation Approach to Feynman Integral Calculus. J. Symbolic Comput. 47, pp 1267-1289, 2012.
- J. Ablinger, J. Blümlein, M. Round, CS. Advanced Computer Algebra Algorithms for the Expansion of Feynman Integrals. In: Loops and Legs in Quantum Field Theory 2012, PoS(2012) , pp. 1-14. 2012.
- CS. Modern Summation Methods for Loop Integrals in Quantum Field Theory: The Packages Sigma, EvaluateMultiSums and SumProduction. In: Proc. ACAT 2013, , J. Phys.: Conf. Ser. 523/012037, pp. 1-17. 2014.
- A. De Freitas, J. Blümlein, CS. Recent Symbolic Summation Methods to Solve Coupled Systems of Differential and Difference Equations. In: Loops and Legs in Quantum Field Theory - LL 2014, J. Blümlein, P. Marquard, T. Riemann (ed.), PoS(LL2014)017, pp. 1-13. 2014.
- J. Ablinger, J. Blümlein, A. de Freitas, CS. A toolbox to solve coupled systems of differential and difference equations. In: Proc. of the 13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology), Nigel Glover, Daniel Maitre, Ben Pecjak (ed.)PoS(RADCOR2015)060, pp. 1-13. 2015.
- J. Ablinger, A. Behring, J. Blümlein, A. de Freitas, CS. Algorithms to solve coupled systems of differential equations in terms of power series. In: Proc. Loops and Legs in Quantum Field Theory - LL 2016, J. Blümlein, P. Marquard, T. Riemann (ed.) (ed.)PoS(LL2016)005, pp. 1-15. 2016.
- J. Blümlein, CS. The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory. Physics Letters B 771, pp. 31-36. 2017.
- J. Middeke, CS. Denominator Bounds for Systems of Recurrence Equations using $\Pi\Sigma$-Extensions. In: Advances in Computer Algebra. WWCA 2016., C. Schneider, E. Zima (ed.), Springer Proceedings in Mathematics & Statistics 226, pp. 149-173. 2018.
- J. Blümlein, CS. Analytic Computing Methods for Precision Calculations in Quantum Field Theory. INTERNATIONAL JOURNAL OF MODERN PHYSICS A (IJMPA) 33(1830015), pp. 1-35. 2018.
- J. Middeke, CS. Towards a Direct Method for Finding Hypergeometric Solutions of Linear First Order Recurrence Systems. 2018. Poster presentation at ISSAC 2018.
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable. Nucl. Phys. B 939, pp. 253-291. 2019.
- Johannes Blümlein, Peter Marquard, Carsten Schneider A refined machinery to calculate large moments from coupled systems of linear differential equations. In: 14th International Symposium on Radiative Corrections (RADCOR2019), D. Kosower, M. Cacciari (ed.), POS(RADCOR2019)078 , pp. 1-13. 2020.

Evaluation of Feynman Integrals



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Feynman integrals

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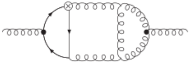
$$Dy = A\,y$$

coupled systems of
linear DE

advanced difference ring theory
(new coupled system solver)

expression in
special functions

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY
(J. Blümlein,
P. Marquard)

LHC at CERN

$$Dy = A\,y$$

coupled systems of linear DE

advanced difference ring theory
(new coupled system solver)

expression in special functions

applicable

Conclusion (used techniques within our RISC–DESY cooperation)

1. symbolic integration
   (method of hyperlogarithms, Almkvist-Zeilberger algorithm)
2. generalized hypergeometric functions (and extensions)
3. Mellin-Barnes techniques
4. **symbolic summation** (WZ-, holonomic, **difference ring methods**)
5. **recurrence solving (so far up to order 50)**
6. integration by parts technique
7. differential equation solving
8. **coupled system solving**
9. method of large moments and guessing (so far up to 10K moments)
10. special function algorithms