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An extension of holonomic sequences: C^2-finite sequences*

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Abstract

Holonomic sequences are widely studied as many objects interesting to mathematicians and computer scientists are in this class. In the univariate case, these are the sequences satisfying linear recurrences with polynomial coefficients and also referred to as D-finite sequences. A subclass are C-finite sequences satisfying a linear recurrence with constant coefficients.

We investigate the set of sequences which satisfy linear recurrence equations with coefficients that are C-finite sequences. These sequences are a natural generalization of holonomic sequences. In this paper, we show that C^2-finite sequences form a difference ring and provide methods to compute in this ring.

Furthermore, we provide an analogous construction for D^2-finite sequences, i.e., sequences satisfying a linear recurrence with holonomic coefficients. We show that these constructions can be iterated and obtain an increasing chain of difference rings.

1 Introduction

Sequences that satisfy a linear recurrence with polynomial coefficients are known under the names holonomic, D-finite or P-recursive. If the recurrence coefficients are just constants, these sequences are also called C-finite or C-recursive. Many interesting combinatorial objects or coefficient sequences of special functions are of this type [4, 16]. In this paper, we define C^2-finite sequences as sequences satisfying a linear recurrence relation with C-finite coefficients. Holonomic and q-holonomic sequences are strictly contained in this set.

For holonomic functions or sequences, closure properties are a basic tool to systematically construct new holonomic objects from given ones and, more importantly, to automatically prove identities on holonomic objects. We set up C^2-finite sequences in a way that allows to derive and implement closure properties. The goal is to develop a toolkit for automated theorem proving as is already available for holonomic sequences and functions [13]. The main computational issue when working with this more general class compared to holonomic sequences is the presence of zero divisors.

To our knowledge, C^2-finite sequences have first been introduced formally in [18] in the context of graph polynomials. [24] gives an overview on different properties of polynomial, C-finite and holonomic sequences and consider the extension under the name X-recursive sequences.

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The setting in these articles is slightly different which leads to complications if one aims at developing an algorithmic approach.

In this paper, we show that C^2-finite sequences form a difference ring with respect to termwise addition and termwise multiplication and present a first step towards setting up the theory of C^2-finite sequences algorithmically. An implementation in SageMath [25] is under development for proof-of-concept and later release. In Section 3, we provide the algebraic characterization of C^2-finite sequences that serves as the theoretical backbone, but cannot be used straightforward in a constructive way. Next, in Section 4, we consider in full detail the computation of the ring operations. Finally, in Section 5, we state some of the classical closure properties such as partial sum or interlacing that can be derived similar to the case of holonomic sequences. In Section 6 we extend our results and show that similar ideas can be used to define D^2-finite sequences. Furthermore, we show that the construction of such sequences can be iterated which yields an increasing chain of difference rings.

This article is an extended version of a conference paper presented at ISSAC 2021 [10]. We expanded Section 4 and provide the algorithms to perform the ring operations in the ring of C^2-finite sequences. Furthermore, we discuss issues related to time complexity and the runtime of our (proof-of-concept) implementation in several examples. In Section 4.1 we give a detailed description of how a recurrence for the subsequence of a C^2-finite sequence can be computed. In Section 6 we extend our results to C^k-finite and also D^k-finite sequences.

2 Preliminaries

In this section, we introduce some notation that is used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \dots, \}$ we denote the set of natural numbers. Let $\mathbb{K}$ be a computable field of characteristic zero and we denote by $\mathbb{K}^\mathbb{N}$ the set of sequences over $\mathbb{K}$. These sequences form a ring with termwise addition and multiplication (i.e., the Hadamard product). The shift operator

$$\sigma \colon \mathbb{K}^\mathbb{N} \to \mathbb{K}^\mathbb{N}, \quad \sigma((a(n))_{n\in\mathbb{N}}) = (a(n+1))_{n\in\mathbb{N}}$$

is an endomorphism on $\mathbb{K}^\mathbb{N}$. A difference subring is a subring R of $\mathbb{K}^\mathbb{N}$ which is closed under shifts, i.e., σ is an endomorphism on R. The noncommutative ring of shift-operators over R is denoted by $R[\sigma]$ and elements $\mathcal{C} = c_0 + c_1\sigma + \cdots + c_r\sigma^r \in R[\sigma]$ act in the natural way on $a \in \mathbb{K}^\mathbb{N}$ as

$$\mathcal{C}a = (c_0(n)a(n) + c_1(n)a(n+1) + \cdots + c_r(n)a(n+r))_{n\in\mathbb{N}}.$$

For a difference subring $R \subseteq \mathbb{K}^\mathbb{N}$, we denote by $R^\times \subseteq R$ the set of sequences which are units in $\mathbb{K}^\mathbb{N}$. These are the sequences which are nonzero everywhere. This is a multiplicatively closed subset of R. Furthermore, $Q(R)$ denotes the localization of R with respect to $R^\times$. We can consider $Q(R)$ as a subring of $\mathbb{K}^\mathbb{N}$ by $((a/b)(n))_{n\in\mathbb{N}} = (a(n)/b(n))_{n\in\mathbb{N}} \in \mathbb{K}^\mathbb{N}$ for $a/b \in Q(R)$. The ring of C-finite sequences is a difference ring and we denote it by $\mathcal{R}_C$.

Definition 2.1. A sequence $a \in \mathbb{K}^\mathbb{N}$ is called C^2*-finite* over $\mathbb{K}$ if there are C-finite sequences $c_0, \dots, c_r \in \mathcal{R}_C$ with coefficients in $\mathbb{K}$ and $c_r \in \mathcal{R}_C^\times$ such that

$$c_0(n)a(n) + c_1(n)a(n+1) + \cdots + c_r(n)a(n+r) = 0,$$

for all $n \in \mathbb{N}$. We call the minimal such r the order of a and denote it by $\operatorname{ord}(a)$.

Note that the set of C^2-finite sequences contains holonomic sequences (and as such C-finite sequences), since polynomial sequences are C-finite. A sequence $a \in \mathbb{L}^\mathbb{N}$ is called q-holonomic over $\mathbb{L} := \mathbb{K}(q)$ (with q transcendental) if a satisfies a linear recurrence relation

$$p_0(q^n)a(n) + \cdots + p_r(q^n)a(n+r) = 0, \quad \text{for all } n \in \mathbb{N},$$

with coefficients $p_i \in \mathbb{L}[x]$ not all zero [15]. As all coefficients $p_i(q^n)$ are C-finite over $\mathbb{L}$, such a q-holonomic sequence a is also C^2-finite over $\mathbb{L}$.

A C^2-finite sequence is described completely by a finite amount of data: the recurrence coefficients $c_0, \dots, c_r \in \mathcal{R}_C$ and initial values $a(0), \dots, a(r-1)$. The recurrence coefficients in turn have a finite description of the same form. This way, C^2-finite sequences can be represented exactly on a computer.

In operator notation, a sequence $a \in \mathbb{K}^{\mathbb{N}}$ is C^2-finite if there is an operator $\mathcal{A} \in \mathcal{R}_C[\sigma]$ with $\operatorname{lc}(\mathcal{A}) \in \mathcal{R}_C^\times$ and $\mathcal{A}a = 0$. We call $\mathcal{A}$ an annihilating operator of a. Let $c \in \mathcal{R}_C$. It is an open problem (the so called *Skolem-Problem* [20]) whether it can be decided algorithmically if $c \in \mathcal{R}_C^\times$. However, even if in practice it may not always be possible to verify formally, usually it is easy to verify empirically.

Instead of working in the ring $\mathbb{K}^{\mathbb{N}}$, we could also work in the ring $\mathcal{S}_{\mathbb{K}} \coloneqq \mathbb{K}^{\mathbb{N}}/J$ for $J \coloneqq \bigcup_{i \in \mathbb{N}} \ker(\sigma^i)$. Two sequences in $\mathcal{S}_{\mathbb{K}}$ are equal if they are equal from some term on [22]. This setting is also used in [18]. Let us write $\pi \colon \mathbb{K}^{\mathbb{N}} \to \mathcal{S}_{\mathbb{K}}$ for the natural projection. We say that $a + J \in \mathcal{S}_{\mathbb{K}}$ is C^2-finite if there is an operator $\mathcal{A} \in \pi(\mathcal{R}_C)[\sigma]$ with $\operatorname{lc}(\mathcal{A}) \in \pi(\mathcal{R}_C)^\times$ and $\mathcal{A}(a + J) = 0 + J$. Equivalently, the sequence $a \in \mathbb{K}^{\mathbb{N}}$ satisfies a C^2-finite recurrence from some term $n_0 \in \mathbb{N}$ on. By shifting the recurrence by n_0, we would get a recurrence which holds for every $n \in \mathbb{N}$. It can be decided whether $\operatorname{lc}(\mathcal{A}) \in \pi(\mathcal{R}_C)^\times$ [3, 5]. Hence, the advantage of working over $\mathcal{S}_{\mathbb{K}}$ is that one can decide if an operator $\mathcal{A}$ is of the desired shape. For practical computations, one is still limited by the Skolem-Problem. Thus, and since in our setting we avoid certain technicalities, we stick to working over the ring $\mathbb{K}^{\mathbb{N}}$ as stated above.

In the setting of [24], a sequence is X-recursive if it satisfies a linear recurrence with arbitrary C-finite coefficients. This includes the cases when the leading coefficient has infinitely many zeros. As a consequence, a sequence may not be uniquely determined by finite data (finitely many initial values) and one can construct examples of X-recursive sequences that grow arbitrarily fast.

We conclude this section by giving three concrete examples of C^2-finite sequences. More examples can be found in [18] and [24].

Example 2.1. Let $(a(n))_{n \in \mathbb{N}}$ count the number of graphs on n labeled nodes (sequence A006125 in the OEIS [9]). Then, $a(n) = 2^{n(n-1)/2}$ and a is C^2-finite as

$$2^n a(n) - a(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Similarly, all sequences $(\alpha^{n^2})_{n \in \mathbb{N}}$ for $\alpha \in \mathbb{K}$ are C^2-finite. These grow faster than holonomic sequences [6, Proposition 1.2.1]. Hence, the set of C^2-finite sequences is a strict generalization of holonomic sequences.

Example 2.2. Let $(f(n))_{n \in \mathbb{N}}$ denote the Fibonacci numbers (with $f(0) = 0, f(1) = 1$). It was observed in [18] that

$$\begin{aligned} f(2n+3)(f(2n+1)f(2n+3) - f(2n+2)^2)f(n^2) \\ +f(2n+2)(f(2n+3) + f(2n+1))f((n+1)^2) \\ -f(2n+1)f((n+2)^2) = 0 \end{aligned}$$

holds for all $n \in \mathbb{N}$. Hence, $(f(n^2))_{n \in \mathbb{N}}$ is C^2-finite (A054783 in the OEIS). In fact, the C-finite coefficients can be simplified and we obtain the simple recurrence:

$$f(2n+3)f(n^2) + f(4n+4)f((n+1)^2) - f(2n+1)f((n+2)^2) = 0.$$

Example 2.3. Let f be as above, l denote the Lucas numbers satisfying the same recurrence as f with initial values $l(0) = 2, l(1) = 1$. Let $\operatorname{Fib}(n, k) \coloneqq \prod_{i=1}^k f(n-i+1)/f(i)$ be the fibonomial coefficient. It has been shown [17, Theorem 1] that

$$a(n) := \sum_{k=0}^{n} \mathrm{Fib}(2n+1, k) = \prod_{k=1}^{n} l(2k), \quad \text{for all } n \in \mathbb{N}.$$

Hence, the sequence a (A294349 in the OEIS) is C^2-finite and satisfies the recurrence

$$l(2n+2)a(n) - a(n+1) = 0, \quad \text{for all } n \in \mathbb{N}.$$

3 Algebraic characterization

For a sequence $a \in \mathbb{K}^{\mathbb{N}}$ and a subring $S \subseteq \mathcal{R}_C$, we consider the module of shifts over the ring $Q(S)$,

$$\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)}$$

where the scalar multiplication is given by the Hadamard product of sequences in $\mathbb{K}^{\mathbb{N}}$. In Theorem 3.3 below, we prove that this module (with $S = \mathcal{R}_C$) is finitely generated, if and only if the sequence is C^2-finite. For this purpose, we first show two auxiliary lemmas.

Lemma 3.1. *Let a be C^2-finite with annihilating operator $\mathcal{A} = c_0 + \dots + c_r\sigma^r$ and let R be the difference ring generated by $c_0, \dots, c_r$. If $R \subseteq S$, then $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)}$ is finitely generated.*

Proof. By assumption we have $\mathrm{lc}(\mathcal{A}) = c_r \in \mathcal{R}_C^{\times}$ and $\mathcal{A}a = 0$. Let $i \in \mathbb{N}$. Then,

$$\sigma^i \mathcal{A} = \sigma^i(c_0)\sigma^i + \dots + \sigma^i(c_r)\sigma^{i+r}$$

and $\mathrm{lc}(\sigma^i \mathcal{A}) = \sigma^i(c_r) \in \mathcal{R}_C^{\times}$. Since $(\sigma^i \mathcal{A})a = \sigma^i(\mathcal{A}a) = 0$, we can write

$$\sigma^{i+r}(a) = -\frac{\sigma^i(c_0)}{\sigma^i(c_r)}\sigma^i(a) - \dots - \frac{\sigma^i(c_{r-1})}{\sigma^i(c_r)}\sigma^{i+r-1}(a).$$

Hence, for all $i \in \mathbb{N}$, the sequence $\sigma^{i+r}a$ is a $Q(R)$-linear combination of the sequences $\sigma^i a, \dots, \sigma^{i+r-1}a$. By induction, $\sigma^{i+r}a$ is a $Q(R)$- and therefore a $Q(S)$-linear combination of $a, \sigma a, \dots, \sigma^{r-1}a$. Thus, the module $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)}$ is generated by $a, \sigma a, \dots, \sigma^{r-1}a$. ☐

Lemma 3.2. *Let $a \in \mathbb{K}^{\mathbb{N}}$ and $S \subseteq \mathcal{R}_C$. If $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)}$ is finitely generated, then a is C^2-finite.*

Proof. As the module is finitely generated, we can write

$$\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)} = \langle b_0, \dots, b_m \rangle_{Q(S)}$$

for some m and some sequences $b_0, \dots, b_m$. There exists an $r \in \mathbb{N}$ such that the elements b_j can be written as $b_j = \sum_{i=0}^{r-1} c_{i,j}\sigma^i a$ for some $c_{i,j} \in Q(S)$. Then, $\sigma^r a$ is a $Q(S)$-linear combination of $b_0, \dots, b_m$, so in particular a linear combination of the sequences $a, \sigma a, \dots, \sigma^{r-1}a$. Hence, there exist sequences $c_0, \dots, c_{r-1} \in S$ and $d_0, \dots, d_{r-1} \in S^{\times}$ with

$$\sigma^r a = \frac{c_0}{d_0}a + \frac{c_1}{d_1}\sigma a + \dots + \frac{c_{r-1}}{d_{r-1}}\sigma^{r-1}a.$$

Clearing denominators shows that a is C^2-finite of order at most r. ☐

Theorem 3.3. *The following are equivalent:*

1. *The sequence a is C^2-finite.*
2. *There exists $\mathcal{A} \in \mathcal{R}_C[\sigma]$ with $\operatorname{lc}(\mathcal{A}) \in \mathcal{R}_C^\times$ and a C^2-finite sequence b with $\mathcal{A}a = b$.*
3. *The module $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)}$ over the ring $Q(\mathcal{R}_C)$ is finitely generated.*

Proof. (1) $\Rightarrow$ (2): We can choose the C^2-finite sequence $b = 0$.

(2) $\Rightarrow$ (1): Since b is C^2-finite, there exists an operator $\mathcal{B} \in \mathcal{R}_C[\sigma]$ with $\operatorname{lc}(\mathcal{B}) \in \mathcal{R}_C^\times$ and $\mathcal{B}b = 0$. Then, $(\mathcal{B}\mathcal{A})a = \mathcal{B}(\mathcal{A}a) = \mathcal{B}b = 0$. Furthermore, $\operatorname{lc}(\mathcal{B}\mathcal{A}) \in \mathcal{R}_C^\times$.

(1) $\Rightarrow$ (3): Follows from Lemma 3.1 with $S = \mathcal{R}_C$.

(3) $\Rightarrow$ (1): Follows from Lemma 3.2 with $S = \mathcal{R}_C$. $\square$

Analogous results like Theorem 3.3 for C-finite and holonomic sequences are often used to show that these sets form rings [16]. In these cases, the base ring is a field and the key step makes use of the fact that submodules of finitely generated modules over fields (i.e., finite dimensional vector spaces) are again finitely generated. This holds more generally for Noetherian rings. However, the rings $\mathcal{R}_C$ and $Q(\mathcal{R}_C)$ are not Noetherian as the next example shows.

Example 3.1. Let $c_k \in \mathcal{R}_C$ with $c_k(n) - c_k(n+k) = 0$ for every $n \in \mathbb{N}$, and $c_k(0) = \cdots = c_k(k-2) = 1, c_k(k-1) = 0$ (i.e., c_k has a 0 at every k-th term and 1 else). Let $L_m \coloneqq \langle c_2, \dots, c_{2^m} \rangle$ be ideals in $\mathcal{R}_C$ for $m \in \mathbb{N}$. Then,

$$L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \cdots$$

is an infinitely properly ascending chain of ideals in $\mathcal{R}_C$. Therefore, $\mathcal{R}_C$ is not a Noetherian ring.

Hence, to use a similar argument for C^2-finite sequences, we construct a Noetherian subring $S \subseteq \mathcal{R}_C$ in the next theorem.

Theorem 3.4. *The set of C^2-finite sequences is a difference ring under termwise addition and termwise multiplication.*

Proof. Let a, b be C^2-finite sequences and $\mathcal{A} = c_0 + c_1\sigma + \cdots + c_{r_1}\sigma^{r_1}$ and $\mathcal{B} = d_0 + d_1\sigma + \cdots + d_{r_2}\sigma^{r_2}$ the corresponding annihilating operators with $c_0, \dots, c_{r_1}, d_0, \dots, d_{r_2} \in \mathcal{R}_C$.

Let $c \in \mathcal{R}_C$ be a C-finite sequence. Then, the $\mathbb{K}$-vector space $\langle \sigma^i c \mid i \in \mathbb{N} \rangle_{\mathbb{K}}$ is finitely generated. Hence, also the $\mathbb{K}$-algebra

$$R_c \coloneqq \mathbb{K}[c, \sigma c, \sigma^2 c, \dots]$$

is finitely generated as an algebra and, in particular, R_c is a Noetherian ring [1, Corollary 7.7].

Now, let $S \subsetneq \mathcal{R}_C$ be the smallest ring containing the Noetherian rings $R_{c_0}, \dots, R_{c_{r_1}}, R_{d_0}, \dots, R_{d_{r_2}}$. This ring S is finitely generated as a ring and therefore, S and $Q(S)$ are Noetherian rings [1, Corollary 7.7 and Proposition 7.3]. By Lemma 3.1, the modules $\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)}$ and $\langle \sigma^i b \mid i \in \mathbb{N} \rangle_{Q(S)}$ are both finitely generated $Q(S)$-modules. Hence, also the modules

$$\langle \sigma^i (a+b) \mid i \in \mathbb{N} \rangle_{Q(S)} \subseteq \langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(S)} + \langle \sigma^i b \mid i \in \mathbb{N} \rangle_{Q(S)}$$

and

$$\langle \sigma^i (ab) \mid i \in \mathbb{N} \rangle_{Q(S)} \subseteq \langle \sigma^i (a) \sigma^j (b) \mid i, j \in \mathbb{N} \rangle_{Q(S)}$$

are finitely generated as they are submodules of finitely generated modules over a Noetherian ring. By Lemma 3.2, the sequences $a + b$ and ab are C^2-finite. Therefore, the set of C^2-finite sequences is a ring.

The operator

$$\tilde{\mathcal{A}} := \sigma(c_0) + \sigma(c_1)\sigma + \cdots + \sigma(c_{r_1})\sigma^{r_1} \in \mathcal{R}_C[\sigma]$$

annihilates σa as

$$\tilde{\mathcal{A}}(\sigma a) = (\tilde{\mathcal{A}}\sigma)a = (\sigma\mathcal{A})a = \sigma(\mathcal{A}a) = 0.$$

Furthermore, we have $\operatorname{lc}(\tilde{\mathcal{A}}) = \sigma(c_{r_1}) \in \mathcal{R}_C^\times$. Hence, the ring of C^2-finite sequences is also closed under shifts. $\square$

In [18, Theorem 1] it was shown that certain sparse subsequences of C-finite sequences are C^2-finite. Example 2.2 given earlier is just a special case of this. We provide an easier proof for a similar result which uses the closed-form representation of C-finite sequences.

Corollary 3.5. *Let c be a C-finite sequence over the field $\mathbb{K}$ and $k, l, m \in \mathbb{N}$. Then, $(c(kn^2 + ln + m))_{n\in\mathbb{N}}$ is C^2-finite over the splitting field $\mathbb{L}$ of the characteristic polynomial of c.*

Proof. By the closed-form representation of C-finite sequences ([16, Theorem 4.1]) c is an $\mathbb{L}$-linear combination of sequences d with $d(n) = n^i\alpha^n$ for $i \in \mathbb{N}$ and $\alpha \in \mathbb{L}$ from some term $n_0 \in \mathbb{N}$ on. It is sufficient to show that $(c(kn^2+ln+m))_{n\in\mathbb{N}}$ satisfies a C^2-finite recurrence for all $n \geq n_0$. This recurrence can be shifted $n \to n + n_0$ to get a C^2-finite recurrence for the entire sequence. We have

$$d(kn^2 + ln + m) = (kn^2 + ln + m)^i(\alpha^k)^{n^2}(\alpha^l)^n\alpha^m.$$

Therefore, with Theorem 3.4, the sequence $(d(kn^2 + ln + m))_{n\in\mathbb{N}}$ is C^2-finite as it is the product of C-finite sequences and the C^2-finite sequence $((\alpha^k)^{n^2})_{n\in\mathbb{N}}$ over $\mathbb{L}$. Since C^2-finite sequences are also closed under $\mathbb{L}$-linear combinations, $(c(kn^2 + ln + m))_{n\in\mathbb{N}}$ is C^2-finite. $\square$

In Section 5.2 we will show that $(c(kn^2 + ln + m))_{n\in\mathbb{N}}$ is even C^2-finite over $\mathbb{K}$ and give an algorithm how such a C^2-finite recurrence can be computed.

4 Ring computations

Classically, closure properties for holonomic functions or sequences are computed using an ansatz method [13]. We describe such an approach for the addition and multiplication of two C^2-finite sequences.

Let a, b be C^2-finite. Then, we have recurrences

$$\begin{aligned}
\tilde{c}_0(n)a(n) + \cdots + \tilde{c}_{r_1-1}(n)a(n + r_1 - 1) + \tilde{c}_{r_1}(n)a(n + r_1) = 0,\\
\tilde{d}_0(n)b(n) + \cdots + \tilde{d}_{r_2-1}(n)b(n + r_2 - 1) + \tilde{d}_{r_2}(n)b(n + r_2) = 0,
\end{aligned}$$

for all $n \in \mathbb{N}$, for $\tilde{c}_0, \ldots, \tilde{c}_{r_1-1}, \tilde{d}_0, \ldots, \tilde{d}_{r_2-1} \in \mathcal{R}_C$ with leading coefficients $\tilde{c}_{r_1}, \tilde{d}_{r_2} \in \mathcal{R}_C^\times$. Therefore,

$$\begin{aligned}
c_0(n)a(n) + \cdots + c_{r_1-1}(n)a(n + r_1 - 1) + a(n + r_1) = 0,\\
d_0(n)b(n) + \cdots + d_{r_2-1}(n)b(n + r_2 - 1) + b(n + r_2) = 0,
\end{aligned}$$

for all $n \in \mathbb{N}$, with $c_0, \ldots, c_{r_1-1}, d_0, \ldots, d_{r_2-1} \in Q(\mathcal{R}_C)$.

To get a recurrence for $a+b$ we make an ansatz of some order s with unknown coefficients $x_0, \dots, x_{s-1} \in Q(\mathcal{R}_C)$:

$$\begin{aligned} x_0(n)(a(n)+b(n)) + \cdots + x_{s-1}(n)(a(n+s-1)+b(n+s-1)) \\ + (a(n+s)+b(n+s)) = 0. \end{aligned}$$

Opposed to the classical approach for C-finite and holonomic sequences, the order of the ansatz s is unknown. Furthermore, the coefficients $x_0, \dots, x_{s-1}$ are in a localized ring and the leading coefficient in the ansatz is 1. We start with an initial value for the order s. If a solution for the coefficients can be found, we can clear the denominators and get a C^2-finite recurrence. Otherwise, if no solution for the coefficients can be found, we increase the order s of the ansatz and repeat this process.

Iterated application of the recurrences shows that each $a(n+j)$ and $b(n+j)$ can be written as $Q(\mathcal{R}_C)$-linear combination of $a(n+i_1)$ for $i_1 = 0, \dots, r_1-1$ and $b(n+i_2)$ for $i_2 = 0, \dots, r_2-1$, respectively. Let $a(n+j) = \sum_{i_1=0}^{r_1-1} u_{j,i_1}(n) a(n+i_1)$ for $j = 0, \dots, s$ for some $u_{j,i_1} \in Q(\mathcal{R}_C)$. Analogously, let $b(n+j) = \sum_{i_2=0}^{r_2-1} v_{j,i_2}(n) b(n+i_2)$ for $j = 0, \dots, s$ for some $v_{j,i_2} \in Q(\mathcal{R}_C)$. Then,

$$\begin{aligned} &\sum_{i_1=0}^{r_1-1} \left(u_{s,i_1}(n) + \sum_{j=0}^{s-1} u_{j,i_1}(n) x_j(n) \right) a(n+i_1) + \\ &\sum_{i_2=0}^{r_2-1} \left(v_{s,i_2}(n) + \sum_{j=0}^{s-1} v_{j,i_2}(n) x_j(n) \right) b(n+i_2) = 0. \end{aligned}$$

Equating the coefficients of $a(n+i_1)$ and $b(n+i_2)$ to zero yields a linear inhomogeneous system. To write it concisely, let us denote

$$u_j^\top = (u_{j,0}, \dots, u_{j,r_1-1}), \qquad v_j^\top = (v_{j,0}, \dots, v_{j,r_2-1}),$$

and

$$w_j = \begin{pmatrix} u_j \\ v_j \end{pmatrix} \in Q(\mathcal{R}_C)^{r_1+r_2}$$

for $j = 0, \dots, s$, and $x^\top = (x_0, \dots, x_{s-1}) \in Q(\mathcal{R}_C)^s$. Now, the system that we obtain from equating the coefficient sequences to zero reads as

$$\begin{pmatrix} w_0, w_1, \dots, w_{s-1} \end{pmatrix} x = -w_s.$$

We call $\begin{pmatrix} w_0, w_1, \dots, w_{s-1} \end{pmatrix} \in Q(\mathcal{R}_C)^{(r_1+r_2)\times s}$ the ansatz matrix of size s for the addition.

To get a recurrence for ab, again we make an ansatz of some order s with unknown coefficients $x_0, \dots, x_{s-1} \in Q(\mathcal{R}_C)$:

$$\begin{aligned} x_0(n)(a(n)b(n)) + \cdots + x_{s-1}(n)(a(n+s-1)b(n+s-1)) \\ + (a(n+s)b(n+s)) = 0. \end{aligned}$$

Using the recurrences for a and b, this equation can be rewritten as

$$\sum_{i_1=0}^{r_1-1} \sum_{i_2=0}^{r_2-1} \left(u_{s,i_1}(n) v_{s,i_2}(n) + \sum_{j=0}^{s-1} u_{j,i_1}(n) v_{j,i_2}(n) x_j(n) \right) a(n+i_1) b(n+i_2) = 0.$$

Equating the coefficients of $a(n+i_1)b(n+i_2)$ to zero yields the linear inhomogeneous system

$$\left(\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{s-1}\right) x = -\tilde{w}_s.$$

where each column $\tilde{w}_j$ is given by the Kronecker product of the vectors u_j, v_j, i.e., $\tilde{w}_j = u_j \otimes v_j$. We call $\left(\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{s-1}\right) \in Q(\mathcal{R}_C)^{(r_1+r_2)\times s}$ the ansatz matrix of size s for the multiplication.

In the next section, we show how the vectors $w_j, \tilde{w}_j$ can be computed. Then, in Section 4.2, we see that the order of the ansatz s can be chosen big enough such that the inhomogeneous system has a solution in $\mathbb{K}^s$ at every term. Finally, from Lemma 4.4 it follows that there is a solution $x \in Q(\mathcal{R}_C)^s$ of the inhomogeneous system.

If one of the C^2-finite sequences has order 1, the inhomogeneous system has a special structure. We use this to derive a bound for the order of the addition of two sequences in Section 4.4.

4.1 Computing the ansatz

Let a be C^2-finite of order r with recurrence

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0,$$

for all $n \in \mathbb{N}$, and for $c_0, \dots, c_{r-1} \in Q(\mathcal{R}_C)$. We write the components of a vector $u_j \in Q(\mathcal{R}_C)^r$ as $u_{j,i}$ for $i = 0, \dots, r-1$. For unit vectors we use the notation $e_j \in Q(\mathcal{R}_C)^r$ for $j = 0, \dots, r-1$. Note that, e.g., $e_0(n)^\top = (1, 0, \dots, 0)$, for all $n \in \mathbb{N}$.

The following lemma shows a straightforward recurrence which can be used to compute the vectors u_j in the ansatz matrix.

Lemma 4.1. *Initialize $u_j := e_j \in Q(\mathcal{R}_C)^r$ with the unit vectors for $j = 0, \dots, r-1$ and define*

$$u_j(n) := -\sum_{k=0}^{r-1} c_k(n+j-r)u_{j+k-r}(n), \qquad \text{for all } n \in \mathbb{N}, \tag{1}$$

inductively for $j \geq r$. These $u_j(n)$ satisfy

$$a(n+j) = \sum_{i=0}^{r-1} u_{j,i}(n)a(n+i), \qquad \text{for all } n \in \mathbb{N}, \tag{2}$$

for all $j \in \mathbb{N}$.

Proof. Shifting the defining recurrence of $a(n)$ yields

$$a(n+j) = -\sum_{k=0}^{r-1} c_k(n+j-r)a(n+j+k-r), \qquad \text{for all } n \in \mathbb{N},$$

for $j \geq r$. We show equation (2) by induction on j. It clearly holds for $j = 0, \dots, r-1$ by the definition of the u_j. Let $n \in \mathbb{N}$ and let us assume that equation (2) holds for $a(n), \dots, a(n+j-1)$. Then,

$$\begin{aligned}\sum_{i=0}^{r-1} u_{j,i}(n)a(n+i) &= -\sum_{i=0}^{r-1}\sum_{k=0}^{r-1} c_k(n+j-r)u_{j+k-r,i}(n)a(n+i) \\ &= -\sum_{k=0}^{r-1} c_k(n+j-r)a(n+j+k-r) = a(n+j).\end{aligned}$$

□

A different way to compute the vectors u_j is to use the companion matrix of a sequence. The companion matrix M_a of the sequence a is defined as

$$M_a := \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{r-1} \end{pmatrix} \in Q(\mathcal{R}_C)^{r\times r}.$$

Lemma 4.2. *Let M_a be the companion matrix of a. Let*

$$u_0 := e_0 = (1, 0, \dots, 0)^\top$$

and define

$$u_j(n) := M_a(n)u_{j-1}(n+1), \qquad \text{for all } n \in \mathbb{N},$$

inductively for $j \geq 1$.

1. *These u_j are identical to the vectors from Lemma 4.1.*
2. *The u_j satisfy equation* (2).

Proof. (1): Clearly $u_j = e_j$ for $j = 0, \dots, r-1$ by the definition of the companion matrix. For $j \geq r$ we show that equation (1) from Lemma 4.1 is satisfied using induction on j. For $j = r$ we have

$$u_r(n)^\top = (-c_0(n), \dots, -c_{r-1}(n)), \qquad \text{for all } n \in \mathbb{N},$$

by the definition of the companion matrix. Therefore,

$$-\sum_{k=0}^{r-1} c_k(n)u_k(n) = -\sum_{k=0}^{r-1} c_k(n)e_k(n) = u_r(n), \qquad \text{for all } n \in \mathbb{N}.$$

Now, we assume that equation (1) from Lemma 4.1 holds for $j-1$, i.e.,

$$u_{j-1}(n) = -\sum_{k=0}^{r-1} c_k(n+j-1-r)u_{j-1+k-r}(n), \tag{3}$$

for all $n \in \mathbb{N}$. Using equation (3) with n shifted by one and the definition of the u_j we have for all $n \in \mathbb{N}$,

$$\begin{aligned} u_j(n) = M_a(n)u_{j-1}(n+1) &= -M_a(n)\sum_{k=0}^{r-1} c_k(n+j-r)u_{j-1+k-r}(n+1) \\ &= -\sum_{k=0}^{r-1} c_k(n+j-r)u_{j+k-r}(n). \end{aligned}$$

(2): Follows directly from part (1) and Lemma 4.1. □

Consider two C^2-finite sequences a, b. To compute the vectors w_j in the ansatz matrix for $a+b$ we can concatenate the vectors we get from Lemma 4.1. Alternatively, following the approach from [11], we can use Lemma 4.2 and compute $w_j(n) = M(n)w_{j-1}(n+1)$, for $n \in \mathbb{N}$, where

$$M = M_a \oplus M_b = \begin{pmatrix} M_a & 0 \\ 0 & M_b \end{pmatrix}$$

is the direct sum of the companion matrices of a and b. If we write $\sigma(w) = (w(n+1))_{n\in\mathbb{N}}$, this can also be written as $w_j = M\sigma(w_{j-1})$.

For the product ab, we have

$$\begin{aligned}(M_a \otimes M_b)\tilde{w}_j &= (M_a \otimes M_b)(u_j \otimes v_j) = (M_a u_j) \otimes (M_b v_j)\\ &= u_{j+1} \otimes v_{j+1} = \tilde{w}_{j+1}.\end{aligned}$$

Hence, we can compute the columns of the ansatz matrix using the Kronecker product $M = M_a \otimes M_b$ of the matrices M_a and M_b.

4.2 Solving the ansatz

In Section 4.1 we have seen constructive ways to compute the ansatz matrix of a specific size s. Lemma 4.3 below yields that this size s can be chosen large enough such that the corresponding inhomogeneous system has a solution at every term. Lemma 4.4 then states that such termwise solvable systems are even solvable in the C-finite sequence ring. For both results we adapt some techniques which were used in [18]. As a consequence, we obtain a (semi-) constructive way to compute the addition and multiplication in the C^2-finite sequence ring.

Lemma 4.3. *Let a, b be C^2-finite sequences. Then, the order s of the ansatz for the sum $a+b$ and the product ab can be chosen in such a way that the corresponding inhomogeneous system has a solution at every term.*

Proof. Let $w_0, w_1, \dots \in Q(\mathcal{R}_C)^r$ (with $r = r_1 + r_2$ if we consider the sum $a+b$ and $r = r_1 r_2$ if we consider the product ab) be the columns of the ansatz matrix. Let $S \subsetneq \mathcal{R}_C$ be the smallest ring containing all $\mathbb{K}$-algebras $\mathbb{K}[c, \sigma c, \sigma^2 c, \dots]$ where c is a coefficient in the annihilating operator of a or b. In the proof of Theorem 3.4 we have seen that $Q(S)$ is a Noetherian ring. Now, define $A_j := (w_0, \dots, w_j) \in Q(S)^{r\times(j+1)}$. Furthermore, let $I_j^{(t)} \trianglelefteq Q(S)$ be the ideals generated by the minors of order t of A_j. For fixed $t \in \{0, \dots, r\}$, these $I_j^{(t)}$ form an increasing chain of ideals. Let $s \in \mathbb{N}$ be such that $I_{s-1}^{(t)} = I_s^{(t)}$ for all $t \in \{0, \dots, r\}$. Then, $A_{s-1}(n)x(n) = -w_s(n)$ has a solution for every n: Suppose

$$t := \operatorname{rank}(A_s(n)) > \operatorname{rank}(A_{s-1}(n)).$$

Then, there exists a nonzero minor $\phi(n)$ of order t of $A_s(n)$. On the other hand, all minors $\phi_0(n) = \cdots = \phi_m(n)$ of order t of $A_{s-1}(n)$ are zero. By the choice of s, the nonzero minor $\phi(n)$ is a $Q(S)$-linear combination of the minors $\phi_0(n), \dots, \phi_m(n)$, a contradiction. Hence, $A_{s-1}(n)$ and $A_s(n)$ have equal rank and, by the Rouché–Capelli theorem, the linear equation has a solution. □

The proof of Lemma 4.3 is not constructive as the Noetherian ring only gives us the existence of the number s. To make this argument constructive we would need to be able to solve instances of the ideal membership problem over $Q(\mathcal{R}_C)$.

The order of the addition and multiplication of C-finite sequences is bounded by the sum and product of the orders of the sequences respectively. Lemma 4.3 shows that these bounds typically, but not necessarily, hold for C^2-finite sequences as well (e.g., if $A_{r-1}(n)$ in the notation of the proof above has full rank for every n). The next example shows that these classical bounds do not work in some cases.

Example 4.1. Consider

$$(-1)^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Actually, a is also C-finite of order 2. Making an ansatz of order 2 for the sequence $g = a + b$ with coefficients $x_0, x_1 \in Q(\mathcal{R}_C)$ yields the linear system

$$\begin{pmatrix} 1 & -(-1)^n \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0(n) \\ x_1(n) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This system is not solvable for even $n \in \mathbb{N}$. Hence, our technique cannot yield a recurrence for g of order 2. However, with an ansatz of order 3 we get the recurrence

$$\left(\tfrac{1}{2}(-1)^{n+1} + \tfrac{1}{2}\right) g(n) + \left(\tfrac{1}{2}(-1)^n + \tfrac{1}{2}\right) g(n+2) + g(n+3) = 0,$$

for every $n \in \mathbb{N}$. Setting up a classical homogeneous ansatz as in [24] yields the recurrence

$$((-1)^n + 1)\, g(n) + 2g(n+1) + (1 - (-1)^n) g(n+2) = 0$$

with a leading coefficient which has infinitely many zeros. Such a recurrence fits in the framework of X-recursive sequences from [24], but it is not a C^2-finite recurrence in our sense.

In order to show how to solve systems over the ring $Q(\mathcal{R}_C)$, we use the Skolem-Mahler-Lech Theorem [5]. It states that the zeros of a sequence $c \in \mathcal{R}_C$ (and therefore also $c \in Q(\mathcal{R}_C)$) are exactly at finitely many arithmetic progressions from some term on. Hence, for a sequence $c \in \mathcal{R}_C$ there exist $n_0, p \in \mathbb{N}$ such that

$$(c(n_0 + pk), \dots, c(n_0 + pk + p - 1))$$

has the same zero-pattern for every $k \in \mathbb{N}$. This number p is called the zero-cycle period of the sequence c.

The main idea for solving linear systems over $Q(\mathcal{R}_C)$ is that the solvability of the system is completely determined by the zeros of the minors of the matrix. By the Skolem-Mahler-Lech Theorem these zeros are exactly at arithmetic progressions. Hence, it is sufficient to solve the system explicitly at those progressions and then interlace these solutions.

Lemma 4.4. *Let $A \in Q(\mathcal{R}_C)^{r\times s}$ and $w \in Q(\mathcal{R}_C)^r$. Suppose the system $A(n)x(n) = w(n)$ has a solution for every $n \in \mathbb{N}$. Then, there is a solution $x \in Q(\mathcal{R}_C)^s$ such that $Ax = w$ in $Q(\mathcal{R}_C)$.*

Proof. All minors of A are sequences in $Q(\mathcal{R}_C)$. Consider the set of all these. By the Skolem-Mahler-Lech Theorem, the zeros of these minors are cyclic. Let $p \in \mathbb{N}$ be the common zero-cycle period of all minors from some term $n_0 \in \mathbb{N}$ on.

We write $A = (w_0, \dots, w_{s-1})$ for $w_0, \dots, w_{s-1} \in Q(\mathcal{R}_C)^r$. Now, for every $m \in \{n_0, \dots, n_0 + p - 1\}$ we can choose a subset $j_m \subseteq \{0, \dots, s-1\}$ such that the vectors $\{w_j(m) \mid j \in j_m\} \subseteq \mathbb{K}^r$ are maximally linearly independent, i.e., they are linearly independent and generate the same subspace as $\{w_0(m), \dots, w_{s-1}(m)\}$. By the choice of n_0 and p this is also true for all $n = m + pk$ for $k \in \mathbb{N}$, i.e., the vectors $\{w_j(m+pk) \mid j \in j_m\} \subseteq \mathbb{K}^r$ are maximally linearly independent for every $k \in \mathbb{N}$. Let us denote by $A_m \in Q(\mathcal{R}_C)^{r\times |j_m|}$ the submatrix of A where we keep the columns w_j with $j \in j_m$.

For every m, we can solve the system

$$A_m(m + pk)x_m(k) = w(m + pk), \qquad \text{for all } k \in \mathbb{N}, \tag{4}$$

using the Moore-Penrose-Inverse: By the choice of m, p, n_0, the matrix $A_m(m + pk)$ has linear independent columns for every $k \in \mathbb{N}$. Therefore, the Gramian matrix $G(k) = A_m(m + pk)^\top A_m(m + pk)$ is regular for every k and $(\det(G(k)))_{k\in\mathbb{N}} \in \mathcal{R}_C^\times$. Now, let

$$x_m(k) = \frac{1}{\det(G(k))} \operatorname{cof}(G(k))\, A_m(m + pk)^\top w(m + pk)$$

where $\mathrm{cof}(\cdot)$ denotes the transposed cofactor matrix. Then, since equation (4) has a termwise solution, $(x_m(k))_{k\in\mathbb{N}} \in Q(\mathcal{R}_C)^{|j_m|}$ satisfies equation (4) by the theory of Moore-Penrose-Inverses. Let $x'_m \in Q(\mathcal{R}_C)^s$ be the vector where we add $0 \in Q(\mathcal{R}_C)$ at the indices $j \in \{0,\dots,s-1\} \setminus \{j_m\}$.

Now, the solution x for the entire system can be computed as the interlacing of $x'_{n_0},\dots,x'_{n_0+p-1}$ from n_0 on and the first n_0 values can be computed explicitly. Then, $x \in Q(\mathcal{R}_C)^s$ as $Q(\mathcal{R}_C)$ is closed under interlacing and specifying finitely many initial values. □

The arithmetic progressions from the Skolem-Mahler-Lech Theorem can be found effectively. Hence, the zero-cycle period of a C-finite sequence can be computed. It is, however, not known whether the index $n_0 \in \mathbb{N}$ such that the zeros beyond this index are cyclic can be found algorithmically (cf. Skolem-Problem [20]). Hence, the proof of Lemma 4.4 is not constructive in general. However, in many cases, this index n_0 can be computed:

- For sequences of order at most 4 the problem is known to be decidable [20, 21].
- If the dominant roots (the roots of the characteristic polynomial with maximal absolute value) of a C-finite sequence satisfy certain conditions, the Skolem-Problem is also known to be decidable. This is for instance the case, if the sequence has a unique dominant root [19, 26, 8].
- Furthermore, automatic proving procedures using cylindrical algebraic decomposition can be used to decide the Skolem-Problem for certain sequences [7, 23].

If these rigorous methods fail, one can use a heuristic approach and check a certain number of initial values to determine any zeros.

Algorithm 1 summarizes the arguments from Section 4.1 and this section. The algorithm computes a recurrence for the addition or multiplication of two C^2-finite sequences a, b of orders r_1, r_2 provided that we can solve linear systems of equations over $Q(\mathcal{R}_C)$. We denote the unit vectors of the respective size by $e_0 \in Q(\mathcal{R}_C)^{r_1+r_2}$ and $\tilde{e}_0 \in Q(\mathcal{R}_C)^{r_1 r_2}$, respectively.

Input : C^2-finite sequences a, b
output: C^2-finite recurrence satisfied by $a+b$ (or ab, respectively)
$M \leftarrow M_a \oplus M_b$ (or $M_a \otimes M_b$ for the multiplication)
$A \leftarrow$ empty matrix
$w \leftarrow e_0 \oplus \tilde{e}_0$ (or $e_0 \otimes \tilde{e}_0$ for the multiplication)
for $s = 0, 1, 2, \dots$ **do**
  if *solution* $x \in Q(\mathcal{R}_C)^s$ *of* $Ax = -w$ *exists* **then**
    return $\sum_{i=0}^{s-1} x_i \sigma^i + \sigma^s$
  else
    $A \leftarrow (A \mid w)$
    $w \leftarrow M\sigma(w)$
  end
end

Algorithm 1: Computing C^2-finite ring operations

Whenever the zeros of the minors of the matrix A can be computed, the proof of Lemma 4.4 gives a procedure to compute a solution. We give this procedure in Algorithm 2. For a sequence $c \in Q(\mathcal{R}_C)$ we will denote by `period_start(c)` and `period_length(c)` the start and the length of the zero-cycle of the sequence c.

Lemma 4.4 also shows a possible algorithm to solve the ideal membership problem in $Q(\mathcal{R}_C)$ from Lemma 4.3: The problem whether $c \in \langle d_1,\dots,d_s\rangle$ for $c, d_1,\dots,d_s \in Q(\mathcal{R}_C)$ is equivalent to

Input : $A \in Q(\mathcal{R}_C)^{r\times s}, w \in Q(\mathcal{R}_C)^r$
output: $x \in Q(\mathcal{R}_C)^s$ with $Ax = w$ if it exists and `false` otherwise
$\Phi \leftarrow$ minors of A
$p \leftarrow \operatorname{lcm}(\texttt{period\_length}(\phi) \mid \phi \in \Phi)$
$n_0 \leftarrow \max(\texttt{period\_start}(\phi) \mid \phi \in \Phi)$
if *$A(m)x(m) = w(m)$ is not solvable for an $m \in \{0, \dots, n_0 - 1\}$* **then**
  return `false`
end
for $m = n_0, \dots, n_0 + p - 1$ **do**
  $j_m \leftarrow$ indices of columns of $A(m)$ which are maximally linearly independent
  $A_m \leftarrow$ matrix built by columns j_m of A
  $A'_m \leftarrow (A_m(m + pk))_{k\in\mathbb{N}}$
  $w' \leftarrow (w(m + pk))_{k\in\mathbb{N}}$
  $G \leftarrow (A'_m)^\top A'_m$
  $x_m \leftarrow \frac{1}{\det(G)} \operatorname{cof}(G)(A'_m)^\top w' \in Q(\mathcal{R}_C)^{|j_m|}$
  $x'_m \leftarrow$ insert 0 in x_m at indices $j \in \{0, \dots, s - 1\} \setminus \{j_m\}$
end
$x \leftarrow$ interlacing of sequences $x'_{n_0}, \dots, x'_{n_0+p-1}$ with prepended terms $x(0), \dots, x(n_0 - 1)$
if $Ax = w$ **then**
  return x
else
  return `false`
end

Algorithm 2: Solving linear systems over $Q(\mathcal{R}_C)$

solving the inhomogeneous equation $dx = c$ with $d = (d_1, \dots, d_s)$ for unknown $x \in Q(\mathcal{R}_C)^s$. With Lemma 4.4 we can compute a possible solution x. Then, we can check with closure properties whether $dx = c$ indeed holds in $Q(\mathcal{R}_C)$. If it does, we have shown that c is in the ideal. If $dx \neq c$, then c is not in the ideal, because otherwise we would have a termwise solution and therefore, with Lemma 4.4, a solution in $Q(\mathcal{R}_C)$.

[18, Lemma 14] states that the components $w_j \in \mathbb{K}^\mathbb{N}$ of a vector $w^\top = (w_0, \dots, w_{r-1})$ are C^2-finite if the vector satisfies a recurrence of the form $w(n+1) = M(n)w(n)$, for every $n \in \mathbb{N}$, with $M \in \mathcal{R}_C^{r\times r}$. Since the existence of the sequence s_n in the proof of Lemma 14 is not guaranteed, their approach seems to not work for certain examples. Our approach, however, is very similar. We use the same idea from Lemma 14 to set up an inhomogeneous linear system over $\mathcal{R}_C$. The proof of Lemma 4.4 is also based on the proof of Lemma 14. The difference in our approach is that we do not assume that the inhomogeneous system has a certain fixed size (which is determined by $\operatorname{ord}(a), \operatorname{ord}(b)$). In our case, the size of this system also depends on the C-finite coefficients in the recurrences. Example 4.1 shows that this flexible approach is sometimes really needed.

4.3 Some remarks on complexity

As a proof-of-concept, we implemented the closure properties of C^2-finite sequences in the open source computer algebra system SageMath [25]. For computations in the base ring of C-finite sequences we use the `ore_algebra` package which provides an efficient implementation of closure properties in this ring [14]. A preliminary version of the package can be obtained from the authors.

In our implementation, we do not use Algorithm 1 verbatim. Instead, we increase the ansatz until the corresponding linear system has a termwise solution for a fixed number of initial terms. We check this using the Rouché–Capelli theorem. This is usually quite fast, since the computation of terms of a C-finite sequence and computations over the field $\mathbb{K}$ are cheap in comparison with the operations in $Q(\mathcal{R}_C)$. Therefore, we solve the system over $Q(\mathcal{R}_C)$ only if these initial terms indicate that the system is in fact solvable.

For solving the linear system we use Algorithm 2 where we guess the zeros of the minors using again some fixed number of initial terms. The most costly step is certainly computing the inverse explicitly. Computing the solutions explicitly can also artificially increase them. It can happen that we get a sequence $x_i = y_i/d_i \in Q(\mathcal{R}_C)$ with $d_i \notin \mathbb{K}$ in the solution although we have $x_i \in \mathcal{R}_C$. We try to detect this by computing some initial terms of x_i and guessing (and proving) a C-finite recurrence from these. Furthermore, after solving the linear system we need to clear the denominators $d_0, \dots, d_{s-1}$ of the solution x. Often, the least common multiple d of $d_0, \dots, d_{s-1}$ can be found by guessing using the termwise least common multiple of $d_0, \dots, d_{s-1}$. If this is not the case, we just use $d = d_0 \cdots d_{s-1}$. Computing such a common multiple d, clearing the denominators and checking whether all these operations using guessing were in fact exact, can also take a significant amount of time.

The following examples were all run on a standard notebook. The computations took from a few seconds to about 30 seconds. The main bottleneck is often the computation of the linear system. As discussed, using Algorithm 2 might also create artificially large solutions which shows in long computations when clearing the denominators. Hence, a more efficient algorithm for solving linear systems over this ring seems vital if we want to compute closure properties for larger sequences.

Example 4.2. In Example 4.1 two linear systems have to be solved. About 70% of the computation time is spent for solving the linear systems. The other 30% are mostly used to compute an ansatz which is big enough and to verify this using the Rouché–Capelli theorem.

Example 4.3. Consider the C^2-finite sequences a, b satisfying the recurrences

$$\begin{aligned} (-1)^n a(n) + 2^n a(n+1) - a(n+2) &= 0, \\ b(n) + 2^n b(n+1) + (-1)^n b(n+2) &= 0. \end{aligned}$$

We compute a C^2-finite recurrence for $g = a + b$ of order 4 and with C-finite coefficients having maximal order 4:

$$\begin{aligned} &(-4 \cdot 4^n - 4\ (-4)^n - 1) g(n) \\ &+ \left(-4 \cdot 8^n + \tfrac{3}{2} \cdot 2^n + \tfrac{3}{2}\ (-2)^n - 4\ (-8)^n\right) g(n+1) \\ &+ \left(-4 \cdot 8^n - \tfrac{3}{2} \cdot 2^n + \tfrac{3}{2}\ (-2)^n + 4\ (-8)^n\right) g(n+3) \\ &+ (4^n - (-4)^n + 1) g(n+4) = 0. \end{aligned}$$

About 70% of the time is used to solve two 4×4 inhomogeneous linear systems. Computing a big enough system takes about 25%.

Example 4.4. Consider the sequences a, b from Example 4.3. We compute a C^2-finite recurrence for $h = ab$ of order 3 and with coefficients of maximal order 2:

$$2h(n) + (-2\ (-4)^n + 1)\, h(n+1) + (-4\ (-4)^n + 2)\, h(n+2) + h(n+3) = 0.$$

Solving the 4×3 inhomogeneous system takes about 80% of the overall time. About 13% of the time is used to compute the linear system.

4.4 Bounds for addition

The order of adding and multiplying C-finite sequences is bounded by the sum and product of the orders of the sequences respectively. Example 4.1 shows that this cannot be achieved in general in our setting for C^2-finite sequences. Furthermore, the proof of Lemma 4.3 does not indicate how large the order for the ansatz of the addition and multiplication should be chosen. However, we can provide explicit a priori bounds in the special case where we add two C^2-finite sequences a, b one of which has order 1. In this case, Lemma 4.6 shows exactly how the order of $a + b$ depends on the coefficients of the recurrences defining a and b.

In this section we assume that a is C^2-finite of order r and b is C^2-finite of order 1 satisfying the recurrences

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0,$$
$$d(n)b(n) + b(n+1) = 0,$$

for all $n \in \mathbb{N}$, with $c_0, \ldots, c_{r-1}, d \in Q(\mathcal{R}_C)$.

Let u_j, v_j be the coefficients for the iterated recurrence of a and b as defined in Lemma 4.1, respectively. Note that $v_0 = 1$ and

$$v_j(n) = v_{j,0}(n) = -d(n+j-1)v_{j-1}(n), \quad \text{for all } n \in \mathbb{N},$$

for $j \geq 1$. Therefore,

$$v_j(n) = (-1)^j d(n)d(n+1)\cdots d(n+j-1) \tag{5}$$

for all $j, n \in \mathbb{N}$.

Let

$$\phi_j := \det(w_0, \ldots, w_{r-1}, w_j) \in Q(\mathcal{R}_C)$$

for $j \geq 0$ with $w_j^\top = (u_j^\top, v_j)$. Let $v = (v_0, \ldots, v_{r-1}) \in Q(\mathcal{R}_C)^r$ and let $I \in \mathbb{K}^{r \times r}$ be the identity matrix. Then,

$$\phi_j(n) = \begin{vmatrix} I & u_j(n) \\ v(n) & v_j(n) \end{vmatrix} = v_j(n) - \sum_{i=0}^{r-1} u_{j,i}(n)v_i(n), \tag{6}$$

for all $n \in \mathbb{N}$.

For $j < r$, we have $\phi_j = 0$, as the matrix has linear dependent columns. For $j = r$, we have $u_r^\top = -(c_0, \ldots, c_{r-1})$ and therefore,

$$\phi_r(n) = v_r(n) + \sum_{i=0}^{r-1} c_i(n)v_i(n), \tag{7}$$

for every $n \in \mathbb{N}$.

Lemma 4.5. *Let $j \geq r$. Then,*

$$\phi_j(n) = -\sum_{k=0}^{r-1} c_k(n+j-r)\phi_{j+k-r}(n) + v_{j-r}(n)\phi_r(n+j-r),$$

for every $n \in \mathbb{N}$.

Proof. With equation (6) and equation (1) we have

$$\begin{aligned}
\phi_j(n) &= v_j(n) - \sum_{i=0}^{r-1} u_{j,i}(n) v_i(n) \\
&= v_j(n) - \sum_{i=0}^{r-1} \left(- \sum_{k=0}^{r-1} c_k(n+j-r) u_{j+k-r,i}(n) \right) v_i(n) \\
&= v_j(n) + \sum_{k=0}^{r-1} c_k(n+j-r) \sum_{i=0}^{r-1} u_{j+k-r,i}(n) v_i(n) \\
&= - \sum_{k=0}^{r-1} c_k(n+j-r) \phi_{j+k-r}(n) + v_j(n) + \sum_{k=0}^{r-1} v_{j+k-r}(n).
\end{aligned}$$

Because of equation (5) we have $v_{j+k-r}(n) = v_k(n+j-r)v_{j-r}(n)$ and $v_j(n) = v_r(n+j-r)v_{j-r}(n)$. Then, equation (7) yields

$$\phi_j(n) = - \sum_{k=0}^{r-1} c_k(n+j-r) \phi_{j+k-r}(n) + v_{j-r}(n) \phi_r(n+j-r).$$

□

We define the order of a sequence $c \in Q(\mathcal{R}_C)$ as the order of the minimal numerator, i.e., $\operatorname{ord}(c) := \min\{\operatorname{ord}(d) \mid c = \frac{d}{e} \in Q(\mathcal{R}_C)\}$.

Lemma 4.6. *Let a, b be C^2-finite of orders r and 1, respectively. Then, $a+b$ has order at most $n_0 + r$ if there exists an $n_0 \in \mathbb{N}$ with $\phi_r(n) = 0$ for $n \geq n_0$. Otherwise, $a+b$ has order at most $\operatorname{ord}(\phi_r) + r$.*

Proof. Let

$$\begin{aligned}
c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) &= 0, \\
d(n)b(n) + b(n+1) &= 0,
\end{aligned}$$

for all $n \in \mathbb{N}$, with $c_0, \ldots, c_{r-1}, d \in Q(\mathcal{R}_C)$.

If $\phi_r(n) = 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, we can shift the sequences by n_0, choose r for the order of the ansatz of $a+b$ and the corresponding system has a solution for every $n \in \mathbb{N}$. Now, shifting the recurrence by n_0 we can specify the initial values $(a+b)(n)$ for $n < n_0$.

Otherwise, we choose the order of the ansatz as $s := \operatorname{ord}(\phi_r) + r$. We show that the corresponding linear system has a solution for every $n \in \mathbb{N}$: If one of the $\phi_r(n), \ldots, \phi_{s-1}(n)$ is nonzero, the system has a solution as we have $r+1$ linearly independent vectors in $\mathbb{K}^{r+1}$. Now, assume that $\phi_r(n) = \cdots = \phi_{s-1}(n) = 0$. By the choice of s, the set $\{\phi_r(n), \ldots, \phi_r(n+s-r-1)\}$ contains a nonzero element $\phi_r(n+s_n-r) \neq 0$ for some $s_n \in \{r, \ldots, s-1\}$. Then, $\phi_{s_n}(n) = 0$ and by Lemma 4.5

$$\begin{aligned}
\phi_{s_n}(n) &= - \sum_{k=0}^{r-1} c_k(n+s_n-r) \phi_{s_n+k-r}(n) + v_{s_n-r}(n) \phi_r(n+s_n-r) \\
&= - \sum_{k=0}^{r-1} c_k(n+s_n-r) 0 + v_{s_n-r}(n) \phi_r(n+s_n-r) = 0.
\end{aligned}$$

Therefore, $v_{s_n-r}(n) = 0$ and with equation (5) we have $v_{s-r}(n) = v_{s_n-r}(n)v_{s-s_n}(n+s_n-r) = 0$. Hence, again with Lemma 4.5, we have

$$\begin{aligned}\phi_s(n) &= -\sum_{k=0}^{r-1} c_k(n+s-r)\phi_{s+k-r}(n) + v_{s-r}(n)\phi_r(n+s-r)\\ &= 0 + 0\phi_r(n+s-r) = 0.\end{aligned}$$

In this case, the system corresponding to the ansatz of order s has a solution as well. Because of Lemma 4.4, we have a recurrence for $a+b$ of order $\operatorname{ord}(\phi_r) + r$. $\square$

Note that the proof of Lemma 4.6 is similar to the proof of Lemma 4.4. In the case where one of the sequences has order 1, Lemma 4.5 gives an explicit relation between the minors. In the proof of Lemma 4.6 we only get the existence of a bound for the order of two C^2-finite sequences with a noetherianity condition. Here, using these relations from Lemma 4.5, we can actually compute a bound.

Example 4.5. In Example 4.1 we have $r = 1$ and

$$\phi_1(n) = \begin{vmatrix} 1 & -(-1)^n \\ 1 & -1 \end{vmatrix} = -1 + (-1)^n,$$

for all $n \in \mathbb{N}$. The sequence ϕ_1 has order 2. Hence, $a + b$ has order at most $1 + \operatorname{ord}(\phi_1) = 3$. Indeed, we have seen a recurrence of order 3 in Example 4.1.

Example 4.6. The order bound in Lemma 4.6 is exact in general. Let $c \in \mathcal{R}_C$ be the cyclic sequence of order m defined by

$$c(n) - c(n+m) = 0, \quad c(0) = -1, c(1) = \cdots = c(m-1) = 1$$

and let a, b be C^2-finite sequence defined by

$$c(n)a(n) - a(n+1) = 0, \quad a(0) = 1, \quad b(n) - b(n+1) = 0, \quad b(0) = 1.$$

Then, we have $\phi_1 = 1 - c$ and $\operatorname{ord}(\phi_1) = m$. Suppose we make an ansatz of order $s < m+1$. With the definition of c, the corresponding linear system at $n = m - s + 1$ is of the form

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_0(n) \\ \vdots \\ x_{s-1}(n) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, the linear system has no solution. For $s = m + 1$ we indeed get a solution for every n as Lemma 4.6 suggests.

5 Further closure properties

For C-finite and holonomic sequences many more closure properties are known. They are often used to construct more complicated sequences from simpler ones. Most of these properties carry over to C^2-finite sequences and can be proven in a very similar way.

Theorem 5.1. *Let $a, a_0, \ldots, a_{m-1}$ be C^2-finite sequences. Then,*

1. *(shifts) $\sigma^k(a)$ is C^2-finite for every $k \in \mathbb{N}$,*
2. *(difference) $\Delta(a) := \sigma(a) - a$ is C^2-finite,*
3. *(partial sums) $b = (\sum_{k=0}^{n} a(k))_{n\in\mathbb{N}}$ is C^2-finite.*
4. *(subsequence) $b = (a(dn))_{n\in\mathbb{N}}$ is C^2-finite for every $d \in \mathbb{N}$,*
5. *$b = (a(\lfloor n/d \rfloor))_{n\in\mathbb{N}}$ is C^2-finite for every $d \in \mathbb{N}$,*
6. *(interlacing) if $b = (b(n))_{n\in\mathbb{N}}$ with $b(n) = a_s(q)$ such that $n = qm + s$ and $0 \le s < m$, then b is C^2-finite.*

Proof. (1), (2): Clear as the set of C^2-finite sequences is a difference ring by Theorem 3.4.

(3): We have $\sigma(b) - b = \sigma(a)$. Therefore, by Theorem 3.3, the sequence b is C^2-finite.

(4): Let $c_0 + c_1\sigma + \cdots + c_r\sigma^r \in \mathcal{R}_C[\sigma]$ be the annihilating operator of a. We have $\sigma^i b = (a(dn + di))_{n\in\mathbb{N}}$, for all $i \in \mathbb{N}$. For $c \in \mathcal{R}_C$ of order s we have

$$(c(dn + j))_{n\in\mathbb{N}} \in \langle (c(dn - s + j)_{n\in\mathbb{N}}, \ldots, (c(dn - 1 + j)_{n\in\mathbb{N}}) \rangle_{\mathbb{K}},$$

for every $j \ge s$. Hence, by induction

$$(c(dn + j))_{n\in\mathbb{N}} \in \langle (c(dn)_{n\in\mathbb{N}}, \ldots, (c(dn + s - 1)_{n\in\mathbb{N}}) \rangle_{\mathbb{K}},$$

for every $j \in \mathbb{N}$. In particular, the algebra

$$\mathbb{K}[(c(dn))_{n\in\mathbb{N}}, (c(dn+1))_{n\in\mathbb{N}}, \ldots]$$

is a Noetherian ring. Let $S \subsetneq \mathcal{R}_C$ be the ring containing the sequences $(c_i(dn + j))_{n\in\mathbb{N}}$, for all $i = 0, \ldots, r$ and $j \in \mathbb{N}$. As the smallest ring containing finitely many Noetherian rings, this ring S is Noetherian. Let $i \in \mathbb{N}$. Using the recurrence for a and induction we have

$$\sigma^i b \in \langle (a(dn))_{n\in\mathbb{N}}, \ldots, (a(dn + r - 1))_{n\in\mathbb{N}} \rangle_{Q(S)}.$$

Therefore, $\langle \sigma^i b \mid i \in \mathbb{N} \rangle_{Q(S)}$ is finitely generated. Hence, by Lemma 3.2, the sequence b is C^2-finite.

(5): Suppose a satisfies $\sum_{i=0}^{r} c_i(n)a(n+i) = 0$, for all $n \in \mathbb{N}$. Then, we also have

$$\sum_{i=0}^{r} c_i(\lfloor n/d \rfloor) a(\lfloor (n + id)/d \rfloor) = \sum_{i=0}^{r} c_i(\lfloor n/d \rfloor) b(n + id) = 0,$$

for all $n \in \mathbb{N}$. Since $(c_i(\lfloor n/d \rfloor))_{n\in\mathbb{N}} \in \mathcal{R}_C$ for all $i = 0, \ldots, r$ and $(c_r(\lfloor n/d \rfloor))_{n\in\mathbb{N}} \in \mathcal{R}_C^{\times}$, the sequence b is C^2-finite.

(6): For all $s = 0, \ldots, m-1$, the sequences $(a_s(\lfloor n/m \rfloor))_{n\in\mathbb{N}}$ are C^2-finite by part (5). Let

$$i_s(n) := \begin{cases} 1 & \text{if } n \equiv s \mod m, \\ 0 & \text{if } n \not\equiv s \mod m. \end{cases}$$

Then, $i_s \in \mathcal{R}_C$ for all $s = 0, \ldots, m-1$. Furthermore,

$$b(n) = \sum_{s=0}^{m-1} i_s(n) a_s(\lfloor n/m \rfloor), \qquad \text{for all } n \in \mathbb{N}.$$

Since the set of C^2-finite sequences is a ring containing $\mathcal{R}_C$, the sequence b is C^2-finite. $\square$

Computing the recurrence for shifts and partial sums is fully constructive. For computing the recurrence of the difference and interlacing we need to be able to add and multiply certain C^2-finite sequences. The ansatz method described in Section 4 can be used to compute a recurrence for the subsequence of a C^2-finite sequence. We describe this approach in more detail in the following section.

5.1 Computing a recurrence for the subsequence

Let a be C^2-finite of order r with recurrence

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0,$$

for all $n \in \mathbb{N}$, and for $c_0, \ldots, c_{r-1} \in Q(\mathcal{R}_C)$. Theorem 5.1.4 suggests the following algorithm to compute a C^2-finite recurrence for the sequence $b = (a(dn))_{n\in\mathbb{N}}$: We make an ansatz

$$x_0(n)b(n) + \cdots + x_{s-1}(n)b(n+s-1) + b(n+s) = 0$$

with unknown coefficients $x_0, \ldots, x_{s-1} \in Q(\mathcal{R}_C)$. Using the recurrence of a, we can express all $b(n+j)$ as

$$b(n+j) = \sum_{i=0}^{r-1} u_{j,i}(n)a(dn+i) \tag{8}$$

for some $u_{j,i} \in Q(\mathcal{R}_C)$ and all $j = 0, \ldots, s$. Then, the ansatz can be rewritten as

$$\sum_{i=0}^{r-1} \left(u_{s,i}(n) + \sum_{j=0}^{s-1} u_{j,i}(n)x_j(n) \right) a(dn+i) = 0.$$

Equating the coefficients of $a(dn+i)$ to zero yields a linear inhomogeneous system. The proof of Lemma 4.3 shows that this linear system is solvable at every term and with Lemma 4.4 we can compute these solutions $x_0, \ldots, x_{s-1} \in Q(\mathcal{R}_C)$ if we know the zeros of the minors of this system.

We write

$$u_j^\top := (u_{j,0}, \ldots, u_{j,r-1}).$$

The next lemma shows how these u_j can be computed.

Lemma 5.2. *Let*

$$u_0 := e_0 = (1, 0, \ldots, 0)^\top$$

and define

$$u_j(n) := M_a(dn) \cdots M_a(dn+d-1)u_{j-1}(n+1), \qquad \text{for all } n \in \mathbb{N},$$

inductively for $j \geq 1$. These u_j satisfy equation (8).

Proof. For convenience we write

$$A(n) := (a(n), \ldots, a(n+r-1)).$$

Then, (8) reads as $b(n+j) = A(dn)u_j(n)$. By definition, u_0 satisfies this equation. Now, suppose u_{j-1} satisfies equation (8), i.e., $b(n+j-1) = A(dn)u_{j-1}(n)$. By the definition of the companion matrix M_a and the defining recurrence of a, we have $A(dn+i)M_a(dn+i) = A(dn+i+1)$. Therefore,

$$\begin{aligned} A(dn)u_j(n) &= A(dn)M_a(dn)\cdots M_a(dn+d-1)u_{j-1}(n+1) \\ &= A(dn+d)u_{j-1}(n+1) = b(n+j) \end{aligned}$$

where we use the induction hypothesis shifted $n \to n+1$ in the last step. $\square$

Hence, choosing $M = M_a(dn)\cdots M_a(dn+d-1)$ and $w = e_0 \in Q(\mathcal{R}_C)^r$ in Algorithm 1 gives an explicit algorithm for computing a recurrence for the subsequence of a C^2-finite sequence.

5.2 Computing a recurrence for the sparse subsequence

Let c be C-finite of order r. In Corollary 3.5 we have shown that sequences $a(n) = c(kn^2+ln+m)$ for $k, l, m \in \mathbb{N}$ are C^2-finite. Using techniques from Theorem 1 of [18] we show how an ansatz method, analogous to the closure properties we already considered, can be used to compute a recurrence for $a(n)$.

Lemma 5.3 below shows that for every $j \in \mathbb{N}$ there exist $u_{j,0}, \ldots, u_{j,r-1} \in \mathcal{R}_C$ such that

$$a(n+j) = \sum_{i=0}^{r-1} u_{j,i}(n)c(kn^2+i). \tag{9}$$

Hence, an ansatz of the form

$$x_0(n)a(n) + \cdots + x_{s-1}(n)a(n+s-1) + a(n+s) = 0$$

with unknown coefficients $x_0, \ldots, x_{s-1} \in Q(\mathcal{R}_C)$ can be equivalently written as

$$\sum_{i=0}^{r-1} \left(u_{s,i}(n) + \sum_{j=0}^{s-1} u_{j,i}(n)x_j(n) \right) c(kn^2+i) = 0.$$

Equating the coefficients of $c(kn^2+i)$ to zero yields a linear inhomogeneous system for the x_j which has a solution with Lemma 4.3 and Lemma 4.4 if the order of the ansatz s is big enough.

Let M_c be the companion matrix of c. Then, Lemma 11 in [18] shows that $\left(M_c^{p(n)}\right)_{n\in\mathbb{N}}$ for linear $p \in \mathbb{N}[n]$ can also be viewed as a matrix of C-finite sequences. These C-finite sequences can be computed using the Cayley-Hamilton theorem or using guessing. The next lemma shows how this can be used to compute the $u_{j,i}$.

Lemma 5.3. *Let $e_{r-1} \coloneqq (0, \ldots, 0, 1) \in Q(\mathcal{R}_C)^r$ and let*

$$u_j(n) = (u_{j,0}(n), \ldots, u_{j,r-1}(n)) = M_c^{2knj+kj^2+ln+lj+m-r+1} e_{r-1}.$$

These $u_{j,i}$ satisfy equation (9).

Proof. Let $C(n) \coloneqq (c(n), \ldots, c(n+r-1))$. By the definition of the companion matrix we have

$$C(n+1) = C(n)M_c \tag{10}$$

for all $n \in \mathbb{N}$. Using $n \to kn^2$ we have $C(kn^2+1) = C(kn^2)M_c$. Repeated application of equation (10) yields

$$C(k(n+j)^2 + l(n+j) + m - r + 1) = C(kn^2)M_c^{2knj+kj^2+ln+lj+m-r+1}.$$

Multiplying by e_{r-1} and using the definition of the $C(n)$ yields identity (9). $\square$

By Lemma 5.3 we have

$$u_0(n) = M_c^{ln+m-r+1} e_{r-1}$$

and

$$u_j(n) = M_c^{k(2n+1)} u_{j-1}(n+1).$$

Hence, choosing $M = M_c^{k(2n+1)}$ and $w = u_0$ in Algorithm 1 yields an algorithm to compute a C^2-finite recurrence for $a(n)$.

Example 5.1. Let f denote the Fibonacci sequence. With Corollary 3.5 and Theorem 5.1, the sequence

$$\left(\sum_{k=0}^{\lfloor n/3 \rfloor} f((2k+1)^2) \right)_{n \in \mathbb{N}}$$

is C^2-finite. The recurrence computed by the suggested algorithms from Theorem 5.1 is of order 9 with coefficients having order at most 12.

6 Extension to C^k- and D^k-finite sequences

For D-finite functions, i.e., formal power series satisfying a linear differential equation with polynomial coefficients, an analogous construction has been carried out [12]: D^2-finite functions satisfying a linear differential equation with D-finite function coefficients. An advantage of this setting is that D-finite functions form an integral domain and one does not have to deal with zero divisors. D^2-finite functions satisfy most closure properties known for D-finite functions (except for the Hadamard product). From this it can be derived that the construction can be iterated to build D^k-finite functions. In this section we proceed similarly and extend our results to the D-finite sequence case and show that the construction of these rings can be iterated as well.

A sequence is called C^0-finite if and only if it is constant and called D^0-finite if and only if it is polynomial.

Definition 6.1. Let $k \geq 1$. A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is called C^k-finite (or D^k-finite) over $\mathbb{K}$ if there are C^{k-1}-finite (or D^{k-1}-finite) sequences $c_0, \dots, c_r$ over $\mathbb{K}$ with $c_r(n) \neq 0$ for all $n \in \mathbb{N}$ such that

$$c_0(n)a(n) + c_1(n)a(n+1) + \cdots + c_r(n)a(n+r) = 0$$

for all $n \in \mathbb{N}$.

Example 6.1. Let $a = (a(n))_{n \in \mathbb{N}}$ with $a(n) = \prod_{k=1}^{n} k!$. The sequence a is D^2-finite satisfying the recurrence

$$(n+1)!\, a(n) - a(n+1) = 0, \quad \text{for all } n \in \mathbb{N}.$$

The sequence is called the superfactorial (A000178 in the OEIS).

Example 6.2. Let $\alpha \in \mathbb{K}$. Every sequence a with $a(n) = \alpha^{n^3}$ is C^3-finite satisfying the recurrence

$$c(n)a(n) - a(n+1) = 0, \quad \text{for all } n \in \mathbb{N},$$

where $c(n) = \alpha^{3n^2+3n+1}$ is C^2-finite.

Example 6.3. Using the same argument as in [18] one can derive a C^3-finite recurrence for $(f(n^3))_{n\in\mathbb{N}}$ where f denotes the Fibonacci numbers:

$$c_0(n)f(n^3) + c_1(n)f((n+1)^3) + c_2(n)f((n+2)^3) = 0, \qquad \text{for all } n \in \mathbb{N},$$

with

$$\begin{aligned}
c_0(n) &= f(3n^2+9n+7)f(3n^2+3n+3)f(3n^2+3n+1) - f(3n^2+9n+7)f(3n^2+3n+2)^2,\\
c_1(n) &= f(3n^2+9n+7)f(3n^2+3n+2) + f(3n^2+9n+6)f(3n^2+3n+1),\\
c_2(n) &= -f(3n^2+3n+1).
\end{aligned}$$

These coefficients c_0, c_1, c_2 are C^2-finite with Theorem 3.5 and clearly $c_2(n) \neq 0$ for all n.

Adapting Section 3 to this more general setting, we will show that the sets of C^k-finite and D^k-finite sequences form difference rings. We denote the set of C^k-finite sequences by $\mathcal{R}_{C^k}$, the set of D-finite sequences by $\mathcal{R}_D$ and the set of D^k-finite sequences by $\mathcal{R}_{D^k}$.

Now, Lemma 3.1, Lemma 3.2 and Theorem 3.3 can be formulated completely analogously for C^k-finite and D^k-finite sequences:

Lemma 6.1. *Let a be C^k-finite (or D^k-finite) with annihilating operator $\mathcal{A} = c_0 + \cdots + c_r\sigma^r$ and let R be the difference ring generated by $c_0, \ldots, c_r$. If $R \subseteq S$, then $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(S)}$ is finitely generated.*

Lemma 6.2. *Let $a \in \mathbb{K}^{\mathbb{N}}$ and S a subset of the set of C^k-finite (or D^k-finite) sequences. If $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(S)}$ is finitely generated, then a is C^k-finite (or D^k-finite).*

The proofs of Lemma 6.1 and Lemma 6.2 are analogous to the proofs of the corresponding Lemmas in Section 3. Using Lemma 6.1 and Lemma 6.2 one can again prove a characterization for C^k-finite and D^k-finite sequences.

Theorem 6.3. *Let $a \in \mathbb{K}^{\mathbb{N}}$.*

1. *The sequence a is C^k-finite if and only if $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(\mathcal{R}_{C^{k-1}})}$ is finitely generated.*
2. *The sequence a is D^k-finite if and only if $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(\mathcal{R}_{D^{k-1}})}$ is finitely generated.*

Similar to the C^2-finite setting, we will use Theorem 6.3 to show that C^k-finite and D^k-finite sequences are a difference ring. Example 3.1 shows that these rings are not Noetherian. Hence, the idea is, again, to restrict the underlying ring to a Noetherian subring.

Lemma 6.4. 1. *Let $\mathcal{A} = \sum_{i=0}^{r} c_i\sigma^i \in \mathcal{R}_{C^k}[\sigma]$. Then, the $\mathbb{K}$-difference-algebra*

$$\mathbb{K}[c_0, \ldots, c_r, \sigma c_0, \ldots, \sigma c_r, \ldots]$$

is contained in a Noetherian ring S.

2. *Let $\mathcal{A} = \sum_{i=0}^{r} c_i\sigma^i \in \mathcal{R}_{D^k}[\sigma]$. Then, the $\mathbb{K}(n)$-difference-algebra*

$$\mathbb{K}(n)[c_0, \ldots, c_r, \sigma c_0, \ldots, \sigma c_r, \ldots]$$

is contained in a Noetherian ring S.

Proof. We use induction on k. For $k = 0$ we have $\mathcal{R}_{C^0} = \mathbb{K}$ and $\mathcal{R}_{D^0} = \mathbb{K}[n]$ which are both Noetherian.

Now, let c be a coefficient of $\mathcal{A}$ and let $\mathcal{C}$ be its annihilator. By induction, the difference-algebra generated by the coefficients of $\mathcal{C}$ is contained in a Noetherian ring S_c. Then, also the localization $Q(S_c)$ is Noetherian. By Lemma 6.1, the module $\langle \sigma^i c \mid i \in \mathbb{N}\rangle_{Q(S_c)}$ is finitely generated. Then, also the difference-algebra $A_c := Q(S_c)[c, \sigma c, \dots]$ is finitely generated and in particular a Noetherian ring containing $\mathbb{K}[c, \sigma c, \dots]$ (or $\mathbb{K}(n)[c, \sigma c, \dots]$ in the D-finite case). Then, S can be chosen as the smallest ring containing the Noetherian rings $A_{c_0}, \dots, A_{c_r}$. This ring S is again Noetherian. $\square$

Theorem 6.5. *The set of C^k-finite and D^k-finite sequences are difference rings under termwise addition and termwise multiplication.*

Proof. Let a, b be C^k-finite (or D^k-finite) sequences and $\mathcal{A} = c_0 + c_1\sigma + \cdots + c_{r_1}\sigma^{r_1}$ and $\mathcal{B} = d_0 + d_1\sigma + \cdots + d_{r_2}\sigma^{r_2}$ the corresponding annihilating operators.

With Lemma 6.4, there is a Noetherian ring S which contains all difference rings generated by $c_0, \dots, c_{r_1}, d_0, \dots, d_{r_2}$. Hence, with Lemma 6.1, the modules

$$\langle \sigma^i(a+b) \mid i \in \mathbb{N}\rangle_{Q(S)} \subseteq \langle \sigma^i a \mid i \in \mathbb{N}\rangle_{Q(S)} + \langle \sigma^i b \mid i \in \mathbb{N}\rangle_{Q(S)}$$

and

$$\langle \sigma^i(ab) \mid i \in \mathbb{N}\rangle_{Q(S)} \subseteq \langle \sigma^i(a)\sigma^j(b) \mid i, j \in \mathbb{N}\rangle_{Q(S)}$$

are finitely generated as they are submodules of finitely generated modules over a Noetherian ring. By Lemma 6.2, the sequences $a + b$ and ab are C^k-finite (or D^k-finite).

The operator

$$\tilde{\mathcal{A}} := \sigma(c_0) + \sigma(c_1)\sigma + \cdots + \sigma(c_{r_1})\sigma^{r_1}$$

annihilates σa. Hence, the ring is also closed under shifts. $\square$

Using the ansatz method described in Section 4 one can reduce the computation of ring operations to linear systems. For instance, for D^2-finite sequences, we need to solve linear systems over the D-finite sequence ring. Since it is not known whether the Skolem-Mahler-Lech Theorem holds for D-finite sequences [2], Lemma 4.4 can, however, not be translated to this case. In practice, the idea of Lemma 4.4 can still be used to solve such linear systems. However, computationally this is very expensive and only works for small examples.

Example 6.4. We define the D-finite sequences

$$\begin{aligned}
(n^2+1)c_0(n) + c_0(n+1) &= 0, & c_0(0) &= 2,\\
(n+7)c_1(n) + (-n-1)c_1(n+1) &= 0, & c_1(0) &= 2,\\
(n+1)d_0(n) - d_0(n+1) &= 0, & d_0(0) &= 1,\\
(n+2)d_1(n) + (-n^2-3)d_1(n+1) &= 0, & d_1(0) &= 4.
\end{aligned}$$

and the D^2-finite sequences

$$\begin{aligned}
c_0(n)a(n) + c_1(n)a(n+1) &= 0, & a(0) &= 3,\\
d_0(n)b(n) + d_1(n)b(n+1) &= 0, & b(0) &= 5.
\end{aligned}$$

By Theorem 6.5, the sequence $h = ab$ is D^2-finite. With the methods introduced in Section 4 we can compute the recurrence

$$e(n)h(n) + h(n+1) = 0, \quad h(0) = 15$$

with

$$(n^6 + 2n^5 + 5n^4 + 8n^3 + 7n^2 + 6n + 3)e(n) + (n^2 + 9n + 14)e(n + 1) = 0, \quad e(0) = -\tfrac{1}{4}.$$

By induction, every C^k-finite sequence is D^k-finite and every D^k-finite sequence is C^{k+1}-finite. Therefore, we get the following chain of rings

$$\mathcal{R}_C \subseteq \mathcal{R}_D \subseteq \mathcal{R}_{C^2} \subseteq \mathcal{R}_{D^2} \subseteq \mathcal{R}_{C^3} \subseteq \cdots$$

Example 6.3 is true more generally. Using Theorem 6.5 we can prove the generalization of Corollary 3.5:

Corollary 6.6. *Let c be a C-finite sequence over the field $\mathbb{K}$ and $p \in \mathbb{N}[n]$. Denote $k := \deg p$. Then, $(c(p(n)))_{n\in\mathbb{N}}$ is C^k-finite over the splitting field $\mathbb{L}$ of the characteristic polynomial of c.*

7 Conclusion and outlook

Summarizing, we showed that C^2-finite sequences form a ring with respect to termwise addition and termwise multiplication. We derived several closure properties and methods to compute with C^2-finite sequences. Furthermore, we extended our results to C^k-finite and D^k-finite sequences and showed that these sets form difference rings.

Guess-and-prove is a common strategy to verify that a sequence is D-finite or to derive a shorter recurrence. It would be desirable to have a guessing routine for C^2-finite sequences. As a naive approach leads to a nonlinear system (see also [24]), it needs to be investigated how this can be solved efficiently.

A useful feature of D-finite sequences is that their generating functions are D-finite as well and that the defining difference and differential equations can be computed from one another. This is often exploited in proofs or simplification of identities. Also, most of the results of Theorem 5.1 would typically be proven by switching between those two representations.

Since D^2-finite functions are not closed under the Hadamard product, there cannot be a one-to-one correspondence to D^2-finite sequences. Still, it seems worthwhile to investigate the relationship between these sets. First ideas on the nature of generating functions of C^2-finite sequences have been presented in [24]. It would be interesting to explore this further and to derive computational properties.

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