SFB Status Seminar 2021, Zoom event

Solving Linear Difference Equations in Difference Rings

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Problem: Solving PLDEs in fields

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Given $0 \neq (a_0, \dots, a_m) \in \mathbb{E}^{m+1}$

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$$\underbrace{a_0\,g+a_1\,\sigma(g)+\dots+a_m\,\sigma^m(g)=\boldsymbol{c_1\,f_1}+\dots+\boldsymbol{c_d\,f_d}}_{\textbf{Parameterized}\text{ Linear Difference Equation (PLDE)}}$$

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Find a basis of

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Definition

PLDEs are solvable in $(\mathbb{E}, \sigma)$ $\Leftrightarrow$ for given $a_i, f_i \in \mathbb{E}$ one can compute a basis of its solution space.

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Definition (Karr, 1981)

Let $(\mathbb{E}, \sigma)$ be a difference field that is built by a tower of field extensions

$$\mathbb{K} = \mathbb{E}_0 \leq \mathbb{E}_1 \leq \cdots \leq \mathbb{E}_e = \mathbb{E}$$

where $\mathbb{E}_i = \mathbb{E}_{i-1}(t_i)$ is a rational function field,

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and for $1 \leq i \leq e$ we have

- $\sigma(t_i) = a_i\, t_i$ with $a_i \in \mathbb{E}^*_{i-1}$ (a product)
- $\sigma(t_i) = t_i + a_i$ with $a_i \in \mathbb{E}_{i-1}$ (a sum)

$(\mathbb{E}, \sigma)$ is called a ΠΣ-field over $\mathbb{K}$ if $\mathrm{const}_\sigma \mathbb{E} = \mathbb{K}$.

Solving PLDEs in $\Pi\Sigma$-fields

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Theorem (Abramov/Bronstein/Petkovšek/CS, JSC 2021)

One can solve PLDEs in a $\Pi\Sigma$-field.

[The first-order case has been solved by Karr'81]

The PLDE solver

- contains creative telescoping as a special case
- is a key step to compute hypergeometric and d'Alembertian solutions
  (see again Abramov/Bronstein/Petkovšek/CS, JSC 2021)

$$\begin{aligned}&\big(1+H_n+nH_n\big)^2\big(3+2n+2H_n+3nH_n+n^2H_n\big)^2G(n)\\ -&(1+n)(3+2n)H_n\big(3+2n+2H_n+3nH_n+n^2H_n\big)^2G(n+1)\\ &\qquad+(1+n)^2(2+n)^3H_n\big(1+H_n+nH_n\big)G(n+2)=0\end{aligned}$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\,H_n\prod_{l=1}^{n}H_l+c_2H_n^2\prod_{l=1}^{n}H_l\mid c_1,c_2\in\mathbb{K}\Big\}$$

Problem: representation of hyp. products

$$y_1 = \prod_{k=1}^{n} \frac{-13122k(1+k)}{(3+k)^3}, \qquad y_2 = \prod_{k=1}^{n} \frac{26244k^2(2+k)^2}{(3+k)^2},$$

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$$\downarrow$$

$$y_1 = \frac{216(-1)^n 2^n (3^n)^8}{(n+1)^2(n+2)^3(n+3)^3 n!}, \qquad y_2 = \frac{9\,(2^n)^2\,(3^n)^8\,(n!)^2}{(n+3)^2},$$

$$y_3 = \frac{15(n+1)^2(n+2)^2(-1)^n(n!)^3}{(n+3)(n+4)(n+5)\,(3^n)^6}, \qquad y_4 = \frac{60(-1)^n 2^n\,(3^n)^4\,n!}{(n+3)(n+4)(n+5)}.$$

in terms of

$$\underbrace{n! = \prod_{k=1}^{n} k, \quad 2^n, \quad 3^n}_{\Pi\Sigma\text{-field}} \quad \text{and} \quad (-1)^n$$

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- elements $e_0 = \frac{1-y}{2}$ and $e_1 = \frac{1+y}{2}$ with
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- In particular, we can decompose the ring as

$$\begin{tikzcd} \mathbb{E} = e_0\,\mathbb{E} \oplus \arrow[r, bend left, "\sigma"] & e_1\,\mathbb{E} \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

Idempotent Difference Rings

Definition

$(\mathbb{E}, \sigma)$ is called idempotent difference ring (IDR) of order λ if there are such idempotent orthogonal elements $e_0, \dots, e_{\lambda-1}$ with

$$\begin{tikzcd}
\mathbb{E} = e_0\,\mathbb{E} \arrow[r, bend left, "\sigma"] & \oplus\; e_1\,\mathbb{E} \arrow[r, bend left, "\sigma"] & \oplus \;\dots \arrow[r, bend left, "\sigma"] & \oplus\; e_{\lambda-2}\,\mathbb{E} \arrow[r, bend left, "\sigma"] & \oplus\; e_{\lambda-1}\,\mathbb{E}\,. \arrow[llll, bend left, "\sigma"]
\end{tikzcd}$$

[M. van der Put, M.F. Singer. Galois theory of difference equations, 1997.
A. Ovchinnikov and M. Wibmer. σ-Galois theory of linear difference equations, 2015,...]

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- is a ring (containing $\mathbb{Q}$)

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(-1)^k & \leftrightarrow & \sigma(y) = -y & y^2 = 1
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α is a primitive λth root of unity $\quad \alpha^k \quad \leftrightarrow \quad \sigma(y) = \alpha\, y \qquad y^\lambda = 1$

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$$\begin{array}{ll} \sigma(p_1) = a_1\, p_1 & a_1 \in \mathbb{K}(x)^* \\ \sigma(p_2) = a_2 p_2 & a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\ \vdots & \\ \sigma(p_e) = a_e p_e & a_e \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^* \end{array}$$

α is a primitive λth root of unity $\quad \alpha^k \leftrightarrow$

$$\sigma(y) = \alpha\, y \qquad y^\lambda = 1$$

(nested) sums $\leftrightarrow$

$$\begin{array}{ll} \sigma(s_1) = s_1 + f_1 & f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][y] \\ \sigma(s_2) = s_2 + f_2 & f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][y][s_1] \\ \sigma(s_3) = s_3 + f_3 & f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][y][s_1][s_2] \\ \vdots & \end{array}$$

Definition: An $R\Pi\Sigma$-ring $(\mathbb{E}, \sigma)$

- is a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{E} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][y][s_1][s_2][s_3] \dots}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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such that $\mathrm{const}_\sigma \mathbb{E} = \{c \in \mathbb{E} | \sigma(c) = c\} = \mathbb{K}$.

DEFINITION

$(\mathbb{E}, \sigma)$ is called idempotent difference ring (IDR) of order λ if there are such idempotent orthogonal elements $e_0, \dots, e_{\lambda-1}$ with

$$\begin{tikzcd} \mathbb{E} = e_0\,\mathbb{E} \arrow[r, bend left, "\sigma"] \oplus & e_1\,\mathbb{E} \arrow[r, bend left, "\sigma"] \oplus & \dots \arrow[r, bend left, "\sigma"] \oplus & e_{\lambda-2}\,\mathbb{E} \arrow[r, bend left, "\sigma"] \oplus & e_{\lambda-1}\,\mathbb{E} \arrow[llll, bend left, "\sigma"] . \end{tikzcd}$$

[M. van der Put, M.F. Singer. Galois theory of difference equations, 1997.
A. Ovchinnikov and M. Wibmer. σ-Galois theory of linear difference equations, 2015,...]

An $R\Pi\Sigma$-ring is such an IDR of order λ

Idempotent Difference Rings

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An $R\Pi\Sigma$-ring is such an IDR of order λ

- each component $e_i\,\mathbb{E}$ is an integral domain

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An $R\Pi\Sigma$-ring is such an IDR of order λ

- each component $e_i\,\mathbb{E}$ is an integral domain
- taking the quotient field $Q(e_i\,\mathbb{E})$ we get a $\Pi\Sigma$-field

$$(Q(e_i\,\mathbb{E}), \sigma^{\lambda})$$

Main strategy for an $R\Pi\Sigma$-ring $(\mathbb{E}, \sigma)$

Given $a_i, f_i \in \mathbb{E}$
find all $c_i \in \mathbb{K}$, $g \in \mathbb{E}$ for

$$\mathbb{E} = e_0\mathbb{E} \oplus \cdots \oplus e_{\lambda-1}\mathbb{E}$$

$$a_0\, g + a_1\sigma^1(g) + \cdots + a_m\, \sigma^\lambda(g) = \boldsymbol{cf}$$

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$$(e_0\, g_0 + \cdots + e_{\lambda-1}g_{\lambda-1}, c_1, \ldots, c_d) \in \mathbb{E} \times \mathbb{K}^d$$

Example

$$\Big[(1+n)(2+n)\Big(\big(2+n+(1+n)\sum_{i=1}^{n}\tfrac{1}{i}\big)(-1)^n-(1+n)^2\sum_{i=1}^{n}\tfrac{(-1)^i}{i}\Big)\Big]\,G(n)$$
$$+\Big[(1+n)(2+n)\Big(\big(2+n+2(1+n)\sum_{i=1}^{n}\tfrac{1}{i}\big)(-1)^n-(1+n)\sum_{i=1}^{n}\tfrac{(-1)^i}{i}\Big)\Big]\,G(n+1)$$
$$+\Big[(1+n)^2(2+n)\Big((-1)^n\sum_{i=1}^{n}\tfrac{1}{i}+n\sum_{i=1}^{n}\tfrac{(-1)^i}{i}\Big)\Big]G(n+2)$$
$$=(2+n)^2+(1+n)\sum_{i=1}^{n}\frac{1}{i}-2(1+n)^3(-1)^n\sum_{i=1}^{n}\frac{(-1)^i}{i}$$

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$$\downarrow$$

Take the $R\Pi\Sigma$-ring $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{Q}(x)[y][s][\bar{s}]$ where

- $\sigma(x)=x+1$,
- $\sigma(y)=-y$,
- $\sigma(s)=s+\frac{1}{x+1}$,
- $\sigma(\bar{s})=\bar{s}+\frac{-y}{x+1}$.

Take $e_0=\frac{1-y}{2}$ and $e_1=\frac{1+y}{2}$.

Example

$$\Big[(1+x)(2+x)(-\bar{s}(1+x)^2+(2+s+x+sx)y)\Big]g$$
$$+\Big[(1+x)(2+x)(-\bar{s}(1+x)+(2+x+2s(1+x))y)\Big]\sigma(g)$$
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$$\uparrow$$

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$$\Big\downarrow \text{linear algebra}$$

$$
\begin{aligned}
b_{0,0}(e_0g_0)+b_{0,1}\sigma^2(e_0g_0)+b_{0,2}\sigma^4(e_0g_0)&=e_0\varphi_0\\
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\begin{aligned}
g_0&=s+c_1+c_2(s+\bar{s}+2x-4sx-2\bar{s}x) && c_1,c_2\in\mathbb{Q}\\
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$$\Big\downarrow \text{combine}$$

$$g=-sy-\kappa_1y+\kappa_2(\bar{s}-2\bar{s}x-sy-2xy-4sxy)\qquad \kappa_1,\kappa_2\in\mathbb{Q}$$

Example (continued)

$$\Big[(1+n)(2+n)\Big(\Big(2+n+(1+n)\sum_{i=1}^{n}\tfrac{1}{i}\Big)(-1)^n-(1+n)^2\sum_{i=1}^{n}\frac{(-1)^i}{i}\Big)\Big]\,G(n)$$
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$$=(2+n)^2+(1+n)\sum_{i=1}^{n}\frac{1}{i}-2(1+n)^3(-1)^n\sum_{i=1}^{n}\frac{(-1)^i}{i}$$

$$\Big\downarrow \texttt{PLDESolver.m}$$

$$-\sum_{i=1}^{n}\tfrac{1}{i}(-1)^n-\kappa_1(-1)^n$$
$$+\kappa_2\Big(-2(-1)^n n-(1+4n)(-1)^n\sum_{i=1}^{n}\tfrac{1}{i}+\sum_{i=1}^{n}\frac{(-1)^i}{i}(1-2n)\Big)$$

Example (continued)

$$
\begin{aligned}
&\big[(1+x)(2+x)(-\bar{s}(1+x)^2+(2+s+x+sx)y)\big]g\\
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$$\Big\downarrow \text{combine}$$

$$g=-sy-\kappa_1y+\kappa_2(\bar{s}-2\bar{s}x-sy-2xy-4sxy) \qquad \kappa_1,\kappa_2\in\mathbb{Q}$$

Shift Projection Matrix

Let $(\mathbb{E}, \sigma)$ be an IDR of order λ with idempotent elements $e_s \in \mathbb{E}$ with $0 \leq s < \lambda$.

Definition (Projection)

$\pi : \mathbb{E} \to \mathbb{E}$ with $\pi(g) \mapsto g_0$ where $g = \sum_{s=0}^{\lambda-1} e_s g_s$ is called a *projection*.

Definition (Shift Projection Matrix)

For $\boldsymbol{a} = (a_0, a_1, \ldots, a_m) \in \mathbb{E}^{m+1}$ we define the $((m+1)\lambda - m) \times (m+1)\lambda$ *shift projection* matrix by

$M_{\sigma,\pi}(\boldsymbol{a}) :=$

$$\begin{pmatrix}
\pi(a_0) & \pi(a_1) & \cdots & \pi(a_m) & 0 & 0 & \cdots & 0 \\
0 & \pi(\sigma(a_0)) & \cdots & \pi(\sigma(a_{m-1})) & \pi(\sigma(a_m)) & 0 & \cdots & 0 \\
\vdots & & \ddots & & & & & \\
0 & 0 & \cdots & 0 & \pi(\sigma^{(m+1)\lambda-m-1}(a_0)) & & \cdots & \pi(\sigma^{(m+1)\lambda-m-1}(a_m))
\end{pmatrix}$$

Example (continued)

$$\begin{aligned}
&\big[(1+x)(2+x)(-\bar{s}(1+x)^2+(2+s+x+sx)y)\big]g\\
+&\big[(1+x)(2+x)(-\bar{s}(1+x)+(2+x+2s(1+x))y)\big]\sigma(g)\\
&\qquad+\big[(1+x)^2(2+x)(\bar{s}x+sy)\big]\sigma^2(g)\\
&\qquad= s(1+x)+(2+x)^2-2\bar{s}(1+x)^3y
\end{aligned}$$

$$\Big\downarrow\, \color{red}{M = M_{\sigma,\pi}\ \textbf{has full rank}}$$

$$b_{0,0}(e_0g_0)+b_{0,1}\sigma^2(e_0g_0)+b_{0,2}\sigma^4(e_0g_0)=e_0\varphi_0$$
$$b_{1,0}(e_1g_1)+b_{0,1}\sigma^2(e_1g_1)+b_{0,2}\sigma^4(e_1g_1)=e_0\varphi_1$$

$$\Big\downarrow\, \Pi\Sigma\text{-solver}$$

$$g_0 = s+c_1+c_2(s+\bar{s}+2x-4sx-2\bar{s}x) \qquad c_1,c_2\in\mathbb{Q}$$
$$g_1 = -s+d_1+d_2(-s+\bar{s}-2x+4sx-2\bar{s}x) \qquad d_1,d_2\in\mathbb{Q}$$

$$\Big\downarrow\, \text{combine}$$

$$g = -sy-\kappa_1y+\kappa_2(\bar{s}-2\bar{s}x-sy-2xy-4sxy) \qquad \kappa_1,\kappa_2\in\mathbb{Q}$$

Summary of our PLDE machninery

Let $(\mathbb{E}, \sigma)$ be an $R\Pi\Sigma$-ring;

$\boldsymbol{a} = (a_0, \ldots, a_m) \in \mathbb{E}^{m+1}$ with $a_m \neq 0$ and $f_i \in \mathbb{E}$.

$M_{\sigma,\pi}(\boldsymbol{a})$ has rank

$$\Downarrow$$

PLDE in the a_i and f_i is solvable

Let $(\mathbb{E}, \sigma)$ be an $R\Pi\Sigma$-ring;

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a_0 or a_m are not zero-divisors (regular) (simple criterion)

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$$\Downarrow \qquad \not\Uparrow$$

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PLDE in the a_i and f_i is solvable

Example: zero-divisors

$$\sigma(x) = x + 1$$
$$\sigma(y) = -y$$

Take the PLDE

$$\frac{1+y}{2} g + x\sigma(g) + x\sigma^2(g) + \frac{1+y}{2}\sigma^3(g) = 0$$

Note: leading and the trailing coefficients are zero-divisors.

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Take the PLDE

$$\frac{1+y}{2} g + x\sigma(g) + x\sigma^2(g) + \frac{1+y}{2}\sigma^3(g) = 0$$

Note: leading and the trailing coefficients are zero-divisors. But:

$$M_{\sigma,\pi}(\boldsymbol{a}) = \begin{pmatrix} 0 & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x{+}1 & x{+}1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x{+}2 & x{+}2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x{+}3 & x{+}3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x{+}4 & x{+}4 & 0 \end{pmatrix} \quad \text{has full rank}$$

$$\Big\downarrow \texttt{PLDESolver.m}$$

$$g = c \cdot y, \qquad c \in \mathbb{Q}$$

Conclusion (further results)

- a PLDE-solver for $R\Pi\Sigma$-rings

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- a PLDE-solver for $R\Pi\Sigma$-rings
- **simplified algorithms** that avoid the heavy $\Pi\Sigma$-field machinery

Main strategy for an $R\Pi\Sigma$-ring $(\mathbb{E}, \sigma)$

Given $a_i, f_i \in \mathbb{E}$
find all $c_i \in \mathbb{K}$, $g \in \mathbb{E}$ for

$$\mathbb{E} = e_0\mathbb{E} \oplus \cdots \oplus e_{\lambda-1}\mathbb{E}$$

$$a_0\, g + a_1\sigma^1(g) + \cdots + a_m\, \sigma^\lambda(g) = \boldsymbol{c}\boldsymbol{f}$$

$$\Big\downarrow\; g = e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}$$

$$b_{0,i}(e_i\, g_i) + b_{1,i}\sigma^\lambda(e_i\, g_i) + \cdots + b_{m,i}\sigma^{\lambda\, m}(e_i\, g_i) = \boldsymbol{\kappa}\boldsymbol{\varphi}$$

$$\Big\downarrow\; \text{solver in } \Pi\Sigma\text{-field } (Q(e_i\, \mathbb{E}), \sigma^\lambda)$$

$$(e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}, c_1, \ldots, c_d) \in \mathbb{E} \times \mathbb{K}^d$$

Simplified solvers

Given $a_i, f_i \in \mathbb{E}$
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$$a_0\, g + a_1\sigma^1(g) + \cdots + a_m\, \sigma^\lambda(g) = \boldsymbol{cf}$$

$$\Big\downarrow\; g = e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}$$

$$b_{0,i}(e_i\, g_i) + b_{1,i}\sigma^\lambda(e_i\, g_i) + \cdots + b_{m,i}{\sigma^\lambda}^m(e_i\, g_i) = \boldsymbol{\kappa\varphi}$$

$$\Big\downarrow\; \textbf{simplified solver in } (e_i\mathbb{E}, \sigma^\lambda)$$

$$(e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}, c_1, \ldots, c_d) \in \mathbb{E} \times \mathbb{K}^d$$

Conclusion (further results)

- a PLDE-solver for $R\Pi\Sigma$-rings
- simplified algorithms that avoid the heavy $\Pi\Sigma$-field machinery
- works also for the **quotient ring of an $R\Pi\Sigma$-ring**

Main strategy for an $R\Pi\Sigma$-ring $(\mathbb{E}, \sigma)$

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$$\Big\downarrow\, g = e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}$$

$$b_{0,i}(e_i\, g_i) + b_{1,i}\sigma^\lambda(e_i\, g_i) + \cdots + b_{m,i}{\sigma^\lambda}^m(e_i\, g_i) = \boldsymbol{\kappa\varphi}$$

$$\Big\downarrow\, \text{solver in } \Pi\Sigma\text{-field } (Q(e_i\, \mathbb{E}), \sigma^\lambda)$$

$$(e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}, c_1, \ldots, c_d) \in \mathbb{E} \times \mathbb{K}^d$$

Bonus: solving PLDEs in quotient rings

Given $a_i, f_i \in \mathbb{E}$
find all $c_i \in \mathbb{K}$, $g \in \mathbb{E}$ for

$$Q(\mathbb{E}) = Q(e_0\mathbb{E}) \oplus \cdots \oplus Q(e_{\lambda-1}\mathbb{E})$$

$$a_0\, g + a_1\sigma^1(g) + \cdots + a_m\, \sigma^\lambda(g) = \boldsymbol{c}\boldsymbol{f}$$

$$\Big\downarrow\; g = e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}$$

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$$\Big\downarrow \; g = e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}$$

$$b_{0,i}(e_i\, g_i) + b_{1,i}\sigma^\lambda(e_i\, g_i) + \cdots + b_{m,i}\sigma^{\lambda m}(e_i\, g_i) = \boldsymbol{\kappa\varphi}$$

$$\Big\downarrow \; \text{solver in } \Pi\Sigma\text{-field } (Q(e_i\,\mathbb{E}), \sigma^\lambda)$$

$$(e_0\, g_0 + \cdots + e_{\lambda-1} g_{\lambda-1}, c_1, \ldots, c_d) \in Q(\mathbb{E}) \times \mathbb{K}^d$$

Conclusion (further results)

- a PLDE-solver for $R\Pi\Sigma$-rings
- simplified algorithms that avoid the heavy $\Pi\Sigma$-field machinery
- works also for the quotient ring of an $R\Pi\Sigma$-ring
- a **general algorithmic framework** for
  - idempotent difference rings
  - for $R\Pi\Sigma$-extensions defined over "computable" difference fields (not only $(\mathbb{K}(x), \sigma)$)

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- also **interesting if the matrix $M_{\sigma,\pi}(\boldsymbol{a})$ does not have full rank**...

Example: not full rank

Take the PLDE

$$(y-1)g + x(y+1)\sigma(g) + (y-1)\sigma^2(g) + x(y+1)\sigma^3(g) = 0$$

The shift projection matrix is

$$M_{\sigma,\pi}(\boldsymbol{a}) = \begin{pmatrix} -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(1+x) & 0 & 2(1+x) & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(3+x) & 0 & 2(3+x) & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 \end{pmatrix}$$

which doesn't have full rank.

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which doesn't have full rank.

Still our machinery gives further inside:

$$g = e_0 g_0 + e_1 g_1 \in \mathbb{E} \text{ is a solution}$$

$$\Downarrow$$

- g_0 satisfies $g_0 + \sigma^2(g_0) = 0$
- g_1 cannot be bounded

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Further (technical) details

Example (continued)

$$\Big[(1+x)(2+x)(-\bar{s}(1+x)^2+(2+s+x+sx)y)\Big]g$$
$$+\Big[(1+x)(2+x)(-\bar{s}(1+x)+(2+x+2s(1+x))y)\Big]\sigma(g)$$
$$+\Big[(1+x)^2(2+x)(\bar{s}x+sy)\Big]\sigma^2(g)$$
$$=s(1+x)+(2+x)^2-2\bar{s}(1+x)^3y$$

$$\Big\downarrow \textbf{linear algebra}$$

$$b_{0,0}(e_0g_0)+b_{0,1}\sigma^2(e_0g_0)+b_{0,2}\sigma^4(e_0g_0)=e_0\varphi_0$$
$$b_{1,0}(e_1g_1)+b_{0,1}\sigma^2(e_1g_1)+b_{0,2}\sigma^4(e_1g_1)=e_0\varphi_1$$

$$\Big\downarrow \Pi\Sigma\text{-solver}$$

$$g_0=s+c_1+c_2(s+\bar{s}+2x-4sx-2\bar{s}x) \qquad c_1,c_2\in\mathbb{Q}$$
$$g_1=-s+d_1+d_2(-s+\bar{s}-2x+4sx-2\bar{s}x) \qquad d_1,d_2\in\mathbb{Q}$$

$$\Big\downarrow \text{combine}$$

$$g=-sy-\kappa_1y+\kappa_2(\bar{s}-2\bar{s}x-sy-2xy-4sxy) \qquad \kappa_1,\kappa_2\in\mathbb{Q}$$

Example (continued)

- Plug $g = e_0 g_0 + e_1 g_1$ into the given equation.

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$$0 = -s(1+x) + 2\bar{s}(1+x)^3 - (2+x)^2 + (1+x)(2+x)(2+s+x+sx-\bar{s}(1+x)^2)g_1 \\ + (1+x)(2+x)(2+x+2s(1+x)-\bar{s}(1+x))\sigma(g_0) + (1+x)^2(2+x)(s+\bar{s}x)\sigma^2(g_1)$$

Example (continued)

- Plug $g = e_0 g_0 + e_1 g_1$ into the given equation.
- Apply σ^j for $j = 0, 1$ and project to the second component:

$$0 = -s(1+x) + 2\bar{s}(1+x)^3 - (2+x)^2 + (1+x)(2+x)(2+s+x+sx-\bar{s}(1+x)^2)g_1$$
$$+ (1+x)(2+x)(2+x+2s(1+x)-\bar{s}(1+x))\sigma(g_0) + (1+x)^2(2+x)(s+\bar{s}x)\sigma^2(g_1),$$

$$0 = -(3+x)^2 - 2(2+x)^3(\bar{s} - \frac{1}{1+x}) - (2+x)(s+\frac{1}{1+x}) + (2+x)(3+x)(-3-s-x-\frac{1}{1+x}$$
$$-(2+x)^2(\bar{s}-\frac{1}{1+x}) - (1+x)(s+\frac{1}{1+x}))\sigma(g_0) + (2+x)(3+x)(-3-x-(2+x)(\bar{s}-\frac{1}{1+x})$$
$$-2(2+x)(s+\frac{1}{1+x}))\sigma^2(g_1) + (2+x)^2(3+x)(-s-\frac{1}{1+x}+(1+x)(\bar{s}-\frac{1}{1+x}))\sigma^3(g_0),$$

Example (continued)

- Plug $g = e_0 g_0 + e_1 g_1$ into the given equation.
- Apply σ^j for $j = 0, 1, 2$ and project to the second component:

$$
\begin{aligned}
0 = & -s(1+x) + 2\bar{s}(1+x)^3 - (2+x)^2 + (1+x)(2+x)(2+s+x+sx-\bar{s}(1+x)^2)g_1 \\
& + (1+x)(2+x)(2+x+2s(1+x)-\bar{s}(1+x))\sigma(g_0) + (1+x)^2(2+x)(s+\bar{s}x)\sigma^2(g_1),
\end{aligned}
$$

$$
\begin{aligned}
0 = & -(3+x)^2 - 2(2+x)^3(\bar{s} - \frac{1}{1+x}) - (2+x)(s + \frac{1}{1+x}) + (2+x)(3+x)(-3-s-x-\frac{1}{1+x} \\
& -(2+x)^2(\bar{s} - \frac{1}{1+x}) - (1+x)(s+\frac{1}{1+x}))\sigma(g_0) + (2+x)(3+x)(-3-x-(2+x)(\bar{s}-\frac{1}{1+x}) \\
& -2(2+x)(s+\frac{1}{1+x}))\sigma^2(g_1) + (2+x)^2(3+x)(-s-\frac{1}{1+x}+(1+x)(\bar{s}-\frac{1}{1+x}))\sigma^3(g_0),
\end{aligned}
$$

$$
\begin{aligned}
0 = & -(4+x)^2 + 2(3+x)^3(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x}) - (3+x)(s + \frac{1}{1+x} + \frac{1}{2+x}) + (3+x)(4+x)(4+s+x+\frac{1}{1+x} \\
& + \frac{1}{2+x} - (3+x)^2(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x}) + (2+x)(s + \frac{1}{1+x} + \frac{1}{2+x}))\sigma^2(g_1) + (3+x)(4+x)(4+x \\
& -(3+x)(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x}) + 2(3+x)(s + \frac{1}{1+x} + \frac{1}{2+x}))\sigma^3(g_0) + (3+x)^2(4+x)(s + \frac{1}{1+x} + \frac{1}{2+x} \\
& + (2+x)(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x}))\sigma^4(g_1),
\end{aligned}
$$

Example (continued)

- Plug $g = e_0 g_0 + e_1 g_1$ into the given equation.
- Apply σ^j for $j = 0, 1, 2, 3$ and project to the second component:

$$
\begin{aligned}
0 = &- s(1+x) + 2\bar{s}(1+x)^3 - (2+x)^2 + (1+x)(2+x)(2+s+x+sx-\bar{s}(1+x)^2)g_1 \\
&+ (1+x)(2+x)(2+x+2s(1+x)-\bar{s}(1+x))\sigma(g_0) + (1+x)^2(2+x)(s+\bar{s}x)\sigma^2(g_1),
\end{aligned}
$$

$$
\begin{aligned}
0 = &- (3+x)^2 - 2(2+x)^3(\bar{s} - \frac{1}{1+x}) - (2+x)(s + \frac{1}{1+x}) + (2+x)(3+x)(-3-s-x-\frac{1}{1+x} \\
&- (2+x)^2(\bar{s} - \frac{1}{1+x}) - (1+x)(s+\frac{1}{1+x}))\sigma(g_0) + (2+x)(3+x)(-3-x-(2+x)(\bar{s}-\frac{1}{1+x}) \\
&- 2(2+x)(s+\frac{1}{1+x}))\sigma^2(g_1) + (2+x)^2(3+x)(-s-\frac{1}{1+x} + (1+x)(\bar{s}-\frac{1}{1+x}))\sigma^3(g_0),
\end{aligned}
$$

$$
\begin{aligned}
0 = &- (4+x)^2 + 2(3+x)^3(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x}) - (3+x)(s+\frac{1}{1+x}+\frac{1}{2+x}) + (3+x)(4+x)(4+s+x+\frac{1}{1+x} \\
&+ \frac{1}{2+x} - (3+x)^2(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x}) + (2+x)(s+\frac{1}{1+x}+\frac{1}{2+x}))\sigma^2(g_1) + (3+x)(4+x)(4+x \\
&- (3+x)(\bar{s}-\frac{1}{1+x}+\frac{1}{2+x}) + 2(3+x)(s+\frac{1}{1+x}+\frac{1}{2+x}))\sigma^3(g_0) + (3+x)^2(4+x)(s+\frac{1}{1+x}+\frac{1}{2+x} \\
&+ (2+x)(\bar{s}-\frac{1}{1+x}+\frac{1}{2+x}))\sigma^4(g_1),
\end{aligned}
$$

$$
\begin{aligned}
0 = &- (5+x)^2 - 2(4+x)^3(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x} - \frac{1}{3+x}) - (4+x)(s+\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}) + (4+x)(5+x)(-5 \\
&- s - x - \frac{1}{1+x} - \frac{1}{2+x} - \frac{1}{3+x} - (4+x)^2(\bar{s} - \frac{1}{1+x} + \frac{1}{2+x} - \frac{1}{3+x}) - (3+x)(s+\frac{1}{1+x}+\frac{1}{2+x} \\
&+ \frac{1}{3+x}))\sigma^3(g_0) + (4+x)(5+x)(-5-x-(4+x)(\bar{s}-\frac{1}{1+x}+\frac{1}{2+x}-\frac{1}{3+x}) - 2(4+x)(s+\frac{1}{1+x}+\frac{1}{2+x} \\
&+ \frac{1}{3+x}))\sigma^4(g_1) + (4+x)^2(5+x)(-s-\frac{1}{1+x}-\frac{1}{2+x}-\frac{1}{3+x}+(3+x)(\bar{s}-\frac{1}{1+x}+\frac{1}{2+x}-\frac{1}{3+x}))\sigma^5(g_0)
\end{aligned}
$$

Example (continued)

This is a linear system

$$M \cdot \begin{pmatrix} \sigma^0(g_1) \\ \sigma^1(g_0) \\ \sigma^2(g_1) \\ \sigma^3(g_0) \\ \sigma^4(g_1) \\ \sigma^5(g_0) \end{pmatrix} = \begin{pmatrix} \pi(\sigma^0(\varphi)) \\ \pi(\sigma^1(\varphi)) \\ \pi(\sigma^2(\varphi)) \\ \pi(\sigma^3(\varphi)) \end{pmatrix}$$

where

$$\varphi = s(1+x) + (2+x)^2 - 2\bar{s}(1+x)^3$$

Example (continued)

$$M = M_{\sigma,\pi}\mathbb{Q}(x, s, \bar{s})^{4\times 6} \text{ has full rank}$$

This is a linear system

$$M \cdot \begin{pmatrix} \sigma^0(g_1) \\ \sigma^1(g_0) \\ \sigma^2(g_1) \\ \sigma^3(g_0) \\ \sigma^4(g_1) \\ \sigma^5(g_0) \end{pmatrix} = \begin{pmatrix} \pi(\sigma^0(\varphi)) \\ \pi(\sigma^1(\varphi)) \\ \pi(\sigma^2(\varphi)) \\ \pi(\sigma^3(\varphi)) \end{pmatrix}$$

where

$$\varphi = s(1+x) + (2+x)^2 - 2\bar{s}(1+x)^3$$

Thus we can solve the system in $6 - 4 = 2$ **variables**

Example (continued)

We get the solutions:

$$
\begin{aligned}
\sigma(g_0) &= \sigma(g_0) \\
\sigma^3(g_0) &= \sigma^3(g_0) \\
\sigma^5(g_0) &= -\frac{1640 + 3485x + 2734x^2 + 1011x^3 + 178x^4 + 12x^5 + 4s(138 + 337x + 304x^2 + 129x^3 + 26x^4 + 2x^5) + \bar{s}(276 + 674x + 608x^2 + 258x^3 + 52x^4 + 4x^5)}{(2+x)(3+x)^2(4+x)(5+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
&\quad - \frac{(98 + 158x + 81x^2 + 16x^3 + x^4 + 2s(24 + 50x + 35x^2 + 10x^3 + x^4) + \bar{s}(24 + 50x + 35x^2 + 10x^3 + x^4))\sigma(g_0)}{(3+x)(4+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
&\quad + \frac{(158 + 253x + 133x^2 + 28x^3 + 2x^4 + 4s(24 + 50x + 35x^2 + 10x^3 + x^4) + 2\bar{s}(24 + 50x + 35x^2 + 10x^3 + x^4))\sigma^3(g_0)}{(3+x)(4+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
g_1 &= \frac{15 + 20x + 8x^2 + x^3 + s(7 + 10x + 3x^2) - 2\bar{s}(-3 - 3x + 3x^2 + 4x^3 + x^4)}{(1+x)(3+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
&\quad - \frac{(5 + 5x + x^2 + \bar{s}(1+x)^2(2+x) + 3s(2 + 3x + x^2))\sigma(g_0)}{5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2)} + \frac{(1+x)(2+x)(s + \bar{s}x)\sigma^3(g_0)}{5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2)} \\
\sigma^2(g_1) &= \frac{5 + 8x + 5x^2 + x^3 - 2\bar{s}(1+x)(2+x)^3 - s(2 + 3x + x^2)}{(2+x)(3+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
&\quad - \frac{(1+x)(1 + s(2+x) + \bar{s}(2+x)^2)\sigma(g_0)}{5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2)} + \frac{(2+x)(-2 - x - s(1+x) + \bar{s}(1+x)^2)\sigma^3(g_0)}{5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2)} \\
\sigma^4(g_1) &= \frac{215 + 254x + 106x^2 + 18x^3 + x^4 - 2\bar{s}(3+x)^2(11 + 18x + 8x^2 + x^3) + s(24 + 41x + 20x^2 + 3x^3)}{(3+x)^2(4+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
&\quad - \frac{(2+x)(-13 - 8x - x^2 + \bar{s}(1+x)(3+x)^2 - s(3 + 4x + x^2))\sigma(g_0)}{(3+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))} \\
&\quad - \frac{(41 + 49x + 18x^2 + 2x^3 - \bar{s}(2+x)^2(3 + 4x + x^2) + 3s(6 + 11x + 6x^2 + x^3))\sigma^3(g_0)}{(3+x)(5 + 5x + x^2 + 2s(2 + 3x + x^2) + \bar{s}(2 + 3x + x^2))}
\end{aligned}
$$

Example (continued)

Finally, we plug this solution into the desired recurrence for $k = 0, 1$:

$$\sum_{i=0}^{2} b_{k,i} (\sigma^{\lambda})^{i} (e_k \, g_k) = e_k \, \varphi_k .$$

Example (continued)

Finally, we plug this solution into the desired recurrence for $k = 0, 1$:

$$\sum_{i=0}^{2} b_{k,i} (\sigma^{\lambda})^i (e_k\, g_k) = e_k\, \varphi_k.$$

$$\Big\downarrow \text{coeff. comparison w.r.t. } 1,\ \sigma(g_0), \sigma^3(g_0)$$

linear system with 3 eq. in 4 unknowns $(b_{k,0}, b_{k,1}, b_{k,2}, \varphi_k)$

Example (continued)

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linear system with 3 eq. in 4 unknowns $(b_{k,0}, b_{k,1}, b_{k,2}, \varphi_k)$

$$\Big\downarrow$$

$$\begin{aligned}
b_{0,0} =& x \left(29 + 12s + 6\bar{s} + 33x + 22sx + 11\bar{s}x + 11x^2 + 12sx^2 + 6\bar{s}x^2 + x^3 + 2sx^3 + \bar{s}x^3\right),\\
b_{0,1} =& -x \left(41 + 24s + 12\bar{s} + 49x + 44sx + 22\bar{s}x + 18x^2 + 24sx^2 + 12\bar{s}x^2 + 2x^3 + 4sx^3 + 2\bar{s}x^3\right),\\
b_{0,2} =& x(2 + x)(3 + x)(2 + 2s + \bar{s} + x + 2sx + \bar{s}x),\\
b_{1,0} =& 29 + 12s - 6\bar{s} + 33x + 22sx - 11\bar{s}x + 11x^2 + 12sx^2 - 6\bar{s}x^2 + x^3 + 2sx^3 - \bar{s}x^3,\\
b_{1,1} =& -41 - 24s + 12\bar{s} - 49x - 44sx + 22\bar{s}x - 18x^2 - 24sx^2 + 12\bar{s}x^2 - 2x^3 - 4sx^3 + 2\bar{s}x^3,\\
b_{1,2} =& (2 + x)(3 + x)(2 + 2s - \bar{s} + x + 2sx - \bar{s}x),\\
\varphi_0 =& -\frac{x \left(292 + 88s + 44\bar{s} + 559x + 212sx + 106\bar{s}x + 387x^2 + 180sx^2 + 90\bar{s}x^2 + 114x^3 + 64sx^3 + 32\bar{s}x^3 + 12x^4 + 8sx^4 + 4\right.}{(1 + x)(2 + x)(4 + x)}\\
\varphi_1 =& \frac{292 + 88s - 44\bar{s} + 559x + 212sx - 106\bar{s}x + 387x^2 + 180sx^2 - 90\bar{s}x^2 + 114x^3 + 64sx^3 - 32\bar{s}x^3 + 12x^4 + 8sx^4 - 4\bar{s}x^4}{(1 + x)(2 + x)(4 + x)}
\end{aligned}$$