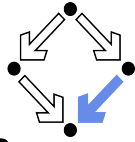


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The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements

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Abstract

We calculate the polarized three-loop singlet anomalous dimensions and splitting functions in QCD in the $\overline{\text{MS}}$ -scheme by using the traditional method of space-like off-shell massless operator matrix elements. This is a gauge-dependent framework. Here one obtains the anomalous dimensions without referring to gravitational currents. We also calculate the non-singlet splitting function $\Delta P_{\text{qq}}^{(2),s,\text{NS}}$ and compare our results to the literature.

1 Introduction

Polarized deep–inelastic scattering allows to reveal the spin and angular momentum structure of nucleons [1]. At sufficiently high scales of the virtuality of the exchanged electro–weak current $Q^2 \gg 1 \text{ GeV}^2$ and for not too small or too large values of Bjorken $x = Q^2/(2p \cdot q)$ [2] the twist–2 contributions dominate over the higher twist contributions [3] and target mass corrections [4]. Because of their strong variation, also the detailed control of the QED radiative corrections is required [5]. The anomalous dimensions of local quark and gluon operators determine the scaling violations of the deep–inelastic scattering structure functions, together with the massless and massive Wilson coefficients, cf. [6,7], by the scale evolution of the parton densities. Precision measurements of the scaling violations therefore require the anomalous dimensions as an input to extract the strong coupling constant $a_s(M_Z^2) = \alpha_s(M_Z^2)/(4\pi)$ [8]. The polarized non–singlet and singlet anomalous dimensions have been calculated to first [9–13], second [14–16] and third order [17–20].

In Ref. [19] the calculation has been performed using the forward Compton amplitude, which required reference to gravitational subsidiary currents for the gluonic sector. In the present paper we use the traditional approach of massless off–shell operator matrix elements (OMEs), allowing the direct calculation. However, the computational framework is gauge dependent, while the anomalous dimensions in minimal subtraction schemes are not. This requires corresponding physical projectors [16]. In the polarized case, unlike the unpolarized case [21], there is no mixing with the so–called alien operators [16, 21–31].

In Refs. [18, 20] the complete polarized three–loop anomalous dimensions $\Delta\gamma_{qq}^{(2),\text{NS}}, \Delta\gamma_{qq}^{(2),\text{PS}}$ and $\Delta\gamma_{qg}^{(2)}$ have already been confirmed using different methods compared to Ref. [17, 19]. In particular massive on–shell OMEs with massless external lines were used in [20], which also covered the contributions $\propto T_F$ of $\Delta\gamma_{gq}^{(2)}$ and $\Delta\gamma_{gg}^{(2)}$. In the following we will recalculate the polarized three–loop anomalous dimensions using massless off–shell OMEs, which will allow us to obtain also the gluonic OMEs $\Delta\gamma_{gq}^{(2)}$ and $\Delta\gamma_{gg}^{(2)}$ in complete form. Unlike the singlet case, the flavor non–singlet description of the structure functions $g_{1,2}^{\text{NS}}(x, Q^2)$ is well explored to three–loop order, also including the heavy flavor contributions [32–34].

The paper is organized as follows. In Section 2 we derive the structure of the physical part of the flavor singlet polarized unrenormalized off–shell OMEs to three–loop order. From their pole terms of $O(1/\varepsilon)$ one can extract the singlet anomalous dimensions. We work in the Larin scheme [16, 35] and perform finally the transformation to the $\overline{\text{M}}$ –scheme [16, 19]. Calculation details are given in Section 3. A central method being applied is the method of arbitrary high moments [36]. In Section 4 we calculate also the polarized non–singlet anomalous dimension $\Delta\gamma_{qq}^{(2),\text{s,NS}}$, cf. also [37], and present the polarized singlet anomalous dimensions in Section 5. Comparisons to the literature are given in Section 6, including a discussion of the small x limit. Section 7 contains the conclusions.

2 The unrenormalized operator matrix elements

The massless off–shell singlet OMEs are defined as expectation values of the local operators

$$O_{q;\mu_1 \dots \mu_N}^{\text{S},5} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{trace terms}, \quad (1)$$

$$O_{g;\mu_1 \dots \mu_N}^{\text{S},5} = 2i^{N-2} \mathbf{S} \text{ tr} \left[\frac{1}{2} \varepsilon^{\mu_1 \alpha \beta \gamma} F_{\beta\gamma}^a D_{\mu_2} \dots D_{\mu_{N-1}} F_{\alpha,a}^{\mu_N} \right] - \text{trace terms} \quad (2)$$

between quark (antiquark) ψ ($\bar{\psi}$) and gluonic states $F_{\mu_1\alpha}^a$ of space-like momentum p , $p^2 < 0$. The OMEs are given by

$$\Delta\hat{A}_{ij} = \langle q(p), j | O_i^{S,(5)} | q(p), j \rangle. \quad (3)$$

Here \mathbf{S} is the symmetry operator, \mathbf{tr} the color trace operator and $D_\mu = \partial_\mu + ig_s t_a A_\mu^a$ the covariant derivative, with A_μ^a the gluon field, ψ the quark field, t_a the generators of $SU(N_c)$, $g_s = \sqrt{4\pi\alpha_s}$, and $F_{\mu\nu}^a$ denotes the field strength tensor of $SU(N_c)$ Yang–Mills theory $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f_{abc} A_\mu^b A_\nu^c$, where f_{abc} denote the $SU(N_c)$ structure constants. The Feynman rules of QCD are given in [38] and for the local operators in [20, 39]. In the present case one more Feynman rule is needed, which we present in Appendix A.

We use the Larin scheme [35]¹ to describe γ^5 in $D = 4 + \varepsilon$ -dimensions and express γ^5 by

$$\gamma^5 = \frac{i}{24} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma, \quad (4)$$

$$\not{\Delta} \gamma^5 = \frac{i}{6} \varepsilon_{\mu\nu\rho\sigma} \Delta^\mu \gamma^\nu \gamma^\rho \gamma^\sigma. \quad (5)$$

The Levi-Civita symbols are now contracted in D dimensions,

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\lambda\tau\gamma} = -\text{Det}[g_\omega^\beta], \quad \beta = \alpha, \lambda, \tau, \gamma; \quad \omega = \mu, \nu, \rho, \sigma. \quad (6)$$

The Larin scheme is one of the consistent calculation schemes in the polarized case. We will later transform to another scheme, the M–scheme. It is needless to say that the calculation of any observable requires to calculate also the Wilson coefficients in the same scheme, which implies that the extracted twist–2 parton density functions are obtained in this scheme, despite being universal w.r.t. the scattering processes in which they are used. Any analytic continuation of γ_5 or the Levi–Civita symbol to D dimensions violates Ward identities. One way to restore these consists in the calculation of scheme–invariant quantities.

The operator matrix elements have the representation

$$\Delta\hat{A}_{qq}^{\text{PS}} = \left[\gamma_5 \not{\Delta} \Delta\hat{A}_{qq}^{\text{PS,phys}} + \gamma_5 \not{p} \frac{\Delta \cdot p}{p^2} \Delta\hat{A}_{qq}^{\text{PS,EOM}} \right] (\Delta \cdot p)^{N-1} \quad (7)$$

$$\Delta\hat{A}_{gg} = \varepsilon_{\mu\nu\alpha\beta} \Delta^\alpha p^\beta \frac{1}{\Delta \cdot p} \Delta\hat{A}_{gg}^{\text{phys}} (\Delta \cdot p)^{N-1} \quad (8)$$

$$\Delta\hat{A}_{gq} = \left[\gamma_5 \not{\Delta} \Delta\hat{A}_{gq}^{\text{phys}} + \gamma_5 \not{p} \frac{\Delta \cdot p}{p^2} \hat{A}_{gq}^{\text{EOM}} \right] (\Delta \cdot p)^{N-1} \quad (9)$$

$$\Delta\hat{A}_{gg,\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} \Delta^\alpha p^\beta \frac{1}{\Delta \cdot p} \Delta\hat{A}_{gg}^{\text{phys}} (\Delta \cdot p)^{N-1}. \quad (10)$$

Here Δ denotes a light–like vector, $\Delta \cdot \Delta = 0$. The following projectors are applied to separate the physical (phys) contributions

$$\Delta\hat{A}_{iq}^{\text{phys}} = -\frac{1}{4(D-2)(D-3)} \varepsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} \left[\not{p} \gamma^\mu \gamma^\nu \Delta\hat{A}_{iq} \right] (\Delta \cdot p)^{-N-1} \quad (11)$$

$$-\frac{p^2}{4(D-2)(D-3)} (\Delta \cdot p)^{-N-2} \varepsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} \left[\not{\Delta} \gamma^\mu \gamma^\nu \Delta\hat{A}_{iq} \right] \quad (12)$$

$$\Delta\hat{A}_{ig}^{\text{phys}} = \frac{1}{(D-2)(D-3)} \varepsilon_{\mu\nu\rho\sigma} \Delta^\rho p^\sigma (\Delta \cdot p)^{-N-1} \Delta\hat{A}_{ig}^{\mu\nu}, \quad (13)$$

¹For other schemes see Refs. [40]. For a discussion of the necessary finite renormalizations see [41].

with $i = q, g$. Since the singlet anomalous dimensions receive only contributions from the unrenormalized OME $\Delta\hat{A}_{qq}^{\text{phys}}$, we will consider only these operator matrix elements in the following. In Mellin N space it has the representation

$$\Delta\hat{A}_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \hat{a}^k S_{\varepsilon}^k \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon k/2} \Delta\hat{A}_{ij}^{(k)}, \quad (14)$$

with the spherical factor

$$S_{\varepsilon} = \exp \left[\frac{\varepsilon}{2} (\gamma_E - \ln(4\pi)) \right], \quad (15)$$

where γ_E is the Euler–Mascheroni number and \hat{a} the bare coupling constant. The free gluon propagator is given by²

$$D_{\mu\nu}^{ab}(k) = \frac{\delta^{ab}}{k^2 + i0} \left[-g_{\mu\nu} + (1 - \hat{\xi}) \frac{k_{\mu}k_{\nu}}{k^2 + i0} \right], \quad (16)$$

which defines the gauge parameter in the $R_{\hat{\xi}}$ gauge. The renormalization of the massless off-shell singlet OMEs encounter the renormalization of the coupling constant and the gauge parameter, as well as that of the local operator. In the following we will deviate from Refs. [16,21] and perform the renormalization of the coupling constant and the gauge parameter and use the resulting expression, $\Delta\tilde{A}$, at $\mu^2 = -p^2$ to extract the anomalous dimensions. In the unrenormalized OME obtained in the diagrammatic calculation the coupling constant and the gauge parameter are renormalized before comparing to \tilde{A} in Eq. (27). The unrenormalized coupling is given by

$$\hat{a} = a \left[1 + \frac{2}{\varepsilon} \beta_0 a + \left(\frac{4}{\varepsilon^2} \beta_0^2 + \frac{1}{\varepsilon} \beta_1 \right) a^2 \right] + O(a^3), \quad (17)$$

where $a \equiv a_s$ denotes the renormalized strong coupling constant in the $\overline{\text{MS}}$ scheme. The expansion coefficients of the QCD β -function are given by [42]³

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F N_F, \quad (18)$$

$$\beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F N_F - 4 C_F T_F N_F. \quad (19)$$

The bare gauge parameter $\hat{\xi}$ is renormalized by

$$\hat{\xi} = \xi Z_3(\xi), \quad (20)$$

where Z_3 is the Z -factor of the gluon propagator, cf. [43–46],

$$Z_3(\xi) = 1 + a \frac{z_{11}}{\varepsilon} + a^2 \left[\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right] + O(a^3), \quad (21)$$

with

$$z_{11} = C_A \left[-\frac{13}{3} + \xi \right] + \frac{8}{3} T_F N_F, \quad (22)$$

²Note a typo in [21], Eq. (2.6).

³Note a typographical error in [21], Eq. (2.13) and [16], Eq. (2.14).

$$z_{22} = C_A^2 \left[-\frac{13}{2} - \frac{17}{6}\xi + \xi^2 \right] + C_A T_F N_F \left[4 + \frac{8}{3}\xi \right], \quad (23)$$

$$z_{21} = C_A^2 \left[-\frac{59}{8} + \frac{11}{8}\xi + \frac{1}{4}\xi^2 \right] + 4 C_F T_F N_F + 5 C_A T_F N_F. \quad (24)$$

The color factors are $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, $T_F = 1/2$ for $SU(N_c)$ and $N_c = 3$ for QCD; N_F denotes the number of massless quark flavors.

In Mellin N space the Z -factor of a local singlet operator reads [39]

$$\begin{aligned} Z_{ij}^S = & \delta_{ij} + a \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a^2 \left[\frac{1}{\varepsilon^2} \left(\frac{1}{2} \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \right) + \frac{1}{2\varepsilon} \gamma_{ij}^{(1)} \right] + a^3 \left[\frac{1}{\varepsilon^3} \left(\frac{1}{6} \gamma_{il}^{(0)} \gamma_{lk}^{(0)} \gamma_{kj}^{(0)} \right. \right. \\ & \left. \left. + \beta_0 \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \frac{4}{3} \beta_0^2 \gamma_{ij}^{(0)} \right) + \frac{1}{\varepsilon^2} \left(\frac{1}{6} \gamma_{il}^{(1)} \gamma_{lj}^{(0)} + \frac{1}{3} \gamma_{il}^{(0)} \gamma_{lj}^{(1)} + \frac{2}{3} \beta_0 \gamma_{ij}^{(1)} + \frac{2}{3} \beta_1 \gamma_{ij}^{(0)} \right) + \frac{1}{3\varepsilon} \gamma_{ij}^{(2)} \right]. \end{aligned} \quad (25)$$

In (25) the terms $\gamma_{ij}^{(k)}$, $k = 0, 1, 2, \dots$ denote the expansion coefficients of the anomalous dimension

$$\gamma_{ij} = \sum_{k=1}^{\infty} a^k \gamma_{ij}^{(k-1)}. \quad (26)$$

The partly renormalized OMEs, $\Delta \tilde{A}_{ij}^{\text{phys}}$, reads

$$\begin{aligned} \Delta \tilde{A}_{ij}^{\text{phys}} = & 1 + a \left[\frac{a_{ij}^{(1,-1)}}{\varepsilon} + a_{ij}^{(1,0)} + a_{ij}^{(1,1)} \varepsilon \right] + a^2 \left[\frac{a_{ij}^{(2,-2)}}{\varepsilon^2} + \frac{a_{ij}^{(2,-1)}}{\varepsilon} + a_{ij}^{(2,0)} \right] \\ & + a^3 \left[\frac{a_{ij}^{(3,-3)}}{\varepsilon^3} + \frac{a_{ij}^{(3,-2)}}{\varepsilon^2} + \frac{a_{ij}^{(3,-1)}}{\varepsilon} \right]. \end{aligned} \quad (27)$$

The expansion coefficients $a_{ij}^{(k,l)}$ are gauge dependent in general. The renormalized OMEs in the Larin scheme are given by

$$\Delta A_{ij}^{\text{phys}} = (\Delta Z_{ik}^S)^{-1} \Delta \tilde{A}_{kj}^{\text{phys}}, \quad (28)$$

expanded to $O(a^3)$ and setting $S_\varepsilon = 1$. The anomalous dimensions are iteratively extracted from the $1/\varepsilon$ pole terms and the other expansion coefficients $a_{ij}^{(k,l)}$ are given in Ref. [47].

The anomalous dimensions in the $\overline{\text{M}}$ -scheme [16, 19] are obtained by the following transformations

$$\gamma_{ij}^{(0),\text{M}} = \gamma_{ij}^{(0),\text{L}} \quad (29)$$

$$\gamma_{qq}^{(1),\text{NS},\text{M}} = \gamma_{qq}^{(1),\text{NS},\text{L}} + 2\beta_0 z_{qq}^{(1)}, \quad (30)$$

$$\gamma_{qq}^{(1),\text{PS},\text{M}} = \gamma_{qq}^{(1),\text{PS},\text{L}}, \quad (31)$$

$$\gamma_{gg}^{(1),\text{M}} = \gamma_{gg}^{(1),\text{L}} + \gamma_{gg}^{(0)} z_{qq}^{(1)}, \quad (32)$$

$$\gamma_{gq}^{(1),\text{M}} = \gamma_{gq}^{(1),\text{L}} - \gamma_{gq}^{(0)} z_{qq}^{(1)}, \quad (33)$$

$$\gamma_{gg}^{(1),\text{M}} = \gamma_{gg}^{(1),\text{L}}. \quad (34)$$

$$\gamma_{qq}^{(2),\text{NS},\text{M}} = \gamma_{qq}^{(2),\text{NS},\text{L}} - 2\beta_0 \left((z_{qq}^{(1)})^2 - 2z_{qq}^{(2),\text{NS}} \right) + 2\beta_1 z_{qq}^{(1)}, \quad (35)$$

$$\gamma_{qq}^{(2),\text{PS},\text{M}} = \gamma_{qq}^{(2),\text{PS},\text{L}} + 4\beta_0 z_{qq}^{(2),\text{PS}}, \quad (36)$$

$$\gamma_{qq}^{(2),\text{M}} = \gamma_{qq}^{(2),\text{L}} + \gamma_{qq}^{(1),\text{M}} z_{qq}^{(1)} + \gamma_{qq}^{(0)} \left(z_{qq}^{(2)} - (z_{qq}^{(1)})^2 \right), \quad (37)$$

$$\gamma_{gq}^{(2),\text{M}} = \gamma_{gq}^{(2),\text{L}} - \gamma_{gq}^{(1),\text{M}} z_{qq}^{(1)} - \gamma_{gq}^{(0)} z_{qq}^{(2)}, \quad (38)$$

$$\gamma_{gg}^{(2),\text{M}} = \gamma_{gg}^{(2),\text{L}}, \quad (39)$$

with [16]

$$z_{qq}^{(1)} = -\frac{8C_F}{N(N+1)}, \quad (40)$$

$$\begin{aligned} z_{qq}^{(2),\text{NS}} = & C_F T_F N_F \frac{16(-3-N+5N^2)}{9N^2(1+N)^2} + C_A C_F \left\{ -\frac{4R_1}{9N^3(1+N)^3} - \frac{16}{N(1+N)} S_{-2} \right\} \\ & + C_F^2 \left\{ \frac{8(2+5N+8N^2+N^3+2N^4)}{N^3(1+N)^3} + \frac{16(1+2N)}{N^2(1+N)^2} S_1 \right. \\ & \left. + \frac{16}{N(1+N)} S_2 + \frac{32}{N(1+N)} S_{-2} \right\}, \end{aligned} \quad (41)$$

$$z_{qq}^{(2),\text{PS}} = 8C_F T_F N_F \frac{(N+2)(1+N-N^2)}{N^3(N+1)^3}, \quad (42)$$

$$z_{qq}^{(2)} = z_{qq}^{(2),\text{NS}} + z_{qq}^{(2),\text{PS}} \quad (43)$$

and

$$R_1 = 103N^4 + 140N^3 + 58N^2 + 21N + 36. \quad (44)$$

3 Details of the calculation

The Feynman diagrams for the massless off-shell OMEs are generated by QGRAF [39, 48] and the Dirac and Lorentz algebra is performed by FORM [49]. The color algebra is carried out by using Color [50]. We resum the local operators into propagators by observing the current crossing relations, cf. [6, 51], see also Ref. [47], for the odd integer moments

$$\sum_{N=0}^{\infty} (\Delta \cdot k)^N (t^N - (-t)^N) \rightarrow \left[\frac{1}{1 - \Delta \cdot k t} - \frac{1}{1 + \Delta \cdot k t} \right]. \quad (45)$$

Through this one obtains a quadratic dependence on the resummation variable t . An expansion in t leads to the moments again.

In the polarized flavor singlet case 125 irreducible diagrams contribute for $\Delta A_{qq}^{(3),\text{PS}}$, 1101 for $\Delta A_{qq}^{(3)}$, 400 for $\Delta A_{gq}^{(3)}$, and 1598 for $\Delta A_{gg}^{(3)}$. For comparison the number of diagrams for $\Delta A_{qq}^{(3),\text{NS}}$ amounts to 559. Finally, 24 diagrams contribute to the part of the forward Compton amplitude, from which $\Delta \gamma_{qq}^{(2),\text{s,NS}}$ is extracted. The total number of irreducible diagrams in the polarized singlet case at three-loop order is larger by a factor of ~ 22 than in the two-loop order. The reducible diagrams are accounted for by wave-function renormalization [44–46], decorating the OMEs at lower order in the coupling constant [16, 47]. The different local operator insertions are resummed using generating functions of the type where t denotes an auxiliary parameter.

In the calculation of the one- and two-loop contributions we also used the package `EvaluateMultiSums` [52]. The irreducible three-loop diagrams are reduced to 252 master integrals

using the code `Crusher` [53] by applying the integration-by-parts relations [54, 55]. The set of master integrals turns out to be the same as in the non-singlet case [18]. We therefore could re-use the large sets of 3000 moments having been already calculated for these master integrals before. For the calculation of the necessary initial values for the difference equations we use the results given in [55, 56] and also the present calculation is based on the method of arbitrary large moments [36], implemented within the package `SolveCoupledSystem` [57] is also used to generate a large number of moments for the massless OMEs. By using the method of guessing [58, 59] and its implementation in `Sage` [60, 61] we determine the difference equations, which correspond to the different color and multiple zeta value [62] factors. To calculate the anomalous dimensions $\Delta\gamma_{ij}^{(2)}$ we generated 3000 odd moments. It turns out that the determination of the largest recurrence requires 462 moments for $\Delta\gamma_{qq}^{(2),PS}$, 989 moments for $\Delta\gamma_{qg}^{(2)}$, 1035 moments for $\Delta\gamma_{gq}^{(2)}$, 1568 moments for $\Delta\gamma_{gg}^{(2)}$. The difference equations are solved by using the package `Sigma` [63, 64], which is based on methods from difference field theory [65]. The treatment of special functions has been performed using functions of the package `HarmonicSums` [66–73, 76]. In this way the anomalous dimensions are obtained. The largest difference equation contributing has order $o = 16$ and degree $d = 304$. These numbers are of the order obtained in the non-singlet case in Ref. [59], where the largest difference equation contributing had order $o = 16$ and degree $d = 192$, and required 1079 moments. A moment based test-run for the polarized anomalous dimensions, like in the unpolarized case, has not been performed.

The overall computation time using the automated chain of codes described amounted to about 18 days of CPU time on `Intel(R) Xeon(R) CPU E5-2643 v4` processors. Again we have only retained the first power of the gauge parameter to have a first check on the renormalization.

The anomalous dimensions, $\Delta\gamma_{ij}$, can all be expressed by harmonic sums [66, 67]. They are recursively defined by

$$S_{b,\bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}, N \in \mathbb{N} \setminus \{0\}. \quad (46)$$

Their Mellin inversion to the splitting functions $\Delta P_{ij}(z)$

$$\Delta\gamma_{ij}(N) = - \int_0^1 dz z^{N-1} \Delta P_{ij}(z) \quad (47)$$

is obtained using routines of the packages `HarmonicSums` and are expressed by harmonic polylogarithms [68], which read

$$H_{b,\bar{a}}(z) = \int_0^z dx f_b(x) H_{\bar{a}}(x), \quad H_{\emptyset} = 1, \quad b, a_i \in \{-1, 0, 1\}, \quad (48)$$

with the alphabet of letters

$$\mathfrak{A}_H = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}. \quad (49)$$

One distinguishes three contributions to the individual splitting functions in z -space, which have a different treatment in the Mellin convolutions,

$$\Delta P(z) = \Delta P^{\delta}(z) + \Delta P^{\text{plu}}(z) + \Delta P^{\text{reg}}(z), \quad (50)$$

with $\Delta P^\delta(z) = p_0 \delta(1-z)$, and $\Delta P^{\text{reg}}(z)$ denotes the regular part in $z \in [0, 1]$. $\Delta P^{\text{plu}}(z)$ is the remaining genuine $+$ -distribution, the Mellin transformation of which is given by

$$\int_0^1 dz (z^{N-1} - 1) \Delta P^{\text{plu}}(z). \quad (51)$$

Also here we reduce the expressions to the algebraic basis, cf. [73], which has the advantage that only the minimal set has to be calculated in numerical applications [74]. We will not present the splitting function in explicit form, since the expressions are rather lengthy. They are given in computer-readable form in an attachment to the present paper. The polarized singlet anomalous dimensions are given in Section 5.

4 The polarized non-singlet anomalous dimension $\Delta\gamma_{\text{qq}}^{(2),s,\text{NS}}$

In our previous paper [18] we had not yet calculated the non-singlet anomalous dimension $\Delta\gamma_{\text{qq}}^{(2),s,\text{NS}}$, which emerges from three-loop order onward. It is best calculated using the vector-axialvector interference term in the forward Compton amplitude, corresponding to the associated structure function $g_5^-(x, Q^2)$, Ref. [51], corresponding to the difference in case of W^- and W^+ charged current scattering. Due to its crossing relations it has even moments. This anomalous dimension has been calculated previously in Ref. [37].

The corresponding gauge boson vertex is parameterized by

$$i \left[v \gamma_\mu + a \frac{i}{6} \varepsilon_{\mu\nu_1\nu_2\nu_3} \gamma^{\nu_1} \gamma^{\nu_2} \gamma^{\nu_3} \right] \quad (52)$$

and we consider the current interference term $\propto a \cdot v$. The projectors for the massless external quark lines of momentum p and boson lines corresponding to a tensor of rank two read

$$P^a = \frac{1}{4} \text{tr} \left[\frac{i}{6} \varepsilon_{\rho\rho_1\rho_2\rho_3} \gamma^{\rho_1} \gamma^{\rho_2} \gamma^{\rho_3} \dots \right]; \quad P_{\mu\nu}^b = -\frac{g_{\mu\nu}}{D-1}. \quad (53)$$

The forward Compton amplitude depends on the invariants $Q^2 = -q^2$ and $p \cdot q = Q^2/(2z) \equiv (Q^2/2)y$. The diagrams can be represented as formal power series in y

$$F(y) = \sum_{N=0}^{\infty} f(N) y^N, \quad (54)$$

cf. e.g. [18], Sect. 2. The IBP-reduction in this case leads to three families and 101 master integrals in total. We consider the differential equations for the individual master integrals $M_k(y, \varepsilon)$, which are computed using the method described in [75]. After insertion of the master integrals into the amplitude and subsequent expansion in the dimensional parameter ε , the anomalous dimension can be determined by using the command `GetMoment[F[y], y, N]` of the package `HarmonicSums` from the pole term $O(1/\varepsilon)$ of the forward Compton amplitude. We obtain

$$\begin{aligned} \Delta\gamma_{\text{NS}}^{(2),s} &= -16 \frac{1 + (-1)^N}{2} N_F \frac{d^{abc} d_{abc}}{N_c} \left[\frac{1}{N^2} + \frac{2Q_1}{N^4(1+N)^4} S_1 + \frac{2(2+3N+3N^2)}{N^2(1+N)^2} \right. \\ &\quad \left. \times [S_3 - 2S_{-3} + 4S_{-2,1}] + \left(\frac{4(2+4N+4N^2+N^3+N^4)}{N^3(1+N)^3} + \frac{8(-1+N)(2+N)}{N^2(1+N)^2} S_1 \right) \right] \end{aligned}$$

$$\times S_{-2} \Big] \quad (55)$$

with

$$Q_1 = 3N^6 + 8N^5 - N^4 - 14N^3 - 29N^2 - 21N - 6. \quad (56)$$

The corresponding splitting function reads

$$\begin{aligned} \Delta P_{\text{NS}}^{(2),s} = & 16N_F \frac{d^{abc}d_{abc}}{N_c} \left\{ (1+x) \left[-4H_0H_{-1}(-4+5H_{-1}) + 8H_{0,-1}(-2+5H_{-1}) \right. \right. \\ & + 18H_{-1}H_0^2 + (-2H_0^2 + 32H_{-1})H_{0,1} + 8H_0H_{0,0,1} - 32H_{0,1,-1} - 32H_{0,-1,1} \\ & - 40H_{0,-1,-1} - (52H_{-1} + 8H_{0,1})\zeta_2 \Big] + (1-x) \left[24H_1 - H_0^2H_1 + 2H_0H_{0,1} \right. \\ & + 4H_0^2H_{0,-1} + 8H_{0,-1}^2 - 8H_0H_{0,0,-1} - 16H_0H_{0,-1,-1} - (20H_1 + 8H_{0,-1})\zeta_2 \Big] \\ & - (1+24x)H_0 - 4xH_0^2 - 6xH_0^3 + \frac{1}{3}xH_0^4 + 2(5+4x)H_{0,1} - 28H_0H_{0,-1} \\ & + 4(1-9x)H_{0,0,1} + 4(5-9x)H_{0,0,-1} + (2(-5+4x) + 2(-3+37x))H_0 \\ & \left. - 2(3+5x)H_0^2\zeta_2 + (2(3+44x) - 16xH_0)\zeta_3 + 2(5+3x)\zeta_2^2 \right\}. \quad (57) \end{aligned}$$

Here ζ_k denotes the Riemann ζ -function at integer argument $k \geq 2$ and the color factor is normalized in the present case to $d^{abc}d_{abc}/N_c = 5/18$ in QCD. The leading small z contribution of $\Delta P_{\text{NS}}^{(2),s}$ is $\propto \ln^2(z)$. This behaviour is, however, not dominant in kinematic regions being accessible at present. The asymptotic behaviour reaches the complete function up to 10% below $z \sim 10^{-11}$ only. One more logarithmic order allows a description below $z \sim 10^{-3}$.

5 The polarized singlet anomalous dimensions

In the following we use the minimal representations in terms of the contributing harmonic sums and harmonic polylogarithms by applying the algebraic relations [73] between the harmonic sums and the harmonic polylogarithms. The polarized singlet anomalous dimensions can be represented by the following 23 harmonic sums up to weight $w = 5$ to three-loop order

$$\left\{ S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_1, S_2, S_3, S_4, S_5, S_{-4,1}, S_{-3,1}, S_{-2,1}, S_{-2,2}, S_{-2,3}, S_{2,-3}, S_{2,1}, S_{3,1}, S_{-3,1,1}, \right. \\ \left. S_{-2,1,1}, S_{-2,2,1}, S_{2,1,-2}, S_{2,1,1}, S_{-2,1,1,1} \right\}. \quad (58)$$

This number is further reduced using also the structural relations [76, 77] to at least 15 sums. In the present case the 10 sums

$$\left\{ S_1, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,1,-2}, S_{-3,1,1}, S_{-2,1,1,1} \right\} \quad (59)$$

suffice.

The splitting functions in z -space depend on the 26 harmonic polylogarithms

$$\left\{ \begin{array}{l} \mathbf{H}_{-1}, \mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_{0,-1}, \mathbf{H}_{0,1}, \mathbf{H}_{0,-1,-1}, \mathbf{H}_{0,-1,1}, \mathbf{H}_{0,0,-1}, \mathbf{H}_{0,0,1}, \mathbf{H}_{0,1,-1}, \mathbf{H}_{0,1,1}, \mathbf{H}_{0,-1,-1,-1}, \mathbf{H}_{0,-1,-1,1}, \\ \mathbf{H}_{0,-1,0,1}, \mathbf{H}_{0,-1,1,-1}, \mathbf{H}_{0,-1,1,1}, \mathbf{H}_{0,0,-1,-1}, \mathbf{H}_{0,0,-1,1}, \mathbf{H}_{0,0,0,-1}, \mathbf{H}_{0,0,0,1}, \mathbf{H}_{0,0,1,-1}, \mathbf{H}_{0,0,1,1}, \mathbf{H}_{0,1,-1,-1}, \\ \mathbf{H}_{0,1,-1,1}, \mathbf{H}_{0,1,1,-1}, \mathbf{H}_{0,1,1,1} \end{array} \right\}. \quad (60)$$

The harmonic sums are defined at the odd integers in the first place and the analytic continuation to $N \in \mathbb{C}$ is performed from there, [76, 78].

We obtain the following expressions for the polarized singlet anomalous dimensions in Mellin- N space, using the shorthand notation $S_{\bar{a}}(N) \equiv S_{\bar{a}}$ from one- to three-loop order. Here we dropped the prefactor $\frac{1}{2}(1 - (-1)^N)$.

$$\Delta\gamma_{qq}^{(0)} = C_F \left[-\frac{2(2 + 3N + 3N^2)}{N(1 + N)} + 8S_1 \right], \quad (61)$$

$$\Delta\gamma_{gg}^{(0)} = -T_F N_F \frac{8(N - 1)}{N(1 + N)}, \quad (62)$$

$$\Delta\gamma_{gq}^{(0)} = -C_F \frac{4(2 + N)}{N(1 + N)}, \quad (63)$$

$$\Delta\gamma_{gg}^{(0)} = T_F N_F \frac{8}{3} + C_A \left[-\frac{2(24 + 11N + 11N^2)}{3N(1 + N)} + 8S_1 \right], \quad (64)$$

$$\Delta\gamma_{qq}^{(1),\text{PS}} = C_F T_F N_F \frac{16(2 + N)(1 + 2N + N^3)}{N^3(1 + N)^3}, \quad (65)$$

$$\begin{aligned} \Delta\gamma_{gg}^{(1)} = & C_F T_F N_F \left[-\frac{8(-1 + N)(2 - N + 10N^3 + 5N^4)}{N^3(1 + N)^3} + \frac{32(N - 1)}{N^2(1 + N)} S_1 \right. \\ & \left. - \frac{16(N - 1)}{N(1 + N)} [S_1^2 - S_2] \right] + C_A T_F N_F \left[-\frac{16P_{20}}{N^3(1 + N)^3} - \frac{64S_1}{N(1 + N)^2} \right. \\ & \left. + \frac{16(N - 1)}{N(1 + N)} [S_1^2 + S_2 + 2S_{-2}] \right], \end{aligned} \quad (66)$$

$$\begin{aligned} \Delta\gamma_{gq}^{(1)} = & C_F \left[T_F N_F \left(\frac{32(2 + N)(2 + 5N)}{9N(1 + N)^2} - \frac{32(2 + N)}{3N(1 + N)} S_1 \right) + C_A \left(-\frac{8P_{33}}{9N^3(1 + N)^3} \right. \right. \\ & \left. \left. + \frac{8(12 + 22N + 11N^2)S_1}{3N^2(1 + N)} - \frac{8(2 + N)S_1^2}{N(1 + N)} + \frac{8(2 + N)S_2}{N(1 + N)} + \frac{16(2 + N)S_{-2}}{N(1 + N)} \right) \right] \\ & + C_F^2 \left[\frac{4(2 + N)(1 + 3N)(-2 - N + 3N^2 + 3N^3)}{N^3(1 + N)^3} - \frac{8(2 + N)(1 + 3N)}{N(1 + N)^2} S_1 \right. \\ & \left. + \frac{8(2 + N)}{N(1 + N)} [S_1^2 + S_2] \right], \end{aligned} \quad (67)$$

$$\Delta\gamma_{gg}^{(1)} = C_F T_F N_F \frac{8P_{40}}{N^3(1 + N)^3} + C_A^2 \left[-\frac{4P_{46}}{9N^3(1 + N)^3} + \left(\frac{8P_{17}}{9N^2(1 + N)^2} - 32S_2 \right) S_1 \right]$$

$$\begin{aligned}
& + \frac{64}{N(1+N)} S_2 - 16S_3 + \left(\frac{64}{N(1+N)} - 32S_1 \right) S_{-2} - 16S_{-3} + 32S_{-2,1} \Big] \\
& + C_A T_F N_F \left[\frac{32P_5}{9N^2(1+N)^2} - \frac{160}{9} S_1 \right], \tag{68}
\end{aligned}$$

$$\begin{aligned}
\Delta\gamma_{qq}^{(2),\text{PS}} = & C_F \left[T_F^2 N_F^2 \left[-\frac{64(2+N)P_{30}}{27N^4(1+N)^4} + \frac{64(2+N)(6+10N-3N^2+11N^3)}{9N^3(1+N)^3} S_1 \right. \right. \\
& - \left. \frac{32(N-1)(2+N)}{3N^2(1+N)^2} [S_1^2 + S_2] \right] + C_A T_F N_F \left[\frac{8P_9}{3N^3(1+N)^3} S_1^2 + \frac{8P_{10}}{3N^3(1+N)^3} S_2 \right. \\
& + \frac{16P_{61}}{27N^5(1+N)^5} + \left(-\frac{16P_{51}}{9N^4(1+N)^4} + \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \right) S_1 \\
& - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 + \frac{16(-58+23N+23N^2)}{3N^2(1+N)^2} S_3 + \left(-\frac{32P_1}{N^3(1+N)^3} \right. \\
& + \left. \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_1 \right) S_{-2} + \frac{32(-10+7N+7N^2)}{N^2(1+N)^2} S_{-3} - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} \\
& \left. - \frac{64(-2+3N+3N^2)}{N^2(1+N)^2} S_{-2,1} - \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \right] \\
& + C_F^2 T_F N_F \left[-\frac{16(2+N)P_{54}}{N^5(1+N)^5} + \left(\frac{16(2+N)P_{27}}{N^4(1+N)^4} - \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \right) S_1 \right. \\
& - \frac{8(N-1)(2+N)(2+3N+3N^2)}{N^3(1+N)^3} S_1^2 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 \\
& - \frac{8(2+N)(14+23N+11N^3)}{N^3(1+N)^3} S_2 - \frac{224(N-1)(2+N)}{3N^2(1+N)^2} S_3 \\
& \left. + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} + \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \right], \tag{69}
\end{aligned}$$

$$\begin{aligned}
\Delta\gamma_{qg}^{(2)} = & C_F T_F^2 N_F^2 \left[\frac{4P_{64}}{27N^5(1+N)^5} + \left(-\frac{32(-24+4N+47N^2)}{27N^2(1+N)} - \frac{32(N-1)}{3N(1+N)} S_2 \right) S_1 \right. \\
& \left. + \frac{32(N-1)(3+10N)}{9N^2(1+N)} S_1^2 - \frac{32(N-1)}{9N(1+N)} S_1^3 + \frac{32(5N-1)}{3N^2(1+N)} S_2 + \frac{320(N-1)}{9N(1+N)} S_3 \right] \\
& + C_A C_F T_F N_F \left[\frac{8P_{29}}{3N^3(1+N)^3} S_2 + \frac{P_{65}}{27N^5(1+N)^5(2+N)} + \left(-\frac{384(N-1)}{N(1+N)} S_{2,1} \right. \right. \\
& + \frac{16P_{60}}{27N^4(1+N)^4(2+N)} + \frac{16(75+14N+18N^2+N^3)S_2}{3N^2(1+N)^2} + \frac{640(N-1)}{3N(1+N)} S_3 \\
& \left. - \frac{192(N-1)}{N(1+N)} \zeta_3 \right) S_1 + \left(-\frac{8P_{25}}{9N^3(1+N)^3} + \frac{160(N-1)}{N(1+N)} S_2 \right) S_1^2 \\
& \left. + \frac{16(3-31N-18N^2+10N^3)}{9N^2(1+N)^2} S_1^3 + \frac{32(N-1)}{3N(1+N)} S_1^4 - \frac{64(N-1)}{N(1+N)} S_2^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{16(N-1)(240-17N+19N^2)}{9N^2(1+N)^2}S_3 + \left(\frac{128(N-1)(-4-N+N^2)}{N^2(1+N)^2(2+N)}S_1 \right. \\
& -\frac{32P_{24}}{N^3(1+N)^3(2+N)} + \frac{192(N-1)}{N(1+N)}S_1^2 \left. \right) S_{-2} + \frac{96(N-1)}{N(1+N)}S_{-2}^2 \\
& + \frac{32(N-1)(2+N)(-1+3N)}{N^2(1+N)^2}S_{-3} + \frac{96(N-1)(4+N+N^2)}{N^2(1+N)^2}S_{2,1} \\
& + \frac{160(N-1)}{N(1+N)}S_{-4} + \frac{64(N-1)}{N(1+N)}S_{3,1} - \frac{128(N-1)^2}{N^2(1+N)^2}S_{-2,1} + \frac{64(N-1)}{N(1+N)}S_{-2,2} \\
& + \frac{192(N-1)}{N(1+N)}S_{2,1,1} - \frac{256(N-1)}{N(1+N)}S_{-2,1,1} - \frac{192(N-1)(-5+3N+3N^2)}{N^2(1+N)^2}\zeta_3 \left. \right] \\
& + C_{AT_F}^2 N_F^2 \left[\frac{16P_{55}}{27N^4(1+N)^4} + \left(\frac{64(23+50N+10N^2+19N^3)}{27N(1+N)^3} - \frac{32(N-1)}{3N(1+N)} \right. \right. \\
& \left. \left. \times S_2 \right) S_1 - \frac{64(-2+5N^2)}{9N(1+N)^2}S_1^2 + \frac{32(N-1)}{9N(1+N)}S_1^3 - \frac{64(-2+6N+5N^2)}{9N(1+N)^2}S_2 \right. \\
& \left. + \frac{64(N-1)}{9N(1+N)}S_3 - \frac{128(-2+5N)}{9N(1+N)}S_{-2} + \frac{128(N-1)}{3N(1+N)}[S_{-3} + S_{2,1}] \right] \\
& + C_A^2 N_F T_F \left[\frac{16P_{34}}{9N^3(1+N)^3}S_2 - \frac{8P_{70}}{27N^5(1+N)^5(2+N)} + \left(-\frac{8P_{56}}{27N^4(1+N)^4} \right. \right. \\
& + \frac{8(-72+181N-48N^2+11N^3)}{3N^2(1+N)^2}S_2 - \frac{704(N-1)}{3N(1+N)}S_3 + \frac{128(N-1)}{N(1+N)}S_{2,1} \\
& + \frac{512(N-1)}{N(1+N)}S_{-2,1} + \frac{192(N-1)}{N(1+N)}\zeta_3 \left. \right) S_1 + \left(\frac{16P_{31}}{9N^3(1+N)^3} - \frac{160(N-1)}{N(1+N)}S_2 \right) S_1^2 \\
& - \frac{8(-24-59N+11N^3)}{9N^2(1+N)^2}S_1^3 - \frac{16(N-1)}{3N(1+N)}S_1^4 - \frac{16(N-1)}{N(1+N)}S_2^2 \\
& - \frac{16(345-428N+11N^3)}{9N^2(1+N)^2}S_3 - \frac{32(N-1)}{N(1+N)}S_4 + \left(\frac{32P_{49}}{9N^3(1+N)^3(2+N)} \right. \\
& - \frac{64(-5+N)(-1+2N)}{N^2(1+N)^2}S_1 - \frac{192(N-1)}{N(1+N)}S_1^2 - \frac{128(N-1)}{N(1+N)}S_2 \left. \right) S_{-2} \\
& - \frac{96(N-1)}{N(1+N)}S_{-2}^2 + \left(-\frac{32(69-92N+11N^3)}{3N^2(1+N)^2} - \frac{512(N-1)}{N(1+N)}S_1 \right) S_{-3} \\
& - \frac{352(N-1)}{N(1+N)}S_{-4} - \frac{32(N-1)(24+11N+11N^2)}{3N^2(1+N)^2}S_{2,1} - \frac{128(N-1)}{N(1+N)}S_{3,1} \\
& - \frac{64(-7+11N)}{N^2(1+N)^2}S_{-2,1} + \frac{448(-1+N)}{N(1+N)}S_{-2,2} + \frac{512(-1+N)}{N(1+N)}S_{-3,1} \\
& - \frac{768(-1+N)}{N(1+N)}S_{-2,1,1} + \frac{96(-1+N)(-8+3N+3N^2)}{N^2(1+N)^2}\zeta_3 \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + C_F^2 T_F N_F \left[-\frac{8P_{26}}{N^3(1+N)^3} S_1^2 + \frac{8P_{28}}{N^3(1+N)^3} S_2 + \frac{P_{63}}{N^4(1+N)^5(2+N)} \right. \\
& + \left(-\frac{8P_{53}}{N^4(1+N)^4} - \frac{8(-6+7N+28N^2+3N^3)}{N^2(1+N)^2} S_2 - \frac{704(N-1)}{3N(1+N)} S_3 \right. \\
& + \left. \left. \frac{256(N-1)}{N(1+N)} S_{2,1} \right) S_1 - \frac{8(N-1)(-10-9N+3N^2)}{3N^2(1+N)^2} S_1^3 - \frac{16(N-1)}{3N(1+N)} S_1^4 \right. \\
& - \frac{48(N-1)}{N(1+N)} S_2^2 - \frac{16(N-1)(-22+27N+3N^2)}{3N^2(1+N)^2} S_3 - \frac{160(N-1)}{N(1+N)} S_4 \\
& + \left(\frac{64P_{21}}{N^2(1+N)^3(2+N)} - \frac{256(N-1)}{N(1+N)^2} S_1 - \frac{128(N-1)}{N(1+N)} S_2 \right) S_{-2} - \frac{64(N-1)}{N(1+N)} S_{-2}^2 \\
& + \left(-\frac{128(N-1)^2}{N^2(1+N)^2} - \frac{256(N-1)}{N(1+N)} S_1 \right) S_{-3} - \frac{320(N-1)}{N(1+N)} S_{-4} - \frac{128(N-1)}{N^2(1+N)^2} S_{2,1} \\
& + \frac{64(N-1)}{N(1+N)} S_{3,1} + \frac{256(N-1)}{N(1+N)^2} S_{-2,1} + \frac{128(N-1)}{N(1+N)} S_{-2,2} + \frac{256(N-1)}{N(1+N)} S_{-3,1} \\
& \left. - \frac{192(N-1)}{N(1+N)} S_{2,1,1} + \frac{96(N-1)(-2+3N+3N^2)}{N^2(1+N)^2} \zeta_3 \right], \tag{70}
\end{aligned}$$

$$\begin{aligned}
\Delta\gamma_{gq}^{(2)} = & C_F^2 \left[C_A \left[\frac{4P_{39}}{9N^3(1+N)^3} S_2 + \frac{P_{73}}{54(N-1)N^5(1+N)^5} + \left(-\frac{4P_{58}}{27N^4(1+N)^4} \right. \right. \right. \\
& - \frac{8(30+203N+177N^2+49N^3)}{3N^2(1+N)^2} S_2 - \frac{640(2+N)}{3N(1+N)} S_3 + \frac{64(2+N)}{N(1+N)} S_{2,1} \\
& + \frac{128(2+N)}{N(1+N)} S_{-2,1} + \left. \left. \frac{288(2+N)}{N(1+N)} \zeta_3 \right) S_1 + \left(\frac{4P_{36}}{9N^3(1+N)^3} - \frac{16(2+N)}{N(1+N)} S_2 \right) S_1^2 \right. \\
& - \frac{8(6+85N+132N^2+50N^3)}{9N^2(1+N)^2} S_1^3 + \frac{8(2+N)(102+65N+29N^2)}{9N^2(1+N)^2} S_3 \\
& + \frac{16(2+N)}{3N(1+N)} S_1^4 - \frac{32(2+N)}{N(1+N)} S_4 + \left(-\frac{16P_{44}}{(N-1)N^3(1+N)^3} + \frac{64(7+3N)}{N(1+N)} S_1 \right. \\
& - \left. \frac{96(2+N)}{N(1+N)} S_1^2 \right) S_{-2} + \left(\frac{32(2+N)(4+3N)}{N(1+N)^2} - \frac{64(2+N)}{N(1+N)} S_1 \right) S_{-3} \\
& + \frac{80(2+N)}{N(1+N)} [S_{-2}^2 + S_{-4}] + \frac{16(2+N)(-6+11N+11N^2)}{3N^2(1+N)^2} S_{2,1} + \frac{224(2+N)}{N(1+N)} S_{3,1} \\
& - \frac{32(2+N)(5+3N)}{N(1+N)^2} S_{-2,1} + \frac{32(2+N)}{N(1+N)} S_{-2,2} - \frac{96(2+N)}{N(1+N)} S_{2,1,1} \\
& \left. - \frac{128(2+N)}{N(1+N)} S_{-2,1,1} - \frac{432(2+N)\zeta_3}{N(1+N)} \right] + T_F N_F \left[\frac{2P_{71}}{27(N-1)N^5(1+N)^5} \right. \\
& + \left(\frac{32(2+N)P_{14}}{27N^3(1+N)^3} + \frac{208(2+N)}{3N(1+N)} S_2 \right) S_1 - \frac{16(2+N)(-3+16N+37N^2)}{9N^2(1+N)^2} S_1^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{80(2+N)}{9N(1+N)} S_1^3 - \frac{16(2+N)(9+46N+67N^2)}{9N^2(1+N)^2} S_2 + \frac{256(2+N)}{9N(1+N)} S_3 \\
& + \left. \frac{256}{(N-1)N^2(1+N)^2} S_{-2} - \frac{64(2+N)}{3N(1+N)} S_{2,1} - \frac{128(2+N)}{N(1+N)} \zeta_3 \right] \\
& + C_F \left[T_F^2 N_F^2 \left[\frac{64(2+N)(3+7N+N^2)}{9N(1+N)^3} + \frac{64(2+N)(2+5N)}{9N(1+N)^2} S_1 \right. \right. \\
& \left. \left. - \frac{32(2+N)}{3N(1+N)} [S_1^2 + S_2] \right] + C_A T_F N_F \left[\frac{8P_{57}}{27(N-1)N^3(1+N)^4} \right. \right. \\
& \left. \left. + \left(-\frac{16P_{37}}{27N^3(1+N)^3} + \frac{80(2+N)S_2}{3N(1+N)} \right) S_1 + \frac{16(18+116N+129N^2+43N^3)}{9N^2(1+N)^2} S_1^2 \right. \right. \\
& \left. \left. - \frac{80(2+N)}{9N(1+N)} S_1^3 + \frac{16(-2+16N+9N^2+N^3)}{3N^2(1+N)^2} S_2 + \frac{512(2+N)}{9N(1+N)} S_3 \right. \right. \\
& \left. \left. + \left(-\frac{64P_7}{3(-1+N)N^2(1+N)^2} + \frac{256(2+N)S_1}{3N(1+N)} \right) S_{-2} + \frac{128(2+N)}{3N(1+N)} [S_{-3} - S_{-2,1}] \right. \right. \\
& \left. \left. + \frac{128(2+N)}{N(1+N)} \zeta_3 \right] + C_A^2 \left[\frac{2P_{35}}{3N^3(1+N)^3} S_2 - \frac{4P_{72}}{27(N-1)N^5(1+N)^5} \right. \right. \\
& \left. \left. + \left(\frac{4P_{62}}{27(N-1)N^4(1+N)^4} - \frac{4(120+158N+141N^2+55N^3)}{3N^2(1+N)^2} S_2 + \frac{128(2+N)}{3N(1+N)} S_3 \right. \right. \right. \\
& \left. \left. + \frac{128(2+N)}{N(1+N)} S_{-2,1} - \frac{96(2+N)}{N(1+N)} \zeta_3 \right) S_1 + \left(-\frac{2P_{38}}{9N^3(1+N)^3} + \frac{48(2+N)}{N(1+N)} S_2 \right) S_1^2 \right. \\
& \left. \left. + \frac{4(24+158N+165N^2+55N^3)}{9N^2(1+N)^2} S_1^3 - \frac{8(2+N)}{3N(1+N)} S_1^4 - \frac{40(2+N)}{N(1+N)} S_2^2 \right. \right. \\
& \left. \left. - \frac{8(-186+295N+528N^2+176N^3)}{9N^2(1+N)^2} S_3 - \frac{16(2+N)}{N(1+N)} S_4 \right. \right. \\
& \left. \left. + \left(-\frac{32P_{12}}{3(N-1)N^2(1+N)^2} S_1 + \frac{16P_{47}}{3(N-1)N^3(1+N)^3} + \frac{96(2+N)}{N(1+N)} S_1^2 \right. \right. \right. \\
& \left. \left. - \frac{64(2+N)}{N(1+N)} S_2 \right) S_{-2} - \frac{16(2+N)}{N(1+N)} S_{-2}^2 + \left(-\frac{16(-126-13N+66N^2+22N^3)}{3N^2(1+N)^2} \right. \right. \\
& \left. \left. - \frac{192(2+N)}{N(1+N)} S_1 \right) S_{-3} - \frac{176(2+N)}{N(1+N)} S_{-4} + \frac{32(-30+13N+33N^2+11N^3)}{3N^2(1+N)^2} S_{-2,1} \right. \\
& \left. \left. - \frac{64(2+N)}{N(1+N)} S_{3,1} + \frac{224(2+N)}{N(1+N)} S_{-2,2} + \frac{256(2+N)}{N(1+N)} S_{-3,1} - \frac{384(2+N)}{N(1+N)} S_{-2,1,1} \right. \right. \\
& \left. \left. + \frac{144(2+N)}{N(1+N)} \zeta_3 \right] \right] + C_F^3 \left[-\frac{2(2+N)P_{19}}{N^3(1+N)^3} S_2 + \frac{P_{66}}{2(N-1)N^5(1+N)^5} \right. \\
& \left. + \left(-\frac{4(2+N)P_{43}}{N^4(1+N)^4} + \frac{4(2+N)(-2+19N+39N^2)}{N^2(1+N)^2} S_2 + \frac{128(2+N)}{3N(1+N)} S_3 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{64(2+N)}{N(1+N)}S_{2,1} - \frac{192(2+N)}{N(1+N)}\zeta_3 \Big) S_1 + \left(\frac{2(2+N)P_3}{N^3(1+N)^3} - \frac{32(2+N)}{N(1+N)}S_2 \right) S_1^2 \\
& + \frac{4(2+N)(-2+3N+15N^2)}{3N^2(1+N)^2}S_1^3 - \frac{8(2+N)}{3N(1+N)}S_1^4 - \frac{24(2+N)}{N(1+N)}S_2^2 \\
& + \frac{64(2+N)(-1+3N^2)}{3N^2(1+N)^2}S_3 - \frac{48(2+N)}{N(1+N)}S_4 + \left(\frac{32P_{22}}{(N-1)N^2(1+N)^3} \right. \\
& \left. - \frac{128(2+N)}{N(1+N)^2}S_1 - \frac{64(2+N)}{N(1+N)}S_2 \right) S_{-2} - \frac{16(2+N)(-2+3N+3N^2)}{N^2(1+N)^2}S_{2,1} \\
& - \frac{96(2+N)}{N(1+N)}S_{-2}^2 + \left(-\frac{64(N-1)(2+N)}{N^2(1+N)^2} - \frac{128(2+N)}{N(1+N)}S_1 \right) S_{-3} \\
& - \frac{160(2+N)}{N(1+N)}[S_{-4} + S_{3,1}] + \frac{128(2+N)}{N(1+N)^2}[S_{-2,1} + (1+N)S_{-3,1}] \\
& + \frac{64(2+N)}{N(1+N)}S_{-2,2} + \frac{96(2+N)}{N(1+N)}S_{2,1,1} + \frac{288(2+N)\zeta_3}{N(1+N)} \Big], \tag{71}
\end{aligned}$$

$$\begin{aligned}
\Delta\gamma_{gg}^{(2)} = & C_A T_F^2 N_F^2 \left[-\frac{16P_8}{27N^2(1+N)^2}S_1 - \frac{4P_{48}}{27N^3(1+N)^3} \right] + C_F \left[T_F^2 N_F^2 \left[-\frac{8P_{59}}{27N^4(1+N)^4} \right. \right. \\
& + \frac{64(N-1)(2+N)(-6-8N+N^2)}{9N^3(1+N)^3}S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^2 \\
& \left. \left. - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 \right] + C_A T_F N_F \left[\frac{8P_6}{N^3(1+N)^3}S_2 - \frac{8P_9}{3N^3(1+N)^3}S_1^2 \right. \right. \\
& + \frac{2P_{77}}{27(N-1)N^5(1+N)^5(2+N)} + \left(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \right. \\
& \left. \left. - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 + 128\zeta_3 \right) S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^3 - \frac{32(34+N+N^2)}{3N^2(1+N)^2} \right. \\
& \left. \times S_3 + \left(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{23}}{(N-1)N^2(1+N)^3(2+N)} \right) S_{-2} \right. \\
& \left. - \frac{192(4-N-N^2)}{N^2(1+N)^2}S_{-3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{128(-8+N+N^2)}{N^2(1+N)^2}S_{-2,1} \right. \\
& \left. \left. - \frac{64(-3+N)(4+N)}{N^2(1+N)^2}\zeta_3 \right] \right] + C_A^3 \left[\frac{64P_{16}}{9N^2(1+N)^2}S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2}S_3 \right. \\
& + \frac{P_{74}}{27(N-1)N^5(1+N)^5(2+N)} + \left(\frac{4P_{69}}{9(N-1)N^4(1+N)^4(2+N)} \right. \\
& \left. - \frac{64P_{17}}{9N^2(1+N)^2}S_2 + 128S_2^2 + \frac{16(-96+11N+11N^2)}{3N(1+N)}S_3 + 192S_4 \right. \\
& \left. + \frac{1024}{N(1+N)}S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1} \right) S_1
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{256(1+3N+3N^2)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1} \right) S_1^2 + \left(-\frac{16P_{41}}{9N^3(1+N)^3} \right. \\
& \left. + 64S_3 + 640S_{-2,1} \right) S_2 - \frac{256}{N(1+N)} S_2^2 - \frac{384}{N(1+N)} S_4 + 64S_5 \\
& + \left(\frac{32P_{52}}{9(N-1)N^3(1+N)^3(2+N)} + \left(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2 \right) \right. \\
& \left. \times S_1 - \frac{512}{N(1+N)} S_2 + 128S_3 - 768S_{2,1} \right) S_{-2} + \left(-\frac{16(24+11N+11N^2)}{3N(1+N)} \right. \\
& \left. + 64S_1 \right) S_{-2}^2 + \left(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)} S_1 + 384S_1^2 - 320S_2 \right) S_{-3} \\
& + \left(-\frac{1024}{N(1+N)} + 512S_1 \right) S_{-4} - 192S_{-5} - 384S_{2,-3} + \frac{1280}{N(1+N)} S_{-2,2} \\
& + 384S_{-2,3} + \frac{1536}{N(1+N)} S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)} S_{-2,1,1} \\
& \left. + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1,1} \right] \\
& + C_F^2 T_F N_F \left[-\frac{4P_{75}}{(N-1)N^5(1+N)^5(2+N)} + \left(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \right. \right. \\
& \left. \left. - \frac{16P_{42}}{N^4(1+N)^4} \right) S_1 + \frac{8(N-1)(2+N)(2+3N+3N^2)}{N^3(1+N)^3} S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \right. \\
& \left. \times S_1^3 - \frac{8(2+N)(2-11N-16N^2+9N^3)}{N^3(1+N)^3} S_2 + \frac{32(10+7N+7N^2)}{3N^2(1+N)^2} S_3 \right. \\
& \left. + \left(-\frac{64(10+N+N^2)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2} S_1 \right) S_{-2} + \frac{256}{N^2(1+N)^2} S_{-3} \right. \\
& \left. - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} - \frac{512}{N^2(1+N)^2} S_{-2,1} + \frac{192(-2-N-N^2)}{N^2(1+N)^2} \zeta_3 \right] \\
& + C_A^2 T_F N_F \left[\frac{32P_4}{9N^2(1+N)^2} S_2 + \frac{32P_{11}}{9N^2(1+N)^2} S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2} S_{-2,1} \right. \\
& + \frac{16P_{13}}{9N^2(1+N)^2} S_3 + \frac{2P_{76}}{27(N-1)N^5(1+N)^5(2+N)} + \left(\frac{1280}{9} S_2 - \frac{64}{3} S_3 \right. \\
& \left. - \frac{8P_{68}}{27(-1+N)N^4(1+N)^4(2+N)} - 128\zeta_3 \right) S_1 + \frac{64}{3} S_{-2}^2 \\
& + \left(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)} S_1 - \frac{32P_{50}}{9(N-1)N^3(1+N)^3(2+N)} \right) S_{-2} \\
& \left. + \frac{128(-3+2N+2N^2)}{N^2(1+N)^2} \zeta_3 \right], \tag{72}
\end{aligned}$$

with the polynomials

$$P_1 = N^4 - 2N^3 - 4N^2 + 15N + 2, \quad (73)$$

$$P_2 = N^4 + 2N^3 - 5N^2 - 6N + 16, \quad (74)$$

$$P_3 = N^4 + 44N^3 + 45N^2 + 38N + 12, \quad (75)$$

$$P_4 = 3N^4 + 6N^3 - 89N^2 - 92N + 12, \quad (76)$$

$$P_5 = 3N^4 + 6N^3 + 16N^2 + 13N - 3, \quad (77)$$

$$P_6 = 3N^4 + 18N^3 + 17N^2 - 46N - 28, \quad (78)$$

$$P_7 = 5N^4 + 9N^3 - 4N^2 - 4N + 6, \quad (79)$$

$$P_8 = 8N^4 + 16N^3 - 19N^2 - 27N + 48, \quad (80)$$

$$P_9 = 11N^4 + 22N^3 + 13N^2 + 2N - 12, \quad (81)$$

$$P_{10} = 11N^4 + 34N^3 + N^2 - 70N - 12, \quad (82)$$

$$P_{11} = 20N^4 + 40N^3 + 11N^2 - 9N + 54, \quad (83)$$

$$P_{12} = 22N^4 + 50N^3 + 5N^2 - 47N + 6, \quad (84)$$

$$P_{13} = 40N^4 + 80N^3 + 73N^2 + 33N + 54, \quad (85)$$

$$P_{14} = 62N^4 - 17N^3 - 76N^2 - 69N - 18, \quad (86)$$

$$P_{15} = 67N^4 + 134N^3 + 49N^2 + 54N - 360, \quad (87)$$

$$P_{16} = 67N^4 + 134N^3 + 49N^2 + 54N - 72, \quad (88)$$

$$P_{17} = 67N^4 + 134N^3 + 67N^2 + 144N + 72, \quad (89)$$

$$P_{18} = 67N^4 + 134N^3 + 109N^2 + 114N - 126, \quad (90)$$

$$P_{19} = 95N^4 + 148N^3 + 35N^2 - 38N - 12, \quad (91)$$

$$P_{20} = N^5 + N^4 - 4N^3 + 3N^2 - 7N - 2, \quad (92)$$

$$P_{21} = 2N^5 + 6N^4 - N^3 - 8N^2 + 15N + 10, \quad (93)$$

$$P_{22} = 2N^5 + 6N^4 + 5N^3 + 4N^2 + 9N - 2, \quad (94)$$

$$P_{23} = 3N^5 + 5N^4 - 33N^3 - 45N^2 + 6N - 16, \quad (95)$$

$$P_{24} = 3N^5 + 14N^4 + 21N^3 + 20N^2 + 10N + 4, \quad (96)$$

$$P_{25} = 8N^5 - 31N^4 + 205N^3 - 59N^2 - 447N - 108, \quad (97)$$

$$P_{26} = 14N^5 + 15N^4 - 19N^3 - 13N^2 - 25N - 20, \quad (98)$$

$$P_{27} = 21N^5 + 9N^4 + 13N^3 - 13N^2 - 22N - 12, \quad (99)$$

$$P_{28} = 26N^5 + 41N^4 - 21N^3 + 21N^2 + 9N - 12, \quad (100)$$

$$P_{29} = 36N^5 - 55N^4 - 243N^3 - 75N^2 - 163N - 108, \quad (101)$$

$$P_{30} = 58N^5 + 7N^4 + 59N^3 + 50N^2 + 3N - 9, \quad (102)$$

$$P_{31} = 67N^5 + 49N^4 - 52N^3 + 164N^2 - 90N - 72, \quad (103)$$

$$P_{32} = 67N^5 + 201N^4 + 67N^3 - 57N^2 + 109N - 189, \quad (104)$$

$$P_{33} = 76N^5 + 271N^4 + 254N^3 + 41N^2 + 72N + 36, \quad (105)$$

$$P_{34} = 85N^5 + 151N^4 - 40N^3 + 164N^2 - 306N - 72, \quad (106)$$

$$P_{35} = 171N^5 + 552N^4 + 343N^3 + 246N^2 + 1052N + 480, \quad (107)$$

$$P_{36} = 305N^5 + 989N^4 + 907N^3 - 5N^2 + 60N + 36, \quad (108)$$

$$P_{37} = 418N^5 + 1525N^4 + 1763N^3 + 650N^2 + 444N + 144, \quad (109)$$

$$P_{38} = 631N^5 + 2524N^4 + 3743N^3 + 3398N^2 + 2844N + 864, \quad (110)$$

$$P_{39} = 725N^5 + 2831N^4 + 3481N^3 + 1699N^2 + 1044N + 324, \quad (111)$$

$$\begin{aligned}
P_{40} &= N^6 + 3N^5 + 5N^4 + N^3 - 8N^2 + 2N + 4, & (112) \\
P_{41} &= 3N^6 + 9N^5 - 584N^4 - 1183N^3 - 275N^2 - 834N - 432, & (113) \\
P_{42} &= 5N^6 + 23N^5 + 11N^4 - 39N^3 - 20N^2 + 16N + 8, & (114) \\
P_{43} &= 9N^6 + 56N^5 + 87N^4 + 54N^3 - 52N^2 - 50N - 12, & (115) \\
P_{44} &= 14N^6 + 39N^5 + 14N^4 - 19N^3 + 6N^2 + 14N + 4, & (116) \\
P_{45} &= 20N^6 + 60N^5 + 11N^4 - 78N^3 - 13N^2 + 36N - 108, & (117) \\
P_{46} &= 48N^6 + 144N^5 + 469N^4 + 698N^3 + 7N^2 + 258N + 144, & (118) \\
P_{47} &= 76N^6 + 217N^5 + 25N^4 - 181N^3 + 106N^2 - 21N - 6, & (119) \\
P_{48} &= 87N^6 + 261N^5 + 249N^4 + 63N^3 - 76N^2 - 64N - 96, & (120) \\
P_{49} &= 94N^6 + 315N^5 + 145N^4 - 63N^3 + 148N^2 - 441N - 18, & (121) \\
P_{50} &= 95N^6 + 285N^5 + 92N^4 - 291N^3 - 97N^2 + 96N - 36, & (122) \\
P_{51} &= 160N^6 + 438N^5 + 364N^4 + 330N^3 + 529N^2 + 321N + 18, & (123) \\
P_{52} &= 325N^6 + 975N^5 + 85N^4 - 879N^3 + 598N^2 - 240N - 72, & (124) \\
P_{53} &= 17N^7 + 9N^6 - 95N^5 - 19N^4 + 76N^3 + 22N^2 + 26N + 28, & (125) \\
P_{54} &= 24N^7 + 33N^6 + 13N^5 - 28N^4 - 31N^3 - 33N^2 - 26N - 8, & (126) \\
P_{55} &= 165N^7 + 330N^6 - 491N^5 - 365N^4 - 136N^3 - 445N^2 - 18N + 144, & (127) \\
P_{56} &= 475N^7 + 833N^6 + 1527N^5 + 2905N^4 - 1342N^3 + 5562N^2 + 3834N + 486, & (128) \\
P_{57} &= 537N^7 + 1200N^6 - 1013N^5 - 2085N^4 + 1720N^3 - 855N^2 - 2468N - 492, & (129) \\
P_{58} &= 1199N^7 + 3523N^6 + 681N^5 - 5953N^4 - 4214N^3 - 4800N^2 - 3168N - 864, & (130) \\
P_{59} &= 33N^8 + 132N^7 + 70N^6 - 612N^5 - 839N^4 + 480N^3 + 712N^2 + 408N + 144, & (131) \\
P_{60} &= 476N^8 + 2297N^7 + 2018N^6 - 4915N^5 - 7324N^4 + 242N^3 - 1218N^2 \\
&\quad - 2700N - 864, & (132) \\
P_{61} &= 914N^8 + 3005N^7 + 3368N^6 + 4349N^5 + 5183N^4 + 548N^3 + 1101N^2 \\
&\quad + 936N + 324, & (133) \\
P_{62} &= 2078N^8 + 8225N^7 + 10475N^6 - 1921N^5 - 10729N^4 - 2560N^3 - 5658N^2 - 7578N \\
&\quad - 2700, & (134) \\
P_{63} &= -5N^9 - 25N^8 + 228N^7 + 926N^6 - 201N^5 - 2377N^4 + 626N^3 + 2788N^2 + 2168N \\
&\quad + 480, & (135) \\
P_{64} &= 99N^9 + 297N^8 - 982N^7 - 662N^6 + 1035N^5 - 3079N^4 + 3448N^3 + 2868N^2 \\
&\quad - 2448N - 1728, & (136) \\
P_{65} &= -1251N^{10} - 6255N^9 - 10972N^8 - 17422N^7 - 1423N^6 + 111905N^5 + 149894N^4 \\
&\quad - 2116N^3 - 37752N^2 + 12384N + 6912, & (137) \\
P_{66} &= -115N^{10} - 383N^9 + 356N^8 + 2762N^7 + 3001N^6 - 471N^5 - 882N^4 + 1068N^3 \\
&\quad + 1352N^2 - 288N - 256, & (138) \\
P_{67} &= 165N^{10} + 825N^9 + 1102N^8 - 578N^7 - 1939N^6 - 239N^5 + 1184N^4 + 448N^3 \\
&\quad - 2456N^2 - 2256N - 864, & (139) \\
P_{68} &= 418N^{10} + 2090N^9 + 3857N^8 + 5096N^7 + 6254N^6 - 808N^5 - 10295N^4 - 5622N^3 \\
&\quad + 2898N^2 + 2376N + 648, & (140) \\
P_{69} &= 735N^{10} + 3675N^9 + 6060N^8 + 6934N^7 + 11743N^6 - 41N^5 - 18290N^4 - 920N^3 \\
&\quad - 8168N^2 - 10656N - 3744, & (141)
\end{aligned}$$

$$P_{70} = 741N^{10} + 3705N^9 + 2650N^8 - 8780N^7 - 12083N^6 - 13127N^5 - 15536N^4 + 3586N^3 - 16128N^2 - 11916N - 3240, \quad (142)$$

$$P_{71} = 1065N^{10} + 6693N^9 + 14084N^8 + 10058N^7 - 3475N^6 - 11707N^5 + 446N^4 + 17132N^3 + 3432N^2 - 6624N - 3456, \quad (143)$$

$$P_{72} = 3321N^{10} + 13584N^9 + 9571N^8 - 17159N^7 - 7838N^6 + 5281N^5 - 20690N^4 - 842N^3 - 5208N^2 - 7884N - 3240, \quad (144)$$

$$P_{73} = 18579N^{10} + 68775N^9 + 3212N^8 - 235282N^7 - 220465N^6 + 54263N^5 + 91994N^4 - 48748N^3 - 24648N^2 + 29664N + 13824, \quad (145)$$

$$P_{74} = -2133N^{12} - 12798N^{11} - 54337N^{10} - 153794N^9 - 137083N^8 + 105398N^7 + 31109N^6 - 29734N^5 + 318684N^4 - 18512N^3 + 82224N^2 + 126720N + 48384, \quad (146)$$

$$P_{75} = N^{12} + 6N^{11} - 27N^{10} - 186N^9 - 197N^8 + 310N^7 + 899N^6 + 1198N^5 + 1020N^4 + 112N^3 - 192N^2 + 64N + 64, \quad (147)$$

$$P_{76} = 699N^{12} + 4194N^{11} + 16447N^{10} + 43214N^9 + 42657N^8 - 19098N^7 - 36963N^6 - 11670N^5 - 45064N^4 - 39392N^3 + 7536N^2 + 8064N + 1728, \quad (148)$$

$$P_{77} = 723N^{12} + 4338N^{11} + 12623N^{10} + 17230N^9 - 8583N^8 - 30018N^7 + 47709N^6 + 75738N^5 + 8776N^4 + 67208N^3 + 4416N^2 - 41184N - 20736. \quad (149)$$

We mention the moment–relation

$$\Delta\gamma_{gg}^{(2)}(N=1) = -2\beta_2, \quad (150)$$

with [42]

$$\beta_2 = \frac{2857}{54}C_A^3 + T_F N_F \left[-\frac{1415}{27}C_A^2 - \frac{205}{9}C_A C_F + 2C_F^2 \right] + (T_F N_F)^2 \left[\frac{158}{27}C_A + \frac{44}{9}C_F \right]. \quad (151)$$

Furthermore, one has

$$\Delta\gamma_{gg}^{(k)}(N=1) = 0, \quad \text{for } k = 0, 1, 2. \quad (152)$$

The expansions have been performed using the `HarmonicSums` command `HarmonicSumsSeries` since (70,72) contain evanescent poles at $N=1$. More generally, also the first moment of the gluonic Wilson coefficient, related to (152), for the structure function $g_1(x, Q^2)$ both for the massless [79, 80] and the massive case in the asymptotic representation to two–loop order [81] vanishes. This is known at one–loop order even for general kinematics [82].

The splitting functions in z –space are obtained by a Mellin inversion, cf. (47), and are given in computer readable form in an attachment to the paper.

6 Comparison to the literature

We confirm the results for the singlet anomalous dimensions calculated in [19], where the on–shell forward Compton amplitude has been used for the computation. The contributions $\propto T_F$ have already been calculated independently as a by–product of the massive on–shell operator matrix elements in Ref. [20], to which we also agree. The comparison to the large N_F expansions of Refs. [83, 84] has been given in our previous paper [20] already, where all these terms are covered.

Let us finally consider the small z limit⁴ of the present results and compare to the predictions given in [86, 87]. In Mellin- N space these terms are given by the most singular contribution expanding around $N = 0$. One obtains at one- and two-loop order

$$\Delta\gamma^{(0),N\rightarrow 0} = -\frac{4}{N} \begin{pmatrix} C_F & -2T_F N_F \\ 2C_F & 4C_A \end{pmatrix} \quad (153)$$

$$\Delta\gamma^{(1),N\rightarrow 0} = -\frac{8}{N^3} \begin{pmatrix} C_F(2C_A - 3C_F - 4T_F N_F) & 2C_F(2C_A + C_F) \\ -2C_F(2C_A + C_F) & 4(2C_A^2 - C_F T_F N_F) \end{pmatrix} \quad (154)$$

which agree with the evolution kernels given in Refs. [86, 87].

At three-loop order we have in the M-scheme

$$\Delta\gamma^{(2),N\rightarrow 0} = -\frac{16}{N^5} \times \begin{pmatrix} C_F(3C_A^2 - 12C_A C_F + 10C_F^2 + T_F N_F(12C_A + 16C_F)) & 2T_F N_F(15C_A^2 + 4C_A C_F - 8C_F T_F N_F) \\ -2C_F(15C_A^2 + 8C_A C_F - 4C_F^2 - 8C_F T_F N_F) & -4(14C_A^3 + T_F N_F(C_A^2 - 12C_A C_F - 2C_F^2)) \end{pmatrix} \quad (155)$$

and find a deviation both in case of $\Delta\gamma_{qq}^{(2)}$ and $\Delta\gamma_{gq}^{(2)}$, already noticed in [88]. The relative deviation amounts to $\sim \pm 3.2\%$, with the difference of the expression in the M-scheme and the result obtained for the infrared evolution equation (IEE), $\delta\Delta\gamma_{ij}^{(2),N\rightarrow 0} = \Delta\gamma_{ij}^{(2),N\rightarrow 0,M} - \Delta\gamma_{ij}^{(2),N\rightarrow 0,IEE}$,

$$\delta\Delta\gamma_{qq}^{(2),N\rightarrow 0} = \frac{48}{N^5} C_F T_F N_F (C_A - C_F), \quad (156)$$

$$\delta\Delta\gamma_{gq}^{(2),N\rightarrow 0} = \frac{64}{N^5} C_F^2 (C_A - C_F). \quad (157)$$

One may consider a different theory but QCD by setting $C_A = C_F$. In this case the prediction [86, 87] for the evolution kernels agrees with the perturbative calculation of the anomalous dimensions.

It has been the group of J. Kodaira [89], who also considered the effective Wilson coefficient in the non-singlet case Ref. [90]⁵, finding that these contributions are suppressed by a further power in N . This also applies to the effective Wilson coefficients in the non-singlet case and therefore the evolution kernels of [90, 91] agree⁶. By considering the expansion of the function to be interpreted as a matrix formulation for the effective Wilson coefficient in [86],

$$\frac{N}{\mathbf{1}N - \frac{1}{8\pi^2}\mathbf{F}_0} = \mathbf{1} + \sum_{k=1}^{\infty} \left(\frac{a_s}{N^2}\right)^k \mathbf{F}_{0,k}. \quad (158)$$

The expansion coefficients $\mathbf{F}_{0,k}$ are 2×2 matrices only depending on color factors. Therefore, in the representation of [86], the Wilson coefficients are suppressed by one power in N , like in the non-singlet case, cf. [89, 90]. To perform a full comparison one has to form two scheme invariant quantities. This is not really possible in pure polarized QCD, since there is only one structure function $g_1(x, Q^2)$ and also considering the Wilson coefficients [79, 80].⁷ In [19] scheme invariant

⁴For a review on the small z predictions of the different evolution kernels see [85].

⁵In using infrared evolution equations the authors of Refs. [86, 90, 91] do not specify the factorization scheme, which complicates comparisons with calculations performed e.g. in the M-scheme.

⁶After correcting Ref. [91] in [92].

⁷At the level of twist-2 the structure function $g_2(x, Q^2)$ is not an independent quantity, but related by the Wandzura-Wilczek relation [93] to the structure function $g_1(x, Q^2)$. One might consider the physical pair $\{g_1(x, Q^2), \partial g_1(x, Q^2)/\partial \ln(Q^2)\}$, cf. [94]. However, the so-called leading powers in N are here not of the same order.

polarized evolution kernels for the contributions to the structure function g_1 and additional fictitious gravitational contributions in the gluonic channels have been formed for which the prediction in [86, 87] hold.

In data analyses or the phenomenological description of the polarized structure functions the consideration of only the leading small z terms in the evolution kernels is numerically not sufficient. Subleading terms dominate over the leading order effects, cf. [85, 87, 92].

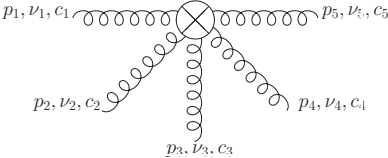
7 Conclusions

We have calculated the polarized three-loop singlet anomalous dimensions $\Delta\gamma_{qq}^{(2),\text{PS}}$, $\Delta\gamma_{qg}^{(2)}$, $\Delta\gamma_{gq}^{(2)}$, $\Delta\gamma_{gg}^{(2)}$, and the non-singlet anomalous dimension $\Delta\gamma_{qq}^{(2),\text{s,NS}}$ in Quantum Chromodynamics and agree with the results of Refs. [19, 20, 37]. The singlet anomalous dimensions have been calculated by using the method of massless off-shell OMEs, which has been applied for this purpose for the first time. The calculation has been fully automated referring to the Larin scheme, performing a finite transformation to the $\overline{\text{M}}$ -scheme for the final results. Both schemes are valid to describe the scaling violations of the polarized structure functions, however, with a different outcome for the polarized parton distribution functions, which are scheme-dependent quantities. Comparing to predictions of the leading small z behaviour one finds deviations for the off-diagonal elements at three-loop order, from which one concludes that the calculation in Ref. [86] is not in the $\overline{\text{M}}$ -scheme starting with three-loop order. To obtain the complete picture, one has to consider also the associated behaviour of the Wilson coefficients and to form scheme-invariant quantities. We also mention that partial checks on the polarized anomalous dimensions are possible from the pole terms of the single- and two-mass massive OMEs to three loop order, cf. [95]. Both the anomalous dimension and splitting functions are given in computer readable form in the file ANOM3pol.m attached to this paper.

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A A new gluonic Feynman rule

The gluonic OME ΔA_{gg} requires a new Feynman rule to calculate the anomalous dimensions in the massless case, containing a local operator with with five external gluon lines, extending the setting given in Refs. [20, 39]. The operator insertion reads in the polarized case



$$= -ig^3 \frac{1 - (-1)^N}{2} \left(\begin{aligned} & f^{c_1 c_2 c_x} f^{c_3 c_x c_y} f^{c_4 c_5 c_y} O_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(p_1, p_2, p_3, p_4, p_5) \\ & + f^{c_1 c_2 c_x} f^{c_4 c_x c_y} f^{c_3 c_5 c_y} O_{\nu_1 \nu_2 \nu_4 \nu_3 \nu_5}(p_1, p_2, p_4, p_3, p_5) + f^{c_1 c_2 c_x} f^{c_5 c_x c_y} f^{c_3 c_4 c_y} O_{\nu_1 \nu_2 \nu_5 \nu_3 \nu_4}(p_1, p_2, p_5, p_3, p_4) \\ & + f^{c_1 c_3 c_x} f^{c_2 c_x c_y} f^{c_4 c_5 c_y} O_{\nu_1 \nu_3 \nu_2 \nu_4 \nu_5}(p_1, p_3, p_2, p_4, p_5) + f^{c_1 c_3 c_x} f^{c_4 c_x c_y} f^{c_2 c_5 c_y} O_{\nu_1 \nu_3 \nu_4 \nu_2 \nu_5}(p_1, p_3, p_4, p_2, p_5) \\ & + f^{c_1 c_3 c_x} f^{c_5 c_x c_y} f^{c_2 c_4 c_y} O_{\nu_1 \nu_3 \nu_5 \nu_2 \nu_4}(p_1, p_3, p_5, p_2, p_4) + f^{c_1 c_4 c_x} f^{c_2 c_x c_y} f^{c_3 c_5 c_y} O_{\nu_1 \nu_4 \nu_2 \nu_3 \nu_5}(p_1, p_4, p_2, p_3, p_5) \\ & + f^{c_1 c_4 c_x} f^{c_3 c_x c_y} f^{c_2 c_5 c_y} O_{\nu_1 \nu_4 \nu_3 \nu_2 \nu_5}(p_1, p_4, p_3, p_2, p_5) + f^{c_1 c_4 c_x} f^{c_5 c_x c_y} f^{c_2 c_3 c_y} O_{\nu_1 \nu_4 \nu_5 \nu_2 \nu_3}(p_1, p_4, p_5, p_2, p_3) \end{aligned} \right)$$

$$\begin{aligned}
& + f^{c_1 c_5 c_x} f^{c_2 c_x c_y} f^{c_3 c_4 c_y} O_{\nu_1 \nu_5 \nu_2 \nu_3 \nu_4}(p_1, p_5, p_2, p_3, p_4) + f^{c_1 c_5 c_x} f^{c_3 c_x c_y} f^{c_2 c_4 c_y} O_{\nu_1 \nu_5 \nu_3 \nu_2 \nu_4}(p_1, p_5, p_3, p_2, p_4) \\
& + f^{c_1 c_5 c_x} f^{c_4 c_x c_y} f^{c_2 c_3 c_y} O_{\nu_1 \nu_5 \nu_4 \nu_2 \nu_3}(p_1, p_5, p_4, p_2, p_3) + f^{c_2 c_3 c_x} f^{c_1 c_x c_y} f^{c_4 c_5 c_y} O_{\nu_2 \nu_3 \nu_1 \nu_4 \nu_5}(p_2, p_3, p_1, p_4, p_5) \\
& + f^{c_2 c_4 c_x} f^{c_1 c_x c_y} f^{c_3 c_5 c_y} O_{\nu_2 \nu_4 \nu_1 \nu_3 \nu_5}(p_2, p_4, p_1, p_3, p_5) + f^{c_2 c_5 c_x} f^{c_1 c_x c_y} f^{c_3 c_4 c_y} O_{\nu_2 \nu_5 \nu_1 \nu_3 \nu_4}(p_2, p_5, p_1, p_3, p_4)
\end{aligned} \tag{159}$$

with

$$\begin{aligned}
O_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(p_1, p_2, p_3, p_4, p_5) = & \\
& \Delta_{\nu_3} \left[\Delta_{\nu_5} \varepsilon_{\Delta \nu_1 \nu_2 \nu_4} - \Delta_{\nu_4} \varepsilon_{\Delta \nu_1 \nu_2 \nu_5} \right] \sum_{m=2}^{N-1} (\Delta \cdot p_1 + \Delta \cdot p_2)^{m-2} (-\Delta \cdot p_4 - \Delta \cdot p_5)^{N-m-1} \\
& + \tilde{O}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(p_1, p_2, p_3, p_4, p_5) + \tilde{O}_{\nu_2 \nu_1 \nu_3 \nu_5 \nu_4}(p_2, p_1, p_3, p_5, p_4) + \tilde{O}_{\nu_4 \nu_5 \nu_3 \nu_2 \nu_1}(p_4, p_5, p_3, p_2, p_1) \\
& + \tilde{O}_{\nu_5 \nu_4 \nu_3 \nu_2 \nu_1}(p_5, p_4, p_3, p_2, p_1)
\end{aligned} \tag{160}$$

and

$$\begin{aligned}
\tilde{O}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(p_1, p_2, p_3, p_4, p_5) = & \\
& -\Delta_{\nu_2} \Delta_{\nu_3} \Delta_{\nu_4} \left[\Delta_{\nu_5} \varepsilon_{\Delta p_5 p_1 \nu_1} + p_5 \cdot \Delta \varepsilon_{\Delta \nu_5 \nu_1 p_1} \right] \sum_{m=2}^{N-3} \sum_{n=m+1}^{N-2} \sum_{o=n+1}^{N-1} (\Delta \cdot p_1)^{m-2} (\Delta \cdot p_1 + \Delta \cdot p_2)^{n-m-1} \\
& \times (-\Delta \cdot p_4 - \Delta \cdot p_5)^{o-n-1} (-\Delta \cdot p_5)^{N-o-1} + \Delta_{\nu_3} \Delta_{\nu_4} \left[\Delta_{\nu_2} \varepsilon_{\Delta p_5 \nu_1 \nu_5} - \Delta_{\nu_1} \varepsilon_{\Delta p_5 \nu_2 \nu_5} \right] \\
& \times \sum_{m=2}^{N-2} \sum_{n=m+1}^{N-1} (\Delta \cdot p_1 + \Delta \cdot p_2)^{m-2} (-\Delta \cdot p_4 - \Delta \cdot p_5)^{n-m-1} (-\Delta \cdot p_5)^{N-n-1}.
\end{aligned} \tag{161}$$

All momenta are inflowing and the symbols f^{abc} denote the structure constants of $SU(N_c)$. For the sums in the Feynman rule it is understood that the upper summation bound is larger or equal than the lower bound.

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