MACMAHON'S PARTITION ANALYSIS XIII: SCHMIDT TYPE PARTITIONS AND MODULAR FORMS

GEORGE E. ANDREWS AND PETER PAULE

Abstract. In 1999, Frank Schmidt noted that the number of partitions of integers with distinct parts in which the first, third, fifth, etc., summands add to n is equal to $p(n)$, the number of partitions of n. The object of this paper is to provide a context for this result which leads directly to many other theorems of this nature and which can be viewed as a continuation of our work on elongated partition diamonds. Again generating functions are infinite products built by the Dedekind eta function which, in turn, lead to interesting arithmetic theorems and conjectures for the related partition functions.

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1. Introduction

In 1999, Frank Schmidt [13] proposed the following problem in the American Mathematical Monthly. We state it as a Theorem.

Theorem 1. *Let $p(n)$ denote the number of partitions of the integer n, and let $f(n)$ denote the number of partitions $a_1 + a_2 + a_3 + \dots$ satisfying $a_1 > a_2 > a_3 > \dots$ and $n = a_1 + a_3 + a_5 + \dots$. For example, $p(5)$ counts the 7 partitions 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$, and $f(5)$ counts the 7 partitions 5, $5+1$, $5+2$, $5+3$, $5+4$, $4+3+1$, and $4+2+1$. Then*

$$p(n) = f(n), \quad n \geq 1. \tag{1.1}$$

Peter Mork's solution was published [10], and eight others were noted as solvers. Ali Uncu [15, Thm. 3.1] proved (1.1) in the context of weighted Rogers-Ramanujan partitions and Dyson crank.

The point of this paper is the observation that this theorem has a very natural setting in MacMahon's Partition Analysis. The advantage of this approach is that it leads to a variety of related theorems of which the following is the most immediate.

Theorem 2. *Let $s(n)$ denote the number of partitions $a_1 + a_2 + a_3 + \dots$ satisfying $a_1 \geq a_2 \geq a_3 \geq \dots$ and $n = a_1 + a_3 + a_5 + \dots$. Let $t(n)$ denote the number of two-color partitions of n. Then*

$$s(n) = t(n), \quad n \geq 1. \tag{1.2}$$

For example, $s(3)$ counts the ten partitions 3, $3+3$, $3+2$, $3+1$, $2+2+1$, $2+2+1+1$, $2+1+1$, $2+1+1+1$, $1+1+1+1+1$, and $1+1+1+1+1+1$. And $t(3)$ counts the ten red and green partitions 3_r, 3_g, 2_r+1_r, 2_g+1_r, 2_r+1_g, 2_g+1_g, $1_r+1_r+1_r$, $1_r+1_r+1_g$, $1_r+1_g+1_g$, $1_g+1_g+1_g$.

The study of MacMahon's Partition Analysis has been the topic of twelve papers we wrote jointly on this topic. Partition Analysis is ideal to study questions of this nature. One of our hopes in launching our study is that we might find new classes of partitions whose generating functions

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are modular forms. In the following sections, a number of Schmidt type partitions arising from partitions on various graphs will be seen to have modular forms as generating functions. For example, in [2], we considered "plane partition diamonds"; i.e., partitions whose parts, a_i, lie on the following graph, Figure 1, with each directed edge indicating $\geq$.

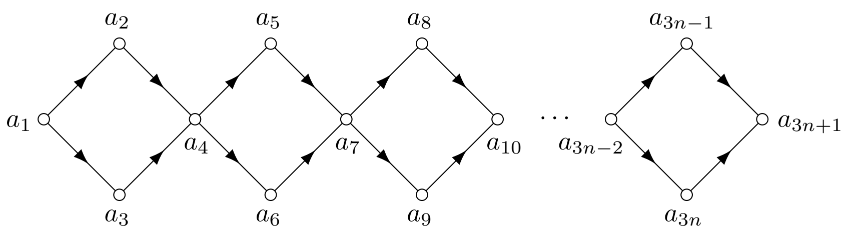

FIGURE 1. A plane partition diamond of length n.

Now, taking such plane partition diamonds of unrestricted length (i.e., $n \to \infty$), and if instead of adding up all the parts, we only add $a_1 + a_4 + a_7 + \cdots + a_{3k+1}$, we will find in Theorem 4 that the generating function is

$$\frac{(-q;q)_\infty}{(q;q)_\infty^3}, \tag{1.3}$$

where

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1}) \text{ and } (A;q)_\infty = \lim_{n\to\infty}(A;q)_n.$$

After giving a short review of Partition Analysis in Section 2, in Section 3 we prove Schmidt's original result, Theorem 1, and also Theorem 2. Moreover, we extend Theorem 2 to partitions with three colors; see Theorem 3

The remainder of our article is devoted to other appealing applications of Schmidt's original idea. In Section 4 we sum the links of partition diamonds and prove Theorem 4. In Section 5 we turn to the aspect of modular functions and related arithmetic properties of the coefficients of the constructed counting functions for Schmidt type partitions. Our primary tool in these investigations is Smoot's implementation [14], the Mathematica package RaduRK, of Radu's Ramanujan-Kolberg algorithm [12]. With this package we prove a variety of identities which, as immediate corollaries, imply divisibility properties. To supplement the "computer proof" of Theorem 5 given in Section 5, in Section 6 we present a classical proof which invokes two identities from the work of Nathan Fine [5]. In Section 7 we sum the links of k-elongated partition diamonds, objects introduced in [4]. In (7.3) we define the corresponding infinite family of generating functions, $(D_k(q))_{k\geq 1}$, counting Schmidt type partitions, and prove a variety of q-series relations and congruences for $k=2$ and $k=3$. In Subsection 7.1, some conjectures concerning congruences living on arithmetic subsequences of Schmidt type partition numbers for $k=2$ are stated.

2. PARTITION ANALYSIS: BASIC FACTS

We shall treat Theorem 1 in great detail to make clear how Partition Analysis is the ideal tool for managing partition questions of this nature. To this end, we prepare by providing some basic facts of MacMahon's method.

The MacMahon operator $\Omega_{\geq}$ is given by

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\dots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\dots,s_r}, \tag{2.1}$$

where the domain of the $A_{s_1,\dots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\dots,s_r}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of $A_{s_1,\dots,s_r}$.

The only application we need for Theorem 1 is

$$\underset{\geq}{\Omega} \frac{\lambda^{-s}}{(1-A\lambda)(1-B\lambda^{-r})} = \frac{A^s}{(1-A)(1-A^rB)}, \tag{2.2}$$

where $r, s \in \mathbb{Z}_{\geq 0}$. This result follows easily,

$$\begin{aligned}\underset{\geq}{\Omega} \frac{\lambda^{-s}}{(1-A\lambda)(1-B\lambda^{-r})} &= \underset{\geq}{\Omega} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A^n B^m \lambda^{n-mr-s} = \sum_{m=0}^{\infty}\sum_{n=mr+s}^{\infty} A^n B^m \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A^{n+mr+s} B^m = \frac{A^s}{(1-A)(1-A^rB)}.\end{aligned}$$

To connect to Partition Analysis, we first consider the generating function of Schmidt partitions of n having exactly one summand a_1. It is given by the coefficient of q^n in

$$\sum_{a_1>a_2\geq 0} x_1^{a_1} x_2^{a_2}\Big|_{x_1=q,x_2=1}.$$

The full generating function in x_1 and x_2 translates into

$$\begin{aligned}\sum_{a_1>a_2\geq 0} x_1^{a_1} x_2^{a_2} = \underset{\geq}{\Omega} \sum_{a_1,a_2\geq 0} x_1^{a_1} x_2^{a_2} \lambda^{a_1-a_2-1} &= \underset{\geq}{\Omega} \frac{\lambda^{-1}}{(1-x_1\lambda)(1-x_2\lambda^{-1})} \\ &= \frac{x_1}{(1-x_1)(1-x_1x_2)},\end{aligned}$$

where the last equality is by (2.2). Consequently, the counting function for the number of Schmidt partitions having exactly one summand is

$$\sum_{a_1>a_2\geq 0} x_1^{a_1} x_2^{a_2}\Big|_{x_1=q,x_2=1} = \frac{q}{(1-q)^2}. \tag{2.3}$$

To determine the closed form of

$$\sum_{a_1>a_2>a_3>a_4\geq 0} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}\Big|_{x_1=q,x_2=1,x_3=q,x_4=1}, \tag{2.4}$$

the counting function for the number of Schmidt partitions $a_1 + a_3$ formed by exactly two summands, we use the Omega package written in the Mathematica system. After placing the package in a directory where we open a Mathematica session, we read it in as follows[1]:

In[1]:= **<< RISC'Omega'**

> Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz

First we translate the x_i-series from (2.4) into what MacMahon called the crude generating function,

In[2]:= **crude = OSum[$x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}$, {$a_1 > a_2, a_2 > a_3, a_3 > a_4, a_4 \geq 0$}, λ]**

Out[2]= $$\underset{\lambda_1,\lambda_2,\lambda_3}{\underset{\geq}{\Omega}} \frac{\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1}}{(1-\lambda_1 x_1)\left(1-\frac{\lambda_2 x_2}{\lambda_1}\right)\left(1-\frac{\lambda_3 x_3}{\lambda_2}\right)\left(1-\frac{x_4}{\lambda_3}\right)}$$

Next, we ask the program to eliminate all the slack variables λ_i,

In[3]:= **OR[crude]**

Out[3]= $$\frac{x_1^3 x_2^2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}$$

The program, in this step, applies the elimination rule (2.2) to eliminate successively λ_1, λ_2, and λ_3. Consequently,

$$\sum_{a_1>a_2>a_3>a_4\geq 0} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}\Big|_{x_1=q,x_2=1,x_3=q,x_4=1} = \frac{q^4}{(1-q)^2(1-q^2)^2}. \tag{2.5}$$

[1]The package is freely available at `https://combinatorics.risc.jku.at/software` upon password request via email to the second named author.

Summarizing, the experiments with the Omega package suggest that

$$\frac{q^{k^2}}{(1-q)^2(1-q^2)^2\dots(1-q^k)^2}, \quad k\geq 1, \tag{2.6}$$

is the counting function for the number of Schmidt partitions $a_1+a_3+\cdots+a_{2k-1}$ formed by exactly k summands. Finally, recall [1, (1.2.3) and (2.2.9)],

$$\sum_{k=0}^{\infty}\frac{q^{k^2}}{(1-q)^2(1-q^2)^2\dots(1-q^k)^2}=\prod_{j=1}^{\infty}\frac{1}{1-q^j}=\sum_{n=0}^{\infty}p(n)q^n. \tag{2.7}$$

This would prove Theorem 1, once the form (2.6) for Schmidt partitions into k parts is proven. This is done in the next section, again by using Partition Analysis.

3. Proof of Theorem 1 and Partitions with two and three colors

For the proof of Theorem 1 and of similar results the following Lemma is crucial.

Lemma 3.1. *For the n-variable generating function for partitions with n parts $\geq s\in\mathbb{Z}_{\geq 0}$ and difference at least $r\in\mathbb{Z}_{\geq 0}$ between parts, and where x_i keeps track of the ith part of the partition,*
(3.1)

$$\sum_{\substack{j_1,\dots,j_n\geq s\\ j_1-j_2\geq r, j_2-j_3\geq r,\dots,j_{n-1}-j_n\geq r}} x_1^{j_1}x_2^{j_2}\dots x_n^{j_n}=\frac{x_1^r(x_1x_2)^r\dots(x_1x_2\dots x_{n-1})^r(x_1x_2\dots x_n)^s}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)\dots(1-x_1x_2\dots x_n)}.$$

Proof. We rewrite the left-hand side of (3.1) using MacMahon's operator $\Omega_{\geq}$, and then successively apply (2.2),

$$\begin{aligned}
&\underset{\geq}{\Omega}\sum_{j_1,\dots,j_n\geq 0} x_1^{j_1}x_2^{j_2}\dots x_n^{j_n}\lambda_1^{j_1-j_2-r}\lambda_2^{j_2-j_3-r}\dots\lambda_{n-1}^{j_{n-1}-j_n-r}\lambda_n^{j_n-s}\\
&=\underset{\geq}{\Omega}\frac{\lambda_1^{-r}\lambda_2^{-r}\dots\lambda_{n-1}^{-r}\lambda_n^{-s}}{(1-x_1\lambda_1)(1-x_2\lambda_2\lambda_1^{-1})\dots(1-x_{n-1}\lambda_{n-1}\lambda_{n-2}^{-1})(1-x_n\lambda_n\lambda_{n-1}^{-1})}\\
&=\frac{x_1^r}{1-x_1}\underset{\geq}{\Omega}\frac{\lambda_2^{-r}\dots\lambda_{n-1}^{-r}\lambda_n^{-s}}{(1-x_1x_2\lambda_2)(1-x_3\lambda_3\lambda_2^{-1})\dots(1-x_{n-1}\lambda_{n-1}\lambda_{n-2}^{-1})(1-x_n\lambda_n\lambda_{n-1}^{-1})}\\
&=\frac{x_1^r(x_1x_2)^r}{(1-x_1)(1-x_1x_2)}\underset{\geq}{\Omega}\frac{\lambda_3^{-r}\dots\lambda_{n-1}^{-r}\lambda_n^{-s}}{(1-x_1x_2x_3\lambda_3)(1-x_4\lambda_4\lambda_3^{-1})\dots(1-x_{n-1}\lambda_{n-1}\lambda_{n-2}^{-1})(1-x_n\lambda_n\lambda_{n-1}^{-1})}\\
&=\frac{x_1^r(x_1x_2)^r\dots(x_1x_2\dots x_{n-1})^r}{(1-x_1)(1-x_1x_2)\dots(1-x_1x_2\dots x_{n-1})}\underset{\geq}{\Omega}\frac{\lambda_n^{-s}}{1-x_1x_2\dots x_n\lambda_n}.
\end{aligned}$$

Finally, another application of (2.2), this time with $B=0$, proves (3.1). □

Corollary 1. *The counting function for Schmidt partitions $a_1+a_3+\cdots+a_{2k-1}$ formed by exactly k summands is of the form as in* (2.6).

Proof. The desired counting function is obtained by the following substitution in (3.1) with $r=1$, $s=0$ and $n=2k$,

$$\begin{aligned}
&\sum_{\substack{j_1,\dots,j_{2k}\geq 0\\ j_1-j_2>0, j_2-j_3>0,\dots,j_{2k-1}-j_{2k}>0}} x_1^{j_1}x_2^{j_2}\dots x_{2k}^{j_{2k}}\Big|_{\substack{x_1=q,x_3=q,\dots,x_{2k-1}=q,\\ x_2=1,x_4=1,\dots,x_{2k}=1}}\\
&=\frac{q^2q^4\dots q^{2k-2}\cdot q^k}{(1-q)^2(1-q^2)^2\dots(1-q^k)^2}=\frac{q^{k^2}}{(1-q)^2(1-q^2)^2\dots(1-q^k)^2},\quad k\geq 1.
\end{aligned}$$

□

Proof of Theorem 1. Theorem 1 is an immediate consequence of Corollary 1 together with (2.7). □

Proof of Theorem 2. Let $s(n,k)$ be the number of partitions $a_1+a_2+a_3+\dots$ satisfying $a_1 \geq a_2 \geq a_3 \geq \dots$ and $n = a_1 + a_3 + \cdots + a_{2k-1}$. Then by Lemma 3.1 with $r = s = 0$ and $n = 2k$,

$$\sum_{n=0}^{\infty} s(n,k)q^n = \sum_{\substack{j_1,\dots,j_{2k}\geq 0\\ j_1-j_2\geq 0, j_2-j_3\geq 0,\dots,j_{2k-1}-j_{2k}\geq 0}} x_1^{j_1}x_2^{j_2}\dots x_{2k}^{j_{2k}}\Big|_{\substack{x_1=q,x_3=q,\dots,x_{2k-1}=q,\\ x_2=1,x_4=1,\dots,x_{2k}=1}}$$
$$= \frac{1}{(1-q)^2(1-q^2)^2\dots(1-q^k)^2}.$$

Letting $k\to\infty$ we obtain the generating function for partitions with two colors. □

We conclude this Section with another application of Lemma 3.1.

Theorem 3. *Let $u(n,k)$ denote the number of partitions $a_1+a_2+\cdots+a_{3k}$ satisfying $a_1 > a_2 > \cdots > a_{3k} \geq 0$ and $n = a_1+a_4+a_7+\cdots+a_{3k-2}$. Let $v(n,k)$ denote the number of three-color partitions of n using exactly k parts of the first color with minimal difference 2 between parts, exactly k parts of the second color with minimal part-difference 1, and maximally k parts of the third color. Then*

$$u(n,k) = v(n,k), \quad n,k\geq 1. \tag{3.2}$$

Proof. By Lemma 3.1 with $r = 1$, $s = 0$ and $n = 3k$,

$$\sum_{n=0}^{\infty} u(n,k)q^n = \sum_{\substack{j_1,\dots,j_{3k}\geq 0\\ j_1-j_2>0, j_2-j_3>0,\dots,j_{3k-1}-j_{3k}>0}} x_1^{j_1}x_2^{j_2}\dots x_{3k}^{j_{3k}}\Big|_{\substack{x_1=q,x_4=q,\dots,x_{3k-2}=q,\\ x_2=1,x_5=1,\dots,x_{3k-1}=1,\\ x_3=1,x_6=1,\dots,x_{3k}=1}}$$
$$= \frac{q^3q^6\dots q^{3(k-1)}\cdot q^k\cdot q^k}{(1-q)^3(1-q^2)^3\dots(1-q^k)^3}$$
$$= \frac{q^{1+3+\cdots+2k-1}}{(1-q)(1-q^2)\dots(1-q^k)}\cdot\frac{q^{1+2+\cdots+k}}{(1-q)(1-q^2)\dots(1-q^k)}\cdot\frac{1}{(1-q)(1-q^2)\dots(1-q^k)}.$$

□

4. Summing Links of Partition Diamonds

In the Introduction we mentioned plane partition diamonds $a_1+a_2+\cdots+a_{3n+1}$ with $a_1\geq a_2\geq\cdots\geq a_{3n+1}\geq 0$ of length n; see Figure 1. Now, for such plane partition diamonds of unrestricted length (i.e., $n\to\infty$) our Schmidt type condition consists of adding up the parts on the nodes linking the diamonds,

$$a_1+a_4+a_7+\cdots+a_{3k+1}.$$

For example, there are 13 such plane partition diamonds that yield 2: four with source 1 as shown in Figure 2, and the nine with source 2, Figure 3.

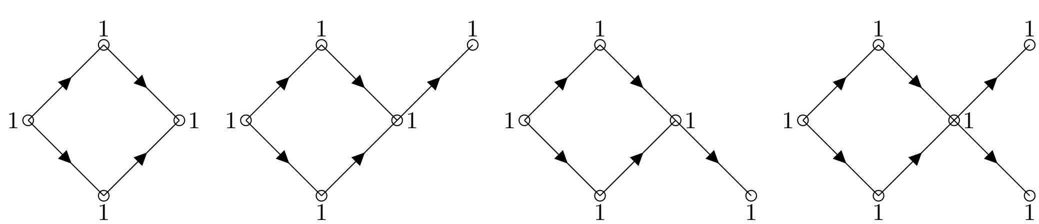

Figure 2. Four plane partition diamonds with source 1.

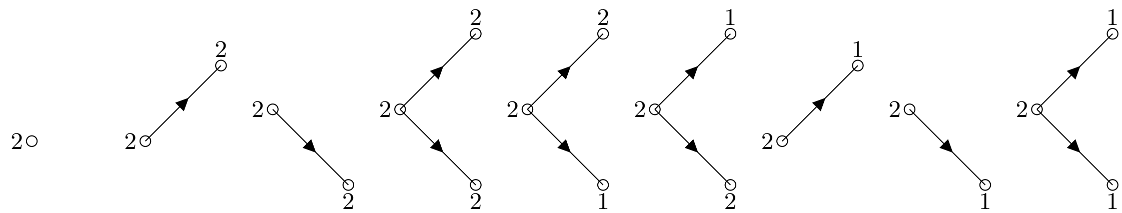

FIGURE 3. Nine plane partition diamonds with source 2.

Theorem 4. *The generating function for the Schmidt type partitions obtained by adding the summands $a_1 + a_4 + \cdots + a_{3k+1}$, $k \geq 0$, at the linking nodes in the plane partition diamonds of unrestricted length is given by*

$$D(q) := \sum_{m=0}^{\infty} d(m)q^m := \frac{(-q;q)_\infty}{(q;q)_\infty^3} = 1 + 4q + 13q^2 + 36q^3 + 90q^4 + 208q^5 + 455q^6 + O(q^7). \tag{4.1}$$

Remark. There is a variety of combinatorial interpretations of the infinite product $D(q)$. For example, it generates four-color partitions in which one of the colors has distinct parts.

Proof. In [2, Thm. 2.1] we proved that the generating function for the plane partition diamonds of length n, where x_i keeps track of the part on the ith node (as indicated in Fig. 1), is given by

$$\frac{1}{(1-X_1)(1-X_2)\ldots(1-X_{3n+1})} \cdot \frac{1-X_1X_3}{1-X_3/x_2} \cdot \frac{1-X_4X_6}{1-X_6/x_5} \cdots \frac{1-X_{3n-2}X_{3n}}{1-X_{3n}/x_{3n-1}}, \tag{4.2}$$

where $X_k = x_1x_2\ldots x_k$, $k \geq 1$.

To obtain those partitions where we consider

$$a_1 + a_4 + a_7 + \cdots + a_{3k+1},$$

we set $x_{3i+1} = q$, and $x_j = 1$ if $j \not\equiv 1 \pmod 3$. As a result, $X_{3i+1} = X_{3i+2} = X_{3i+3} = q^{i+1}$, $i \geq 0$, and (4.2) becomes

$$\Big(\prod_{i=1}^{n} \frac{1}{(1-q^i)^3}\Big) \frac{1}{1-q^{n+1}} \cdot \frac{1-q^2}{1-q} \cdot \frac{1-q^4}{1-q^2} \cdots \frac{1-q^{2n}}{1-q^n},$$

and letting $n \to \infty$, we obtain the product (4.1). □

Corollary 2. *Let $d(m)$ be the number of partitions considered in Theorem 4. Then for $m \geq 0$,*

$$4 \mid d(2m+1) \quad \textit{and} \quad d(2m) \equiv p(m) \pmod 4.$$

Proof.

$$\begin{aligned}
\sum_{m=0}^{\infty} d(m)q^m &= \prod_{i=1}^{\infty} \frac{1-q^{2i}}{(1-q^i)^4} = \prod_{i=1}^{\infty} \frac{1-q^{2i}}{1-4q^i+6q^{2i}-4q^{3i}+q^{4i}} \\
&\equiv \prod_{i=1}^{\infty} \frac{1-q^{2i}}{(1-q^{2i})^2} \pmod 4 \\
&= \prod_{i=1}^{\infty} \frac{1}{1-q^{2i}} = \sum_{m=1}^{\infty} p(m)q^{2m}.
\end{aligned}$$

□

In the next section, this corollary will be put into a broader context involving modular functions.

5. Witness Identities for Partition Diamond Congruences

As pointed out in the Introduction, one of the goals of our study of this type of partitions, enabled by the usage of Partition Analysis, is to give combinatorial constructions of modular forms which, in turn, will give rise to new arithmetical theorems. In this section will illustrate this aspect with the modular form $D(q)$ introduced in Theorem 4.

Our first result is the first statement of Corollary 2 in the form of a witness identity.

Theorem 5. *Let $d(m)$ be the number of partitions considered in Theorem 4. Then*

$$\sum_{m=0}^{\infty} d(2m+1)q^m = 4\frac{\left(q^2;q^2\right)_\infty^2 \left(q^4;q^4\right)_\infty^4}{(q;q)_\infty^9}. \tag{5.1}$$

Today, identities like (5.1) can be proven with computer algebra programs which implement Radu's Ramanujan-Kolberg algorithm [12]. We will apply the Mathematica package RaduRK by Nicolas Smoot [14] which is very convenient to use.[2] To prepare for its usage, follow the installation instructions given in [14], and invoke it within a Mathematica session as follows:

In[4]:= **<< RaduRK`**

> math4ti2: Mathematica interface to 4ti2 (`http://www.4ti2.de`)
> © 2017, Ralf Hemmecke `<ralf@hemmecke.org>`
> © 2017, Silviu Radu `<sradu@risc.jku.at>`

> RaduRK: Ramanujan–Kolberg Program Version 3.0 2021 written by Nicolas Smoot `<nicolas.smoot@risc.jku.at>` © Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz

Before running the program, one needs to set the two global key variables q and t:

In[5]:= **{SetVar1[q], SetVar2[t]}**

Out[5]= {q, t}

Proof. The algorithmic proof of (5.1) is done with the procedure call

In[6]:= **RK[4, 2, {−4, 1}, 2, 1]**

After a few seconds, Smoot's package delivers the proof in the form,

Out[6] =

N:	4
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{2,(-4,1)\}$
m:	2
$P_{m,r}(j)$:	$\{1\}$
$f_1(q)$:	$\frac{(q;q)_\infty^9}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^4}$
t:	$\frac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$
AB:	$\{1\}$
$\{p_g(t)$: $g \in$AB$\}$	$\{4\}$
Common Factor:	4

□

The interpretation of the output is as follows:

• The assignment $\{M,(r_\delta)_{\delta\mid M}\} = \{2,(-4,1)\}$ comes from the second and third entry of the procedure call `RK[4,2,{-4,1},2,1]`; this corresponds to specifying $M = 2$ and $(r_\delta)_{\delta\mid 2} = (r_1,r_2) = (-4,1)$ such that

$$\sum_{n=0}^{\infty} d(n)q^n = \prod_{\delta\mid M}(q^\delta;q^\delta)_\infty^{r_\delta} = \frac{(q^2,q^2)_\infty}{(q,q)_\infty^4}.$$

In the output expression $P_{m,r}(j)$ the abbreviation $r := (r_\delta)_{\delta\mid M}$ is used; i.e., here $r = (-4,1)$.

[2]The package is freely available at `https://combinatorics.risc.jku.at/software` upon password request via email to the second named author.

• The last two entries in the procedure call `RK[4,2,{-4,1},2,1]` correspond to the assignment $m = 2$ and $j = 1$, which means that we are interested in the generating function

$$\sum_{n=0}^{\infty} d(mn+j)q^n = \sum_{n=0}^{\infty} d(2n+1)q^n.$$

In the output expression $P_{m,r}(j)$ these parameters m and j are used; i.e., here $P_{m,r}(j) = P_{2,r}(1)$ with $r = (-4, 1)$.

• The first entry in the procedure call `RK[4,2,{-4,1},2,1]` corresponds to specifying $N = 4$, which fixes the space of modular functions the program will work with:

$$M(\Gamma_0(N)) := \text{ the algebra of modular functions for } \Gamma_0(N).$$

• The output $P_{m,r}(j) = P_{2,(-4,1)}(1) = \{1\}$ means that there exists an infinite product

$$f_1(q) = \frac{(q;q)_\infty^9}{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^4}$$

such that

$$f_1(q) \sum_{n=0}^{\infty} d(2n+1)q^n \in M(\Gamma_0(N)) \text{ with } N = 4.$$

• The output

$$t = \frac{1}{q} \frac{(q;q)_\infty^8}{(q^4;q^4)_\infty^8}, \quad \mathrm{AB} = \{1\}, \quad \text{and} \quad \{p_g(t) : g \in \mathrm{AB}\} = \{4\} \tag{5.2}$$

presents a solution to the following task: find a modular function $t \in M(\Gamma_0(N))$ and polynomials $p_g(t)$ such that

$$f_1(q) \sum_{n=0}^{\infty} d(2n+1)q^n = \sum_{g \in \mathrm{AB}} p_g(t) \cdot g. \tag{5.3}$$

In general, the elements of the finite set AB constitute a $\mathbb{C}[t]$-module basis of $M(\Gamma_0(N))$, resp. of a large subspace of $M(\Gamma_0(N))$. The elements g of AB are $\mathbb{C}$-linear combinations of modular functions in $M(\Gamma_0(N))$ which are representable in infinite product form such as $f_1(q)$ and t. In the specific case under consideration, the program delivers (5.2), which means,

$$f_1(q) \sum_{n=0}^{\infty} d(2n+1)q^n = 4 \cdot 1.$$

This is (5.1).

Remark. For the definition of notions such as $\Gamma_0(N)$ or $M(\Gamma_0(N))$, together with a general introduction to Radu's Ramanujan-Kolberg algorithm, see [11]. For the correctness proof and details of the algorithm, resp. of the implementation, see [12], resp. [14].

The next result is a witness identity which implies the second statement of Corollary 2.

Theorem 6. *Let $d(m)$ be the number of partitions considered in Theorem 4. Then*

$$\sum_{m=0}^{\infty} d(2m)q^m = 4\,q \frac{(q^2;q^2)_\infty^4 (q^8;q^8)_\infty^4}{(q;q)_\infty^9 (q^4;q^4)_\infty^2} + \frac{(q^4;q^4)_\infty^{10}}{(q;q)_\infty^9 (q^8;q^8)_\infty^4}. \tag{5.4}$$

To see that this indeed implies the second statement of Corollary 2, one uses $(1-x)^4 \equiv (1-x^2)^2 \pmod 4$ repeatedly to obtain,

$$\begin{aligned}\frac{(q^4;q^4)_\infty^{10}}{(q;q)_\infty^8(q^8;q^8)_\infty^4} &= \frac{(q^4;q^4)_\infty^{10}}{((q;q)_\infty^4)^2(q^8;q^8)_\infty^4} \equiv \frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4} \pmod 4\\ &\equiv \frac{(q^4;q^4)_\infty^{10}}{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4} \pmod 4\\ &= \frac{((q^4;q^4)_\infty^4)^2}{(q^8;q^8)_\infty^4} \equiv \frac{(q^8;q^8)_\infty^4}{(q^8;q^8)_\infty^4} \equiv 1 \pmod 4.\end{aligned}$$

Proof. For the algorithmic proof of (5.4) we again use Smoot's package,

In[7]:= **RK[8, 2,{−4, 1}, 2, 0]**

which, after a few seconds, computes the constituents of the witness identity (5.4) in the form,

Out[7] =

N:	8
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{2,(-4,1)\}$
m:	2
$P_{m,r}(j)$:	$\{0\}$
$f_1(q)$:	$\frac{(q;q)_\infty^9(q^4;q^4)_\infty^2}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4}$
t:	$\frac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}$
AB:	$\{1\}$
$\{p_g(t):\ g\in\text{AB}\}$	$\{4+t\}$
Common Factor:	None

□

Remark. Notice that this time $N = 8$; i.e., the witness identity of the form,

$$f_1(q)\sum_{n=0}^{\infty} d(2n)q^n = (4+t)\cdot 1,$$

is an identity involving modular functions in $M(\Gamma_0(8))$. Trying the package with $N = 4$ gives,

In[8]:= **RK[4, 2,{−4, 1}, 2, 0]**
Select another N.

The next result is a Ramanujan-Kolberg relation which, as a witness identity, is a q-series refinement of the fact that $4 \mid a(2m+1)$.

Theorem 7. *Let $d(m)$ be the number of partitions considered in Theorem 4. Then*

$$\begin{aligned}&\sum_{m=0}^{\infty} d(4m+1)q^m \sum_{m=0}^{\infty} d(4m+3)q^m\\ &= 16\,q^5\frac{(q^2;q^2)_\infty^{16}(q^8;q^8)_\infty^{22}}{(q;q)_\infty^{37}(q^4;q^4)_\infty^{7}}\cdot(4+t)(4+3t)(64+528t+108t^2+3t^3),\end{aligned} \tag{5.5}$$

where

$$t = \frac{(q^4;q^4)_\infty^{12}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}.$$

Corollary 3. *For the number $d(m)$ of partitions considered in Theorem 4,*

$$4 \mid d(4m+1) \quad \textit{and} \quad 4 \mid d(4m+3), \quad m \geq 0.$$

For the algorithmic proof of (5.5) we again use Smoot's package.

Proof. Choosing $m = 4$ and $j = 1$ as the last two entries in the procedure call,

In[9]:= **RK[8, 2,{−4, 1}, 4, 1]**

the program produces the Ramanujan-Kolberg type identity (5.5) as follows:

Out[9] =

N:	8
$\{M, (r_\delta)_{\delta\mid M}\}$:	$\{2, (-4, 1)\}$
m:	4
$P_{m,r}(j)$:	$\{1, 3\}$
$f_1(q)$:	$\frac{(q;q)_\infty^{37}(q^4;q^4)_\infty^{19}}{q^6(q^2;q^2)_\infty^{20}(q^8;q^8)_\infty^{30}}$
t:	$\frac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}$
AB:	$\{1\}$
$\{p_g(t)\colon\ g \in \mathrm{AB}\}$	$\{16t(4+t)(4+3t)(64+528t+108t^2+3t^3)\}$
Common Factor:	16

□

Remark. Again the relation involves modular functions in $M(\Gamma_0(N))$ with $N = 8$. But now, according to the output $P_{m,r}(j) = \{1, 3\}$, the witness identity involves a product of generating functions,

$$f_1(q) \prod_{k\in P_{m,r}(j)} d(4n+k)q^n = f_1(q) \sum_{n=0}^{\infty} d(4n+1)q^n \sum_{n=0}^{\infty} d(4n+3)q^n = p_1(t)\cdot 1,$$

with the polynomial $p_1(t)$ as given in the output `Out[9]`. Identities involving products in this form were first studied in systematic manner by Kolberg [8]. The entry "Common Factor" in the output refers to the common factor 16 of all the coefficients of $p_1(t)$.

Equipped with the RaduRK package, one can derive numerous identities of Ramanujan-Kolberg type. We restrict to state immediate corollaries of three identities which we derived analogously to the examples above, using the procedure call `RK[10, 2, {-4, 1}, 5, j]` and choosing $j \in \{1, 2, 3\}$. Notice that $N = 10$; i.e., the corresponding Ramanujan-Kolberg relations involve modular functions for $\Gamma_0(10)$.

Corollary 4. *Let $d(m)$ be the number of partitions considered in Theorem 4. Then*

$$\sum_{m=0}^{\infty} d(5m)q^m \sum_{m=0}^{\infty} d(5m+1)q^m \equiv q\frac{(q^2;q^2)_\infty^{14}(q^5;q^5)_\infty^{50}}{(q;q)^{50}(q^{10};q^{10})^{20}} \pmod 2, \tag{5.6}$$

$$\sum_{m=0}^{\infty} d(5m+3)q^m \equiv q\frac{(q^2;q^2)_\infty^{6}(q^5;q^5)_\infty^{20}}{(q;q)_\infty^{24}(q^{10};q^{10})_\infty^{5}} \pmod 4, \tag{5.7}$$

and

$$\sum_{m=0}^{\infty} d(5m+2)q^m \sum_{m=0}^{\infty} d(5m+4)q^m \equiv q^2\frac{(q^2;q^2)_\infty^{12}(q^5;q^5)_\infty^{40}}{(q;q)^{48}(q^{10};q^{10})^{10}} \pmod 2. \tag{5.8}$$

6. A Classical Proof of Theorem 5

Let $d(m)$ be the number of partitions considered in Theorem 4. In Section 5 we presented a "computer-proof" of the infinite product representation of $\sum_{m\geq 0} d(2m+1)q^m$ stated in Theorem 5. We find it instructive to show how this identity, (5.1), can be proven with classical means. To this end, recall

$$D(q) = \frac{(-q;q)_\infty}{(q;q)_\infty^3}, \quad \text{and define } \psi(q) := \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Proof of (5.1). We begin by "summing the odd part",

$$
\begin{aligned}
2\sum_{m\geq 0} d(2m+1)q^{2m+1} = D(q)-D(-q) &= \frac{(-q;q)_\infty}{(q;q)_\infty^3} - \frac{(q;-q)_\infty}{(-q;-q)_\infty^3}\\
&= \frac{(-q;q^2)_\infty(-q^2;q^2)_\infty}{(q;q^2)_\infty^3(q^2;q^2)_\infty^3} - \frac{(q;q^2)_\infty(-q^2;q^2)_\infty}{(-q;q^2)_\infty^3(q^2;q^2)_\infty^3}\\
&= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty^3}\Big(\frac{(-q;q^2)_\infty}{(q;q^2)_\infty^3} - \frac{(q;q^2)_\infty}{(-q;q^2)_\infty^3}\Big)\\
&= \frac{(-q^2;q^2)_\infty(-q;q^2)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty^7}\Big(\psi(q)^4-\psi(-q)^4\Big).
\end{aligned}
$$

To simplify further, we recall two identities arising as special cases of a more general framework studied by Nathan Fine. The first one is [5, p. 76, (31.51)],

$$\psi(q)^4 = \frac{1}{q}\sum_{N\geq 1}\sigma(2N-1)q^N, \tag{6.1}$$

where σ is the sum of divisors function.[3] The second identity is [5, p. 76, (31.54)],

$$\sum_{N\geq 1}\sigma(4N-1)q^N = 4q\,\frac{(q^4;q^4)_\infty^4}{(q;q^2)_\infty^2}. \tag{6.2}$$

Now, by "summing the even part",

$$
\begin{aligned}
\psi(q)^4-\psi(-q)^4 &= \frac{2}{q}\sum_{N\geq 1}\sigma(4N-1)q^{2N} \qquad \text{(by (6.1))}\\
&= 8q\,\frac{(q^8;q^8)_\infty^4}{(q^2;q^4)_\infty^2} \qquad \text{(by (6.2))}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
2\sum_{m\geq 0} d(2m+1)q^{2m+1} &= 8q\,\frac{(-q^2;q^2)_\infty(-q;q^2)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty^7}\,\frac{(q^8;q^8)_\infty^4}{(q^2;q^4)_\infty^2}\\
&= 8q\,(-q;q)_\infty(q;q^2)_\infty\,\frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^2}\\
&= 8q\,\frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^2}\\
&= 8q\,\frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^2}\,(-q^2;q^2)_\infty^2(q^2;q^4)_\infty^2\\
&= 8q\,\frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^9}\,(-q^2;q^2)_\infty^2(q^2;q^2)_\infty^2 = 8q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^9}.
\end{aligned}
$$

This completes the proof of (5.1). □

Remark. Mike Hirschhorn showed us an alternative classical proof. Namely, multiplying formula (19.2.5) in [7] with $\psi(q^2)^2$, and then using [7, (1.5.13)] gives (5.1) with q replaced by q^2.

7. Summing Links of k-elongated Partition Diamonds

In this section we will continue the theme of summing links of partition diamonds. To this end, we consider k-elongated plane partition diamonds introduced in [4]. The case $k = 1$ corresponds to using square-shaped building blocks of plane partition diamonds as depicted in Figure 1.

Instead of glueing squares together as in Figure 1, we take as building blocks k-elongated partition diamonds, which are configurations as shown in Fig. 4.

The result of gluing n such k-elongated diamonds together is shown in Figure 5.

[3] This identity was already known by Legendre [9, p. 134].

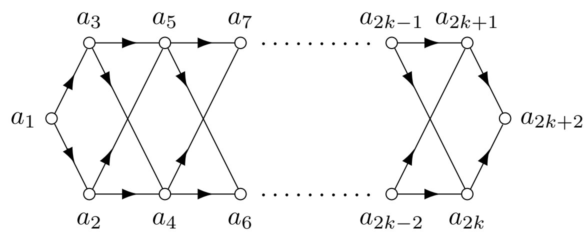

FIGURE 4. A k-elongated partition diamond of length 1.

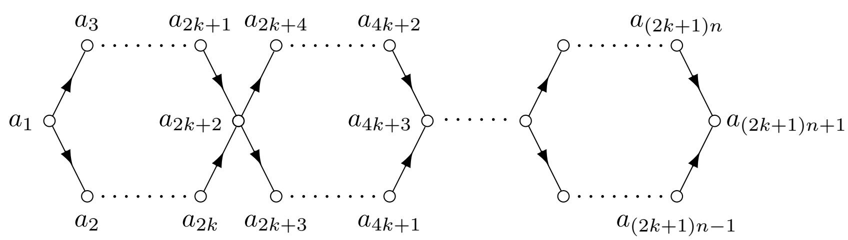

FIGURE 5. A k-elongated partition diamond of length n.

The corresponding generating function is defined as

$$h_{n,k}(x_1,\ldots,x_{(2k+1)n+1}) := \sum_{(a_1,\ldots,a_{(2k+1)n+1})\in H_{n,k}} x_1^{a_1}x_2^{a_2}\cdots x_{(2k+1)n+1}^{a_{(2k+1)n+1}},$$

where

$$H_{n,k} := \{(a_1,\ldots,a_{(2k+1)n+1}) \in \mathbb{Z}_{\geq 0}^{(2k+1)n+1} : \text{ the } a_i \text{ satisfy the order relations in Figure 5}\}.$$

As a consequence, the generating function of partition numbers produced by summing the links of k-elongated plane partition diamonds of length n (Figure 5) is obtained by the substitutions

$$x_1 = x_{2k+2} = x_{4k+3} = \cdots = x_{(2k+1)n+1} = q, \text{ and } x_j = 1 \text{ if } j \not\equiv 1 \pmod{2k+1}.$$

For $n,k \in \mathbb{Z}_{\geq 1}$ we define

$$D_{n,k}(q) := h_{n,k}(x_1,\ldots,x_{(2k+1)n+1})\Big|_{\substack{x_1=q,x_{2k+2}=q,\ldots,x_{(2k+1)n+1}=q,\\ x_j=1 \text{ if } j\not\equiv 1 \pmod{2k+1}}}. \tag{7.1}$$

By [4, Thm. 6] we know,

$$h_{n,k}(x_1,x_2,\ldots,x_{(2k+1)n+1}) = \prod_{j=1}^{(2k+1)n+1}\frac{1}{1-X_j}$$
$$\times \prod_{i=0}^{n-1}\prod_{\ell=1}^{k}\frac{1-X_{(2k+1)i+2\ell-1}X_{(2k+1)i+2\ell+1}}{1-\frac{X_{(2k+1)i+2\ell+1}}{x_{(2k+1)i+2\ell}}}, \tag{7.2}$$

where $X_0 := 1$ and $X_m := x_1x_2\cdots x_m$, $m \geq 1$.

The substitution as in (7.1) gives

$$X_{(2k+1)i+1} = X_{(2k+1)i+2} = \cdots = X_{(2k+1)i+2k+1} = q^{i+1}, \quad i = 0,\ldots,n-1.$$

Consequently, (7.2) implies

$$D_{n,k}(q) = \frac{1}{(q;q)_n^{2k+1}}\cdot\frac{1}{1-q^{n+1}}\cdot\frac{(q^2;q^2)_n^k}{(q;q)_n^k},$$

which in the limit $n\to\infty$ turns into

$$D_k(q) := \sum_{m=0}^{\infty} d_k(m)q^m := \lim_{n\to\infty} D_{n,k}(q) = \frac{(q^2;q^2)_\infty^k}{(q;q)_\infty^{3k+1}}. \tag{7.3}$$

Notice that for $k=1$ we have

$$D_1(q) = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^4} = D(q); \text{ i.e., } d_1(m) = d(m), \quad m \geq 0.$$

The remaining part of this section presents various observations on $D_k(q)$, respectively on the partition numbers $d_k(m)$, for $k=2$ and $k=3$.

7.1. **Arithmetic properties of $d_2(m)$.** This subsection is devoted to a study of arithmetic properties of the coefficients of the generating function for 2-elongated plane partitions,

$$D_2(q) = \sum_{m=0}^{\infty} d_2(m) q^m = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^7} \tag{7.4}$$
$$= 1 + 7q + 33q^2 + 126q^3 + 419q^4 + 1260q^5 + 3509q^6 + 9185q^7 + 22842q^8 + O(q^9).$$

In this case the building blocks, 2-elongated diamonds of length 1, are of hexagonal shape. Divisibility by powers of 3 seem to be among the most striking properties of the $d_2(m)$.

We begin with a witness identity.

Theorem 8. *Let $d_2(m)$ be the number of partitions as in* (7.4)*. Then*

$$f_1 \cdot \sum_{m=0}^{\infty} d_2(3m+2) q^m = 3(8+t)(1728 + 288t + 11t^2), \tag{7.5}$$

where

$$f_1 = \frac{1}{q^3} \frac{(q;q)_\infty^{19} (q^2;q^2)_\infty (q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^{21}} \quad \text{and} \quad t = \frac{1}{q} \frac{(q;q)_\infty^5 (q^3;q^3)_\infty}{(q^2;q^2)_\infty (q^6;q^6)_\infty^5}.$$

Corollary 5. *Let $d_2(m)$ be the number of partitions as in* (7.4)*. Then*

$$3 \mid d_2(3m+2), \quad m \geq 0.$$

The algorithmic proof of (7.5) is by Smoot's package:

Proof. Choosing $N=6$, and $m=3$ and $j=2$ as the last two entries in the procedure call `RK[6, 2, {-7, 2}, 3, 2]`, produces (7.5) as a relation between modular functions for $\Gamma_0(6)$. □

Another corollary of Theorem 8 concerns the general distribution of even and odd partitions numbers $d_2(3m+2)$.

Corollary 6. *Let $d_2(m)$ be the number of partitions as in* (7.4)*. Then*

$$\sum_{m=0}^{\infty} d_2(3m+2) q^m \equiv \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}} \pmod 2. \tag{7.6}$$

Proof. Notice that 33 is the only odd coefficient occuring in the polynomial on the right-hand side of (7.5), which implies,

$$\sum_{m=0}^{\infty} d_2(3m+2) q^m \equiv \frac{t^3}{f_1} \pmod 2$$
$$= \frac{(q^6;q^6)_\infty^6}{(q;q)_\infty^4 (q^2;q^2)_\infty^4 (q^3;q^3)_\infty^3} \equiv \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}} \pmod 2.$$

The last equivalence is by $(1-x)^2 \equiv 1 - x^2 \pmod 2$. □

In a similar fashion, relation (7.5) implies further identities of the form (7.6), for example, with respect to mod 4 and mod 8. This principle applies in general: identities as (7.5) often give rise to equivalences such as (7.6), provided the coefficients of the polynomial in t show sufficiently "nice" patterns when taken modulo suitable integers. To illustrate this aspect in the given context, we restrict to showing only a small sample of such results.

For instance, the procedure calls `RK[6, 2, {-7, 2}, 3, 0]` and `RK[6, 2, {-7, 2}, 3, 1]` deliver relations implying the following equivalences as immediate consequences.

Corollary 7. *Let $d_2(m)$ be the number of partitions as in* (7.4). *Then*

$$\sum_{m=0}^{\infty} d_2(3m)q^m \equiv \frac{(q;q)_\infty^2 (q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^5} \pmod{16}, \tag{7.7}$$

$$\sum_{m=0}^{\infty} d_2(3m+1)q^m \equiv 7 \cdot \frac{(q^6;q^6)_\infty^3}{(q;q)_\infty^5 (q^2;q^2)_\infty^3} \pmod{64}, \tag{7.8}$$

and

$$\sum_{m=0}^{\infty} d_2(3m+1)q^m \equiv \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5} \pmod{3}. \tag{7.9}$$

The procedure calls `RK[6, 2, {-7, 2}, 9, 5]` and `RK[6, 2, {-7, 2}, 9, 8]` deliver relations with polynomials in t of degree 14, which imply the following divisibilities.

Corollary 8. *Let $d_2(m)$ be the number of partitions as in* (7.4). *Then*

$$9 \mid d_2(9m+5), \quad m \geq 0, \tag{7.10}$$

and

$$27 \mid d_2(9m+8), \quad m \geq 0. \tag{7.11}$$

The procedure calls `RK[6, 2, {-7, 2}, 27, 8]`, `RK[6, 2, {-7, 2}, 27, 17]`, and `RK[6, 2, {-7, 2}, 27, 26]` deliver relations with polynomials in t of degree 48, which imply the following divisibilities.

Corollary 9. *Let $d_2(m)$ be the number of partitions as in* (7.4). *Then*

$$3^5 \mid d_2(3^3m+8), \quad m \geq 0, \tag{7.12}$$

$$3^3 \mid d_2(3^3m+17), \quad m \geq 0, \tag{7.13}$$

and

$$3^3 \mid d_2(3^3m+26), \quad m \geq 0. \tag{7.14}$$

Applying the `RaduRK` package, based on computer algebra, to sequences $d_2(3^k m + j)$ for $k \geq 4$ runs up against computational limits when using a standard laptop. Hence we conclude this subsection with several conjectures made on the basis of numerical computations.

Conjecture 1. Let $d_2(m)$ be the number of partitions as in (7.4). Then for $m \geq 0$,

$$3^5 \mid d_2(3^4m+j) \ \text{ for } \ j \in \{8, 35, 62, 71\}, \tag{7.15}$$

and

$$3^4 \mid d_2(3^4m+44). \tag{7.16}$$

Conjecture 2. Let $d_2(m)$ be the number of partitions as in (7.4). Then for $m \geq 0$,

$$3^5 \mid d_2(3^5m+j) \ \text{ for } \ j \in \{8, 35, 62, 89, 116, 143, 152, 170, 197, 224, 233\}, \tag{7.17}$$

and

$$3^6 \mid d_2(3^5m+71). \tag{7.18}$$

Owing to lack of numerical evidence the following conjecture concerning an infinite Ramanujan type family of divisibilities is more daring.

Conjecture 3. Let $d_2(m)$ be the number of partitions as in (7.4). Then for $m \geq 0$,

$$3^k \mid d_2(3^km+j_k) \ \text{ for all } \ k \geq 1, \tag{7.19}$$

where the integers j_k are chosen such that $8j_k \equiv 1 \pmod{3^k}$ and $1 < j_k < 3^k$. The first j_k are $2, 8, 17, 71, 152$, etc.

Note added in proof. Ralf Hemmecke, using his implementation of Radu's Ramanujan-Kolberg algorithm from the QEta package [6], succeeded to produce a computer proof of Conjecture 1 for the case $j = 71$. In addition, Nicolas Smoot succeeded to prove the instance $j = 152$ of Conjecture 2. Moreover, Smoot was able to find a refinement of Conjecture 3 together with a proof.

7.2. **Arithmetic properties of $d_3(m)$.** This subsection is devoted to a study of arithmetic properties of the coefficients of the generating function for 3-elongated plane partitions

(7.20)
$$D_3(q) = \sum_{m=0}^{\infty} d_3(m)q^m = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^{10}}$$
$$= 1 + 10q + 62q^2 + 300q^3 + 1235q^4 + 4522q^5 + 15130q^6 + 47084q^7 + 137990q^8 + O(q^9).$$

In this case the building blocks, 3-elongated diamonds of length 1, are of octagonal shape.

We present a couple of a witness identities which again imply various congruences.

Theorem 9. *Let $d_3(m)$ be the number of partitions as in* (7.20)*. Then*

$$f_1 \cdot \sum_{m=0}^{\infty} d_3(2m+1)q^m = 2(16 + 40t + 5t^2), \tag{7.21}$$

where

$$f_1 = \frac{1}{q^2}\frac{(q;q)_\infty^{22}(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^{10}} \quad \text{and} \quad t = \frac{1}{q}\frac{(q^4;q^4)_\infty^{12}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}.$$

Corollary 10. *Let $d_3(m)$ be the number of partitions as in* (7.20)*. Then*

$$2 \mid d_3(2m+1), \quad m \geq 0.$$

The algorithmic proof of (7.21) by Smoot's package is done as follows:

Proof. Choosing $N = 8$, and $m = 2$ and $j = 1$ as the last two entries in the procedure call `RK[8, 2, {-10, 3}, 2, 1]`, produces (7.21) as a relation between modular functions for $\Gamma_0(8)$. □

Theorem 10. *Let $d_3(m)$ be the number of partitions as in* (7.20)*. Then*

$$f_1 \cdot \sum_{m=0}^{\infty} d_3(2m)q^m = t(80 + 40t + t^2), \tag{7.22}$$

where

$$f_1 = \frac{1}{q^3}\frac{(q;q)_\infty^{22}(q^4;q^4)_\infty^{11}}{(q^2;q^2)_\infty^{12}(q^8;q^8)_\infty^{14}} \quad \text{and} \quad t = \frac{1}{q}\frac{(q^4;q^4)_\infty^{12}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}.$$

Proof. Choosing $N = 8$, and $m = 2$ and $j = 0$ as the last two entries in the procedure call `RK[8, 2, {-10, 3}, 2, 0]`, produces (7.22) as a relation between modular functions for $\Gamma_0(8)$. □

Corollary 11. *Let $d_3(m)$ be the number of partitions as in* (7.20)*. Then*

$$\sum_{m=0}^{\infty} d_3(2m)q^m \equiv \frac{1}{(q;q)_\infty^2} \pmod 2. \tag{7.23}$$

Proof. Notice that the only odd coefficient occuring in the polynomial on the right-hand side of (7.21) is 1, the coefficient of t^6. This implies,

$$\sum_{m=0}^{\infty} d_3(2m)q^m \equiv \frac{t^3}{f_1} \pmod 2$$
$$= \frac{(q^4;q^4)_\infty^{25}}{(q;q)_\infty^{22}(q^8;q^8)_\infty^{10}} \equiv \frac{1}{(q;q)_\infty^2} \pmod 2.$$

The last equivalence is by $(1-x)^2 \equiv 1 - x^2 \pmod 2$. □

Remark. It is interesting to note that the right-hand side of (7.23) leads us back to our first result on Schmidt type partitions, Theorem 2. Moreover, as already remarked, the structure of the polynomial $t(80+40t+t^2)$ gives rise to further equivalences, e.g., modulo 5 and 8.

Theorem 11. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$\sum_{m=0}^{\infty} d_3(3m)q^m \cdot \sum_{m=0}^{\infty} d_3(3m+1)q^m \cdot \sum_{m=0}^{\infty} d_3(3m+2)q^m \equiv q^7 \frac{(q^3;q^3)_\infty^4 (q^9;q^9)_\infty^{24}}{(q;q)^{40}} \pmod 2. \tag{7.24}$$

The proof is obtained from the Kolberg type relation involving modular functions for $\Gamma_0(18)$ which is computed with the procedure call `RK[18, 2, {-10, 3}, 3, 0]`. The respective polynomial in t is of degree 41. For the generating functions $\sum_{m=0}^{\infty} d_3(4m+j)q^m$ the respective relations are between modular functions for $\Gamma_0(8)$. The case $j=0$ gives no divisibility.

Theorem 12. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$\begin{aligned} f_1 \cdot \sum_{m=0}^{\infty} d_3(4m+1)q^m = 2t(&1228800 + 8597504t + 12672000t^2 + 6344448t^3 + 1263680t^4 \\ &+ 95120t^5 + 2036t^6 + 5t^7), \end{aligned} \tag{7.25}$$

where

$$f_1 = \frac{1}{q^8} \frac{(q;q)_\infty^{45}(q^4;q^4)_\infty^{19}}{(q^2;q^2)_\infty^{23}(q^8;q^8)_\infty^{34}} \quad \textit{and} \quad t = \frac{1}{q}\frac{(q^4;q^4)_\infty^{12}}{(q^2;q^2)_\infty^4 (q^8;q^8)_\infty^8}.$$

Proof. Choosing $N=8$, and $m=4$ and $j=1$ as the last two entries in the procedure call `RK[8, 2, {-10, 3}, 4, 1]`, produces (7.25) as a relation between modular functions for $\Gamma_0(8)$. □

In view of Corollary 10, the divisibility $2 \mid d_3(4m+1)$, implied by (7.25), is no news.

Theorem 13. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$\begin{aligned} f_1 \cdot \sum_{m=0}^{\infty} d_3(4m+2)q^m = 2(4+t)(&16384 + 1216512t + 5182464t^2 + 5201152t^3 + 1695936t^4 \\ &+ 188848t^5 + 6108t^6 + 31t^7), \end{aligned} \tag{7.26}$$

where

$$f_1 = \frac{1}{q^8} \frac{(q;q)_\infty^{43}(q^4;q^4)_\infty^{15}}{(q^2;q^2)_\infty^{17}(q^8;q^8)_\infty^{34}} \quad \textit{and} \quad t = \frac{1}{q}\frac{(q^4;q^4)_\infty^{12}}{(q^2;q^2)_\infty^4 (q^8;q^8)_\infty^8}.$$

Proof. Choosing $N=8$, and $m=4$ and $j=2$ as the last two entries in the procedure call `RK[8, 2, {-10, 3}, 4, 2]`, produces (7.26) as a relation between modular functions for $\Gamma_0(8)$. □

Corollary 12. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$2 \mid d_3(4m+2), \quad m \geq 0.$$

Theorem 14. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$\begin{aligned} f_1 \cdot \sum_{m=0}^{\infty} d_3(4m+3)q^m = 4t(&81920 + 2084864t + 6087680t^2 + 5054720t^3 + 1586112t^4 \\ &+ 198000t^5 + 8396t^6 + 75t^7), \end{aligned} \tag{7.27}$$

where

$$f_1 = \frac{1}{q^8} \frac{(q;q)_\infty^{45}(q^4;q^4)_\infty^{25}}{(q^2;q^2)_\infty^{25}(q^8;q^8)_\infty^{38}} \quad \textit{and} \quad t = \frac{1}{q}\frac{(q^4;q^4)_\infty^{12}}{(q^2;q^2)_\infty^4 (q^8;q^8)_\infty^8}.$$

Proof. Choosing $N=8$, and $m=4$ and $j=3$ as the last two entries in the procedure call `RK[8, 2, {-10, 3}, 4, 3]`, produces (7.26) as a relation between modular functions for $\Gamma_0(8)$. □

Corollary 13. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$4 \mid d_3(4m+3), \quad m \geq 0.$$

The degrees of the polynomials $p(t)$ grow fast when increasing k in the relations

$$f_1 \cdot \sum_{m=0}^{\infty} d_3(km+j)q^m = p(t).$$

Consequently, we restrict to presenting two further results related to $k = 5$.

Theorem 15. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$\begin{aligned} f_1 \cdot \sum_{m=0}^{\infty} d_3(5m+1)q^m = 5t(&335544320 + 4143972352t + 18433966080t^2 + 46252687360t^3 \\ &+ 74878812160t^4 + 86169354240t^5 + 74399891456t^6 + 50069135360t^7 \\ &+ 26613713920t^8 + 11145157120t^9 + 3615576320t^{10} + 864877856t^{11} \\ &+ 133950780t^{12} + 10750435t^{13} + 341960t^{14} + 2900t^{15} + 2t^{16}), \end{aligned} \tag{7.28}$$

where

$$f_1 = \frac{1}{q^{17}} \frac{(q;q)_\infty^{46}(q^5;q^5)_\infty^{20}}{(q^2;q^2)_\infty^{4}(q^{10};q^{10})_\infty^{55}} \quad and \quad t = \frac{1}{q}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5}.$$

Proof. Choosing $N = 10$, and $m = 5$ and $j = 1$ as the last two entries in the procedure call `RK[10, 2, {-10, 3}, 5, 1]`, produces (7.28) as a relation between modular functions for $\Gamma_0(10)$. □

Corollary 14. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$5 \mid d_3(5m+1), \quad m \geq 0.$$

Theorem 16. *Let $d_3(m)$ be the number of partitions as in (7.20). Then*

$$f_1 \cdot \sum_{m=0}^{\infty} d_3(5m+3)q^m \cdot \sum_{m=0}^{\infty} d_3(5m+4)q^m = p(t), \tag{7.29}$$

where

$$f_1 = \frac{1}{q^{33}} \frac{(q;q)_\infty^{92}(q^5;q^5)_\infty^{40}}{(q^2;q^2)_\infty^{8}(q^{10};q^{10})_\infty^{110}}, t = \frac{1}{q}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5},$$

and

$$\begin{aligned} p(t) = 25t^2(&117093590311632896 + 2842897264777625600t + 29684351043905781760t^2 \\ &+ 183865787974376488960t^3 + 773262866324631060480t^4 + 2383555914920808677376t^5 \\ &+ 5664084337113767608320t^6 + 10753875441422748876800t^7 + 16753765696748178636800t^8 \\ &+ 21860867201684629094400t^9 + 24274755761068247613440t^{10} + 23223590260152310169600t^{11} \\ &+ 19321344609893325209600t^{12} + 14073379704545057177600t^{13} + 9014270525233220812800t^{14} \\ &+ 5088865294954182737920t^{15} + 2532462032626332467200t^{16} + 1108705968828389785600t^{17} \\ &+ 425105816599919001600t^{18} + 141700146400976076800t^{19} + 40608572249636413440t^{20} \\ &+ 9846762031189683200t^{21} + 1974753256674540800t^{22} + 317218144362572800t^{23} \\ &+ 39075625930290400t^{24} + 3492651955227376t^{25} + 212202881089575t^{26} + 8134117807260t^{27} \\ &+ 179717975960t^{28} + 2032304980t^{29} + 9576646t^{30} + 14820t^{31}). \end{aligned}$$

Proof. Choosing $N = 10$, and $m = 5$ and $j = 3$ as the last two entries in the procedure call `RK[10, 2, {-10, 3}, 5, 3]`, produces (7.29) as a Kolberg type relation between modular functions for $\Gamma_0(10)$. □

Corollary 15. *Let $d_3(m)$ be the number of partitions as in (7.20). Then for all $m \geq 0$,*

$$5 \mid d_3(5m+3) \quad and \quad 5 \mid d_3(5m+4).$$

8. Conclusion

This paper hopefully will spur efforts to find further natural arithmetic/combinatorial objects generated by modular forms. Applications most often arise from the combinatorial side with subsequent important information being supplied by the fact that the generating functions are modular forms. The richness of results found from these few instances considered here suggests that much awaits.

Concerning the topical area of this paper, the Conjectures 1, 2, and 3 stated in Subsection 7.1 seem to be particularly challenging, especially the infinite family (7.19) of Ramanujan type congruences. A related open problem is the question about the possible existence of other such families in the context of Schmidt type partition numbers $d_k(n)$.

Many of the presented results were proven with the use of computer algebra. Nevertheless, in order to obtain more substantial mathematical insight, classical proofs, such as that one of Theorem 5, given in Section 6, would be desirable.

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[G.E.A.] Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

Email address: `gea1@psu.edu`

[P.P.] Research Institute for Symbolic Computation, Johannes Kepler University Linz, 4040 Linz, Austria

Email address: `Peter.Paule@risc.jku.at`