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RUTGERS EXPERIMENTAL MATHEMATICS SEMINAR

Difference Ring Algorithms for Symbolic Summation and Challenging Applications

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FWF Der Wissenschaftsfonds.

Outline

1. A warm-up example
2. The difference ring machinery for symbolic summation
3. Challenging applications

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\color{blue}{\sum_{j=0}^{\infty}}\color{black}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

↑ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \underbrace{+\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!}+\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!}+\frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= **SigmaLimit[res, {n}, a]**

Out[4]= $$\frac{1}{n!}\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n) f(n, k) + c_1(n)\, f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n,\ c_1(n) = (n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) \Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n) f(n,k) + \sum_{k=1}^{a} c_1(n) f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n) \sum_{k=1}^{a} f(n,k) + c_1(n) \sum_{k=1}^{a} f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n) f(n, k) + c_1(n)\, f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n)\, \mathsf{A}(n) + c_1(n)\, \mathsf{A}(n+1)}$$

$$||\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = {\color{blue}0} \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a] + S[1,n] - S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a] + S[1,n] - S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n] + 1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\left\{\left\{0, \frac{1}{n(n+1)}\right\}, \left\{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\right\}\right\}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\left\{\left\{0, \frac{1}{n(n+1)}\right\}, \left\{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\right\}\right\}$

Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) &= \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}\\ &= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}\end{aligned}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)}\Big)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Part 2: The difference ring machinery for symbolic summation

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

$$\begin{aligned}&\big(1+S_1(n)+nS_1(n)\big)^2\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2A(n)\\&-(1+n)(3+2n)S_1(n)\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2A(n+1)\\&\qquad+(1+n)^2(2+n)^3S_1(n)\big(1+S_1(n)+nS_1(n)\big)A(n+2)=0\end{aligned}$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\,S_1(n)\prod_{l=1}^{n}S_1(l)+c_2S_1(n)^2\prod_{l=1}^{n}S_1(l)\mid c_1,c_2\in\mathbb{K}\Big\}$$

$$-2(1+n)^3(3+n)n!^2A(n) + (1+n)\big(8+9n+2n^2\big)n!A(1+n) - A(2+n) = 0$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{ c_1 \prod_{i=1}^{n} i! + c_2\Big(-2^n n! \prod_{i=1}^{n} i! + \tfrac{3}{2} \prod_{i=1}^{n} i! \sum_{i=1}^{n} 2^i i!\Big) \mid c_1, c_2 \in \mathbb{K} \Big\}$$

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3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

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$$\|$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$\|$$

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$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

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$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

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In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}}[s]}_{\text{polynomial ring}}$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{aligned} \mathrm{ev}' : \quad & \mathbb{Q}(x) \times \mathbb{N} && \to \quad \mathbb{Q} \\ & \left(\tfrac{p(x)}{q(x)}, n\right) && \mapsto \quad \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\text{ev} : \quad \mathbb{Q}(x)[s] \times \mathbb{N} \quad \rightarrow \quad \mathbb{Q}$$

$$\mathbf{ev(s, n) = S_1(n)}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$$\begin{array}{llll} \mathrm{ev} : & \mathbb{Q}(x)[s] \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, n\right) & \mapsto & \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) S_1(n)^i \end{array} \qquad \mathbf{ev(s,n) = S_1(n)}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

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Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) = \langle 0,1,2,3,\dots\rangle\langle 0,1,\tfrac{1}{2},\tfrac{1}{3},\dots\rangle = \langle 0,1,1,1,\dots\rangle$$

$$\langle 0,1,1,1,\dots\rangle = \langle 1,1,1,1,\dots\rangle$$

$$\tau(x\,\tfrac{1}{x}) = \tau(1) = \langle 1,1,1,1,\dots\rangle$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

$$||$$

$$\langle 0, 1, 1, 1, \dots \rangle$$

$$||$$

$$\tau(x\,\tfrac{1}{x}) = \tau(1) \quad = \quad \langle 1, 1, 1, 1, \dots \rangle$$

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1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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$$\sigma : \quad \mathbb{Q}(x)[s] \to \mathbb{Q}(x)[s] \qquad s \mapsto s + \frac{1}{x+1}$$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Simplify

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$$s \mapsto s + \frac{1}{x+1}$$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\operatorname{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

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$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

In this example:

$$\mathrm{const}_{\sigma}\mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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τ is an injective difference ring homomorphism:

$$\begin{array}{ccc} \mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\ \downarrow \tau & = & \downarrow \tau \\ \mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S\quad} & \mathbb{K}^{\mathbb{N}}/\sim \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

built on Karr's DR theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
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$$\tau(\sigma(s)) = \langle 1, 1+\tfrac{1}{2}, 1+\tfrac{1}{2}+\tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1+\tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(n)\rangle_{n\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

$$\boxed{\begin{array}{ccccc} (\mathbb{A},\sigma) & \simeq & (\tau(\mathbb{A}),S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim,S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}] & & \end{array}}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\begin{array}{ccccc} (\mathbb{A}, \sigma) & \simeq & (\tau(\mathbb{A}), S) & \leq & (\mathbb{K}^{\mathbb{N}} / \sim, S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}] & & \end{array}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

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Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$\sum_{k=0}^{a} S_1(k) = ?$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0) = (a+1)S_1(a+1) - (a+1)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\, S_1(k) - k$

$$\Updownarrow$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\, s - x$

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$A(\lambda) = B(\lambda)$ for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$

(δ can be computed explicitly)

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$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

- and such that

the arising sums and products in $B(k)$ (except the alternating sign) are **algebraically independent**
(i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Application 2: Canonical representations

A_1 A_2 expressions in a term algebra

Application 2: Canonical representations

$$\mathrm{ev}(A_1, n) \quad \stackrel{?}{=} \quad \mathrm{ev}(A_2, n)$$

expressions in a term algebra

Application 2: Canonical representations

expressions in a term algebra

$$A_1 \qquad\qquad A_2$$

$\downarrow$ SigmaReduce $\downarrow$

$$B_1 \qquad\qquad B_2$$

$\forall n \geq 0 \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \qquad \mathsf{ev}(B_2, n) = \mathsf{ev}(A_2, n)$

Application 2: Canonical representations

$$
\begin{array}{ccc}
A_1 & & A_2 \\
\big\downarrow & \text{SigmaReduce} & \big\downarrow \\
B_1 & & B_2
\end{array}
$$

expressions in a term algebra

$$\forall n \geq 0 \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \qquad = \qquad \mathsf{ev}(B_2, n) = \mathsf{ev}(A_2, n)$$

$$\Updownarrow \qquad \text{canonical simplifier}$$

$$B_1 = B_2$$

Application 2: Canonical representations

$\mathrm{ev}(A_1,n) \quad \stackrel{?}{=} \quad \mathrm{ev}(A_2,n)$ expressions in a term algebra

$\downarrow$ SigmaReduce $\downarrow$

$B_1 \qquad\qquad B_2$

$\forall n\geq 0\;\; \mathrm{ev}(A_1,n)=\mathrm{ev}(B_1,n) \quad = \quad \mathrm{ev}(B_2,n)=\mathrm{ev}(A_2,n)$

$\Updownarrow$ canonical simplifier

$B_1=B_2$

Application 2: Canonical representations

$\mathsf{ev}(A_1, n) \quad \stackrel{?}{=} \quad \mathsf{ev}(A_2, n)$ expressions in a term algebra

$\downarrow$ SigmaReduce $\downarrow$

$B_1 \qquad\qquad B_2$

$$\forall n \geq 0 \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \quad = \quad \mathsf{ev}(B_2, n) = \mathsf{ev}(A_2, n)$$

$\Updownarrow$ canonical simplifier

$$B_1 = B_2$$

Application 3: We solve the zero-recognition problem.

$A_1(k)$ evaluates to 0 from a certain point on $\quad \Leftrightarrow \quad B_1 = 0$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

Part 3: Challenging applications

Part 3: Challenging applications in number theory

Example: a challenging email

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
-Doron
```

[arose in the bounds on the run time of the simplex algorithm on a polytope]

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks
suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \boxed{\sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}}$$

with

$$S_1(j) := \sum_{i=1}^{j} \frac{1}{i}$$

The inner sum

$$\boxed{k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}}$$

`Sigma.m`

Recurrence finder

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}$$

The inner sum

$$k^2\mathbf{A}(k)-(k+1)(2k+1)\mathbf{A}(k+1)+(k+1)(k+2)\mathbf{A}(k+2)=\frac{1}{k+1}$$

Sigma.m **Recurrence solver** Sigma.m

$$\mathbf{A}(k)=\sum_{j=1}^{\infty}\frac{S_1(j)}{j(j+k)} \in \left\{ c_1\frac{S_1(k)}{k}+c_2\frac{1}{k}+\frac{kS_1(k)^2-2S_1(k)+kS_2(k)}{2k^2} \,\middle|\, c_1,c_2\in\mathbb{R}\right\}$$

where

$$S_2(k)=\sum_{i=1}^{k}\frac{1}{i^2}$$

The inner sum

$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

Sigma.m

Recurrence solver

Sigma.m

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)} = \boxed{0\frac{S_1(k)}{k} + \zeta(2)\frac{1}{k} + \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k)}{2k^2}}$$

where

$$S_2(k) = \sum_{i=1}^{k} \frac{1}{i^2} \qquad \zeta(z) = \sum_{i=1}^{\infty} \frac{1}{i^z}$$

Simplify

$$\sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}.$$

$$||$$

$$\frac{kS_1(k)^2 - 2S_1(k) + kS_2(k) + 2k\,\zeta(2)}{2k^2}$$

Simplify

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}.$$

$$||$$

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \times \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k) + 2k\zeta(2)}{2k^2}$$

||telescoping + limit calculations

$$-4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5)$$

$$||$$

$$0.999222... \neq 1$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \, \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \, \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}\,.$$

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Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

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and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}}\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-m+j}{n}\frac{(t-j)_n}{n!}\,.$$

Then

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Theorem (CS, Sigma, Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m\Big(t+\frac{n}{2}\Big)\frac{(t-n)_m}{m!}\,\frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}}\,\frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}}\left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}}\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-m+j}{n}\frac{(t-j)_n}{n!}\,.$$

Then

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

with

$$\alpha_0(n,m)=(2n-m)^5,$$

$$\alpha_1(n,m)=-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3$$
$$+30n^2m-14nm^2+2m^3+8n^2-10nm+2m^2-4n+m),$$

$$\alpha_2(n,m)=-(2n-m-1)^3(4n-m)(m+2).$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

$$
\begin{aligned}
S(n,m) = \sum_{j=0}^{n}\sum_{\nu=1}^{\infty}\Bigg(&\frac{\binom{n}{j}^2\binom{j-m+2n}{n}(1+\nu)_{-m+2n}(1-j+\nu+n)_{-1+n}}{(1+\nu+n)_n(1+\nu+n)_{-m+2n}(\nu+n)^4(\nu-m+2n)^3}\\
&\times\Bigg((\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)\\
&\quad+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad-\nu(j-\nu-n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)\\
&\quad-S_1(\nu-m+3n)-S_1(-j+\nu+n)+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad+\nu(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)\\
&\quad-S_1(\nu-m+3n)-S_1(-j+\nu+n)+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad-(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)+2S_1(\nu+n)\\
&\quad-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad+\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{(j-\nu-2n)^2}-S_2(\nu)+2S_2(\nu+n)\\
&\quad-S_2(\nu+2n)-S_2(\nu-m+3n)-S_2(-j+\nu+n)\\
&\quad+S_2(\nu-m+2n)+S_2(-j+\nu+2n)\Big)
\end{aligned}
$$

$$+ 4(j+n)(\nu+n) - 3(\nu+n)^2 + n(-m+n) - j(m+2n)\Big)$$

$$- 2(\nu+n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} - S_1(\nu)$$

$$+ 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$+ S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big)$$

$$+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)$$

$$- 3(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} - S_1(\nu)$$

$$+ 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$+ S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big)$$

$$+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)$$

$$- (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n}$$

$$- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$+ S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big)$$

$$+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)$$

$$\times \big(- S_1(\nu+n) + S_1(\nu+2n)\big)$$

$$+ (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n}$$

$$- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$
\begin{aligned}
&+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&\times\big(-S_1(\nu)+S_1(\nu-m+2n)\big)\\
&-(\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}\\
&-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&\times\big(-S_1(\nu+n)+S_1(\nu-m+3n)\big)\\
&+(\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}\\
&-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3\\
&-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&\times\Big(-\frac{1}{-j+\nu+2n}-S_1(-j+\nu+n)+S_1(-j+\nu+2n)\Big)\Big)\Big)
\end{aligned}
$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \text{ Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+\,a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$

$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \text{ Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$
$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$\Big\downarrow \text{ Sigma.m with Holonomic-DR approach}$$

$$(2n-m)^5S(n,m)$$
$$-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3+30n^2m-14nm^2$$
$$+2m^3+8n^2-10nm+2m^2-4n+m)S(n,m+1)$$
$$-(2n-m-1)^3(4n-m)(m+2)S(n,m+2)=R(n,m)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

SigmaReduce

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$

$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

Finally, check 2 initial values: another round of non-trivial summation...

Part 3: Challenging applications in combinatorics

On January 22, 2020 I received the following email by Doron Zeilberger:

```
Dear Carsten,

I (and Shalosh) just posted a paper

https://arxiv.org/abs/2001.06839

with a challenge to you (see the middle of page 4)

Can you (and Sigma) extend theorem 5 of that paper
to the general case with k absent-minded passengers?
....
If you and Sigma can do the fourth moment, and derive
the asymptotic in n (with a fixed but arbitrary k), I will
donate $100$ to the OEIS in your honor.
...
Best wishes,
Doron
```

On January 22, 2020 I received the following email by Doron Zeilberger:

```
Dear Carsten,

I (and Shalosh) just posted a paper

https://ar

with

Can
to the
....
If you and Si                                        nd derive
the asymptotic in n (wi          but arbitrary k), I will
donate $100$ to the OEIS in your honor.
...
Best wishes,
Doron
```

This email provoked various heavy calculations by means of computer algebra that solved fully the above challenge (based on beautiful results of Doron).
In the following only the symbolic summation aspect is illustrated.

$n \geq 2$ passengers take step-wise their seats in a plane with n seats.

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$$\Big\downarrow \text{[Henze/Last:arXiv:1809.10192]}$$

The expected value for the passengers sitting in the wrong seat is

$$E(X_n) = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n}$$

and the variance is

$$V(X_n) = \frac{k(n-1)}{n^2} + \sum_{i=1}^{-k+n} \frac{(1-i-k+n)\left(1-\frac{1-i-k+n}{1-i+n}\right)}{1-i+n}$$
$$+ 2\left(\frac{(k-1)k}{2(n-1)n^2} + \sum_{i=1}^{k}\sum_{j=1}^{-k+n} \frac{\frac{1-j-k+n}{-j+n} - \frac{1-j-k+n}{1-j+n}}{n}\right)$$

In[6]:= $\mathbf{E = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n};}$

In[7]:= $\mathbf{EvaluateMultiSum[V, \{\}, \{k, n\}, \{1, 2\}, \{n, Infinity\}]}$

In[6]:= $$\mathbf{E = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n};}$$

In[7]:= **EvaluateMultiSum[V, {}, {k, n}, {1, 2}, {n, Infinity}]**

Out[7]= $$\frac{-k S[1,k] + k S[1,n] + k(n-1)}{n}$$

In[6]:= $$\mathbf{E}=\frac{k(n-1)}{n}+\sum_{i=1}^{-k+n}\frac{k}{1-i+n};$$

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Out[7]= $$\frac{-kS[1,k]+kS[1,n]+k(n-1)}{n}$$

In[8]:= $$\mathbf{V}=\frac{k(n-1)}{n^2}+\sum_{i=1}^{-k+n}\frac{(1-i-k+n)\left(1-\frac{1-i-k+n}{1-i+n}\right)}{1-i+n}+2\left(\frac{(k-1)k}{2(n-1)n^2}+\sum_{i=1}^{k}\sum_{j=1}^{-k+n}\frac{\frac{1-j-k+n}{-j+n}-\frac{1-j-k+n}{1-j+n}}{n}\right);$$

In[9]:= $\mathbf{EvaluateMultiSum[V, \{\}, \{k, n\}, \{1, 2\}, \{n, Infinity\}]}$

In[6]:= $\mathbf{E} = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n};$

In[7]:= EvaluateMultiSum[V, {}, {k, n}, {1, 2}, {n, Infinity}]

Out[7]= $\frac{-kS[1,k] + kS[1,n] + k(n-1)}{n}$

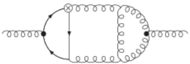
In[8]:= $V = \frac{k(n-1)}{n^2} + \sum_{i=1}^{-k+n} \frac{(1-i-k+n)\left(1 - \frac{1-i-k+n}{1-i+n}\right)}{1-i+n} + 2\left(\frac{(k-1)k}{2(n-1)n^2} + \sum_{i=1}^{k}\sum_{j=1}^{-k+n} \frac{\frac{1-j-k+n}{-j+n} - \frac{1-j-k+n}{1-j+n}}{n}\right);$

In[9]:= EvaluateMultiSum[V, {}, {k, n}, {1, 2}, {n, Infinity}]

Out[9]= $-\frac{k(2+n)S[1,k]}{n} + \frac{k(2+n)S[1,n]}{n} + k^2S[2,k] - k^2S[2,n] + \frac{2k - k^2 - 2n - 2kn + 2k^2n + 2n^2 - kn^2}{(n-1)n^2}$

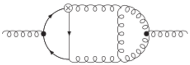
Part 3: Challenging applications in particle physics

Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals



behavior of particles

$$\longrightarrow \quad \int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

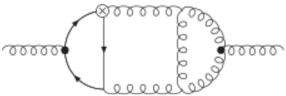
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \; dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \quad \Big|\Big|$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

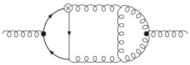
$$\Bigg[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Bigg]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times\left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{N-3-j}dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

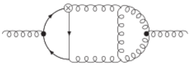
Evaluation of Feynman Integrals



behavior of particles ⟶ $\int \Phi(N, \epsilon, x)\, dx$

Feynman integrals

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

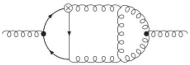
Feynman integrals

DESY

$$\sum f(N,\epsilon,k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



behavior of particles → $\int \Phi(N,\epsilon,x)\,dx$ Feynman integrals

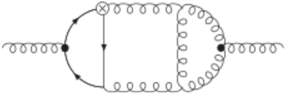
↓ **DESY**

$\sum f(N,\epsilon,k)$ complicated multi-sums

advanced difference ring theory (Sigma-package) →

expression in special functions

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad \Big\|$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

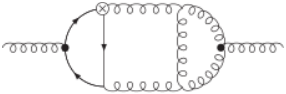
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

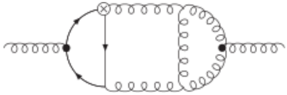
$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times [x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-\left(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s)\right)$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\big)S_1(N)^2$$
$$+ \big(- \frac{4(13N+5)}{N^2(N+1)^2} + \big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\big)S_2(N) + \big(\frac{29}{3} - (-1)^N\big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big)S_1(N) + \big(\frac{3}{4} + (-1)^N\big)S_2(N)^2$$
$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\big)$$
$$+ \big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\big)S_2(N) + S_{-2}(N)\big(10S_1(N)^2 + \big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(- 22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\big)$$
$$+ \big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\big)S_3(N) + \big(\frac{19}{2} - 2(-1)^N\big)S_4(N) + \big(- 6 + 5(-1)^N\big)S_{-4}(N)$$
$$+ \big(- \frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(- 17 + 13(-1)^N\big)S_{3,1}(N)$$
$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$+\Big(- \ldots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

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$$+ \Big(\ldots \frac{\ldots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big) S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big) S_3(N)$$

$$+ \big(2 + \ldots 28 S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big) \ldots {}^N\big) S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(\ldots \Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big) S_2(N) + S_{-2}(N)\big(10 S_1(N)^2 + \Big(\frac{\ldots(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big) S_2(N) - \frac{16}{N(N+1)}\big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big) S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big) S_4(N) + \big(-6 + 5(-1)^N\big) S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big) S_{2,1}(N) + \big(20 + 2(-1)^N\big) S_{2,-2}(N) + \big(-17 + 13(-1)^N\big) S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big) S_{-3,1}(N) + \big(3 - 5(-1)^N\big) S_{2,1,1}(N)$$

$$+ 32 S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3 S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(\dots \frac{\dots 1)^N(2N+1)}{\dots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \dots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big) \dots N)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(- \dots\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)} \dots$$

$$+ \frac{4(3N - \dots}{N(N+ \dots} \dots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N \dots}{N(\dots} \dots N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(- \dots}{\dots} \dots\Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(\dots}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

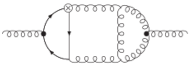
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

DESY

$$\sum f(N,\epsilon,k)$$

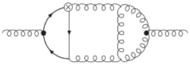
complicated
multi-sums

advanced difference ring theory

(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

DESY

LHC at CERN

applicable

$$\sum f(N,\epsilon,k)$$

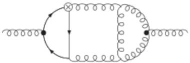
complicated multi-sums

advanced difference ring theory

(Sigma-package)

expression in special functions

Evaluation of Feynman Integrals



behavior of particles

→ $\int \Phi(N,\epsilon,x)\,dx$

Feynman integrals

- What did the universe look like in the first second
- Do the 4 fundamental forces unite at high energies?
- Do the properties of the new particle agree with the predicted Higgs-Boson?

↓ **DESY**

$\sum f(N,\epsilon,k)$

complicated multi-sums

← **advanced difference ring theory**

(Sigma-package)

expression in special functions

applicable