Contiguous Relations and Creative Telescoping

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Abstract This article presents an algorithmic theory of contiguous relations. Contiguous relations, first studied by Gauß, are a fundamental concept within the theory of hypergeometric series. In contrast to Takayama's approach, which for elimination uses non-commutative Gröbner bases, our framework is based on parameterized telescoping and can be viewed as an extension of Zeilberger's creative telescoping paradigm based on Gosper's algorithm. The wide range of applications include elementary algorithmic explanations of the existence of classical formulas for non-terminating hypergeometric series such as Gauß, Pfaff-Saalschütz, or Dixon summation. The method can be used to derive new theorems, like a non-terminating extension of a classical recurrence established by Wilson between terminating ${}_4F_3$-series. Moreover, our setting helps to explain the non-minimal order phenomenon of Zeilberger's algorithm.

1 Preamble

A first version of this article [22] has been produced about twenty years ago. At the occasion of the Wolfgang Pauli Centre Workshop "Antidifferentiation and the Calculation of Feynman Amplitudes" at DESY (Deutsches Elektronen-Synchrotron, Zeuthen, October 4–9, 2020) the organizers Johannes Blümlein and Carsten Schneider asked (and encouraged!) me to speak about this work. Without their initiative this updated version of [22] would have never appeared.

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2 Introduction

Contiguous relations are a fundamental concept within the theory of hypergeometric series and orthogonal polynomials; see, for instance, [1]. An example in connection with the thematical scope of this volume is [13], which is contained in this book and whose subsection 2.2 is devoted to contiguous relations for multivariate hypergeometric functions.

As often, the story begins with Gauß [10]. Let

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} z^k, $$

where $(x)_k$ is the shifted factorial

$$ (x)_k = x(x+1)\cdots(x+k-1) \text{ if } k \geq 1 \text{ and } (x)_0 = 1. $$

Gauß defined two such ${}_2F_1$ series as *contiguous*, if two of the parameters are pairwise equal, and if the third pair differs by 1. In particular, Gauß showed that a ${}_2F_1$ series and any two others contiguous to it are linearly related. For instance,

$$ \begin{aligned} &(a-c)\,{}_2F_1\left(\begin{matrix} a-1, b \\ c \end{matrix}; z\right) + (c - 2a - (b-a)z)\,{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) \\ &+ a(1-z)\,{}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right) = 0, \end{aligned} \tag{1} $$

is the first entry [10, 7.1] in Gauß' list of fifteen ($= 6 \cdot 5/2$) fundamental contiguous relations. Moreover, in Section 11 of [10] Gauß describes how to obtain relations between

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right), {}_2F_1\left(\begin{matrix} a+\lambda, b+\mu \\ c+\nu \end{matrix}; z\right) \text{ and } {}_2F_1\left(\begin{matrix} a+\lambda', b+\mu' \\ c+\nu' \end{matrix}; z\right), $$

where the $\lambda, \lambda', \mu, \mu', \nu, \nu'$ are integers taken from $\{-1, 0, 1\}$. This gives in total 325 ($= 26 \cdot 25/2$) relations.[†]

Today Gauß' notion of contiguous relations is used in a more general sense. Namely, two ${}_pF_q$ series, i.e.,

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, $$

[†] The number of these relations can be reduced further by taking symmetries (e.g., swapping a and b) into account.

are said to be *contiguous* if their parameters differ by integers. A first systematic textbook treatment of contiguous relations was presented by E. Rainville [25]. For an excellent up-to-date account the reader is referred to the book by G.E. Andrews, R. Askey, and R. Roy [1].

In particular, this book has played an influential role for the work presented in this article. Section 3.12 of [1] is devoted to summation, in particular, to a comparison of the classic method of contiguous relations to the W–Z method, more precisely, to Zeilberger's algorithm [40] also called *creative telescoping*. It is pointed out that "the W–Z method is an effective algorithm for discovering useful instances of contiguous relations." Moreover, it is explained that a specific use of contiguous relations, called *Pfaff's method*, can serve as a valuable alternative. Namely, "There is a somewhat different summation method due to Pfaff. This method is less algorithmic than the W–Z method. However, it spreads out the algebraic complications to systems of recurrences. Consequently, it may provide new summations in addition to the one we wish to prove and it may allow the required algebra to be considerably simpler than that required by the W–Z method. Pfaff's method rather resembles the W–Z method; however, it allows the various additional parameters in the summation to play an important role" [1, p. 171].

In order to illustrate this point, various examples are given. We consider one of these, namely Bailey's summation of a balanced ${}_4F_3$ series,

$$S(n) = S(n,a,b) = {}_4F_3\left(\begin{matrix} a/2, (a+1)/2, b+n, -n \\ b/2, (b+1)/2, a+1 \end{matrix}; 1\right) = \frac{(b-a)_n}{(b)_n}; \quad n \geq 0; \tag{2}$$

see [1, (3.11.7)]. By $F(n,k)$ we denote the kth summand of this ${}_4F_3$ series.

As a note, when dealing with such hypergeometric series we always assume that the summand terms are well-defined; this convention is made explicit also in (16). In view of the fact that the given sum is terminating at n, in this particular case, $a+1+\ell \neq 0$, $b/2+\ell \neq 0$, and $(b+1)/2+\ell \neq 0$ for $0 \leq \ell \leq n$.

Following the presentation in [1], we first prove (2) by applying Zeilberger's algorithm[‡] which computes the telescoping summand recurrence

$$\begin{aligned}(n+1)(-n-b+a)\,F(n,k) + (-a^2+ab-a+2nb+3b+2+4n+2n^2)\,F(n+1,k)\\ -(b+n+1)(a+n+2)\,F(n+2,k) = \Delta_k\,G(n,k-1),\end{aligned} \tag{3}$$

where the (forward) difference operator is defined as usual as

$$\Delta_k\, f(k) = f(k+1) - f(k), \tag{4}$$

and where

[‡] We are using the Mathematica package "fastZeil" presented in [19]; it is freely available as described at `https://combinatorics.risc.jku.at/software`

$$G(n,k) = -\frac{(a+1+2k)(a+2k)(b+n+k)(n+1)}{(b+n)(n-k+1)} F(n,k). \tag{5}$$

Summing (3) over k from 0 to n implies a recurrence for the sum $S(n)$,

$$\begin{aligned}&(n+1)(-n-b+a)\,S(n) + (-a^2+ab-a+2nb+3b+2+4n+2n^2)\,S(n+1)\\ &-(b+n+1)(a+n+2)\,S(n+2) = 0.\end{aligned} \tag{6}$$

Since $S(0) = 1$, $S(1) = (b-a)/b$, and since $(b-a)_n/(b)_n$ satisfies (6), the summation (2) is proven. We note that owing to the constraint on b, the denominator on the right-hand side of (5) is non-zero for $0 \le k \le n$.

Notice that Zeilberger's algorithm has not produced a minimal recurrence for $S(n)$. This is in contrast to the closed form representation $(b-a)_n/(b)_n$ which corresponds to a recurrence of order 1. The reason for this phenomenon will be explained below.

Again following [1], Pfaff's method works as follows. By subtracting the summands term by term one finds that

$$S(n,a,b) - S(n-1,a,b) = \frac{a(1-b-2n)}{b(b+1)} T(n-1,a+2,b+2), \tag{7}$$

where

$$T(n,a,b) = {}_4F_3\left(\begin{matrix}a/2,(a+1)/2,b+n-1,-n\\ b/2,(b+1)/2,a\end{matrix};1\right).$$

Then inspection of $T(n,a,b)$ for some concrete values of n leads to the conjecture

$$T(n,a,b) = \frac{(b-a)_n}{(b+2n-1)(b)_{n-1}}. \tag{8}$$

Next one repeats this step by subtracting $S(n-1,a,b)$ from $T(n,a,b)$, which yields,

$$T(n,a,b) - S(n-1,a,b) = -\frac{(a+n)(b+n-1)}{b(b+1)} T(n-1,a+2,b+2). \tag{9}$$

The proof is then completed by observing that (7) and (9), together with the initial values $S(0,a,b) = T(0,a,b) = 1$, completely define $S(n,a,b)$ and $T(n,a,b)$, and by verifying that the right sides of (2) and (8) satisfy the same recurrences and initial values.

Summarizing, as pointed out in [1, Sect. 3.12], both proofs rely on contiguous relations. The Zeilberger output recurrence (6) is of the form

$$\begin{aligned}&c_0 \cdot {}_4F_3\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix};1\right) + c_1 \cdot {}_4F_3\left(\begin{matrix}a_1,a_2,a_3+1,a_4-1\\ b_1,b_2,b_3\end{matrix};1\right)\\ &+ c_2 \cdot {}_4F_3\left(\begin{matrix}a_1,a_2,a_3+2,a_4-2\\ b_1,b_2,b_3\end{matrix};1\right) = 0,\end{aligned} \tag{10}$$

where the ${}_4F_3$ parameters a_i and b_j are taken as in the ${}_4F_3$ series in (2), and where the c_l are the coefficients in the recurrence (6). The existence of this relation is predicted by Theorem 1C in Section 10. To compute the relation as in (3), the rational function version (116) applies.

The proof by Pfaff's method relies on *two* contiguous relations. Namely, if the ${}_4F_3$ parameters a_i and b_j are taken again as in the ${}_4F_3$ series in (2), then

$$\begin{aligned} c_0 \cdot {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right) + c_1 \cdot {}_4F_3\left(\begin{matrix} a_1, a_2, a_3 - 1, a_4 + 1 \\ b_1, b_2, b_3 \end{matrix}; 1\right) \\ + c_2 \cdot {}_4F_3\left(\begin{matrix} a_1 + 1, a_2 + 1, a_3, a_4 + 1 \\ b_1 + 1, b_2 + 1, b_3 + 1 \end{matrix}; 1\right) = 0 \end{aligned} \tag{11}$$

with $c_0 = 1$, $c_1 = -1$ and $c_2 = -a(1 - b - 2n)/(b(b + 1))$ corresponds to (7), and

$$\begin{aligned} c_0 \cdot {}_4F_3\left(\begin{matrix} a_1, a_2, a_3 - 1, a_4 \\ b_1, b_2, b_3 - 1 \end{matrix}; 1\right) + c_1 \cdot {}_4F_3\left(\begin{matrix} a_1, a_2, a_3 - 1, a_4 + 1 \\ b_1, b_2, b_3 \end{matrix}; 1\right) \\ + c_2 \cdot {}_4F_3\left(\begin{matrix} a_1 + 1, a_2 + 1, a_3, a_4 + 1 \\ b_1 + 1, b_2 + 1, b_3 + 1 \end{matrix}; 1\right) = 0 \end{aligned} \tag{12}$$

with $c_0 = 1$, $c_1 = -1$ and $c_2 = (a + n)(b + n - 1)/(b(b + 1))$ corresponds to (9). The existence of both of these contiguous relations is implied by Theorem 1B in Section 9.

We note that proving (2) by Pfaff's method leads in a direct manner to the discovery and the proof of a 'companion summation', namely (8). Another difference between the methods is that, when executing the 'Pfaff proof', the relations (11) and (12) in [1] have been derived 'by hand', whereas (10) was delivered automatically by Zeilberger's algorithm. This latter aspect of 'hand-computation' will be removed by the main theorems in this article. In particular, we will see that:

Any contiguous relation between terminating and most of the contiguous relations between non-terminating hypergeometric series can be found automatically by the computer.

In particular, this means that in its essence Pfaff's method is as algorithmic as the W–Z method. For example, if we do not know the coefficients in (11) and (12), we simply compute them by the algorithm described in Section 3.

It is important to note that conceptually even more is true. As explained in Section 4, contiguous relations can be found automatically by the same mechanism which is applied to find Zeilberger recurrences, namely, creative telescoping [40].

Zeilberger's algorithm [39] is based on the observation that a straightforward extension of Gosper's algorithm [11] for *indefinite* hypergeometric summation (hypergeometric telescoping) can be used for automatic *definite* hypergeometric summation (creative telescoping). For the sake of completeness of the presentation, the essence of creative telescoping, the *parameterized Gosper algorithm*, is briefly sketched in Section 3; in Section 7 we show how the RISC package `fastZeil`, which implements this extended Gosper algorithm, can be brought into action. For further information on Zeilberger's algorithm, creative telescoping, and the W–Z method, the reader is referred to the book [23] by M. Petkovšek, H.S. Wilf, and D. Zeilberger.

Before listing the contents of this article, we illustrate this new application of creative telescoping by having another look at Bailey's summation (2).

First of all we note that running Zeilberger's algorithm with order 1 results in an empty output. This proves that there does not exist a contiguous relation of the form

$$c_0\, S(n,a,b) + c_1\, S(n-1,a,b) = 0.$$

To overcome this issue, in the spirit of creative telescoping, we introduce a further shift – but not with respect to n as we would do when using Zeilberger's algorithm! Instead we shift one of the parameters, say a, and take as a new ansatz,

$$c_0\, S(n,a,b) + c_1\, S(n-1,a,b) + c_2\, S(n-1,a-1,b) = 0. \tag{13}$$

Then we apply the package described in Section 7 which computes indeed a relation of type (13) with the coefficients

$$\begin{aligned} c_0 &= -(a+n)(n+b-1)(2n+b-2), \\ c_1 &= (n+b-a-1)\Big((a^2+a(2n-1)+n(2n+b-2)\Big), \text{ and} \\ c_2 &= a(a-b)(1+a-b). \end{aligned} \tag{14}$$

This provides a new proof of Bailey's summation (2), since (13) with the values c_l from (14) together with the initial value $S(0,a,b) = 1$ completely define $S(n,a,b)$, and it is easy to verify that $(b-a)_n/(b)_n$ satisfies the same recurrence and initial value.

As described in Section 4, all such contiguous relations can be computed automatically by creative telescoping via *telescoping contiguous relations*; see Theorem 1 and Theorem 2. For instance, in order to obtain (13), the corresponding telescoping contiguous relation for all $k \geq 0$ is computed as

$$\begin{aligned} &c_0\, \frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k k!} + c_1\, \frac{(a_1)_k(a_2)_k(a_3-1)_k(a_4+1)_k}{(b_1)_k(b_2)_k(b_3)_k k!} \\ &+ c_2\, \frac{(a_1)_k(a_2-1)_k(a_3-1)_k(a_4+1)_k}{(b_1)_k(b_2)_k(b_3-1)_k k!} = \Delta_k\, C(k)\, \frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k k!} \end{aligned} \tag{15}$$

with c_l as in (14),

$$a_1 = \frac{a}{2},\ a_2 = \frac{a+1}{2},\ a_3 = b+n,\ a_4 = -n,\ b_1 = \frac{b}{2},\ b_2 = \frac{b+1}{2},\ b_3 = a+1,$$

and

$$C(x) = \frac{x(x+a)(2x+b-2)(2x+b-1)(b+n-1)(a+2n-1)}{n(2x+a-1)(x+b+n-1)}.$$

Summing both sides of (15) over k from 0 to n results in (13). We want to stress that the existence of (15), and thus that of (13), is predicted by Theorem 1C in Section 10.

There is quite some literature where contiguous relations are used. Most often this usage is more or less of implicit nature, for instance, as part of a method or a derivation. Much less literature can be found where general aspects of how to compute contiguous relations are treated; but there is still some. In addition to the books [25] and [1], there are articles such as [35], [36], [26], [27], or [9]; the latter devoted to q-series summation. In particular we want to stress the pioneering work of Takayama [35], where for elimination in difference-differential operator rings, non-commutative Gröbner bases methods are introduced. This theme reoccurs in the context of Zeilberger's holonomic systems approach to special functions identities [41]; see, for instance, the work of Chyzak [5] and of Koutschan [16], and also the references given there.

Despite all this work, we feel that our viewpoint and methods described in this article have particular advantages. Based on difference equations our approach is elementary and connects directly to Zeilberger's extension of Gosper's algorithm. An independent development in this direction is [3]. This article mentions connections to Karr-Schneider summation theory which also applies here: Basically all what we describe can be algorithmically realized using Schneider's Sigma package [30]. Further references to Schneider's work are given in Section 3.1.

Nevertheless, there are other aspects like summation theory for non-terminating hypergeometric series. In our setting, the existence of fundamental summation theorems like the Gauß ${}_2F_1$ or the Pfaff-Saalschütz ${}_3F_2$ formulas find natural explanations; see the Sections 8.2 and 10.2. This, in particular, includes contiguous relations between non-terminating hypergeometric series. Another spin-off concerns the fact that our setting in many cases admits explanations of the phenomenon why Zeilberger's algorithm does not always deliver the minimal recurrence and why 'creative symmetrizing' sometimes can help; see the Sections 11.2 and 11.3. For related work see [6]. Finally, all what we say in this article, including the explanations for non-minimality of Zeilberger orders, carries over to the case of q-hypergeometric series and q-contiguous relations.

The organization of the rest of this paper is as follows. In Section 3 a brief description of the parameterized Gosper algorithm is given. This section is kept as short as possible since this algorithm is essentially the same as when used as the computational engine in Zeilberger's algorithm for proving definite hypergeometric summation identities.

Section 4 presents Theorem 1 which states the existence of telescoping contiguous relations for terms which are summands of hypergeometric ${}_pF_q$-series with argument $z = 1$ or when $p \neq q + 1$. In Section 5 a detailed proof of Theorem 1 is given.

The examples presented in Section 6 should give a first impression of the variety of potential applications for telescoping contiguous relations and the methods described. In this section the examples are enriched with some details for proper illustration of the method, a theme which is continued further in Section 7. There we give a brief description of the computer algebra package we have used for our algorithmic applications.

The Sections 8, 9, and 10 describe three somehow disjoint refinements of Theorem 1 which imply the existence of contiguous relations for hypergeometric ${}_{q+1}F_q$-series with argument $z = 1$. Illustrating examples concern formulas such as the non-terminating versions of Gauß' ${}_2F_1$ and the Pfaff-Saalschütz ${}_3F_2$ summations, but include also the examples from the Introduction as Bailey's ${}_4F_3$-series summation.

Finally, Section 11 presents further, more involved applications. Using parameterized telescoping, we derive a generalization of a result which arose in the classical work by James Wilson on hypergeometric recurrences and contiguous relations. In addition, we discuss non-minimality of Zeilberger recurrences from telescoping contiguous relations point of view. In particular, we explain why 'creative symmetrizing' in some instances successfully reduces the order. This discussion includes a new (algorithmic) proof of the non-terminating version of Dixon's well-poised ${}_3F_2$-series. The concluding Section 12 points to the fact that all what has been said in this article carries over to q-hypergeometric series and to q-contiguous relations.

3 The Parameterized Gosper Algorithm

Zeilberger, [39] and [40], was the first who discovered that Gosper's algorithm [11] finds a straightforward extension that can be used for creative telescoping. In other words, Zeilberger observed that Gosper's algorithm for indefinite hypergeometric summation can be used to solve also definite hypergeometric summation problems. On this basis, Wilf and Zeilberger developed a rich theoretical framework which, for instance, includes also W–Z pairs and companion identities; see [23].

We present the essence of creative telescoping in the form of an input/output description of the corresponding parameterized extension of Gosper's algorithm. To this end we need to introduce a few definitions.

Throughout this article, p and q denote fixed non-negative integers; Δ_k is the difference operator defined in (4). The parameters a_i, b_j, and the argument z range over the complex numbers; for z we assume $z \neq 0$, unless explicitly mentioned otherwise.

Remark. As in the computer algebra examples presented, for the purpose of symbolic computation the a_i, b_j, and z usually are taken as indeterminates; i.e., instead of $\mathbb{K} = \mathbb{C}$, one takes

$$\mathbb{K} = \mathbb{C}(a_1, \ldots, a_p, b_1, \ldots, b_q, z);$$

or, even more precisely,

$$\mathbb{K} = \mathbb{F}(a_1, \ldots, a_p, b_1, \ldots, b_q, z),$$

where the field $\mathbb{F}$ is a computable algebraic extension of $\mathbb{Q}$ depending on extra parameters involved.

However, when seeing $\mathbb{K}$ in this article, the reader should feel free to interpret it as $\mathbb{K} = \mathbb{C}$.

In contrast to complex variables like a_i, b_j, or z, the variable x will always denote an indeterminate. As usual, with $\mathbb{K}$ as the coefficient domain, $\mathbb{K}[x]$ is the ring of polynomials in x; $\mathbb{K}(x)$ is its quotient field, the rational functions in x.

Throughout, $\mathbb{N} := \mathbb{Z}_{\geq 0}$ is the set of non-negative integers. The variables n and k always denote non-negative integers; i.e., $n, k \in \mathbb{N}$.

Following [12], for $k \in \mathbb{N}$ we will use the notation

$${}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k := \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{16}$$

When dealing with such a term we always assume it is well-defined; this means, $b_j + \ell \neq 0$ for $0 \leq \ell \leq k-1$ and all j. The analogous convention applies to hypergeometric series,

$${}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k .$$

The notation ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ and ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)_k$ will be used within text lines.

Definition 1 A sequence $t(k)$ over $\mathbb{K}$ is called a *hypergeometric term* if there exists a rational function $\rho \in \mathbb{K}(x)$ such that $t(k+1) = \rho(k) \cdot t(k)$ for all sufficiently large k.

Definition 2 Two hypergeometric terms $s(k)$ and $t(k)$ over $\mathbb{K}$ are *similar* if there exists a rational function $\rho \in \mathbb{K}(x)$ such that $s(k) = \rho(k)t(k)$ for all sufficiently large k.

3.1 The Parameterized Gosper Algorithm

Input. Hypergeometric terms $t(k)$, $t_0(k)$, $\ldots$, $t_d(k)$ over $\mathbb{K}$ where each $t_l(k)$, $0 \leq l \leq d$, is similar to $t(k)$.

Remark. We want to comment on the way how to actually input a hypergeometric term. According to Definition 1, this, for instance, can be done by specifying a homogeneous first-order recurrence with polynomial coefficients plus an initial value. Alternatively, one can give an expression in closed form, for instance, in the form of a hypergeometric term as in (16).

Output. All hypergeometric terms $g(k)$ over $\mathbb{K}$ and all tuples $(c_0,\ldots,c_d) \in \mathbb{K}^{d+1}$ such that for all sufficiently large k,

$$c_0\, t_0(k) + \cdots + c_d\, t_d(k) = g(k+1) - g(k) \quad (= \Delta_k g(k))\,. \tag{17}$$

One can show that each such $g(k)$ must be of the form

$$g(k) = r(k)\, t(k) \tag{18}$$

where $r \in \mathbb{K}(x)$ is a rational function which is computed by the algorithm.

Note. Gosper's algorithm is the special case $d = 0$ with $t(k) = t_0(k)$. Let $t(k) = f(n,k)$ be a term which is hypergeometric with respect to k and n (plus mild side conditions), then Zeilberger's algorithm is the special case with $t_l(k) = f(n+l,k)$ for $0 \le l \le d$, and $\mathbb{K}$, for instance, chosen as $\mathbb{K} = \mathbb{C}(n)$. For more detailed descriptions of these algorithms see, for instance, [23].

We also note that the zero term $g(k) = 0$ together with $(c_0,\ldots,c_d) = (0,\ldots,0)$ always form a solution to (17). All solutions $(c_0,\ldots,c_d,g(k))$ form a vector space over $\mathbb{K}$, hence the output of the parameterized Gosper algorithm can be given in terms of a basis.

Independent Verification

It is important to note that running the algorithm delivers all the information necessary to prove the correctness of the telescoping recurrence (17) *independently* from the steps of the algorithm. Namely, suppose we want to verify (17) for certain c_l and $g(k)$, where $g(k)$ is given as in (18) by the rational function $r \in \mathbb{K}(x)$.

Since all terms $t_l(k)$ are similar to $t(k)$, the left hand side of (17) can be written as a rational function multiple of $t(k)$. Due to

$$g(k+1) - g(k) = \left(r(k+1)\frac{t(k+1)}{t(k)} - r(k) \right) t(k)$$

we can divide both sides of (17) by $t(k)$, and checking (17) then reduces to checking the resulting equality of rational functions. Wilf and Zeilberger [37] call $r(x)$ the *certificate*.

Remark. As already mentioned, the parameterized Gosper algorithm (creative telescoping) is the driving engine of Zeilberger's algorithm. It is described in detail in a slightly different form in [23]. — It is interesting to note that the parameterized Gosper algorithm, in much more general form, has been used extensively by M. Karr [14, 15] in his difference field approach to symbolic summation. However, Karr has never linked it to *definite* summation. In the framework of difference fields and, more recently, difference rings, this step has been carried out, accompanied by other substantial theoretical and algorithmical enhancements, by Carsten Schneider in [31], [32]; [33], [34]; see also the references given there.

4 Telescoping Contiguous Relations for $z \neq 1$ or $p \neq q + 1$

This section contains the first main theorem of the paper, Theorem 1. It states the existence of telescoping contiguous relations with respect to non-negative integer shifts if $z \neq 1$ or $p \neq q + 1$ (or both).

Despite the existence of telescoping contiguous relations apriori is independent from the question whether they involve summands of convergent series, for applications such as taking the infinite sum over such summands,

$$
{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right)_k,
$$

we need to consider the conditions for the convergence of such series in case they are non-terminating; i.e., where none of the a_i is zero or a negative integer.

According to [1, Th. 2.1.1] such series converge absolutely for all z if $p \leq q$ and for $|z| < 1$ if $p = q + 1$; if $p > q + 1$ they diverge for all $z \neq 0$. The remaining case, "The case $|z| = 1$ when $p = q + 1$ is of great interest"; see [1, p. 62]. According to [1, Th. 2.1.2], the ${}_{q+1}F_q$-series with $|z| = 1$ converges absolutely if

$$
\mathrm{Re}\Big(\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_j\Big) > 0. \tag{19}
$$

For $|z| = 1$ and $z \neq 1$ it converges conditionally if

$$
0 \geq \mathrm{Re}\Big(\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_j\Big) > -1, \tag{20}
$$

and it diverges if this real part is less or equal -1.

We want to note explicitly that if $z = 1$ and $p = q + 1$, the criteria for the existence of telescoping contiguous relations, given by the Theorems 1A, 1B, and 1C, come in the form of refinements of (19).

Theorem 1 *Suppose $z \neq 1$ or $p \neq q + 1$. Let $d = \max\{p, q + 1\}$. For $0 \leq l \leq d$ let $(\alpha_1^{(l)}, \ldots, \alpha_p^{(l)}, \beta_1^{(l)}, \ldots, \beta_q^{(l)})$ be pairwise different tuples with non-negative integer entries. Then there exist $c_0, \ldots, c_d$ in $\mathbb{K}$, not all 0, and a polynomial $C(x) \in \mathbb{K}[x]$ such that for all $k \geq 0$,*

$$\sum_{l=0}^{d} c_l \cdot {}_pF_q \left(\begin{matrix} a_1 + \alpha_1^{(l)}, \ldots, a_p + \alpha_p^{(l)} \\ b_1 - \beta_1^{(l)}, \ldots, b_q - \beta_q^{(l)} \end{matrix} ; z \right)_k = \Delta_k \, C(k) \, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k . \tag{21}$$

Moreover, $C(0) = 0$, and if $C(x) \neq 0$, for the polynomial degree of $C(x)$ one has

$$\deg C(x) \leq q + 1 - d + \max_{0 \leq l \leq d} \{\alpha_1^{(l)} + \cdots + \alpha_p^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)}\}; \tag{22}$$

in addition, if $p \leq q + 1$,

$$\lim_{k \to \infty} C(k) {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k = 0. \tag{23}$$

If $p > q$ the limit (23) *is valid if one of the a_i is a non-positive integer.*

Proof. Since $C(x)$ is a polynomial, the limit (23) is immediate from classical asymptotics as Theorem 2.2.1 in [1]. The rest of Theorem 1 is proven in Section 5. □

Remark. According to the convergence criteria stated above, when summing (21) over k from 0 to ∞, imposing $|z| < 1$ or $p \leq q$ guarantees the absolute convergence of ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ and of the series

$${}_pF_q \left(\begin{matrix} a_1 + \alpha_1^{(l)}, \ldots, a_p + \alpha_p^{(l)} \\ b_1 - \beta_1^{(l)}, \ldots, b_q - \beta_q^{(l)} \end{matrix} ; 1 \right), \quad l \in \{0, \ldots, d\}.$$

If $p = q+1$ and $z \neq 1$ in such applications of Theorem 1, one needs to check whether the criteria for absolute or conditional convergence are satisfied. An example for $z = -1$ and $p = 2 = q+1$ is provided by Kummer's summation formula (152). There we also show that, alternatively, Kummer's identity can be derived as a limiting case of Dixon summation (150) which can be obtained from Theorem 1A with $p = 3 = q + 1$ and argument $z = 1$.

Remark. In Theorem 1 one can allow arbitrary integer parameters instead of restricting to non-negative integers. More precisely, for arbitrary parameters $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ this gives a relation,

$$\sum_{l=0}^{d} c_l \cdot {}_pF_q\left(\begin{matrix} a_1+\alpha_1^{(l)}, \ldots, a_p+\alpha_p^{(l)} \\ b_1-\beta_1^{(l)}, \ldots, b_q-\beta_q^{(l)} \end{matrix}; z\right)_k = \Delta_k\, R(k)\, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right)_k, \quad (24)$$

with a *rational* function $R(x) \in \mathbb{K}(x)$ instead of a *polynomial* $C(x) \in \mathbb{K}[x]$. For this extension, in view of (30) and (31), it is important to notice that because of possible poles of the $R(x)$, not all integer choices of $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ are admissible.

Definition 3 The relations (21) and (24) are called *telescoping contiguous relations*.

The next corollary shows that the restriction to non-negative integer shifts in Theorem 1 is not an essential one.

Corollary 1 *Any telescoping contiguous relation of the form* (21) *and, if poles of* $R(x)$ *cause no problem, in the version of* (24) *can be computed by the parameterized Gosper algorithm.*

Proof. It suffices to prove the statement with respect to the form (24). For the parameterized Gosper algorithm described in Section 4, take as input

$$t_l(k) = {}_pF_q\left(\begin{matrix} a_1+\alpha_1^{(l)}, \ldots, a_p+\alpha_p^{(l)} \\ b_1-\beta_1^{(l)}, \ldots, b_q-\beta_q^{(l)} \end{matrix}; z\right)_k$$

and

$$t(k) = {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right)_k$$

with $\mathbb{K} = \mathbb{C}(a_1, \ldots, a_p, b_1, \ldots, b_q, z)$ as the field of constants. Note that $t(k)$ and the $t_l(k)$ are hypergeometric terms; in addition, all terms $t_l(k)$ are similar to $t(k)$ as required. The parameterized Gosper algorithm finds *all* $(c_0, \ldots, c_d) \in \mathbb{K}^{d+1}$ and $R(x) \in \mathbb{K}(x)$ such that for $g(k) = R(k)t(k)$ the tuple $(c_0, \ldots, c_d, g(k))$ satisfies (17). Therefore the solution $(c_0, \ldots, c_d, R(k)t(k))$ as in (24) will be found by the algorithm. □

Before proving Theorem 1 in Section 5, we present some immediate consequences. Further applications are given in Section 6.

4.1 Telescoping Contiguous Relations for $z \neq 1$ and $(p, q) = (1, 0)$

Suppose $z \neq 1$ and $(p, q) = (1, 0)$. In this case, $d = \max\{p, q+1\} = 1$. According to Theorem 1 there exist c_0 and c_1, not all 0, and a polynomial $C(x)$ with $C(0) = 0$ such that for all $k \geq 0$,

$$c_0\, t_0(k) + c_1\, t_1(k) = \Delta_k\, C(k)\, t(k), \tag{25}$$

where

$$t(k) = t_0(k) = {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right)_k \quad \text{and} \quad t_1(k) = {}_1F_0\left(\begin{matrix} a+\alpha \\ - \end{matrix}; z\right)_k$$

with $\alpha \in \mathbb{Z}_{>0}$ According to (22),

$$\deg C(x) \leq q + 1 - d + \alpha = \alpha.$$

For fixed α, the c_j and $C(x)$ can be computed automatically as described in Section 7. For example, for $\alpha = 1$ one obtains (25) with $c_0 = a$, $c_1 = a(z-1)$, and $C(x) = x$. For $|z| < 1$ one can sum the resulting telescoping relation over k from 0 to infinity, which using (23) gives,

$$a \cdot {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right) - a(1-z) \cdot {}_1F_0\left(\begin{matrix} a+1 \\ - \end{matrix}; z\right) = 0; \tag{26}$$

this is in accordance with the binomial expansion

$${}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right) = (1-z)^a.$$

4.2 Telescoping Contiguous Relations for $z \neq 1$ and $(p, q) = (2, 1)$

Suppose $z \neq 1$ and $(p, q) = (2, 1)$. In this case, $d = \max\{p, q+1\} = 2$. According to Theorem 1 there exist c_0, c_1 and c_2, not all 0, and a polynomial $C(x)$ with $C(0) = 0$ such that for all $k \geq 0$,

$$c_0\, t_0(k) + c_1\, t_1(k) + c_2\, t_2(k) = \Delta_k\, C(k)\, t(k), \tag{27}$$

where $t(k) = t_0(k)$ and

$$t_0(k) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)_k, \; t_1(k) = {}_1F_0\left(\begin{matrix} a+\alpha_1, b+\beta_1 \\ c-\gamma_1 \end{matrix}; z\right)_k, \; t_2(k) = {}_2F_1\left(\begin{matrix} a+\alpha_2, b+\beta_2 \\ c-\gamma_2 \end{matrix}; z\right)_k.$$

Here the $(\alpha_l, \beta_l, \gamma_l)$ for $l = 1$ and $l = 2$ are different triples of non-negative integers, each with entries not all 0. For $|z| < 1$ one can sum the telescoping relation (27) over k from 0 to infinity, which using (23) gives,

$$c_0 \cdot {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) + c_1 \cdot {}_1F_0\left(\begin{matrix} a+\alpha_1, b+\beta_1 \\ c-\gamma_1 \end{matrix}; z\right) + c_2 \cdot {}_2F_1\left(\begin{matrix} a+\alpha_2, b+\beta_2 \\ c-\gamma_2 \end{matrix}; z\right) = 0. \tag{28}$$

For fixed α_l,β_l, and γ_l, the c_j and $C(x)$ can be computed automatically as described in Section 7; concrete examples are given there.

We want to conclude this section with the remark that Theorem 1 together with our implementation of parameterized telescoping presented in Section 7 (or any other implementation meeting the specification given in Section 3) settles the existence and the computation of the general three term contiguous ${}_2F_1$-relations treated by Gauß in [10].

5 Proof of Theorem 1

In this section we prove Theorem 1. To this end we make use of several elementary facts which are presented in the form of lemmas.

5.1 Preparatory Lemmas

Definition 4 Let x be an indeterminate, $c \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$, and $m \in \mathbb{N}$. Define $\mu_0(c;x) := 1$, and

$$\mu_m(c;x) := \left(1+\frac{x}{c}\right)\left(1+\frac{x}{c+1}\right)\cdots\left(1+\frac{x}{c+m-1}\right), \qquad m \geq 1.$$

For the degree of the polynomial $\mu_m(c;x) \in \mathbb{K}[x]$ we have

$$\deg \mu_m(c;x) = m. \tag{29}$$

Lemma 1 *Let* $(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q) \in \mathbb{N}^{p+q}$ *such that for all* $i \in \{1,\ldots,p\}$ *and* $j \in \{1,\ldots,q\}$,

$$a_i+\alpha_i \notin \{1,\ldots,\alpha_i\} \;\; \textit{and} \;\; b_j-\beta_j \notin \mathbb{Z}_{\leq 0}.$$

Then for all $k \geq 0$,

$$\begin{aligned}&{}_pF_q\left(\begin{matrix}a_1+\alpha_1,\ldots,a_p+\alpha_p\\ b_1-\beta_1,\ldots,b_q-\beta_q\end{matrix};z\right)_k\\ &=\prod_{i=1}^{p}\mu_{\alpha_i}(a_i;k)\cdot\prod_{j=1}^{q}\mu_{\beta_j}(b_j-\beta_j;k)\cdot{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)_k.\end{aligned}$$

Proof. For $k \geq 0$,

$$(a+1)_k = \left(1 + \frac{k}{a}\right)(a)_k \tag{30}$$

and

$$\frac{1}{(b-1)_k} = \left(1 + \frac{k}{b-1}\right)\frac{1}{(b)_k}. \tag{31}$$

The lemma is proven by iterated application of (30) and (31). □

For

$$t(k) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}\frac{z^k}{k!} \tag{32}$$

let us consider all pairs $P(x) \in \mathbb{K}[x]$ and $R(x) \in \mathbb{K}(x)$ such that for all $k \geq 0$,

$$P(k)\,t(k) = \Delta_k R(k)\,t(k). \tag{33}$$

It turns out that if (33) holds, then $R(x)$ has to be a polynomial, too. More precisely, all such pairs can be characterized as follows. We note that the essence of this characterization is closely related to what is called the Gosper, resp. Gosper-Petkovšek, form and to the author's concept of greatest factorial factorization; see [20].

Lemma 2 *Suppose $P(x) \in \mathbb{K}[x]$ and $R(x) \in \mathbb{K}(x)$ satisfy the relation*

$$P(k)\,{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)_k = \Delta_k\,R(k)\,{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)_k, \quad k \geq 0. \tag{34}$$

Then there exists a polynomial $P_1(x) \in \mathbb{K}[x]$ such that

$$P(x) = z\prod_{i=1}^{p}(x+a_i)\cdot P_1(x+1) - x\prod_{j=1}^{q}(x+b_j-1)\cdot P_1(x) \tag{35}$$

and

$$R(x) = x\prod_{j=1}^{q}(x+b_j-1)\cdot P_1(x). \tag{36}$$

Vice versa, if $P(x)$ and $R(x)$ are of the form (35) *and* (36) *with $P_1(x)$ being an arbitrary polynomial in $\mathbb{K}[x]$, then relation* (34) *is satisfied.*

Proof. First we prove the statement assuming that

$$a_i \notin \mathbb{Z} \text{ and } a_i - b_j \notin \mathbb{N}, \ \ i \in \{1, \dots, p\}, j \in \{1, \dots, q\}.$$

Then, since (34) is a finite sum, analytic continuation proves the statement without these restrictions.

With $t(k)$ as in (32), relation (34) turns into $P(k)t(k) = R(k+1)t(k+1) - R(k)t(k)$. Dividing out $t(k)$ results in an equality between rational functions; in other words, relation (34) is equivalent to the relation

$$\frac{P(x) + R(x)}{R(x+1)} = \frac{(x+a_1)\cdots(x+a_p)}{(x+b_1)\cdots(x+b_q)} \frac{z}{x+1}. \tag{37}$$

Suppose $P(x)$ and $R(x)$ are of the form (35) and (36) with $P_1(x)$ being an arbitrary polynomial in $\mathbb{K}[x]$. It is easily verified that then relation (37) is satisfied. Therefore it remains to prove the other direction of the lemma.

To this end, suppose that (34) holds for some $P(x) \in \mathbb{K}[x]$ and $R(x) = R_1(x)/R_2(x) \in \mathbb{K}(x)$ with R_1 and R_2 being coprime polynomials in $\mathbb{K}[x]$. So we can rewrite relation (37) as

$$(x+1) \prod_{j=1}^{q} (x+b_j) \cdot \overline{P}(x)\, R_2(x+1) = z \prod_{i=1}^{p} (x+a_i) \cdot R_1(x+1)\, R_2(x) \tag{38}$$

where

$$\overline{P}(x) = R_1(x) + P(x) R_2(x). \tag{39}$$

In the following we will use that $\gcd(R_1(x), R_2(x)) = 1$, $\gcd\left(\overline{P}(x), R_2(x)\right) = 1$, and the fact that if $h(x) \in \mathbb{K}[x]$ is irreducible then $\gcd(h(x), h(x+l)) = 1$ for all non-zero integers l.

Suppose that $x + b \mid R_2(x)$ where $b = 1$ or $b = b_j$ for some $j \in \{1, \dots, q\}$. Then $x + b + 1$ divides $R_2(x+1)$, and (38) implies that $x + b + 1 \mid R_2(x)$. By iterating this observation we obtain that $x + b + l \mid R_2(x)$ for all $l \in \mathbb{N}$, a contradiction to $R_2(x)$ being a non-zero polynomial. Consequently $(x+1) \prod_{j=1}^{q} (x+b_j)$ must divide $R_1(x+1)$; in other words, there exists a polynomial $P_1(x) \in \mathbb{K}[x]$ such that

$$R_1(x) = x \prod_{j=1}^{q} (x + b_j - 1) \cdot P_1(x). \tag{40}$$

By an analogous reasoning one can show that $\prod_{i=1}^{p} (x+a_i)$ must divide $\overline{P}(x)$; this means, there exists a polynomial $Q(x) \in \mathbb{K}[x]$ such that

$$\overline{P}(x) = \prod_{i=1}^{p}(x + a_i) \cdot Q(x). \tag{41}$$

By (40) and (41), equation (38) reduces to

$$Q(x)\, R_2(x + 1) = z\, P_1(x + 1)\, R_2(x). \tag{42}$$

Without loss of generality we may assume that the leading coefficient of $R_2(x)$ is equal to 1. The next proof step will show that $R_2(x)$ must have degree 0 which implies that

$$R_2(x) = 1 \qquad \text{and} \qquad Q(x) = z\, P_1(x + 1). \tag{43}$$

For proving this, suppose that an arbitrary irreducible polynomial $h(x)$ divides $R_2(x)$. Then $h(x + 1) \mid R_2(x + 1)$, and (42) implies that $h(x + 1) \mid R_2(x)$. Iterating this observation we obtain that $h(x + l) \mid R_2(x)$ for all $l \in \mathbb{N}$. Therefore $R_2(x)$ can only have irreducible factors which are constants, and (43) is proved.

Finally, equation (39) together with (40) and (41) imply (35). Since $R_2(x) = 1$ we have $R(x) = R_1(x)$, and equation (36) is nothing but relation (40). This completes the proof of the lemma. □

We are interested in polynomial solutions $P(x)$ and $R(x)$ to (34) which are minimal with respect to their degree in x; see also Lemma 3.

Corollary 2 *The minimal non-trivial choice for* $P(x) \in \mathbb{K}[x]$ *and* $R(x) \in \mathbb{K}(x)$ *such that* (34) *holds is the following:*

$$P(x) = z \prod_{i=1}^{p}(x + a_i) - x \prod_{j=1}^{q}(x + b_j - 1) \tag{44}$$

and

$$R(x) = x \prod_{j=1}^{q}(x + b_j - 1). \tag{45}$$

Proof. Immediate from Lemma 2 choosing $P_1(x) = 1$. □

The minimally chosen polynomials $P(x)$ and $Q(x)$ from Corollary 2 will play a fundamental role which gives rise to the following definition.

Definition 5 To any hypergeometric term ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)_k$ we associate the polynomials

$$ {}_pP_q(x) := z\prod_{i=1}^{p}(x+a_i) - x\prod_{j=1}^{q}(x+b_j-1) \tag{46}$$

and

$$ {}_pR_q(x) := x\prod_{j=1}^{q}(x+b_j-1). \tag{47}$$

For the proof of Theorem 1 we need a bit more than the minimal non-trivial choice specified in Corollary 2; we also need the cases where $P_1(x)=x^n$.

Definition 6 For any non-negative integer n,

$$ {}_pP_q^{(n)}(x) := z\,(x+1)^n\prod_{i=1}^{p}(x+a_i) - x^{n+1}\prod_{j=1}^{q}(x+b_j-1) \in \mathbb{K}[x]. \tag{48}$$

Notice that ${}_pP_q^{(0)}(x) = {}_pP_q(x)$.

Lemma 3 *Suppose $z\neq 1$ or $p\neq q+1$. Then*

$$\deg {}_pP_q^{(n)}(x) = \max\{n+p, n+1+q\} = n+\max\{p,q+1\} = n+\deg {}_pP_q(x), \tag{49}$$

and for the leading coefficient,

$$\operatorname{lcf} {}_pP_q^{(n)}(x) = \begin{cases} -1 & ,\ \mathit{if}\quad p\leq q \\ z-1 & ,\ \mathit{if}\quad p=q+1 \\ z & ,\ \mathit{if}\quad p>q+1 \end{cases}. \tag{50}$$

Proof. Immediate by inspection. □

Lemma 4 ("Reduction Lemma")

Suppose $z\neq 1$ or $p\neq q+1$. Fix $n\in\mathbb{N}$. Let

$$d = \deg {}_pP_q(x)\ \ \mathit{and}\ \ c_n = \operatorname{lcf} {}_pP_q^{(n)}(x)$$

be the degree and the leading coefficient, respectively, of the polynomial ${}_pP_q^{(n)}(x)$*. Then there exists a polynomial* ${}_pQ_q^{(n)}[x] \in \mathbb{K}[x]$ *with* $\deg {}_pQ_q^{(n)}(x) \leq n + d - 1$ *such that for all* $k \geq 0$*,*

$$c_n \cdot k^{n+d}\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k = \left({}_pQ_q^{(n)}(k) + \Delta_k\, k^n\, {}_pR_q(k) \right) {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k . \tag{51}$$

Proof. Let $P_1(x) = x^n$. Then according to Lemma 2 we have for all $k \geq 0$,

$${}_pP_q^{(n)}(k)\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k = \Delta_k\, k^n\, {}_pR_q(k)\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k . \tag{52}$$

By Lemma 3 we have $\deg {}_pP_q^{(n)}(x) = n + d$; i.e., choosing $c_n := \operatorname{lcf} {}_pP_q^{(n)}(x) \in \mathbb{K}$ we can define

$${}_pQ_q^{(n)}(x) := c_n \cdot x^{n+d} - {}_pP_q^{(n)}(x) \in \mathbb{K}[x]$$

where $\deg {}_pQ_q^{(n)}(x) \leq n + d - 1$. Hence Lemma 4 follows from (52) after replacing ${}_pP_q^{(n)}(k)$ by $c_n \cdot k^{n+d} - {}_pQ_q^{(n)}(k)$. □

The Reduction Lemma implies the following result in a straightforward manner.

Corollary 3 *Suppose* $z \neq 1$ *or* $p \neq q + 1$*. Let* $d = \deg {}_pP_q(x)$*. For any fixed* $n \in \mathbb{N}$ *there exist polynomials* $u_n(x)$ *and* $v_n(x)$ *in* $\mathbb{K}[x]$ *with*

$$\deg u_n(x) \leq d - 1 \quad \textit{and} \quad \deg v_n(x) = n \tag{53}$$

such that for all $k \geq 0$*,*

$$\begin{aligned} k^{n+d}\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k &= u_n(k)\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k \\ &\quad + \Delta_k\, v_n(k)\, {}_pR_q(k)\, {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right)_k . \end{aligned} \tag{54}$$

Proof. The proof proceeds by induction on n. For $n = 0$ we invoke Lemma 4 with $n = 0$; i.e., we can choose $u_0(x) = 1/c \cdot {}_pQ_q^{(0)}(x)$ with $\deg u_0(x) \leq d - 1$ and $v_0(x) = 1/c$ where $c = \operatorname{lcf} {}_pP_q^{(0)}(x)$.

For proving the induction step, let $t(k)$ be as in (32). According to Lemma 4 we have with $c = \operatorname{lcf} {}_pP_q^{(n+1)}(x)$ that for all $k \geq 0$,

$$k^{n+1+d}\, t(k) = \frac{1}{c} \cdot {}_pQ_q^{(n+1)}(k)\, t(k) + \Delta_k \frac{1}{c} \cdot k^{n+1}\, {}_pR_q(k)\, t(k). \tag{55}$$

Since $\deg {}_pQ_q^{(n+1)}(x) \leq n + d$, the polynomial ${}_pQ_q^{(n+1)}(x)$ can be written in the form

$${}_pQ_q^{(n+1)}(x) = \sum_{j=0}^{n+d} Q_j x^j \quad \text{with} \quad Q_j \in \mathbb{K}.$$

For all m with $0 \leq m \leq n$ we apply the induction hypothesis to $Q_{m+d}\, k^{m+d}\, t(k)$, and thus by (54) we obtain polynomials $u(x)$ and $v(x)$ in $\mathbb{K}[x]$ with $\deg u(x) \leq d - 1$ and $\deg v(x) \leq n$ such that for all $k \geq 0$,

$${}_pQ_q^{(n+1)}(k)\, t(k) = \sum_{j=0}^{d-1} Q_j k^j\, t(k) + u(k)\, t(k) + \Delta_k\, v(k)\, {}_pR_q(k)\, t(k). \tag{56}$$

Finally combining (56) with (55) we obtain the polynomials

$$u_{n+1}(x) = \frac{1}{c}\Big(u(x) + \sum_{j=0}^{d-1} Q_j x^j\Big) \quad \text{and} \quad v_{n+1}(x) = \frac{1}{c}\left(x^{n+1} + v(x)\right),$$

which satisfy (53) and (54) for $n + 1$ instead of n. This completes the proof of Corollary 3. □

We shall utilize Corollary 3 in the following form. We note explicitly that if $0 \in \mathbb{K}[x]$ is the zero polynomial, we use the convention $\deg 0 = -1$.

Corollary 4 *Suppose $z \neq 1$ or $p \neq q + 1$. Let $d = \deg {}_pP_q(x)$. For any $M \in \mathbb{K}[x]$ with $\deg M(x) = m$ there exist polynomials $U(x)$ and $V(x)$ in $\mathbb{K}[x]$ with*

$$\deg U(x) \leq d - 1 \quad \textit{and} \quad \deg V(x) = \max\{m - d, -1\} \tag{57}$$

such that for all $k \geq 0$,

$$\begin{aligned} M(k)\, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right)_k &= U(k)\, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right)_k \\ &\quad + \Delta_k\, V(k)\, {}_pR_q(k)\, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right)_k. \end{aligned} \tag{58}$$

Proof. Without loss of generality we can rewrite $M(x)$ into the form

$$M(x) = \sum_{j=0}^{d-1} M_j x^j + \sum_{n=0}^{m-d} M_{n+d} x^{n+d}$$

with coefficients M_i in $\mathbb{K}$.

If $m < d$ the second sum is zero. This means, in this case we can choose $U(x) = M(x)$ and $V(x) = 0$, and both, (57) and (58), are satisfied. In particular we have that $\deg V(x) = \deg 0 = \max\{m - d, -1\} = -1$.

In order to prove the corollary also for $m \geq d$, let $t(k)$ be as in (32). By invoking Corollary 3 we obtain that for all $k \geq 0$,

$$\begin{aligned} M(k)\, t(k) = &\sum_{j=0}^{d-1} M_j k^j \, t(k) \\ &+ \sum_{n=0}^{m-d} M_{n+d}\, u_n(k)\, t(k) + \Delta_k \sum_{n=0}^{m-d} M_{n+d}\, v_n(k)\, {}_pR_q(k)\, t(k). \end{aligned}$$

But this proves Corollary 4 since we can choose,

$$U(x) = \sum_{j=0}^{d-1} M_j x^j + \sum_{n=0}^{m-d} M_{n+d}\, u_n(x) \;\text{ and }\; V(x) = \sum_{n=0}^{m-d} M_{n+d}\, v_n(x),$$

and it is easily verified that both, (57) and (58), are satisfied. □

5.2 Proof of Theorem 1

With the results of the preceding subsection we are ready to prove Theorem 1.

Let $t(k)$ be as in (32). First we prove the statement of Theorem 1 assuming that the condition to apply Lemma 1 holds; namely, for $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$, $l \in \{0, \ldots, d\}$,

$$a_i + \alpha_i^{(l)} \notin \{1, \ldots, \alpha_i^{(l)}\} \;\text{ and }\; b_j - \beta_j^{(l)} \notin \mathbb{Z}_{\leq 0}.$$

Then, analytic continuation proves the statement without these restrictions an the a_i.[§]

According to Lemma 1 the left hand side of (21), with unspecified $c_l \in \mathbb{K}$ which will be specialized further in a later step, can be rewritten as

$$\Big(\sum_{l=0}^{d} c_l\, M_l(k)\Big)\, t(k)$$

where

[§] The conditions on the $b_j - \beta_j^{(l)}$ remain valid as being those for ${}_pF_q$ bottom parameters.

$$M_l(x) = \prod_{i=1}^{p} \mu_{\alpha_i^{(l)}}(a_i; x) \prod_{j=1}^{q} \mu_{\beta_j^{(l)}}(b_j - \beta_j^{(l)}; x) \in \mathbb{K}[x],$$

with $\mu_m(c, x)$ as in Definition 4 of Section 5.1.

According to Corollary 4 there exist polynomials $U_l(x)$ and $V_l(x)$ in $\mathbb{K}[x]$ with

$$\deg U_l(x) \le d - 1 \quad \text{and} \quad \deg V_l(x) = \max\{\deg M_l(x) - d, -1\} \tag{59}$$

such that for all $k \ge 0$,

$$\Big(\sum_{l=0}^{d} c_l\, M_l(k)\Big)\, t(k) = \Big(\sum_{l=0}^{d} c_l\, U_l(k)\Big)\, t(k) + \Delta_k \Big(\sum_{l=0}^{d} c_l\, V_l(k)\Big)\, {}_pR_q(k)\, t(k). \tag{60}$$

If we can choose $c_l \in \mathbb{K}$, not all zero, such that

$$\sum_{l=0}^{d} c_l\, U_l(x) = 0, \tag{61}$$

then Theorem 1 is proven. Namely, using these specific solutions $c_0, \ldots, c_d$, not all 0, we can set

$$C(x) := \Big(\sum_{l=0}^{d} c_l\, V_l(x)\Big)\, {}_pR_q(x). \tag{62}$$

And, choosing the c_l and $C(x)$ this way, (60) is nothing but (21); moreover, we have $C(0) = 0$ owing to ${}_pR_q(0) = 0$. In addition, if $C \neq 0$ the degree estimate (22) holds which can be seen as follows. From (62) and (59) we have that

$$\deg C(x) \le \deg {}_pR_q(x) - d + \max_{0 \le l \le d}\{\deg M_l(x)\},$$

and (22) is implied by $\deg {}_pR_q(x) = q + 1$ together with

$$\deg M_l(x) = \alpha_1^{(l)} + \cdots + \alpha_p^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)},$$

according to (29).

Finally we show that a non-trivial choice of c_l satisfying (61) indeed exists. To this end we define for $0 \le m \le d - 1$ and $0 \le l \le d$,

$$u_{m,l} := \text{ coefficient of } x^m \text{ in } U_l(x).$$

This gives rise to a $d \times (d + 1)$ matrix U via

$$U := (u_{m,l}) = \begin{pmatrix} u_{0,0} & u_{0,1} & \cdots & u_{0,d} \\ \vdots & \vdots & \ddots & \vdots \\ u_{d-1,0} & u_{d-1,1} & \cdots & u_{d-1,d} \end{pmatrix}. \tag{63}$$

Now finding all c_l satisfying (61) is equivalent to finding all solutions $(c_0, \ldots, c_d) \in \mathbb{K}^{d+1}$ to the homogeneous nullspace problem

$$\begin{pmatrix} u_{0,0} & u_{0,1} & \cdots & u_{0,d} \\ \vdots & \vdots & \ddots & \vdots \\ u_{d-1,0} & u_{d-1,1} & \cdots & u_{d-1,d} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since we have $d+1$ unknowns and d equations, there exists a solution $(c_0, \ldots, c_d) \in \mathbb{K}^{d+1}$ where the c_l are not all 0. This completes the proof of Theorem 1. □

Remark. In symbolic summation various articles describe algorithms that split the summand into a summable and a non-summable part. Then computing a recurrence only for the non-summable part often yields a speed-up. For the hypergeometric case this has been considered, e.g., in [4]. The approach presented in this subsection is different: the main goal is to find optimal estimates on the shift-set and to guarantee that the certificate $C(x)$ is a polynomial. As pointed out by the anonymous referee, the approach of Section 5.2 might also yield a refined method to compute the parameterized telescoping solution, and that it would be interesting to check if the underlying system to be solved is simpler than the system one has to solve in the standard parameterized Gosper method.

5.3 Connection to Differential Equations

This section is not necessary for understanding the flow of the arguments. Nevertheless, we feel that we should at least mention how things are related to the classical hypergeometric differential equations.

We begin by recalling a fact which is straight-forward. In this section we suppose that $p \le q+1$.

Lemma 5 *Let D_x be the differential operator with respect to x. For $n, k \in \mathbb{N}$ such that $n \ge k$,*

$$(xD_x)^n \, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right)_k = k^n \, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right)_k. \tag{64}$$

Next, for the choice $P(x) = {}_pP_q(x)$ and $R(x) = {}_pR_q(x)$, we sum both sides of (34) over all k from 0 to ∞ to obtain,

$$\begin{aligned}\sum_{k=0}^{\infty} {}_pP_q(k)\, {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)_k &= -{}_pR_q(0) + \lim_{k\to\infty} {}_pR_q(k)\, {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)_k\\ &= 0, \end{aligned} \tag{65}$$

where the last equality is owing to (23) and (36). Setting $\theta = zD_z$ and using Lemma 5, the left hand side of (65) turns into

$$\big(z\prod_{i=1}^{p}(\theta + a_i) - \theta\prod_{j=1}^{q}(\theta + b_j - 1)\big)\, {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right) \tag{66}$$

which is the non-trivial side of the classic homogeneous differential equation for ${}_pF_q$; see, for instance, [25, §47, eq. (2)].

Thus we can summarize as follows: *Relation* (34) *with the minimal choice* $P(x) = {}_pP_q(x)$ *and* $R(x) = {}_pR_q(x)$ *can be considered as a finite, telescoping version of the homogeneous differential equation for* ${}_pF_q$ *series.*

6 Applications of Theorem 1

Example. We begin by considering one of Gauß' fifteen classical contiguous relations [10, 7.2],

$$(b-a)\, {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right) + a\, {}_2F_1\left(\begin{matrix}a+1,b\\ c\end{matrix};z\right) - b\, {}_2F_1\left(\begin{matrix}a,b+1\\ c\end{matrix};z\right) = 0. \tag{67}$$

We have $p = 2$, $q = 1$, and thus $d = \deg {}_2P_1(x) = 2$; in addition, $c \notin \mathbb{Z}_{\le 0}$, and $|z| < 1$ as a condition for convergence.

Following the proof of Theorem 1 in Section 5.2, let us determine complex numbers c_0, c_1, and c_2, and a polynomial $C(x) \in \mathbb{C}[x]$ with $C(0) = 0$ and

$$\deg C(x) \le q + 1 - d + \max\{0, 1\} = 1 + 1 - 2 + \max\{0, 1\} = 1,$$

such that for all $k \ge 0$,

$$\begin{aligned}&c_0 \cdot {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)_k + c_1 \cdot {}_2F_1\left(\begin{matrix}a+1,b\\ c\end{matrix};z\right)_k + c_2 \cdot {}_2F_1\left(\begin{matrix}a,b+1\\ c\end{matrix};z\right)_k\\ &= \Delta_k\, C(k)\, {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)_k. \end{aligned} \tag{68}$$

As in the proof of Lemma 2, w.l.o.g. we may assume that $a \notin \{0\}$ and $b \notin \{0\}$. Let

$$t(k) = \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{69}$$

According to Lemma 1 the left hand side of (68) can be written as

$$\Big(\sum_{l=0}^{2} c_l\, M_l(k)\Big)\, t(k)$$

where

$$M_0(x) = 1, \quad M_1(x) = 1 + \frac{x}{a}, \quad \text{and} \quad M_2(x) = 1 + \frac{x}{b}.$$

Hence, according to Corollary 4, to establish the relation (60) we can choose

$$U_l(x) = M_l(x) \quad \text{and} \quad V_l(x) = 0, \qquad 0 \le l \le 2. \tag{70}$$

Then (62) implies that $C(x) = 0$; i.e., $C(0) = 0$ and $\deg C(x) = -1 \le 1$.

Finally we have to choose $c_l \in \mathbb{C}$, not all 0, such that $\sum_{l=0}^{2} c_l\, U_l(x) = 0$. Because of (70) we obtain according to (63),

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/a & 1/b \end{pmatrix}.$$

It is easily verified that

$$(c_0, c_1, c_2) = (b - a, a, -b)$$

generates the one-dimensional nullspace of

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/a & 1/b \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently we obtain as the desired telescoping contiguous relation

$$(b-a)\,{}_2F_1\begin{pmatrix} a, b \\ c \end{pmatrix}; z\Big)_k + a\,{}_2F_1\begin{pmatrix} a+1, b \\ c \end{pmatrix}; z\Big)_k - b\,{}_2F_1\begin{pmatrix} a, b+1 \\ c \end{pmatrix}; z\Big)_k = 0. \tag{71}$$

Note that the right hand side is 0 which is due to $C(x) = 0$; this means, in this case the contiguous relation (67) is already true when restricted to the kth summand. Of course, as any (telescoping) contiguous relation, the equality (71) can be verified independently from its derivation. Namely, after dividing both sides by $t(k)$, relation (71) reduces to

$$b - a + a\,\frac{a+k-1}{a} - b\,\frac{b+k-1}{b} = 0. \tag{72}$$

Example. As a second example we again consider relation (1),

$$(a+1-c)\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)+((a+1-b)z-2(a+1)+c)\ {}_2F_1\left(\begin{matrix}a+1,b\\c\end{matrix};z\right)$$
$$+(1-z)(a+1)\,{}_2F_1\left(\begin{matrix}a+2,b\\c\end{matrix};z\right)=0, \tag{73}$$

which is the first of Gauß' fifteen fundamental contiguous relations with a replaced by $a+1$. As explained in [1, (2.5.19)], this relation gives rise to a set of orthogonal polynomials. — As in the previous example we have $p=2$, $q=1$, and thus $d=\deg {}_2P_1(x)=2$; in addition, $c\notin\mathbb{Z}_{\le 0}$, and $|z|<1$ as a condition for convergence.

Again by following the proof of Theorem 1 in Section 5.2, we determine complex numbers c_0, c_1, and c_2 and a polynomial $C(x)\in\mathbb{C}[x]$ with $C(0)=0$ and

$$\deg C(x)\le 1+1-2+\max\{0,1,2\}=2, \tag{74}$$

and such that for all $k\ge 0$,

$$c_0\cdot\ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)_k+c_1\cdot\ {}_2F_1\left(\begin{matrix}a+1,b\\c\end{matrix};z\right)_k+c_2\cdot\ {}_2F_1\left(\begin{matrix}a+2,b\\c\end{matrix};z\right)_k$$
$$=\Delta_k\,C(k)\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)_k. \tag{75}$$

As in the proof of Lemma 2, w.l.o.g. we may assume that $a\notin\{0,-1\}$. Let $t(k)$ be as in (69).

According to Lemma 1 the left hand side of (75) can be written as

$$\Big(\sum_{l=0}^{2}c_l\,M_l(k)\Big)\,t(k)$$

where

$$M_0(x)=1,\quad M_1(x)=1+\frac{x}{a},\quad\text{and}$$
$$M_2(x)=\left(1+\frac{x}{a}\right)\left(1+\frac{x}{a+1}\right)=1+\frac{2a+1}{a(a+1)}x+\frac{1}{a(a+1)}x^2.$$

From Lemma 4 (with $n=0$) we obtain that for all $k\ge 0$,

$$(z-1)\cdot k^2\,t(k)={}_2Q_1^{(0)}(k)\,t(k)+\Delta_k\,{}_2R_1(k)\,t(k) \tag{76}$$

where

$${}_2Q_1^{(0)}(x)=\operatorname{lcf}{}_2P_1(x)\cdot x^2-{}_2P_1(x)=-\left((a+b)z-c+1\right)x-abz,$$

and

$${}_2R_1(x)=x(x+c-1).$$

Utilizing (76) we obtain

$$\begin{aligned}
&\Big(\sum_{l=0}^{2} c_l\, M_l(k)\Big)\, t(k)\\
&= \left(c_0\, M_0(k) + c_1\, M_1(k) + c_2 \left(1 + \frac{2a+1}{a(a+1)} k\right)\right) t(k) + c_2 \frac{k^2}{a(a+1)} t(k)\\
&= \left(c_0\, U_0(k) + c_1\, U_1(k) + c_2\, U_2(k)\right) t(k) + \Delta_k\, C(k)\, t(k)
\end{aligned}$$

where

$$\begin{aligned}
U_0(x) &= 1 \quad (= M_0(x)), \quad U_1(x) = 1 + \frac{x}{a}, \quad (= M_1(x)), \quad \text{and}\\
U_2(x) &= \frac{(a-b+1)z-a-1}{(a+1)(z-1)} + \frac{(a-b+1)z-2a-2+c}{a(a+1)(z-1)}\, x. \qquad (77)
\end{aligned}$$

and

$$C(x) = \frac{c_2}{a(a+1)(z-1)}\, {}_2R_1(x). \qquad (78)$$

Now (78) implies that $C(0) = 0$ and $\deg C(x) = 2$ in accordance with (74).

Finally we have to choose $c_l \in \mathbb{C}$, not all 0, such that $\sum_{l=0}^{2} c_l\, U_l(x) = 0$. Because of (77) we obtain according to (63),

$$U = \begin{pmatrix} 1 & 1 & \frac{(a-b+1)z-a-1}{(a+1)(z-1)} \\ 0 & \frac{1}{a} & \frac{(a-b+1)z-2a-2+c}{a(a+1)(z-1)} \end{pmatrix}.$$

It is easily verified that

$$(c_0, c_1, c_2) = (a(a-c+1), a((a-b+1)z-2a-2+c), a(a+1)(1-z)) \qquad (79)$$

generates the one-dimensional nullspace of

$$U = \begin{pmatrix} 1 & 1 & \frac{(a-b+1)z-a-1}{(a+1)(z-1)} \\ 0 & \frac{1}{a} & \frac{(a-b+1)z-2a-2+c}{a(a+1)(z-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently, by choosing the c_l as in (79), and $C(x)$ with substituting $c_2 = a(a+1)(1-z)$ into (78), we obtain as the desired telescoping contiguous relation (75) from which (73) is obtained as usual by summation over all $k \geq 0$.

In practice, as stated and proven in Corollary 1, the coefficients c_l and the polynomial $C(x)$ are computed by the Parameterized Gosper Algorithm; see Section 7 and the examples presented subsequently.

7 A Package for Computing Telescoping Contiguous Relations

In Section 4 we began to explain that telescoping and classical contiguous relations can be computed automatically — up to restrictions imposed by computational complexity — by creative telescoping. Each computer algebra package that implements Zeilberger's algorithm is in its essence based on the parameterized Gosper algorithm which executes creative telescoping. Consequently each of these packages can be easily adapted to contiguous relations computations.

Already at the time of the prototype version [22] of this article, Axel Riese has carried out such an adaption within the Paule–Schorn [19] package `fastZeil`, written in the Mathematica system and available from the Web at

`https://combinatorics.risc.jku.at/software`

To use this package, follow the installation instructions, open a Mathematica session, and read in the package as follows:

```
In[1]:= << RISC`fastZeil`
```

Fast Zeilberger Package version 3.61 written by Peter Paule, Markus Schorn, and Axel Riese © RISC-JKU

For better readability, we write the rising factorials in 'pretty print' format:

In[2]:= $(x\_)_{k\_}$ **:= Pochhammer[x, k]**

In[3]:= $\{(x)_0, (x)_1, (x)_2, (x)_5\}$

Out[3]= $\{1, x, x(1+x), x(1+x)(2+x)(3+x)(4+x)\}$

7.1 Computer discovery and proof of (67)

To do the example (67), resp. (68), we consider the problem to compute c_0, c_1, c_2, not all 0, and a polynomial $C(x)$ such that for all $k \geq 0$,

$$c_0\, t_0(k) + c_1\, t_1(k) + c_2\, t_2(k) = \Delta_k\, C(k)\, t(k), \tag{80}$$

where $t(k) = t_0(k)$, and

$$t_0(k) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)_k, t_1(k) = {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right)_k, t_2(k) = {}_2F_1\left(\begin{matrix} a, b+1 \\ c \end{matrix}; z\right)_k.$$

To invoke the package, we need to make explicit use of the similarity[¶] between the $t_j(k)$:

[¶] Recall Definition 2.

In[4]:= **t[a_, b_, c_, k_] := $\frac{(a)_k (b)_k}{(c)_k k!} z^k$**

In[5]:= **ra = t[a + 1, b, c, k]/t[a, b, c, k] //FullSimplify**

Out[5]= $\frac{a+k}{a}$

In[6]:= **rb = t[a, b + 1, c, k]/t[a, b, c, k] //FullSimplify**

Out[6]= $\frac{b+k}{b}$

Hence we have,

$$t_1(k) = \mathrm{ra} \cdot t(k) = \frac{a+k}{a} \cdot t(k) \text{ and } t_2(k) = \mathrm{rb} \cdot t(k) = \frac{b+k}{b} \cdot t(k).$$

To solve the problem related to (80), we call parameterized telescoping as follows:

In[7]:= **Gosper[t[a, b, c, k], {k, n1, n2}, Parameterized → {1, ra, rb}]**

If '-n1+n2' is a natural number, then: :

Out[7]= $\{\mathrm{Sum}[(a-b)F_0(k) - aF_1(k) + bF_2(k), \{k, n1, n2\}] = 0\}$

This output has to be interpreted as follows: in the setting

$$F_0(k) = t_0(k), F_1(k) = t_1(k), \text{ and } F_2(k) = t_2(k),$$

one has for all $n_j \in \mathbb{N}$ such that $n_1 \leq n_2$:

$$(a-b)\sum_{k=n_1}^{n_2} t(k) - a\sum_{k=n_1}^{n_2} t_1(k) + b\sum_{k=n_1}^{n_2} t_2(k) = 0. \tag{81}$$

For $n = n_1 = n_2$ this is (71).

Remark. As already mentioned, despite being a trivial relation on the summand level, relation (71), resp. (81), cannot be handled with the standard Zeilberger algorithm owing to the fact that we have shifts in *two* parameters: $a \longrightarrow a+1$ and $b \longrightarrow b+1$.

7.2 Computer discovery and proof of (73)

To do the example (73), resp. (75), we consider the problem to compute c_0, c_1, c_2, not all 0, and a polynomial $C(x)$ such that for all $k \geq 0$,

$$c_0\, t_0(k) + c_1\, t_1(k) + c_2\, t_2(k) = \Delta_k\, C(k)\, t(k), \tag{82}$$

where $t(k) = t_0(k)$, and

$$t_0(k) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)_k, t_1(k) = {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right)_k, t_2(k) = {}_2F_1\left(\begin{matrix} a+2, b \\ c \end{matrix}; z\right)_k.$$

To invoke the package, we need again use similarity between the $t_j(k)$:

In[8]:= $\mathbf{t[a\_, b\_, c\_, k\_] := \frac{(a)_k (b)_k}{(c)_k k!} z^k}$

In[9]:= **ra1 = t[a + 1, b, c, k]/t[a, b, c, k] //FullSimplify**

Out[9]= $\frac{a+k}{a}$

In[10]:= **ra2 = (t[a + 2, b, c, k]/t[a + 1, b, c, k] //FullSimplify) ***
(t[a + 1, b, c, k]/t[a, b, c, k] //FullSimplify)//Factor

Out[10]= $\frac{(a+k)(1+a+k)}{a(1+a)}$

Hence we have,

$$t_1(k) = \mathrm{ra1} \cdot t(k) = \frac{a+k}{a} \cdot t(k) \text{ and } t_2(k) = \mathrm{ra2} \cdot t(k) = \frac{(a+k)(1+a+k)}{a(1+a)} \cdot t(k).$$

To solve the problem related to (82), we again call parameterized telescoping:

In[11]:= **Gosper[t[a, b, c, k], {k, 0, n}, Parameterized → {1, ra1, ra2}]**

If ‘n’ is a natural number, then: :

Out[11]= $\{\mathrm{Sum}[-a(a-c+1)F_0(k) - a(az-2a-bz+c+z-2)F_1(k) + a(a+1)(z-1)F_2(k), \{k, 0, n\}] = \frac{(a+n)(b+n)z^{n+1}(a)_n(b)_n}{n!(c)_n}\}$

This output has to be interpreted as follows: taking

$$F_0(k) = t_0(k), F_1(k) = t_1(k), \text{ and } F_2(k) = t_2(k),$$

one has for all $n \in \mathbb{N}$:

$$\begin{aligned} a(a+1-c)\sum_{k=0}^{n} t(k) + a((a+1-b)z - 2(a+1) + c)\sum_{k=0}^{n} t_1(k) \\ + a(a+1)(1-z)\sum_{k=0}^{n} t_2(k) = -(n+1)(c+n) \cdot t(n+1). \end{aligned} \tag{83}$$

For $n \to \infty$ this relation becomes Gauß’ relation (73) since the right-hand side turns to zero owing to the limit property (23).

Finally, we note that subtracting from (83) the case $n-1$ results in

$$\begin{aligned} & a(a+1-c)t(n) + a((a+1-b)z - 2(a+1) + c)t_1(n) + a(a+1)(1-z)t_2(n) \\ & = -\Delta_n n(c+n-1)t(n), \end{aligned}$$

which confirms the choice of the c_i as in (79). With these c_i we obtained the desired telescoping contiguous relation of the form (75) with $C(x) = -x(c+x-1)$.

8 Telescoping Contiguous Relations for $z = 1$: Case A

An important class of contiguous relations concerns the case $z = 1$ and $p = q+1 \geq 1$; i.e., involving summands of the form

$$ {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k . \tag{84}$$

To establish versions of Theorem 1 for this situation, we need to refine further.

Definition 7 (Case-A condition) We say that the complex parameters in (84) satisfy the Case-A condition, if

$$\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i - q \notin \mathbb{Z}_{\geq 0}. \tag{85}$$

This section gives a Case-A version of Theorem 1; in the Sections 9 and 10 corresponding theorems, Theorem 1B and Theorem 1C, for other parameter conditions, Case-B and Case-C, respectively, are presented.

Theorem 1A. *Suppose $z = 1$ and $p = q + 1$. Let the complex parameters a_i and b_j satisfy the Case-A condition* (85)*. For $0 \leq l \leq q$ let $(\alpha_1^{(l)}, \ldots, \alpha_{q+1}^{(l)}, \beta_1^{(l)}, \ldots, \beta_q^{(l)})$ be pairwise different tuples with non-negative integer entries.*

Then there exist $c_0, \ldots, c_q$ in $\mathbb{K}$, not all 0, and a polynomial $C(x) \in \mathbb{K}[x]$ such that for all $k \geq 0$,

$$\sum_{l=0}^{q} c_l \cdot {}_{q+1}F_q\left(\begin{matrix} a_1 + \alpha_1^{(l)}, \ldots, a_{q+1} + \alpha_{q+1}^{(l)} \\ b_1 - \beta_1^{(l)}, \ldots, b_q - \beta_q^{(l)} \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k . \tag{86}$$

Moreover, $C(0) = 0$, and if $C(x) \neq 0$, for the polynomial degree of $C(x)$ one has

$$\deg C(x) \leq 1 + M \ \ \textit{where} \ \ M := \max_{0 \leq l \leq q} \{\alpha_1^{(l)} + \cdots + \alpha_{q+1}^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)}\}; \tag{87}$$

in addition,

$$\mathrm{Re}\Big(\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i\Big) > M \Rightarrow \lim_{k\to\infty} C(k)\, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k = 0. \tag{88}$$

Remark. According to [1, Thm. 2.1.2], the condition on the left-hand side of (88) is exactly the condition needed for the absolute convergence of all series

$$ {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(l)},\ldots,a_{q+1}+\alpha_{q+1}^{(l)} \\ b_1-\beta_1^{(l)},\ldots,b_q-\beta_q^{(l)} \end{matrix};1\right), \quad l\in\{0,\ldots,q\}. $$

Proof. Observe that when $z=1$,

$$ {}_{q+1}P_q(x) = x^q\Big(\sum_{i=1}^{q+1} a_i - \sum_{j=1}^{q} b_j + q\Big) + O(x^{q-1}), $$

hence $\deg {}_{q+1}P_q(x) = q$ and $\deg {}_{q+1}P_q^{(n)}(x) = n+q$. Using these degree estimates the statement is proven analogously to the proof of Theorem 1. The limit (88) follows from (87) by using,

$$ {}_{q+1}F_q\left(\begin{matrix} a_1,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix};1\right) \sim \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_{q+1})} k^{-1+\sum_i a_i-\sum_j b_j}, \quad k\to\infty; \tag{89} $$

see [1, proof of Thm. 2.1.2] for this asymptotic estimate. □

Remark. As in Theorem 1 one can allow arbitrary integer parameters instead of restricting to non-negative integers. More precisely, for arbitrary parameters $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ this gives a relation,

$$ \sum_{l=0}^{q} c_l\cdot {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(l)},\ldots,a_{q+1}+\alpha_{q+1}^{(l)} \\ b_1+\beta_1^{(l)},\ldots,b_q+\beta_q^{(l)} \end{matrix};1\right)_k = \Delta_k\, R(k)\, {}_{q+1}F_q\left(\begin{matrix} a_1,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix};1\right)_k, \tag{90} $$

with a *rational* function $R(x)\in\mathbb{K}(x)$ instead of a *polynomial* $C(x)\in\mathbb{K}[x]$. Again, as in (24), it is important to notice that because of possible poles of the $R(x)$, not all integer choices of $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ are admissible.

Definition 8 Also the relations (86) and (90) are called *telescoping contiguous relations*.

Corollary 5 *Any telescoping contiguous relation of the form* (86) *and, if poles of* $R(x)$ *cause no problem, in the version of* (90) *can be computed by the parameterized Gosper algorithm.*

Proof. Analogous to that for Corollary 1. □

We present some illustrating applications.

8.1 Telescoping Contiguous Relations for $z = 1$ and $(p, q) = (1, 0)$

Suppose $z = 1$ and $(p, q) = (1, 0)$. According to Theorem 1 there exist $c_0 \neq 0$ and a polynomial $C(x)$ with $C(0) = 0$ such that for all $k \geq 0$,

$$c_0 \cdot {}_1F_0\left(\begin{matrix} a+\alpha \\ - \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; 1\right)_k . \tag{91}$$

For all $\alpha \in \mathbb{Z}_{\geq 0}$ this is true if the case-A condition,

$$\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i - q = -a \notin \mathbb{Z}_{\geq 0}, \tag{92}$$

holds. It turns out that $C(x) = x/(a + \alpha)$, and summing (91) over k from 0 to n produces a telescoping hypergeometric sum,

$$\sum_{k=0}^{n} {}_1F_0\left(\begin{matrix} a+\alpha \\ - \end{matrix}; 1\right)_k = \sum_{k=0}^{n} (-1)^k \binom{-(a+\alpha)}{k} = \frac{n+1}{a+\alpha}\,\frac{(a)_{n+1}}{(n+1)!}. \tag{93}$$

Independently from Theorem 1A, this is obtained — including a confirmation of the Case-A condition — by our implementation of parameterized telescoping:

In[12]:= Gosper$\left[\frac{(\mathbf{a}+\boldsymbol{\alpha})_{\mathbf{k}}}{\mathbf{k!}}, \{\mathbf{k}, \mathbf{0}, \mathbf{n}\}\right]$

If ‘n’ is a natural number and $a + \alpha \neq 0$, then: :

Out[12]= Sum$\left[\frac{(a+\alpha)_k}{k!}, \{k, 0, n\}\right] = \frac{(a+\alpha+n)(a+\alpha)_n}{n!(a+\alpha)}$

Finally, we remark that applying the limit formula (88) the relation (93) turns into

$$\sum_{k=0}^{\infty} \frac{(a+\alpha)_k}{k!} = \sum_{k=0}^{\infty} (-1)^k \binom{-(a+\alpha)}{k} = 0,$$

matching $(1 - 1)^{-(a+\alpha)} = 0$.

8.2 Computer Proof of Gauß’ ${}_2F_1$ Summation

Most of the classical non-terminating ${}_pF_q$ series summation formulas can be proved using contiguous relations. With the means of telescoping contiguous relations the essential part of these proofs can be done automatically by the computer.

For example, let us take Gauß’ ${}_2F_1$ summation theorem [1, Thm. 2.2.2]:

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0. \tag{94}$$

The condition on the real part is to guarantee absolute convergence; see [1, Thm. 2.1.2]. We will follow a variant of the proof idea presented in [1, 2.2]. Its key ingredient is the contiguous relation

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{(c-1)(c-a-b-1)}{(c-a-1)(c-b-1)} {}_2F_1\left(\begin{matrix} a, b \\ c-1 \end{matrix}; 1\right). \tag{95}$$

Once this relation is found, the rest of the proof of (94) follows by unfolding (95),

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) &= \frac{(c-1)(c-a-b-1)}{(c-a-1)(c-b-1)} \frac{(c-2)(c-a-b-2)}{(c-a-2)(c-b-2)} {}_2F_1\left(\begin{matrix} a, b \\ c-2 \end{matrix}; 1\right) = \ldots \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot \frac{\Gamma(c-a-n)\Gamma(c-b-n)}{\Gamma(c-n)\Gamma(c-a-b-n)} {}_2F_1\left(\begin{matrix} a, b \\ c-n \end{matrix}; 1\right), \end{aligned}$$

and by observing that

$$\lim_{n\to\infty} \frac{\Gamma(c-a-n)\Gamma(c-b-n)}{\Gamma(c-n)\Gamma(c-a-b-n)} {}_2F_1\left(\begin{matrix} a, b \\ c-n \end{matrix}; 1\right) = 1.$$

Using the parameterized Gosper algorithm, relation (95) is found automatically as follows. For the ansatz

$$c_0 \cdot {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k + c_1 \cdot {}_2F_1\left(\begin{matrix} a, b \\ c-1 \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \tag{96}$$

the algorithm computes

$$c_0 = (c-a-1)(c-b-1),\ c_1 = -(c-1)(c-a-b-1), \quad \text{and } C(x) = x(x+c-1). \tag{97}$$

Consequently, $\deg C(x) \leq 1 + M$ where $M = 1$. To ensure convergence of all series involved, one has to require also $\mathrm{Re}(c-a-b) > 1 = M$. This allows to apply (88) after summing (96) over k from 0 to ∞, which gives the desired (95).

9 Telescoping Contiguous Relations for $z = 1$: Case B

The next refinement of Theorem 1 concerns the following violation of the Case-A condition

Definition 9 (Case-B condition) The parameters $a_1, \ldots, a_{q+1}$ and $b_1, \ldots, b_q$ satisfy the Case-B condition, if

$$\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i - q \in \mathbb{Z}_{\geq 1}. \tag{98}$$

Hence the remaining violation of the Case-A condition is when $\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i = q$; this is Case-C which is treated in Section 10.

Theorem 1B. *Suppose $z = 1$ and $p = q+1 \geq 2$. Let the complex parameters a_i and b_j satisfy the Case-B condition* (98) *and let $d = q-1$ or $d = q$. For $0 \leq l \leq d$ let $(\alpha_1^{(l)}, \dots, \alpha_{q+1}^{(l)}, \beta_1^{(l)}, \dots, \beta_q^{(l)})$ be pairwise different tuples with non-negative integer entries. Suppose,*

$$M := \max_{0 \leq l \leq d} \{\alpha_1^{(l)} + \cdots + \alpha_{q+1}^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)}\} < \sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i. \tag{99}$$

Then for $d = q-1$ or $d = q$ there exist $c_0, \dots, c_d$ in $\mathbb{K}$, not all 0, and a polynomial $C(x) \in \mathbb{K}[x]$ such that for all $k \geq 0$,

$$\sum_{l=0}^{d} c_l \cdot {}_{q+1}F_q \left(\begin{matrix} a_1 + \alpha_1^{(l)}, \dots, a_{q+1} + \alpha_{q+1}^{(l)} \\ b_1 - \beta_1^{(l)}, \dots, b_q - \beta_q^{(l)} \end{matrix} ; 1 \right)_k = \Delta_k\, C(k)\, {}_{q+1}F_q \left(\begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} ; 1 \right)_k. \tag{100}$$

Moreover, $C(0) = 0$, and

$$\deg C(x) \leq 1 + \sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i. \tag{101}$$

In addition, owing to (89)*, if strict inequality in* (101) *holds,*

$$\lim_{k\to\infty} C(k)_p F_q \left(\begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} ; 1 \right)_k = 0; \tag{102}$$

otherwise,

$$\lim_{k\to\infty} C(k)_{q+1} F_q \left(\begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} ; 1 \right)_k = \textit{leading coefficient of } C(x) \cdot \frac{\Gamma(b_1) \dots \Gamma(b_q)}{\Gamma(a_1) \dots \Gamma(a_{q+1})}. \tag{103}$$

Remark. According to [1, Thm. 2.1.2], the condition (99) is exactly the condition needed for the absolute convergence of all series

$${}_{q+1}F_q \left(\begin{matrix} a_1 + \alpha_1^{(l)}, \dots, a_{q+1} + \alpha_{q+1}^{(l)} \\ b_1 - \beta_1^{(l)}, \dots, b_q - \beta_q^{(l)} \end{matrix} ; 1 \right), \quad l \in \{0, \dots, q-1\}.$$

Proof. Let $k_0 := \sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i$, and suppose that $k_0 - q = \alpha \in \mathbb{Z}_{\geq 1}$. Observe that

$$\deg{}_{q+1}P_q(x) = q, \ldots, \deg{}_{q+1}P_q^{(\alpha-1)}(x) = q + \alpha - 1, \deg{}_{q+1}P_q^{(\alpha)}(x) = q + \alpha - 1, \tag{104}$$

but

$$\deg{}_{q+1}P_q^{(\alpha+1)}(x) = q + \alpha + 1.$$

Owing to (104), one can find a polynomial $P_1(x)$ with $\deg P_1(x) = \alpha$ such that

$$P(x) = \prod_{i=1}^{q+1}(x+a_i) \cdot P_1(x+1) - x\prod_{j=1}^{q}(x+b_j-1) \cdot P_1(x) = c \cdot x^{q-1} + O(x^{q-2}), \tag{105}$$

where c is some constant in $\mathbb{K}$. Analogously to the proof of Theorem 1, for

$$t_l(k) := {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(l)}, \ldots, a_{q+1}+\alpha_{q+1}^{(l)} \\ b_1-\beta_1^{(l)}, \ldots, b_q-\beta_q^{(l)} \end{matrix}; 1\right)_k \text{ and } t(k) := {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)$$

one carries out the following transformation into a telescoping form:

$$\begin{aligned} \sum_{l=0}^{d} c_l t_l(k) &= \sum_{l=0}^{d} c_l \prod_{i=1}^{q+1} \mu_{\alpha_i^{(l)}}(a_i;k) \prod_{j=1}^{q} \mu_{\beta_j^{(l)}}(b_j-\beta_j^{(l)};k)t(k) \\ &= \sum_{l=0}^{d} c_l M_l(k)t(k) \\ &= \big(\sum_{l=0}^{d} c_l\, U_l(k)\big)\, t(k) + \Delta_k \big(\sum_{l=0}^{d} c_l\, V_l(k)\big)\, {}_pR_q(k)\, t(k). \end{aligned}$$

For each $l \in \{0, \ldots, d\}$: if the constant c in (105) is non-zero, then $\deg U_l(x) \leq q-2$; otherwise, $\deg U_l(x) \leq q-1$. In the first case one has $d := q-1$; otherwise $d := q$. For both cases, with the same argument as in Theorem 1, there is a choice of the c_l, not all zero, such that $\sum_{l=0}^{d} c_l\, U_l(x) = 0$. Notice that for the transformation into a telescoping form we used the condition (99) for the estimate,

$$\deg M_l(x) = \alpha_1^{(l)} + \cdots + \alpha_{q+1}^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)} \leq M < k_0 = q + \alpha,$$

together with (104) and (105). To prove the remaining statements, one again uses the arguments as in the proof of Theorem 1. □

Remark. Again one can allow arbitrary integer parameters instead of restricting to non-negative integers. More precisely, for arbitrary parameters $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ this gives a relation,

$$\sum_{l=0}^{d} c_l \cdot {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(l)}, \ldots, a_{q+1}+\alpha_{q+1}^{(l)} \\ b_1+\beta_1^{(l)}, \ldots, b_q+\beta_q^{(l)} \end{matrix}; 1\right)_k = \Delta_k\, R(k)\, {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k, \tag{106}$$

with a *rational* function $R(x) \in \mathbb{K}(x)$ instead of a *polynomial* $C(x) \in \mathbb{K}[x]$. Again, as in (24), it is important to notice that because of possible poles of the $R(x)$, not all integer choices of $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ are admissible.

For the sake of better visibility, we state the criterion for the choice of d, which emerged from the proof, as a particular corollary.

Corollary 6 *Let c be the constant as in* (105). *Then a criterion for the choice of d in Theorem 1B is this:*

$$d = \begin{cases} q-1, & \text{if } c \neq 0 \\ q, & \text{if } c = 0 \end{cases}. \tag{107}$$

Definition 10 Also the relations (100) and (106) are called *telescoping contiguous relations*.

Corollary 7 *Any telescoping contiguous relation of the form* (86) *and, if poles of $R(x)$ cause no problem, in the version of* (90) *can be computed by the parameterized Gosper algorithm.*

Proof. Analogous to that for Corollary 1. □

An example is provided by the existence of the contiguous relation (11), representing the Pfaff relation (7), which is predicted by Theorem 1B with $d = 2 = q - 1$ and with $M = 4$ for the maximum of the shift sums; moreover, connecting to the proof of Theorem 1B, $k_0 = 5$. With the same data, Theorem 1B also implies the existence of the contiguous relation (12), representing the Pfaff relation (9).

The next subsections show two more illustrating examples.

9.1 Telescoping Contiguous Relations for $z = 1$ and $(p, q) = (2, 1)$

The minimal choice for Theorem 1B is $q = 1$; i.e., assuming that the constant c in Corollary 6 is non-zero, we seek for a relation,

$$c_0 \cdot {}_2F_1\left(\begin{matrix} a_1+\alpha_1, a_2+\alpha_2 \\ b_1-\beta_1 \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; 1\right)_k, \quad k \geq 0.$$

The Case-B condition is $b_1 - a_1 - a_2 - 1 := \alpha \in \mathbb{Z}_{\geq 1}$; hence $\alpha := 1$ is again a minimal choice. Finally, we need condition (99) to be satisfied. This means, $M := \alpha_1 + \alpha_2 + \beta_1 < b_1 - a_1 - a_2 = 2$, and $(\alpha_1, \alpha_2, \beta_1) = (0, 0, 0)$ is a minimal choice. Then Theorem 1B tells us that the sum

$$\sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k}{(a_1 + a_2 + 2)_k \, k!}$$

has a closed form by telescoping. In view of (99), other admissible choices are $(\alpha_1, \alpha_2, \beta_1) = (1, 0, 0)$ or $(\alpha_1, \alpha_2, \beta_1) = (0, 0, 0)$, which says that the sums

$$\sum_{k=0}^{n} \frac{(a_1 + 1)_k (a_2)_k}{(a_1 + a_2 + 2)_k \, k!} \quad \text{and} \quad \sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k}{(a_1 + a_2 + 1)_k \, k!}$$

are also telescoping.[‖] This can be confirmed by running Gosper's algorithm, for example:

In[13]:= $\mathrm{Gosper}\left[\frac{(\mathbf{a_1})_\mathbf{k} (\mathbf{a_2})_\mathbf{k}}{(\mathbf{a_1} + \mathbf{a_2} + \mathbf{1})_\mathbf{k} \mathbf{k!}}, \{\mathbf{k}, \mathbf{0}, \mathbf{n}\}\right]$

If 'n' is a natural number and $a_1 a_2 \neq 0$, then: :

Out[13]= $\mathrm{Sum}\left[\frac{(a_1)_k (a_2)_k}{(a_1 + a_2 + 1)_k k!}, \{k, 0, n\}\right] = \frac{(a_1 + n)(a_2 + n)}{a_1 a_2} \frac{(a_1)_n (a_2)_n}{(a_1 + a_2 + 1)_n n!}$

The algorithm computes $C(x) = x(x + a_1 + a_2)/(a_1 a_2)$, which means that we have equality in the bound estimate (101), and the limit (103) of Theorem 1B implies,

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(a_1 + a_2 + 1)_k \, k!} = \frac{1}{a_1 a_2} \frac{\Gamma(a_1 + a_2 + 1)}{\Gamma(a_1)\Gamma(a_2)} = \frac{\Gamma(a_1 + a_2 + 1)}{\Gamma(a_1 + 1)\Gamma(a_2 + 1)}.$$

Notice that this is a telescoping special case of Gauß' summation formula (94).

10 Telescoping Contiguous Relations for $z = 1$: Case C

It remains to consider the final possibility for a violation of the Case-A condition, namely, when $\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i = q$.

Definition 11 (Case-C condition) The parameters $a_1, \ldots, a_{q+1}$ and $b_1, \ldots, b_q$ satisfy the Case-C condition, if

$$\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i = q. \tag{108}$$

[‖] Notice that the right-hand sum is obtained by replacing a_1 with $a_1 - 1$ in the left sum.

Theorem 1C. *Suppose $z = 1$ and $p = q + 1 \geq 2$. Let the complex parameters a_i and b_j satisfy the Case-C condition* (108) *and let*

$$d := \begin{cases} q-1, & \text{if } \deg_{q+1}P_q(x) = q-1 \\ q, & \text{if } \deg_{q+1}P_q(x) < q-1 \end{cases}. \tag{109}$$

For $0 \leq l \leq d$ let $(\alpha_1^{(l)}, \ldots, \alpha_{q+1}^{(l)}, \beta_1^{(l)}, \ldots, \beta_q^{(l)})$ be pairwise different tuples with non-negative integer entries. Suppose,

$$M := \max_{0 \leq l \leq d} \{\alpha_1^{(l)} + \cdots + \alpha_{q+1}^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)}\} < q. \tag{110}$$

Then for d there exist $c_0, \ldots, c_d$ in $\mathbb{K}$, not all 0*, and a polynomial $C(x) \in \mathbb{K}[x]$ such that for all $k \geq 0$,*

$$\sum_{l=0}^{d} c_l \cdot {}_{q+1}F_q\left(\begin{matrix} a_1 + \alpha_1^{(l)}, \ldots, a_{q+1} + \alpha_{q+1}^{(l)} \\ b_1 - \beta_1^{(l)}, \ldots, b_q - \beta_q^{(l)} \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k. \tag{111}$$

Moreover, $C(0) = 0$, and

$$\deg C(x) \leq 1 + q. \tag{112}$$

In addition, owing to (89)*, if strict inequality in* (112) *holds,*

$$\lim_{k\to\infty} C(k)_p F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k = 0; \tag{113}$$

otherwise,

$$\lim_{k\to\infty} C(k)_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k = \text{leading coefficient of } C(x) \cdot \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_{q+1})}. \tag{114}$$

Remark. According to [1, Thm. 2.1.2], the condition (99) is exactly the condition needed for the absolute convergence of all series

$${}_{q+1}F_q\left(\begin{matrix} a_1 + \alpha_1^{(l)}, \ldots, a_{q+1} + \alpha_{q+1}^{(l)} \\ b_1 - \beta_1^{(l)}, \ldots, b_q - \beta_q^{(l)} \end{matrix}; 1\right), \quad l \in \{0, \ldots, q-1\}.$$

Proof. Observe that

$$\deg_{q+1}P_q(x) \leq q-1, \quad \text{but} \quad \deg_{q+1}P_q^{(n)}(x) = q+n, \quad n \geq 1; \tag{115}$$

hence the degree $q - 1$ provides a natural bound. Analogously to the proof of Theorem 1, for

$$t_l(k) := {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(l)},\dots,a_{q+1}+\alpha_{q+1}^{(l)} \\ b_1-\beta_1^{(l)},\dots,b_q-\beta_q^{(l)} \end{matrix};1\right)_k \text{ and } t(k) := {}_{q+1}F_q\left(\begin{matrix} a_1,\dots,a_{q+1} \\ b_1,\dots,b_q \end{matrix};1\right)$$

one carries out the following transformation into a telescoping form:

$$\begin{aligned}\sum_{l=0}^{q-1} c_l t_l(k) &= \sum_{l=0}^{q-1} c_l \prod_{i=1}^{q+1} \mu_{\alpha_i^{(l)}}(a_i;k) \prod_{j=1}^{q} \mu_{\beta_j^{(l)}}(b_j-\beta_j^{(l)};k)t(k) \\ &= \sum_{l=0}^{q-1} c_l M_l(k) t(k) \\ &= \big(\sum_{l=0}^{q-1} c_l\, U_l(k)\big)\, t(k) + \Delta_k \big(\sum_{l=0}^{q-1} c_l\, V_l(k)\big)\, {}_pR_q(k)\, t(k).\end{aligned}$$

For each $l \in \{0,\dots,d\}$: if ${}_{q+1}P_q(x) = q-1$ then $\deg U_l(x) \le q-2$; otherwise, $\deg U_l(x) \le q-1$. In the first case one has $d := q-1$; otherwise $d := q$. For both cases, with the same argument as in Theorem 1, there is a choice of the c_l, not all zero, such that $\sum_{l=0}^{d} c_l\, U_l(x) = 0$. Notice that for the transformation into a telescoping form we used the condition (110) for the estimate,

$$\deg M_l(x) = \alpha_1^{(l)} + \cdots + \alpha_{q+1}^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)} \le M < q.$$

To prove the remaining statements, one again uses the arguments as in the proof of Theorem 1. □

Remark. Again one can allow arbitrary integer parameters instead of restricting to non-negative integers. More precisely, for arbitrary parameters $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ this gives a relation,

$$\sum_{l=0}^{q-1} c_l \cdot {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(l)},\dots,a_{q+1}+\alpha_{q+1}^{(l)} \\ b_1+\beta_1^{(l)},\dots,b_q+\beta_q^{(l)} \end{matrix};1\right)_k = \Delta_k\, R(k)\, {}_{q+1}F_q\left(\begin{matrix} a_1,\dots,a_{q+1} \\ b_1,\dots,b_q \end{matrix};1\right)_k, \tag{116}$$

with a *rational* function $R(x) \in \mathbb{K}(x)$ instead of a *polynomial* $C(x) \in \mathbb{K}[x]$. Again, as in (24), it is important to notice that because of possible poles of the $R(x)$, not all integer choices of $\alpha_i^{(l)}$ and $\beta_j^{(l)}$ are admissible.

Definition 12 Also the relations (111) and (116) are called *telescoping contiguous relations*.

Corollary 8 *Any telescoping contiguous relation of the form* (111) *and, if poles of* $R(x)$ *cause no problem, in the version of* (116) *can be computed by the parameterized Gosper algorithm.*

Proof. Analogous to that for Corollary 1. □

An example is provided by the existence of the contiguous relation (10), representing the Zeilberger output recurrence (6), which is predicted by Theorem 1C with $d = \deg {}_4P_3(x) = 2 = q - 1$ and with $M = 2$ for the maximum of the shift sums. With the same data, Theorem 1C implies the contiguous relation (15) representing the new mixed relation (13) for Bailey's summation. To compute the relation as in (15), one uses the rational function variation (116) of Theorem 1C.

The next subsections show two more illustrating examples.

10.1 Telescoping Contiguous Relations for $z = 1$ and $(p, q) = (2, 1)$

The minimal choice for Theorem 1C is $q = 1$. Let us seek for a relation,

$$c_0 \cdot {}_2F_1\left(\begin{matrix} a_1 + \alpha_1, a_2 + \alpha_2 \\ b_1 - \beta_1 \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; 1\right)_k, \quad k \geq 0.$$

To guarantee existence, according to Theorem 1C we consider,

$$\begin{aligned} \deg {}_2P_1(x) &= \deg\Big((x + a_1)(x + a_2) - x(x + b_1 - 1)\Big) \\ &= \deg\Big((a_1 + a_2 - b_1 + 1)x + a_1 a_2\Big) = \deg a_1 a_2 = 0, \end{aligned}$$

invoking the Case-C condition, $b_1 - a_1 - a_2 = 1$. Hence, if $a_1 a_2 \neq 0$ then $\deg {}_2P_1(x) = 0 = q - 1$, and we can call Theorem 1C with $d = q - 1 = 0$. Moreover, we need $M := \alpha_1 + \alpha_2 + \beta_1 < q = 1$, thus $(\alpha_1, \alpha_2, \beta_1) = (0, 0, 0)$ is the only choice, and Theorem 1C tells us that the sum,

$$\sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k}{(a_1 + a_2 + 1)_k\, k!} = \frac{(a_1 + 1)_n (a_2 + 1)_n}{(a_1 + a_2 + 1)_n\, n!}, \tag{117}$$

has a closed form by telescoping; for the evaluation see `Out[13]`. In other words, this minimal case coincides with that of Theorem 1B presented in Section 9.1.

10.2 Telescoping Contiguous Relations for $z = 1$ and $(p, q) = (3, 2)$

We proceed with $q = 2$ as the next to minimal choice for Theorem 1C. Let us seek for a relation,

$$c_0 \cdot {}_3F_2\left(\begin{matrix} a_1+\alpha_1^{(0)}, a_2+\alpha_2^{(0)}, a_3+\alpha_3^{(0)} \\ b_1-\beta_1^{(0)}, b_2-\beta_1^{(0)} \end{matrix}; 1\right)_k + c_1 \cdot {}_3F_2\left(\begin{matrix} a_1+\alpha_1^{(1)}, a_2+\alpha_2^{(1)}, a_3+\alpha_3^{(1)} \\ b_1-\beta_1^{(1)}, b_2-\beta_1^{(1)} \end{matrix}; 1\right)_k$$
$$= \Delta_k\, C(k)\, {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; 1\right)_k, \quad k \geq 0.$$

To guarantee existence, according to Theorem 1C, we consider,

$$\deg {}_3P_2(x) = \deg\Big((x+a_1)(x+a_2)(x+a_3) - x(x+b_1-1)(x+b_2-1)\Big)$$
$$= \deg\Big((a_1+a_2+a_3-b_1-b_2+2)x^2 + (a_1a_2+a_1a_3+a_2a_3-(b_1-1)(b_2-1))x + a_1a_2a_3\Big)$$
$$= \deg\Big((a_1a_2+a_1a_3+a_2a_3-(b_1-1)(b_2-1))x + a_1a_2a_3\Big)$$

invoking the Case-C condition, $b_1+b_2-a_1-a_2-a_3=2$. Hence we assume that

$$a_1a_2+a_1a_3+a_2a_3-(b_1-1)(b_2-1) \neq 0,$$

because then $\deg {}_3P_2(x) = 1$, and we can call Theorem 1C with $d = q-1 = 1$. Moreover, we need to have,

$$M := \max_{0\leq l\leq 1}\{\alpha_1^{(l)}+\alpha_2^{(l)}+\alpha_3^{(l)}+\beta_1^{(l)}+\beta_2^{(l)}\} < q = 2.$$

Thus, $(\beta_2^{(0)}, \alpha_3^{(1)}) = (1,1)$ and all other parameters equal to zero, is an admissible choice. For this choice, Theorem 1C tells us that a non-trivial telescoping relation of the form

$$c_0 \cdot {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2-1 \end{matrix}; 1\right)_k + c_1 \cdot {}_3F_2\left(\begin{matrix} a_1, a_2, a_3+1 \\ b_1, b_2 \end{matrix}; 1\right)_k$$
$$= \Delta_k\, C(k)\, {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; 1\right)_k, \quad k \geq 0,$$

exists provided that $b_1+b_2-a_1-a_2-a_3=2$. As described in Section 7, the RISC package `fastZeil` by applying parameterized telescoping computes:

$$c_0 = -a_3(1+a_3-b_1)(1+a_1+a_2+a_3-b_1), c_1 = a_3(1+a_1+a_3-b_1)(1+a_2+a_3-b_1)$$

and

$$C(x) = x(x+b_1-1)(x+a_1+a_2+a_3-b_1+1);$$

for the computation b_2 is replaced by $a_1+a_2+a_3-b_1+2$. Applying (114) one obtains in the limit $k \to \infty$ the relation:

$$c_0 \cdot {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, a_1+a_2+a_3-b_1+1 \end{matrix}; 1\right) + c_1 \cdot {}_3F_2\left(\begin{matrix} a_1, a_2, a_3+1 \\ b_1, a_1+a_2+a_3-b_1+2 \end{matrix}; 1\right)$$
$$= \frac{\Gamma(b_1)\Gamma(a_1+a_2+a_3-b_1+2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}. \tag{118}$$

Another special case is obtained by replacing a_3 with $-n \in \mathbb{Z}_{\leq 0}$ and then taking the limit $k \longrightarrow \infty$,

$$S(n) = \frac{(b_1 - a_1 + n - 1)(b_1 - a_2 + n - 1)}{(b_1 + n - 1)(b_1 - a_1 - a_2 + n - 1)} S(n-1), \quad n \geq 1. \tag{119}$$

where

$$S(n) := {}_3F_2\left(\begin{matrix} a_1, a_2, -n \\ b_1, a_1 + a_2 - b_1 - (n+1) \end{matrix}; 1\right).$$

Unfolding the relation (119) gives the celebrated Pfaff-Saalschütz formula [1, Thm. 2.2.6],

$$S(n) = \frac{(b_1 - a_1)_n (b_1 - a_2)_n}{(b_1)_n (b_1 - a_1 - a_2)_n}. \tag{120}$$

Finally, setting $b_1 = -n$ in $S(n)$ gives the telescoping special case (117) of Gauß' summation formula.

11 Further Applications

The examples presented in this section should deepen the impression of a wide spectrum of potential applications of telescoping contiguous relations and the methods described. Using parameterized telescoping, we derive a generalization, Theorem 2, of a result which arose in the classical work by James Wilson on hypergeometric recurrences and contiguous relations. Two further subsections discuss non-minimality of Zeilberger recurrences from telescoping contiguous relations point of view. In particular, we explain why 'creative symmetrizing' in some instances successfully reduces the order. This discussion includes a new (algorithmic) proof of the non-terminating version of Dixon's well-poised ${}_3F_2$-series.

11.1 Generalizing a Theorem by James A. Wilson

As mentioned, in [1] various approaches for deriving contiguous relations are described, for instance, by integration or by using Wilson's method [38]. We choose an example that is given in [1, (3.7.5)] for explaining Wilson's technique, namely

$$\begin{aligned} &fg\,{}_4F_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; 1\right) - (f-a)(g-a)\,{}_4F_3\left(\begin{matrix} a, b+1, c+1, d+1 \\ e+1, f+1, g+1 \end{matrix}; 1\right) \\ &\quad + \frac{a(e-b)(e-c)(e-d)}{e(e+1)}\,{}_4F_3\left(\begin{matrix} a+1, b+1, c+1, d+1 \\ e+2, f+1, g+1 \end{matrix}; 1\right) = 0, \end{aligned} \tag{121}$$

where *one of the upper parameters is a negative integer*, and where

$$a + b + c + d + 1 = e + f + g. \tag{122}$$

Throughout this section we assume (122) to hold. To connect to classical terminology we remark that Wilson's contiguous relation is between *balanced* ${}_4F_3$-series; i.e., as in (122) the sum of the top parameters plus 1 equals the sum of the bottom parameters. More generally, if the parameters of a ${}_{q+1}F_q$-series satisfy the relation

$$\sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i = m \tag{123}$$

for $m \in \mathbb{Z}$, the series is called m-balanced.

To fit (121) into our framework we set

$$a_1 = a, a_2 = b, a_3 = c, a_4 = d, b_1 = e + 2, b_2 = f + 1, b_3 = g + 1,$$

which translates (121) into the telescoped version,

$$\begin{aligned} &c_0\, {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1 - 2, b_2 - 1, b_3 - 1 \end{matrix}; 1\right)_k + c_1\, {}_4F_3\left(\begin{matrix} a_1, a_2 + 1, a_3 + 1, a_4 + 1 \\ b_1 - 1, b_2, b_3 \end{matrix}; 1\right)_k \\ &\quad + c_2\, {}_4F_3\left(\begin{matrix} a_1 + 1, a_2 + 1, a_3 + 1, a_4 + 1 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k = \Delta_k C(k) {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k, \ k \geq 0, \end{aligned} \tag{124}$$

with

$$b_1 + b_2 + b_3 - (a_1 + a_2 + a_3 + a_4) - q = e + 2 + f + 1 + g + 1 - (a + b + c + d) - 3 = 2.$$

Consequently, this turns the series into ones which are 5-balanced and the Case-B condition (98) is satisfied. Moreover, in view of

$$M := \max_{0 \leq l \leq q-1} \{\alpha_1^{(l)} + \cdots + \alpha_{q+1}^{(l)} + \beta_1^{(l)} + \cdots + \beta_q^{(l)}\} = 4 < \sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i = 5,$$

the existence of (124) is guaranteed by Theorem 1B with $d = q - 1 = 2$, and where the coefficients c_j and the polynomial $C(x)$ can be computed by parameterized telescoping.

We remark explicitly that to this end, instead of renaming the variables, one can work directly in the original setting (121). More precisely, using our package we compute coefficients c_0, c_1, c_2, not all 0, and a polynomial $C(x)$ such that for all $k \geq 0$,

$$c_0\, t_0(k) + c_1\, t_1(k) + c_2\, t_2(k) = \Delta_k\, C(k)\, t(k), \tag{125}$$

where

$$t(k) = {}_4F_3\left(\begin{matrix} a, b, c, d \\ e+2, f+1, g+1 \end{matrix}; 1\right)_k,$$
$$t_0(k) = {}_4F_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; 1\right)_k,$$
$$t_1(k) = {}_4F_3\left(\begin{matrix} a, b+1, c+1, d+1 \\ e+1, f+1, g+1 \end{matrix}; 1\right)_k, \quad \text{and}$$
$$t_2(k) = {}_4F_3\left(\begin{matrix} a+1, b+1, c+1, d+1 \\ e+2, f+1, g+1 \end{matrix}; 1\right)_k.$$

For running the program we supply the rational functions r_l that are induced by the hypergeometric similarity relations

$$t_l(k) = r_l(k)\, t(k);$$

recall Definition 2. More concretely:

In[14]:= $\mathbf{t[k\_] := \frac{(a)_k(b)_k(c)_k(d)_k}{(e+2)_k(f+1)_k(g+1)_k k!}}$

In[15]:= $\mathbf{r0 = \frac{(a)_k(b)_k(c)_k(d)_k}{k!(e)_k(f)_k(g)_k}\frac{1}{t[k]}}$ **//FullSimplify**

Out[15]= $\frac{(k+e)(k+e+1)(k+f)(k+g)}{e(e+1)fg}$

In[16]:= $\mathbf{r1 = \frac{(a)_k(b+1)_k(c+1)_k(d+1)_k}{k!(e+1)_k(f+1)_k(g+1)_k}\frac{1}{t[k]}}$ **//FullSimplify**

Out[16]= $\frac{(k+b)(k+c)(k+d)(k+e+1)}{bcd(e+1)}$

In[17]:= $\mathbf{r2 = \frac{(a+1)_k(b+1)_k(c+1)_k(d+1)_k}{k!(e+2)_k(f+1)_k(g+1)_k}\frac{1}{t[k]}}$ **//FullSimplify**

Out[17]= $\frac{(k+a)(k+b)(k+c)(k+d)}{abcd}$

In[18]:= **RatFuMults = {r0, r1, r2} /. g → a + b + c + d − e − f + 1**

After these preparations we are ready to compute the desired telescoping relation:

In[19]:= **Gosper[t[k] /. g → a + b + c + d − e − f + 1, {k, 0, n − 1}, Parameterized → RatFuMults]**

If ‘n’ is a natural number, then: :

Out[19]= Sum[bcde(e + 1)f(a + b + c + d − e − f + 1)t₀(k) + bcde(e + 1)(a − f)(b + c + d − e − f + 1)t₁[k] − abcd(b − e)(c − e)(d − e)t₂[k], {k, 0, n − 1}]
$= -n(n+e+1)(n+f)(n+a+b+c+d-e-f+1)\left(bcd - bce - bde - ben - cde - cen - den - en^2\right)$
$\frac{(a)_n(b)_n(c)_n(d)_n}{n!(e+2)_n(f+1)_n(a+b+c+d-e-f+2)_n}$

In other words, the telescoping relation (125) is constituted by

$$c_0 = bcde(e+1)fg, \quad c_1 = -bcde(e+1)(f-a)(g-a),$$
$$c_2 = -abcd(b-e)(c-e)(d-e),$$

and the polynomial

$$C(x) = -x(x+e+1)(x+f)(x+g)\left(bcd - bce - bde - cde - (b+c+d)ex - ex^2\right).$$

We remark that after executing the call `In[19]` for parameterized telescoping, the program allows to retrieve the polynomial $C(x)$ explicitly with

```
In[20]:= show[R]

Out[20]= k(1 + e + k)(1 + a + b + c + d − e − f + k)(f + k)(−bcd + bce + bde + cde + (b + c + d)ek + ek^2)
```

Obviously, if one of the entries a, b, c, d is a negative integer, relation (125) in the limit $k \to \infty$ implies Wilson's relation (121). But applying the limit property (103), the telescoping relation (125) as a "bonus" implies a generalization of Wilson's (121), *which does not require that one of the upper parameters is a negative integer*:

Theorem 2 *If* $a + b + c + d + 1 = e + f + g$ *then*

$$\begin{aligned} &fg\,{}_4F_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; 1\right) - (f-a)(g-a)\,{}_4F_3\left(\begin{matrix} a, b+1, c+1, d+1 \\ e+1, f+1, g+1 \end{matrix}; 1\right) \\ &\quad + \frac{a(e-b)(e-c)(e-d)}{e(e+1)}\,{}_4F_3\left(\begin{matrix} a+1, b+1, c+1, d+1 \\ e+2, f+1, g+1 \end{matrix}; 1\right) \\ &= \frac{\Gamma(e+1)\Gamma(f+1)\Gamma(g+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c+1)\Gamma(d+1)}. \end{aligned} \tag{126}$$

11.2 Non-Minimality of Zeilberger Recurrences

Bailey's summation (2) already has shown that Zeilberger's algorithm does not always deliver a recurrence of minimal order for the sum in question. Another such example is the summation

$$S_d(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{d\,k}{n} = (-d)^n, \quad n \geq 0, \tag{127}$$

where d is any positive integer.

We remark that this evaluation is an immediate consequence of the following elementary fact which is implied by the binomial theorem. For any choice of complex numbers a_i,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_0 + a_1\,k + \cdots + a_n\,k^n) = (-1)^n n!\,a_n;$$

see, for instance, [12, Ch. 5].

However, in [19] it has been pointed out that the Zeilberger recurrence for $S_d(n)$, $d \geq 2$, is of order $d-1$. For instance, for $d=3$ by running Zeilberger's algorithm one obtains

$$2(2n+3)\,S_3(n+2) + 3(5n+7)\,S_3(n+1) + 9(n+1)\,S_3(n) = 0$$

as the output recurrence for the sum $S_3(n)$.

In order to consider the problem from contiguous relations point of view, we translate $S_3(n)$ for $n = 3m$ into hypergeometric notation. One can easily verify that

$$S_3(3m) = (-1)^m \binom{3m}{m} T(m) \qquad (m \geq 0) \tag{128}$$

where

$$T(m) = \sum_{k=0}^{2m} \frac{(-2m)_k (m+1/3)_k (m+2/3)_k}{(1/3)_k (2/3)_k\, k!} = {}_3F_2\left(\begin{matrix} -2m, m+1/3, m+2/3 \\ 1/3, 2/3 \end{matrix}; 1\right). \tag{129}$$

According to (127) we have for $m \geq 0$,

$$T(m) = (-1)^m \binom{3m}{m}^{-1} (-3)^{3m}. \tag{130}$$

The fact that the Zeilberger recurrence for $T(m)$ is of order 2 tells us that there is no contiguous relation with $c_l \in \mathbb{C}(m)$ of the form

$$c_0 \cdot {}_3F_2\left(\begin{matrix} -2m, m+1/3, m+2/3 \\ 1/3, 2/3 \end{matrix}; 1\right) + c_1 \cdot {}_3F_2\left(\begin{matrix} -2m-2, m+4/3, m+5/3 \\ 1/3, 2/3 \end{matrix}; 1\right) = 0,$$

where in the second ${}_3F_2$-series m is replaced by $m+1$.

However, one can try another ansatz for a (telescoping) contiguous relation, for instance,

$$\begin{aligned} &c_0\, {}_3F_2\left(\begin{matrix} -2m, m+1/3, m+2/3 \\ 1/3, 5/3 \end{matrix}; 1\right)_k + c_1\, {}_3F_2\left(\begin{matrix} -2m, m+1/3, m+2/3 \\ 1/3, 2/3 \end{matrix}; 1\right)_k \\ &+ c_2\, {}_3F_2\left(\begin{matrix} -2m-2, m+4/3, m+5/3 \\ 1/3, 2/3 \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_3F_2\left(\begin{matrix} -2m-2, m+1/3, m+2/3 \\ 1/3, 5/3 \end{matrix}; 1\right)_k . \end{aligned} \tag{131}$$

Summing (131) over k from 0 to $2m+2$ would then give the recurrence

$$c_0\, P(m) + c_1\, T(m) + c_2\, T(m+1) = 0, \quad m \geq 0, \tag{132}$$

where

$$P(m) = {}_3F_2\left(\begin{matrix}-2m, m+1/3, m+2/3\\ 1/3, 5/3\end{matrix}; 1\right) \tag{133}$$

is a *balanced* series which owing to the Pfaff-Saalschütz formula (120) evaluates to

$$P(m) = 0 \ \text{ for } \ m \geq 1. \tag{134}$$

Running the parameterized Gosper algorithm shows that a formula of type (131) indeed exists. Our package computes that (131), and thus (132), holds for

$$\begin{aligned}
c_0 &= 9m(3m-1)(21m^2+27m+8),\\
c_1 &= 18(m+1)(2m+1)(9m+1),\\
c_2 &= -(3m+1)(3m+2)(9m+1), \quad \text{and}\\
C(x) &= \frac{x(9x^2-4)\tilde{C}(x)}{2(m+1)(3m+2)(3m+4)}
\end{aligned}$$

where

$$\tilde{C}(x) = 3x(162m^3+405m^2+261m+40)-(3m+1)(189m^3+549m^2+555m+184).$$

Summarizing, in contrast to Zeilberger's algorithm the contiguous relations approach allows additional integer shifts in other parameters. So in the present example this enables one to invoke the Pfaff-Saalschütz evaluation (134) to zero, which finally has led to the desired order 1 recurrence for $T(m)$, namely

$$\frac{T(m+1)}{T(m)} = 18\frac{(m+1)(2m+1)}{(3m+1)(3m+2)},$$

which together with $T(0) = 1$ proves (130). *In other words, why Zeilberger's algorithm sometimes misses to compute the minimal recurrence simply is explained by the fact that this algorithm searches only within a restricted subclass of contiguous relations.*

Remark. We want to note explicitly that this example is remarkable also with regard to the existence of (131). Renaming the variables as follows,

$$a_1 = -2m-2, a_2 = m+1/3, a_3 = m+2/3, b_1 = 1/3, b_2 = 5/3,$$

turns (131) into

$$\begin{aligned}
&c_0 \cdot {}_3F_2\left(\begin{matrix}a_1+2, a_2, a_3\\ b_1, b_2\end{matrix}; 1\right)_k + c_1 \cdot {}_3F_2\left(\begin{matrix}a_1+2, a_2, a_3\\ b_1, b_2-1\end{matrix}; 1\right)_k\\
&+ \ c_2 \cdot {}_3F_2\left(\begin{matrix}a_1, a_2+1, a_3+1\\ b_1, b_2-1\end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_3F_2\left(\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; 1\right)_k.
\end{aligned} \tag{135}$$

In this case,

$b_1 + b_2 - (a_1 + a_2 + a_3) - q = 1/3 + 5/3 - (-2m - 2 + m + 1/3 + m + 2/3) - 2 = 1,$

which matches the Case-B condition (98). But

$$M := \max_{0\le l\le 2}\{\alpha_1^{(l)} + \alpha_2^{(l)} + \alpha_3^{(l)} + \beta_1^{(l)} + \beta_2^{(l)}\} = 3 < \sum_{j=1}^{2} b_j - \sum_{i=1}^{3} a_i = 3$$

violates the requirement (99), and thus the existence of (131) cannot be derived from the generic form of Theorem 1B with $d = q = 2$. Nevertheless, the following refinement for $q = 2$ applies.

Corollary 1B. *Suppose $z = 1$ and $p = q + 1 = 3$. Let the complex parameters a_i and b_j satisfy the Case-B condition* (98)*. For $0 \le l \le 2$ let $(\alpha_1^{(l)}, \alpha_2^{(l)}, \alpha_3^{(l)}, \beta_1^{(l)}, \beta_2^{(l)})$ be pairwise different tuples with non-negative integer entries such that*

$$M := \max_{0\le l\le 2}\{\alpha_1^{(l)} + \alpha_2^{(l)} + \alpha_3^{(l)} + \beta_1^{(l)} + \beta_2^{(l)}\} = 3. \tag{136}$$

Then there exist c_0, c_1, c_2 in $\mathbb{K}$, not all 0, and a polynomial $C(x) \in \mathbb{K}[x]$ such that for all $k \ge 0$,

$$\sum_{l=0}^{2} c_l \cdot {}_3F_2\left(\begin{matrix} a_1 + \alpha_1^{(l)}, a_2 + \alpha_2^{(l)}, a_3 + \alpha_3^{(l)} \\ b_1 - \beta_1^{(l)}, b_2 - \beta_2^{(l)} \end{matrix}; 1\right)_k = \Delta_k\, C(k)\, {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; 1\right)_k. \tag{137}$$

Proof. To prove the statement one modifies the proof of Theorem 1; we restrict to presenting a sketch. In the case of Corollary 1B we have $k_0 := \sum_{j=1}^{q} b_j - \sum_{i=1}^{q+1} a_i = 3$; this means, $k_0 - q = \alpha$ with $q = 2$ and $\alpha = 1$. Observe that

$$\deg {}_3P_2(x) = 2, \deg {}_3P_2^{(1)}(x) = 2, \text{ but } \deg {}_3P_2^{(2)}(x) = 4. \tag{138}$$

Owing to (138), one can find a polynomial $P_1(x)$ with $\deg P_1(x) = 1$, or $\deg P_1(x) = 0$, such that

$$P(x) = \prod_{i=1}^{3}(x + a_i) \cdot P_1(x + 1) - x \prod_{j=1}^{2}(x + b_j - 1) \cdot P_1(x) = p_1 x + p_0, \tag{139}$$

where $p_0, p_1 \in \mathbb{K}$ are not both zero. Suppose $p_1 \ne 0$. Then for $j = 1, 2$ there are $\gamma_j \in \mathbb{K}$ and polynomials $C_j(x) \in \mathbb{K}[x]$ such that

$$k^j\, t(k) = \gamma_j\, t(k) + \Delta_k C_j(x) t(k), \tag{140}$$

where $t(k) = {}_3F_2(a_1, a_2, a_3; b_1, b_2; 1)_k$. Again with the notation used in the proof of Theorem 1B, the left-hand side of (137) turns into,

$$\sum_{l=0}^{2} c_l t_k(k) = \sum_{l=0}^{2} c_l M_l(k) t(k),$$

with polynomials $M_l(x)$ of the form,

$$M_l(x) = \gamma_{l,0} + \gamma_{l,1} x + \gamma_{l,2} x^2 + \gamma_{l,3} x^3, \quad l = 0, 1, 2.$$

Finally, owing to (140),

$$\sum_{l=0}^{2} c_l M_l(k) t(k) = \big(\sum_{l=0}^{2} c_l\, U_l(k)\big)\, t(k) + \Delta_k\, \big(\sum_{l=0}^{2} c_l\, V_l(k)\big)\, {}_pR_q(k)\, t(k),$$

with $U_l(x)$ of the form $U_l(x) = u_{l,0} + u_{l,3} x^3$. Hence there exist $c_0, c_1, c_2 \in \mathbb{K}$, not all zero, such that $\sum_{l=0}^{2} c_l\, U_l(x) = 0$. □

11.3 Creative Symmetrizing Revisited

The discussion in Section 11.2 has shed new light on the fact that Zeilberger's algorithm does not always deliver a minimal recurrence for a given sum. In several such instances, by the method of 'creative symmetrizing', introduced in [18], it is possible to transform the original sum in such a way that for the transformed version Zeilberger's recurrence is minimal. In this section we shall see that the contiguous relations point of view can help to understand why creative symmetrizing can help.

As an illustrating example we consider a sum which Helmut Prodinger [24] has brought to our attention; namely for $n \geq 1$ let

$$S(n) = \sum_{k=1}^{n} (-1)^k \binom{n}{k}^2 \binom{n}{k-1}. \tag{141}$$

Carlitz [2], using Pfaff-Saalschütz summation, gave the evaluation

$$S(2m) = (-1)^m \frac{(3m)!}{(m!)^2\, (m-1)!\, (2m+1)} \tag{142}$$

for $m \geq 1$. However, he did not mention what happens if n is odd.

Before applying contiguous relations we explain what creative symmetrizing is about. Let $A(m) = S(2m)$ for $m \geq 1$. Again Zeilberger's algorithm does not deliver the first order recurrence corresponding to (142); rather than this it outputs,

$$
\begin{aligned}
&- 18(2m+1)(3m+1)(3m+2)(4m+7)(6m+5)(6m+7)\,A(m)\\
&- 12(2m+3)(4m+5)(36m^4+180m^3+341m^2+290m+90)\,A(m+1)\\
&- 2(m+1)(m+2)(2m+3)(2m+5)^2(4m+3)\,A(m+2) = 0,
\end{aligned}
$$

which together with the corresponding certificate, which is too huge to be displayed here, is sufficient to prove (142).

However, as observed by Axel Riese [28] creative symmetrizing reduces the order to the minimal one. Namely, consider

$$
\begin{aligned}
2A(m) &= \sum_{k=1}^{2m}(-1)^k\binom{2m}{k}^2\binom{2m}{k-1} + \sum_{k=1}^{2m}(-1)^{2m+1-k}\binom{2m}{2m+1-k}^2\binom{2m}{2m-k}\\
&= \sum_{k=1}^{2m}(-1)^k\binom{2m}{k}^2\binom{2m}{k-1}\left(1-\binom{2m}{k-1}\binom{2m}{k}^{-1}\right)\\
&= \sum_{k=1}^{2m}(-1)^k\,\frac{2m-2k+1}{2m-k+1}\binom{2m}{k}^2\binom{2m}{k-1}.
\end{aligned}
$$

This way we obtain an equivalent but transformed sum presentation $a(m)$ of $A(m)$, where

$$
a(m) = \sum_{k=1}^{2m}(-1)^k\,\frac{2m-2k+1}{2(2m-k+1)}\binom{2m}{k}^2\binom{2m}{k-1}. \tag{143}
$$

Now, when we take the summand of $a(m)$ as input for Zeilberger's algorithm, it outputs as recurrence for the sum $a(m)$,

$$
3(2m+1)(3m+1)(3m+2)\,a(m) + m(m+1)(2m+3)\,a(m+1) = 0 \tag{144}
$$

which, in view of $a(1) = A(1) = S(2) = -2$ immediately implies (142).

We note that in the odd case, i.e., if $n = 2m+1$, Zeilberger's algorithm again gives a second order recurrence; but creative symmetrizing also helps here. Namely, analogous to above, for $B(m) = S(2m+1)$, $m \geq 0$, one obtains an equivalent but transformed sum presentation $b(m)$, where

$$
b(m) = \sum_{k=1}^{2m+1}(-1)^k\,\frac{m+1}{2(2m-k+2)}\binom{2m+1}{k}^2\binom{2m+1}{k-1}. \tag{145}
$$

For this rearrangement, Zeilberger's algorithm again outputs the minimal recurrence, namely

$$
-3(3m+4)(3m+5)\,b(m) - (m+2)^2\,b(m+1) = 0. \tag{146}
$$

Consequently, since $b(0) = B(0) = S(1) = -1$, we obtain for $m \geq 0$,

$$
S(2m+1) = b(m) = (-1)^{m+1}\,\frac{(3m+2)!}{2\,(m+1)!^2\,m!} \tag{147}
$$

as a closed form evaluation for the odd case.

Summarizing, we have seen that creative symmetrizing, i.e., rearranging the summation by combining the first and the last summand, the second and the term before the last one, a.s.o., resulted in an order reduction of Zeilberger's output recurrence.

In the remaining part of this section we show that an explanation of this phenomenon is provided by the contiguous relations point of view.

To this end, let us consider the odd case (the even case can be treated analogously) and rewrite $B(m) = S(2m+1)$ into hypergeometric notation, i.e.,

$$B(m) = -(2m+1)^2\,{}_3F_2\left(\begin{matrix}-(2m+1), -2m, -2m\\ 2, 2\end{matrix}; 1\right). \tag{148}$$

The ${}_3F_2$ series is nearly-poised, this means, the second top and the first bottom parameter add up to the same number as the third top and the second bottom parameter; in the given example this is $-2m+2$. The series would be well-poised, if the remaining top parameter increased by 1 would be the same number. As we will explain below, well-poised series behaves "more nicely" with respect to (telescoping) contiguous relations.

First, we point out that creative symmetrizing converts the sum representation (145) into a (terminating) well-poised series; namely,

$$b(m) = -(m+1)(2m+1)\,{}_3F_2\left(\begin{matrix}-2m, -(2m+1), -(2m+1)\\ 2, 2\end{matrix}; 1\right). \tag{149}$$

In fact, this well-poised ${}_3F_2$ is the special case $a = -2m, b = c = -(2m+1)$ of Dixon's summation formula [1],

$${}_3F_2\left(\begin{matrix}a, b, c\\ a+1-b, a+1-c\end{matrix}; 1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+\frac{a}{2}-b-c)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)}, \tag{150}$$

where $\mathrm{Re}(1+\frac{1}{2}-b-c) > 0$. The substitution $a = -2m, b = c = -(2m+1)$ gives a hypergeometric term on the right-hand side of (150), hence $b(m)$ satisfies an order 1 recurrence.

Second, we explain why well-poised series behave better with respect to (telescoping) contiguous relations than nearly-poised series. Namely, Theorem 1A with $d = q = 2$ and parameterized telescoping gives,

$$\begin{aligned} &c_0 \cdot {}_3F_2\left(\begin{matrix}a, b, c\\ a+1-b, a+1-c\end{matrix}; 1\right)_k + c_1 \cdot {}_3F_2\left(\begin{matrix}a+1, b, c\\ a+2-b, a+2-c\end{matrix}; 1\right)_k \\ &+ c_2 \cdot {}_3F_2\left(\begin{matrix}a+2, b, c\\ a+3-b, a+3-c\end{matrix}; 1\right)_k = \Delta_k R(k)\,{}_3F_2\left(\begin{matrix}a, b, c\\ a+1-b, a+1-c\end{matrix}; 1\right)_k, \end{aligned} \tag{151}$$

where

$$\begin{aligned} c_0 &= -a(1+a-b)(2+a-b)(2+a-2b-2c)(1+a-c)(2+a-c),\\ c_1 &= 0,\\ c_2 &= a(1+a)(2+a-2b)(2+a-2c)(1+a-b-c)(2+a-b-c), \end{aligned}$$

and

$$R(k) = p(k)\frac{(-2-a+b)(-1-a+b)(-2-a+c)(-1-a+c)k}{(1+a-b+k)(1+a-c+k)}$$

with

$$\begin{aligned} p(k) = {} & -2-a+3a^2+2a^3+4b+ab-2a^2b-2b^2+4c+ac-2a^2c-6bc-abc\\ & +2b^2c-2c^2+2bc^2+3ak+3a^2k-2abk-2ack+ak^2. \end{aligned}$$

Remark. Notice that in view of the pattern of the shifts of the bottom parameters in the ${}_3F_2$-series, we applied Theorem 1A in the version of (106) which gives a *rational* function $R(x) \in \mathbb{K}(x)$ instead of a *polynomial* $C(x) \in \mathbb{K}[x]$.

Inspection of the coefficients c_j reveals the crucial feature of the well-poised property: it puts c_1 to zero!

Besides its relevance for our example, the fact that $c_1 = 0$ allows a proof of Dixon's identity along the same lines as our proof of Gauß' summation formula (94). Using the abbreviation,

$$F(a,b,c) := {}_3F_2\left(\begin{matrix} a, b, c \\ a+1-b, a+1-c \end{matrix}; 1\right),$$

in the limit $k \longrightarrow \infty$ relation (151) turns into

$$\begin{aligned} F(a,b,c) &= \frac{(1+a)(2+a-2b)(2+a-2c)(1+a-b-c)(2+a-b-c)}{(1+a-b)(2+a-b)(2+a-2b-2c)(1+a-c)(2+a-c)}F(a+2,b,c)\\ &= \frac{(a)_{2n}(\frac{a}{2}-b+1)_n(\frac{a}{2}-c+1)_n(a-b-c+1)_{2n}}{(\frac{a}{2})_n(\frac{a}{2}-b-c+1)_n(a-b+1)_{2n}(a-c+1)_{2n}}F(a+2n,b,c). \end{aligned}$$

Finally, applying

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^{x-1}}{(x)_n} \quad \text{and} \quad \lim_{n\to\infty}F(a+2n,b,c) = 1$$

proves (150).

Remark. Connecting to the remarks given after the statement of Theorem 1 in Section 4, we note that Dixon's identity (150) in the limit $c \longrightarrow \infty$ gives

$$ {}_2F_1\left(\begin{matrix} a, b \\ a+1-b \end{matrix}; -1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)}, \quad \mathrm{Re}(b) < 1, \tag{152}$$

which is Kummer's summation theorem [1, Cor. 3.1.2], Alternatively, one can compute the telescoping relation of the form,

$$
\begin{aligned}
&c_0 \cdot {}_2F_1\left(\begin{matrix}a, b\\ a+1-b\end{matrix}; -1\right)_k + c_1 \cdot {}_2F_1\left(\begin{matrix}a+1, b\\ a+2-b\end{matrix}; -1\right)_k \\
&+ c_2 \cdot {}_2F_1\left(\begin{matrix}a+2, b\\ a+3-b\end{matrix}; -1\right)_k = \Delta_k R(k) {}_2F_1\left(\begin{matrix}a, b\\ a+1-b\end{matrix}; -1\right)_k,
\end{aligned}
\tag{153}
$$

where $c_0 = -(a+1-b)(a+2-b)$, $c_1 = 0$, $c_2 = (a+1)(a+2-2b)$ and $R(x) = (a+1-b)(a+2-b)(b-1)x/(a(a+1-b+x))$. As with (151), this relation is a recurrence with shifts in a only, hence it can be computed already with Zeilberger's algorithm. However, we want to emphasize that its existence is predicted by Theorem 1 applied with the condition $p = 2 = q+1$ and $z = -1$. Finally note that Kummer's summation follows by taking the limit $k \to \infty$ in (153), and by iterating the resulting relation as we did to obtain the Dixon sum. To arrive at (152), one has to apply the binomial theorem [1, (2.1.6)] in the form,

$$
{}_1F_0\left(\begin{matrix}b\\ -\end{matrix}; -1\right) = 2^{-b}.
$$

12 Conclusion: q-Case

There are many variations like Corollary 1B of the method presented. Such variations depend on the particular application, needless to say. But even when facing a problem not generically covered by one of the theorems in this article, using a computer algebra implementation of parameterized telescoping could lead to the desired (telescoping) contiguous relation. Still this algorithmic possibility does not make tables of such relations obsolete. An excellent reference in this regard is [17], a huge collection of hypergeometric series summation and transformation identities including contiguous relations; most importantly, the table look-up is greatly supported by coming in the form of a Mathematica package.

Another aspect is that all what has been said in this article carries over to q-hypergeometric series and to q-contiguous relations. We are planning to treat the q-case in a subsequent paper.

As a kind of a "preview": already at the time of [22], Axel Riese has implemented a q-version of the algorithm for computing (telescoping) contiguous relations described in Section 4. This extension of his Mathematica package `qZeil` [21] allows to derive automatically (telescoping) q-contiguous relations, for example, those of Heine [8, Exercise 1.9]. Also in the scope are q-functional relations like

$$F(a,b;t) = \frac{1-atq}{1-t} + \frac{(1-aq)(b-atq)}{(1-bq)(1-t)} tqF(aq,bq;tq)$$

where

$$F(a,b;t) = 1 + \sum_{n=1}^{\infty} \frac{(1-aq)(1-aq^2)\cdots(1-aq^n)}{(1-bq)(1-bq^2)\cdots(1-bq^n)} t^n;$$

see, for instance, the book by N.J. Fine [7, (4.1)].

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