SFB AEC-Meeting, JKU, August 10, 2021

Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation

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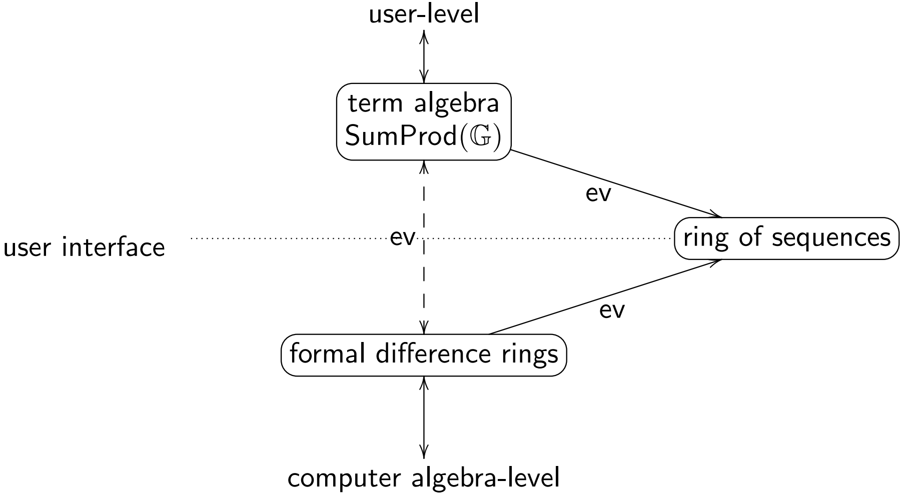
Der Wissenschaftsfonds.

Outline of the talk:



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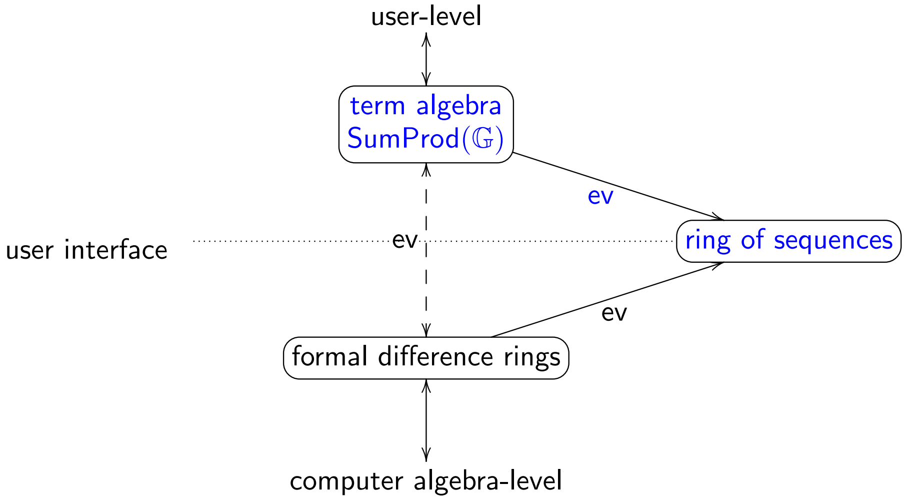

Outline of the talk:

Part 1: A term algebra for nested sums over hypergeometric products

Part 2: A canonical simplifier (justified by difference ring theory)

Part 3: Construction of appropriate difference rings

The ground field (throughout this talk): $\mathbb{G} = \mathbb{K}(x)$

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- For any element $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x]$ where $q \neq 0$ and p, q being coprime we define

$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases}$$

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- We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further,

$$Z(f) = \max(L(\tfrac{1}{p}), L(\tfrac{1}{q})) \qquad \text{if } f \neq 0.$$

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Example: For

$$f = \frac{p}{q} = \frac{x-4}{(x-3)(x-1)}$$

we get

$$(\mathrm{ev}(f, n))_{n \geq 0} = (-\tfrac{4}{3}, \underline{0}, 2, \underline{0}, 0, \tfrac{1}{8}, \dots) \in \mathbb{Q}^{\mathbb{N}}$$

For $n \geq L(f) = 4$ no poles arise;

for $n \geq Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) = \max(4, 5) = 5$ no zeroes arise.

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- We define

$$\mathcal{R} = \{r \in \mathbb{K} \setminus \{1\} \mid r \text{ is a root of unity}\}$$

with the function $\mathrm{ord} : \mathcal{R} \to \mathbb{Z}_{\geq 1}$ where

$$\mathrm{ord}(r) = \min\{n \in \mathbb{Z}_{\geq 1} \mid r^n = 1\}.$$

$\mathbb{G} \quad \longrightarrow \quad$ SumProd($\mathbb{G}$) (nested sums over hypergeometric products)

Let $\oslash$, $\oplus$, $\odot$, Sum, Prod and RPow be operations with the signatures

$$\begin{array}{llll} \oslash: & \mathrm{SumProd}(\mathbb{G}) \times \mathbb{Z} & \rightarrow & \mathrm{SumProd}(\mathbb{G}) \\ \oplus: & \mathrm{SumProd}(\mathbb{G}) \times \mathrm{SumProd}(\mathbb{G}) & \rightarrow & \mathrm{SumProd}(\mathbb{G}) \\ \odot: & \mathrm{SumProd}(\mathbb{G}) \times \mathrm{SumProd}(\mathbb{G}) & \rightarrow & \mathrm{SumProd}(\mathbb{G}) \\ \mathrm{Sum}: & \mathbb{N} \times \mathrm{SumProd}(\mathbb{G}) & \rightarrow & \mathrm{SumProd}(\mathbb{G}) \\ \mathrm{Prod}: & \mathbb{N} \times \mathrm{SumProd}(\mathbb{G}) & \rightarrow & \mathrm{SumProd}(\mathbb{G}) \\ \mathrm{RPow}: & \mathcal{R} & \rightarrow & \mathrm{SumProd}(\mathbb{G}). \end{array}$$

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Prod$^*(\mathbb{G})$= the smallest set that contains 1 with the following properties:

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Prod$^*(\mathbb{G})$= the smallest set that contains 1 with the following properties:

1. If $r \in \mathcal{R}$ then RPow$(r) \in$ Prod$^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$ and $l \in \mathbb{N}$ with $l \geq Z(f)$ then Prod$(l, f) \in$ Prod$^*(\mathbb{G})$.
3. If $p, q \in$ Prod$^*(\mathbb{G})$ then $p \odot q \in$ Prod$^*(\mathbb{G})$.
4. If $p \in$ Prod$^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p \owedge z \in$ Prod$^*(\mathbb{G})$.

Furthermore, we define

$$\Pi(\mathbb{G}) = \{\text{RPow}(r) \mid r \in \mathcal{R}\} \cup \{\text{Prod}(l, f) \mid f \in \mathbb{G}, l \in \mathbb{N}\}.$$

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Example: In $\mathbb{G} = \mathbb{Q}(x)$ we get

$$P = (\underbrace{\text{Prod}(1, x)}_{\in\Pi(\mathbb{G})}{}^{\textcircled{\wedge}}(-2)) \odot \underbrace{\text{RPow}(-1)}_{\Pi(\mathbb{G})} \in \text{Prod}^*(\mathbb{G}).$$

$\mathbb{G}$ $\longrightarrow$ SumProd($\mathbb{G}$) (nested sums over hypergeometric products)

SumProd($\mathbb{G}$) = the smallest set containing $\mathbb{G} \cup \text{Prod}^*(\mathbb{G})$ with:

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3. For all $f \in \text{SumProd}(\mathbb{G})$ and $k \in \mathbb{Z}_{\geq 1}$ we have $f^{\wedge\!\!\!\bigcirc} k \in \text{SumProd}(\mathbb{G})$.
4. For all $f \in \text{SumProd}(\mathbb{G})$ and $l \in \mathbb{N}$ we have $\text{Sum}(l, f) \in \text{SumProd}(\mathbb{G})$.

$\mathbb{G} \quad \longrightarrow \quad \text{SumProd}(\mathbb{G})$ (nested sums over hypergeometric products)

SumProd$(\mathbb{G})$ = the smallest set containing $\mathbb{G} \cup \text{Prod}^*(\mathbb{G})$ with:

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Furthermore, the **set of nested sums over hypergeometric products** is given by

$$\Sigma(\mathbb{G}) = \{\text{Sum}(l, f) \mid l \in \mathbb{N} \text{ and } f \in \text{SumProd}(\mathbb{G})\}$$

and the **set of nested sums and hypergeometric products** is given by

$$\Sigma\Pi(\mathbb{G}) = \Sigma(\mathbb{G}) \cup \Pi(\mathbb{G}).$$

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Example

With $\mathbb{G} = \mathbb{K}(x)$ we get, e.g., the following expressions:

$$E_1 = \text{Sum}(1, \text{Prod}(1, x)) \in \Sigma(\mathbb{G}) \subset \text{SumProd}(\mathbb{G}),$$
$$E_2 = \text{Sum}(1, \tfrac{1}{x+1} \odot \text{Sum}(1, \tfrac{1}{x^3}) \odot \text{Sum}(1, \tfrac{1}{x})) \in \Sigma(\mathbb{G}) \subset \text{SumProd}(\mathbb{G}),$$
$$E_3 = (E_1 \oplus E_2) \odot E_1 \in \text{SumProd}(\mathbb{G}).$$

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1. For $f, g \in \mathsf{SumProd}(\mathbb{G})$, $k \in \mathbb{Z} \setminus \{0\}$ ($k > 0$ if $f \notin \mathsf{Prod}^*(\mathbb{G})$) we set

$$\mathrm{ev}(f \circledwedge k, n) := \mathrm{ev}(f, n)^k,$$
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2. for $r \in \mathcal{R}$ and $\mathsf{Sum}(l, f), \mathsf{Prod}(\lambda, g) \in \mathsf{SumProd}(\mathbb{G})$ we define

$$\mathrm{ev}(\mathsf{RPow}(r), n) := \prod_{i=1}^{n} r = r^n,$$
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Note: $\Pi(\mathbb{G})$ defines all hypergeometric products (which evaluate to sequences with non-zero entries).

ev applied to $f \in \mathsf{SumProd}(\mathbb{G})$ represents a sequence.
f can be considered as a simple program and $\mathrm{ev}(f,n)$ with $n \in \mathbb{N}$ executes it (like an interpreter/compiler) yielding the nth entry of the represented sequence.

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For $E_i \in \mathsf{SumProd}(\mathbb{K}(x))$ with $i = 1,2,3$ we get

$$E_1(n) = \mathrm{ev}(E_1,n) = \mathrm{ev}(\mathsf{Sum}(1,\mathsf{Prod}(1,x)),n) = \sum_{k=1}^{n}\prod_{i=1}^{k} i = \sum_{k=1}^{n} k!,$$

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$$E_3(n) = (E_1(n) + E_2(n)) E_1(n)$$

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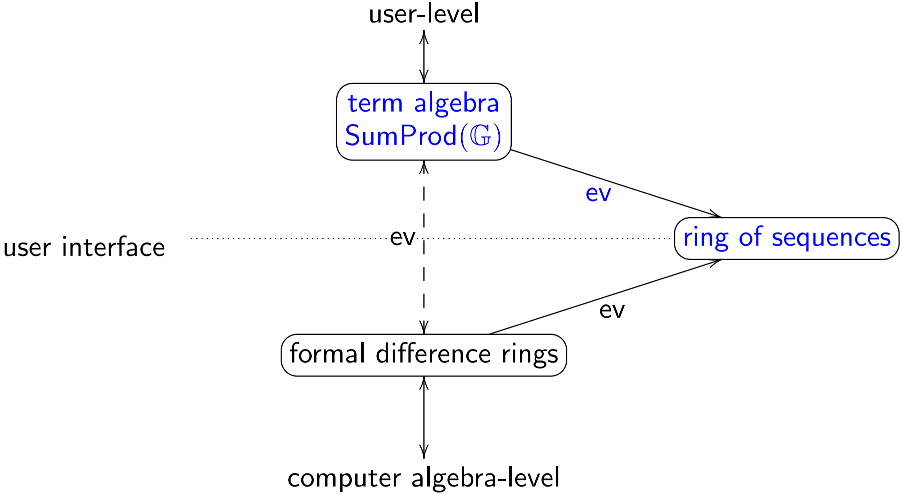

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- An expression $A \in \mathsf{SumProd}(\mathbb{G})$ is in **reduced representation** if

$$A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \cdots \oplus (f_r \odot P_r) \tag{1}$$

with $f_i \in \mathbb{G}^*$ and

$$P_i = (a_{i,1} \circledwedge z_{i,1}) \odot (a_{i,2} \circledwedge z_{i,2}) \odot \cdots \odot (a_{i,n_i} \circledwedge z_{i,n_i})$$

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for $1 \leq i \leq r$ where

- $a_{i,j} = \mathsf{Sum}(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in \mathsf{SumProd}(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z}_{\geq 1}$,
- $a_{i,j} = \mathsf{Prod}(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in \mathsf{Prod}^*(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z} \setminus \{0\}$,
- $a_{i,j} = \mathsf{RPow}(f_{i,j})$ with $f_{i,j} \in \mathcal{R}$ and $1 \leq z_{i,j} < \operatorname{ord}(r_{i,j})$

such that the following properties hold:

1. for each $1 \leq i \leq r$ and $1 \leq j < j' < n_i$ we have $a_{i,j} \neq a_{i,j'}$;
2. for each $1 \leq i < i' \leq r$ with $n_i = n_j$ there does not exist a $\sigma \in S_{n_i}$ with $P_{i'} = (a_{i,\sigma(1)} {}^{\owedge} z_{i,\sigma(1)}) \odot (a_{i,\sigma(2)} {}^{\owedge} z_{i,\sigma(2)}) \odot \cdots \odot (a_{i,\sigma(n_i)} {}^{\owedge} z_{i,\sigma(n_i)})$.

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    - A is in reduced representation as given in (1);
    - $l \geq \max(L(f_1), \ldots, L(f_r))$ (i.e., no poles occur);

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$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases}$$

- We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further,

$$Z(f) = \max(L(\tfrac{1}{p}), L(\tfrac{1}{q})) \qquad \text{if } f \neq 0.$$

Definition

- An expression $A \in \mathsf{SumProd}(\mathbb{G})$ is in **reduced representation** if

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with $f_i \in \mathbb{G}^*$

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Lemma

For any $A \in \mathit{SumProd}(\mathbb{G})$, there is a $B \in \mathit{SumProd}(\mathbb{G})$ in sum-product reduced representation and $\lambda \in \mathbb{N}$ such that

$$A(n) = B(n) \quad \forall n \geq \lambda.$$

Key-Definitions: Let $W \subseteq \Sigma\Pi(\mathbb{G})$.

SumProd$(W, \mathbb{G}) =$ the set of elements from SumProd$(\mathbb{G})$ which are in reduced representation and the arising sums/products are taken from W.

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- W is called **shift-closed over** $\mathbb{G}$ if for any $A \in$ SumProd$(W, \mathbb{G})$, $s \in \mathbb{Z}$ there are $B \in$ SumProd$(W, \mathbb{G})$ and $\delta \in \mathbb{N}$ such that

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- W is called **shift-stable over** $\mathbb{G}$ if for any product or sum in W the multiplicand or summand is built by sums and products from W.
- W is called **canonical reduced over** $\mathbb{G}$ if for any $A, B \in$ SumProd$(W, \mathbb{G})$ with
$$A(n) = B(n) \quad \forall n \geq \delta$$
for some $\delta \in \mathbb{N}$ the following holds: A and B are the same up to permutations of the operands in $\oplus$ and $\odot$.

Definition

$W \subseteq \Sigma\Pi(\mathbb{G})$ is called σ-**reduced over** $\mathbb{G}$ if

1. the elements in W are in sum-product reduced form,
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In particular, $A \in \mathsf{SumProd}(W, \mathbb{G})$ is called σ-**reduced (w.r.t.** W**)** if W is σ-reduced over $\mathbb{G}$.

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Problem SigmaReduce: Compute a σ-reduced representation

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$,
$B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$
such that for all $1 \leq i \leq r$ we get

$$A_i(n) = B_i(n) \quad n \geq \delta_i.$$

Application: Canonical representations in term algebras

A_1 A_2 in $\mathsf{SumProd}(\mathbb{G})$

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$$\mathsf{ev}(A_1, n) \quad \stackrel{?}{=} \quad \mathsf{ev}(A_2, n) \quad \text{in } \mathsf{SumProd}(\mathbb{G})$$

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A_1 $\qquad\qquad$ A_2 $\qquad$ in SumProd($\mathbb{G}$)

$\downarrow$ σ-reduced w.r.t. a W $\downarrow$

B_1 $\qquad\qquad$ B_2

$\forall n \in \mathbb{N}$ $\mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n)$ $\qquad$ $\mathsf{ev}(B_2, n) = \mathsf{ev}(A_2, n)$

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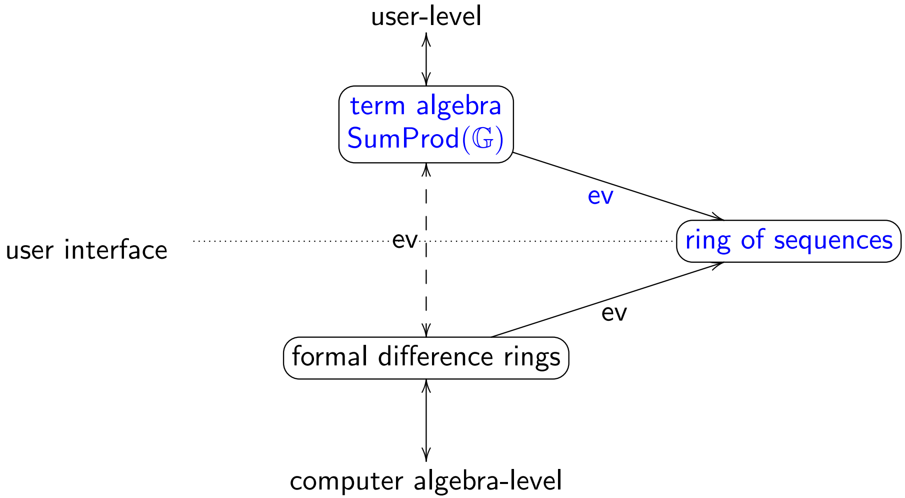
$\Updownarrow$ canonical simplifier

$B_1 = B_2$

Outline of the talk:



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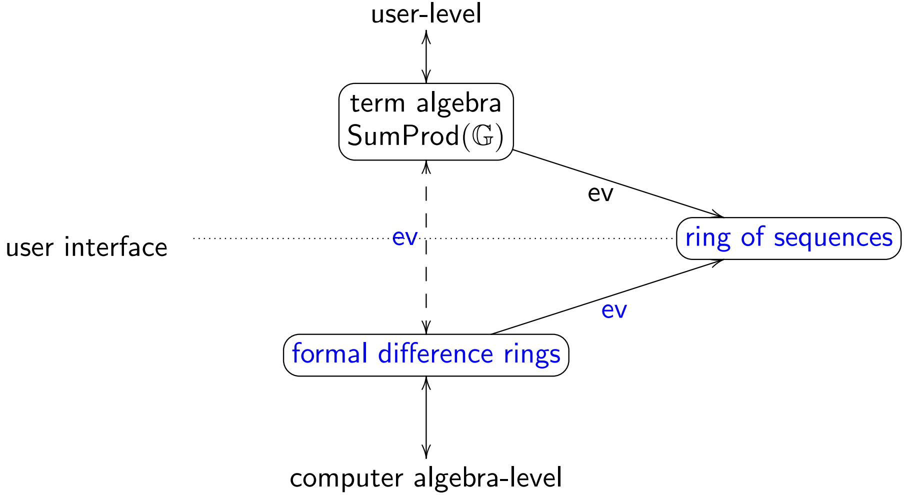

Outline of the talk:

Part 1: A term algebra for nested sums over hypergeometric products

Part 2: A canonical simplifier (justified by difference ring theory)

Part 3: Construction of appropriate difference rings

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

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$$\begin{aligned} \mathrm{ev} : \quad & \mathbb{Q}(x)[s] \times \mathbb{N} && \to \quad \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i \, s^i, n \right) && \mapsto \quad \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) H_n^i \end{aligned} \qquad \mathbf{ev(s, n) = H_n}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

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$$\begin{array}{rccc} \tau: & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f,n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

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It is an injective ring homomorphism (ring embedding):

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Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

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ev and σ interact:

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\text{ev}(\sigma(s), n) = \text{ev}(s + \tfrac{1}{x+1}, n) = H_n + \tfrac{1}{n+1} = \text{ev}(s, n+1)$$

$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\begin{array}{ccc} \mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\ \downarrow \tau & = & \downarrow \tau \\ \mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S \quad} & \mathbb{K}^{\mathbb{N}}/\sim \end{array}$$

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τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle H_n \rangle_{n \geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}} / \sim, S)$$

General construction

$$H \in \mathsf{SumProd}(\mathbb{G})$$

$$\Big\downarrow$$

- a formal ring $\mathbb{A} \supseteq \mathbb{G} \supseteq \mathbb{K}$ with $h \in \mathbb{A}$;
- an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ with $H(n) = \text{ev}(h, n)$;
- a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ with $H(n+1)$ with $\sigma(h)$.

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

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Sk!=(k+1)k!

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

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Sk!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

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hypergeometric products $\leftrightarrow$ $\sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$

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$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]$$

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$$\sigma(x) = x + 1$$

hypergeometric	$\leftrightarrow$	$\sigma(p_1) = a_1\, p_1$	$a_1 \in \mathbb{K}(x)^*$
products		$\sigma(p_2) = a_2 p_2$	$a_2 \in \mathbb{K}(x)^*$

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$$\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}]$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x) = x+1$$

hypergeometric products $\leftrightarrow$

$$\begin{aligned}\sigma(p_1) &= a_1\,p_1 & a_1 &\in \mathbb{K}(x)^*\\ \sigma(p_2) &= a_2p_2 & a_2 &\in \mathbb{K}(x)^*\\ &\vdots\\ \sigma(p_e) &= a_ep_e & a_e &\in \mathbb{K}(x)^*\end{aligned}$$

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- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z]$$

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$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 \qquad && a_1 \in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 && a_2 \in \mathbb{K}(x)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e && a_e \in \mathbb{K}(x)^* \end{aligned}$$

$$(-\mathbf{1})^{\mathbf{k}} \quad \leftrightarrow \quad \sigma(\mathbf{z}) = -\mathbf{z} \qquad \mathbf{z}^{\mathbf{2}} = \mathbf{1}$$

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hypergeometric products	$\leftrightarrow$	$\sigma(p_1)=a_1\,p_1$ $\sigma(p_2)=a_2p_2$ $\vdots$ $\sigma(p_e)=a_ep_e$	$a_1\in\mathbb{K}(x)^*$ $a_2\in\mathbb{K}(x)^*$ $a_e\in\mathbb{K}(x)^*$
γ is a primitive λth root of unity $\quad\gamma^{\mathbf{k}}$	$\leftrightarrow$	$\sigma(\mathbf{z})=\gamma\,\mathbf{z}$	$\mathbf{z}^{\lambda}=\mathbf{1}$

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- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1]}$$

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hypergeometric products $\leftrightarrow$

$$\begin{aligned}\sigma(p_1) &= a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2) &= a_2p_2 && a_2\in\mathbb{K}(x)^*\\ &\vdots\\ \sigma(p_e) &= a_ep_e && a_e\in\mathbb{K}(x)^*\end{aligned}$$

γ is a primitive λth root of unity $\quad \gamma^{\mathbf{k}} \leftrightarrow \sigma(\mathbf{z}) = \gamma\,\mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$

$$H_{k+1} = H_k + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{x+1}$$

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$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1]$$

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γ is a primitive λth root of unity $\gamma^{\mathbf{k}}$	$\leftrightarrow$	$\boldsymbol{\sigma(\mathbf{z}) = \gamma\, \mathbf{z}}$	$\mathbf{z}^{\lambda} = \mathbf{1}$
(nested) sum	$\leftrightarrow$	$\sigma(s_1) = s_1 + f_1$	$f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z]$

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- a ring

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$$\sigma(s_2) = s_2 + f_2 \qquad f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1]$$
$$\sigma(s_3) = s_3 + f_3 \qquad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2]$$
$$\vdots$$

Definition (Evaluation function)

Take $(\mathbb{A}, \sigma)$ with a subfield $\mathbb{K}$ of $\mathbb{A}$ with $\sigma|_{\mathbb{K}} = \text{id}$.

1. $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ is called **evaluation function** for $(\mathbb{A}, \sigma)$ if for all $f, g \in \mathbb{A}$, $c \in \mathbb{K}$ and $l \in \mathbb{Z}$ there exists a $\lambda \in \mathbb{N}$ with

$$\forall n \geq \lambda : \text{ev}(c, n) = c, \tag{2}$$

$$\forall n \geq \lambda : \text{ev}(f + g, n) = \text{ev}(f, n) + \text{ev}(g, n), \tag{3}$$

$$\forall n \geq \lambda : \text{ev}(f\, g, n) = \text{ev}(f, n)\ \text{ev}(g, n), \tag{4}$$

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2. $L : \mathbb{A} \to \mathbb{N}$ is called o-**function** if for any $f, g \in \mathbb{A}$ with $\lambda = \max(L(f), L(g))$ the properties (3) and (4) hold and for any $f \in \mathbb{A}$ and $l \in \mathbb{Z}$ with $\lambda = L(f) + \max(0, -l)$ property (5) holds.

Connection between SumProd($\mathbb{G}$) and hypergeometric APS-extension

$(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{G}\langle t_1\rangle\dots\langle t_e\rangle$ a hypergeometric APS-extension of $(\mathbb{G},\sigma)$
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$\forall n\geq L(t_i):$
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$W=\{T_1,\dots,T_e\}\subseteq\Sigma\Pi(\mathbb{G})$ is sum-product reduced and
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In particular, if $f\in\mathbb{E}\setminus\{0\}$, then we can take the "unique"
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Definition

For $f\in\mathbb{E}$ we also write expr$(f)=F$ for this particular F.

Connection between SumProd($\mathbb{G}$) and hypergeometric APS-extension

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$\mathrm{ev}(t_i,n)=T_i(n)\in\Sigma\Pi(\mathbb{G})$

$W=\{T_1,\dots,T_e\}\subseteq\Sigma\Pi(\mathbb{G})$ is sum-product reduced and shift stable: sums/products in T_i are from $\{T_1,\dots,T_{i-1}\}$.

Example

For $f=x+\frac{x+1}{x}s^4\in\mathbb{Q}(x)[s]$ we obtain

$$\text{expr}(f)=F=x\oplus(\tfrac{x+1}{x}\odot(\text{Sum}(1,\tfrac{1}{x})^{\circledwedge}4)\in\text{Sum}(\mathbb{Q}(x)))$$

with $F(n)=\mathrm{ev}(f,n)$ for all $n\geq 1$.

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$\downarrow$ $\quad$ $\forall n\geq L(t_i)$:
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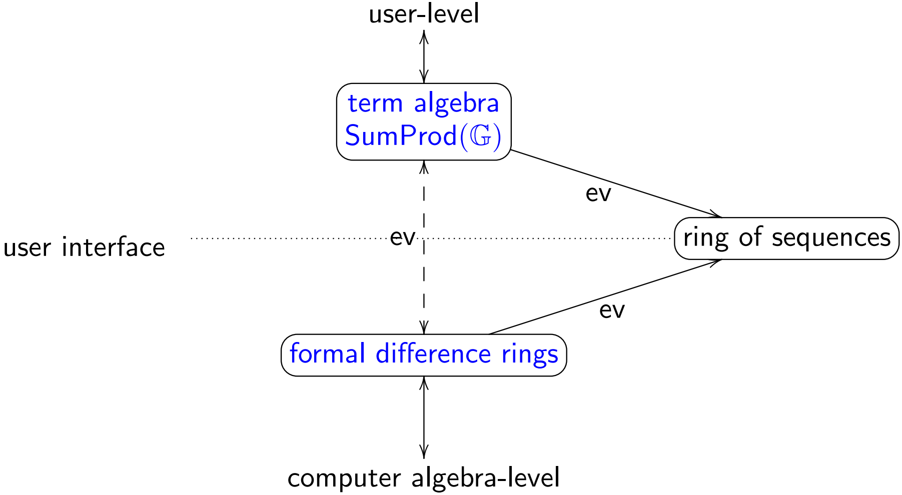
shift stable: sums/products in T_i are from $\{T_1,\dots,T_{i-1}\}$.

Outline of the talk:



Difference ring theory in action

Let $(\mathbb{E}, \sigma)$ be a hypergeometric APS-extension of $(\mathbb{G}, \sigma)$ with $\text{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ and let $\tau : \mathbb{E} \to \mathbb{K}^{\mathbb{N}}/\sim$ be the $\mathbb{K}$-homomorphism given by

$$\tau(f) = (\text{ev}(f, n))_{n \geq 0}.$$

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Lemma

Let $W = \{T_1, \ldots, T_e\} \in \Sigma\Pi(\mathbb{G})$ with $T_i = expr(t_i)$. Then:

$$W \text{ is canonical reduced} \quad \Leftrightarrow \quad \tau \text{ is injective.}$$

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Let $W=\{T_1,\dots,T_e\}\in\Sigma\Pi(\mathbb{G})$ *with* $T_i=$ *expr*(t_i). *Then:*

$$W \textit{ is canonical reduced} \quad\Leftrightarrow\quad \tau \textit{ is injective.}$$

Using difference ring theory we get the following crucial property:

Theorem

$$\tau \textit{ is injective} \quad\Leftrightarrow\quad \mathrm{const}_\sigma\mathbb{E}=\mathbb{K}.$$

Example

For our difference field $\mathbb{G} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$ and $\mathrm{const}_\sigma \mathbb{K} = \mathbb{K}$ we have $\mathrm{const}_\sigma \mathbb{K}(x) = \mathbb{K}$.

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Definition

A hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ is called **hypergeometric $R\Pi\Sigma$-extension** if

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A hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ is called **hypergeometric $R\Pi\Sigma$-extension** if

$$\mathrm{const}_\sigma\mathbb{E} = \mathbb{K}.$$

Theorem

Let $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ be in sum-product reduced representation and shift-stable, i.e., for each $1 \leq i \leq e$ the arising sums and products in T_i are contained in $\{T_1, \dots, T_{i-1}\}$. Then the following is equivalent:

1. *There is a hypergeometric $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ equipped with an evaluation function ev with $T_i = \mathit{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.*
2. *W is σ-reduced over $\mathbb{G}$.*

This yields a strategy (actually the only strategy for shift-stable sets).

A strategy to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.

Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

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1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \dots \langle t_e \rangle$ equipped with $\text{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with

$$A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}$$

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A strategy to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ equipped with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with
$$A_i(n) = \mathrm{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}$$
2. Set $W = \{T_1, \dots, T_e\}$ with $T_i := \mathsf{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.

This yields a strategy (actually the only strategy for shift-stable sets).

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A strategy to solve Problem SigmaReduce

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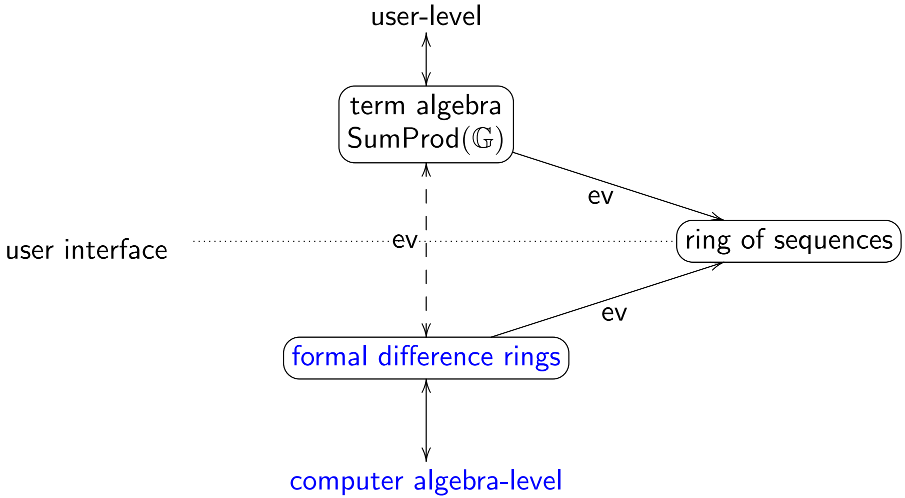
4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

Outline of the talk:



Outline of the talk:

Part 1: A term algebra for nested sums over hypergeometric products

Part 2: A canonical simplifier (justified by difference ring theory)

Part 3: Construction of appropriate difference rings

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 && a_1 \in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 && a_2 \in \mathbb{K}(x)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e && a_e \in \mathbb{K}(x)^* \end{aligned}$$

γ is a primitive λth root of unity $\gamma^{\mathbf{k}}$ $\leftrightarrow$

$$\sigma(\mathbf{z}) = \gamma\, \mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$$

(nested) sum $\leftrightarrow$

$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 && f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 && f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 && f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

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such that $\text{const}_\sigma\mathbb{E} = \mathbb{K}$

Represent sums (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant set

$$\text{const}_\sigma\mathbb{A} := \{k \in \mathbb{A} \mid \sigma(k) = k\}.$$

Note 1: $\text{const}_\sigma\mathbb{A}$ is a ring that contains $\mathbb{Q}$

Note 2: We always take care that $\text{const}_\sigma\mathbb{A}$ is a field

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A hypergeometric $R\Pi\Sigma$-extension of $(\mathbb{K}(x), \sigma)$ is

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The hypergeometric case

- Take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma|_{\mathbb{K}} = \text{id}$ and $\sigma(x) = x + 1$.
- Let $\alpha_1, \dots, \alpha_r \in \mathbb{K}(x)^*$

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Note: There are similar results for the q-rational, multi-basic and mixed case

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This yields a strategy (actually the only strategy for shift-stable sets).

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Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \dots \langle t_e \rangle$ equipped with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with
$$A_i(n) = \mathrm{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}$$
2. Set $W = \{T_1, \dots, T_e\}$ with $T_i := \mathsf{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.
3. Set $B_i := \mathsf{expr}(a_i) \in \mathsf{SumProd}(W, \mathbb{G})$ for $1 \leq i \leq u$.
4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

This yields a strategy (actually the only strategy for shift-stable sets).

An algorithm to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.

Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ equipped with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with
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4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

Conclusion. All results can be generalized to the following setting:

- the mixed multibasic hypergeometric case:
  $\mathbb{G} := \mathbb{K}(x, x_1, \ldots, x_v)$ with $\mathbb{K} = K(q_1, \ldots, q_v)$ For $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x, x_1, \ldots, x_v]$ where $q \neq 0$ and p, q being coprime we define

$$\operatorname{ev}(f, k) = \begin{cases} 0 & \text{if } q(k, q_1^k, \ldots, q_v^k) = 0 \\ \frac{p(k, q_1^k, \ldots, q_v^k)}{q(k, q_1^k, \ldots, q_v^k)} & \text{if } q(k, q_1^k, \ldots, q_v^k) \neq 0. \end{cases}$$

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- simple products: $\mathrm{Prod}^*(\mathbb{G})$ is the smallest set that contains 1 with:

1. If $r \in \mathcal{R}$ then $\mathrm{RPow}(r) \in \mathrm{Prod}^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$, $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\mathrm{Prod}(l, f \quad) \in \mathrm{Prod}^*(\mathbb{G})$.
3. If $p, q \in \mathrm{Prod}^*(\mathbb{G})$ then $p \odot q \in \mathrm{Prod}^*(\mathbb{G})$.
4. If $p \in \mathrm{Prod}^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p^{\textcircled{\wedge}} z \in \mathrm{Prod}^*(\mathbb{G})$.

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For further details see
Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. To appear in: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation, 2021. Springer, arXiv:2102.01471 [cs.SC]