Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz
Outline of the talk:

- User interface
- Formal difference rings
- Computer algebra-level
- Term algebra $\text{SumProd}(G)$
- User-level
Outline of the talk:

- Term algebra: SumProd($G$)
- User interface
- Formal difference rings
- Computer algebra-level
- Ring of sequences

Diagram:

- User-level
- Term algebra: SumProd($G$)
  - Ev
  - Ev
  - Ev
- Formal difference rings
  - Ev
- Computer algebra-level
  - Ring of sequences
Outline of the talk:

Part 1: A term algebra for nested sums over hypergeometric products

Part 2: A canonical simplifier (justified by difference ring theory)

Part 3: Construction of appropriate difference rings
The ground field (throughout this talk): $\mathbb{G} = \mathbb{K}(x)$
The ground field (throughout this talk): $G = K(x)$

For any element $f = \frac{p}{q} \in G$ with $p, q \in K[x]$ where $q \neq 0$ and $p, q$ being coprime we define

$$ev(f, k) = \begin{cases} 
0 & \text{if } q(k) = 0 \\
\frac{p(k)}{q(k)} & \text{if } q(k) \neq 0.
\end{cases}$$
The ground field (throughout this talk): $\mathbb{G} = \mathbb{K}(x)$

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$$ev(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases}$$

- We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further,

$$Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) \quad \text{if } f \neq 0.$$
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$$Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) \quad \text{if } f \neq 0.$$  

**Example:** For

$$f = \frac{p}{q} = \frac{x - 4}{(x - 3)(x - 1)}$$

we get

$$(\text{ev}(f, n))_{n \geq 0} = (-\frac{4}{3}, 0, 2, 0, 0, 1, 8, \ldots) \in \mathbb{Q}^\mathbb{N}$$

For $n \geq L(f) = 4$ no poles arise;

for $n \geq Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) = \max(4, 5) = 5$ no zeroes arise.
The ground field (throughout this talk): $\mathbb{G} = \mathbb{K}(x)$

- For any element $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x]$ where $q \neq 0$ and $p, q$ being coprime we define

$$\text{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases}$$

- We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further,

$$Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) \quad \text{if } f \neq 0.$$ 

- We define

$$\mathcal{R} = \{ r \in \mathbb{K} \setminus \{1\} \mid r \text{ is a root of unity} \}$$

with the function $\text{ord}: \mathcal{R} \to \mathbb{Z}_{\geq 1}$ where

$$\text{ord}(r) = \min\{ n \in \mathbb{Z}_{\geq 1} \mid r^n = 1 \}.$$
\( G \rightarrow \text{SumProd}(G) \) (nested sums over hypergeometric products)

Let \( \ominus, \oplus, \odot, \text{Sum}, \text{Prod} \) and \( \text{RPow} \) be operations with the signatures

\[
\begin{align*}
\ominus & : \text{SumProd}(G) \times \mathbb{Z} & \rightarrow & \text{SumProd}(G) \\
\oplus & : \text{SumProd}(G) \times \text{SumProd}(G) & \rightarrow & \text{SumProd}(G) \\
\odot & : \text{SumProd}(G) \times \text{SumProd}(G) & \rightarrow & \text{SumProd}(G) \\
\text{Sum} & : \mathbb{N} \times \text{SumProd}(G) & \rightarrow & \text{SumProd}(G) \\
\text{Prod} & : \mathbb{N} \times \text{SumProd}(G) & \rightarrow & \text{SumProd}(G) \\
\text{RPow} & : \mathcal{R} & \rightarrow & \text{SumProd}(G).
\end{align*}
\]
Let $\odot$, $\oplus$, $\otimes$, Sum, Prod and RPow be operations with the signatures

\[
\begin{align*}
\odot : & \quad \text{SumProd}(G) \times \mathbb{Z} \rightarrow \text{SumProd}(G) \\
\oplus : & \quad \text{SumProd}(G) \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\otimes : & \quad \text{SumProd}(G) \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\text{Sum} : & \quad \mathbb{N} \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\text{Prod} : & \quad \mathbb{N} \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\text{RPow} : & \quad \mathcal{R} \rightarrow \text{SumProd}(G).
\end{align*}
\]

$\text{Prod}^*(G) =$ the smallest set that contains 1 with the following properties:
Part 1: A term algebra for nested sums over hypergeometric products

Let $\otimes$, $\oplus$, $\odot$, $\text{Sum}$, $\text{Prod}$ and $\text{RPow}$ be operations with the signatures

\[
\begin{align*}
\otimes &: \text{SumProd}(G) \times \mathbb{Z} &\rightarrow &\text{SumProd}(G) \\
\oplus &: \text{SumProd}(G) \times \text{SumProd}(G) &\rightarrow &\text{SumProd}(G) \\
\odot &: \text{SumProd}(G) \times \text{SumProd}(G) &\rightarrow &\text{SumProd}(G) \\
\text{Sum} &: \mathbb{N} \times \text{SumProd}(G) &\rightarrow &\text{SumProd}(G) \\
\text{Prod} &: \mathbb{N} \times \text{SumProd}(G) &\rightarrow &\text{SumProd}(G) \\
\text{RPow} &: \mathcal{R} &\rightarrow &\text{SumProd}(G).
\end{align*}
\]

\(\text{Prod}^*(G)\) is the smallest set that contains 1 with the following properties:

1. If \(r \in \mathcal{R}\) then \(\text{RPow}(r) \in \text{Prod}^*(G)\).
2. If \(f \in \text{G}^*\) and \(l \in \mathbb{N}\) with \(l \geq Z(f)\) then \(\text{Prod}(l,f) \in \text{Prod}^*(G)\).
3. If \(p,q \in \text{Prod}^*(G)\) then \(p \odot q \in \text{Prod}^*(G)\).
4. If \(p \in \text{Prod}^*(G)\) and \(z \in \mathbb{Z} \setminus \{0\}\) then \(p \otimes z \in \text{Prod}^*(G)\).

Furthermore, we define

\[
\Pi(G) = \{\text{RPow}(r) \mid r \in \mathcal{R}\} \cup \{\text{Prod}(l,f) \mid f \in \text{G}, l \in \mathbb{N}\}.
\]
Part 1: A term algebra for nested sums over hypergeometric products

Let $\boxdot$, $\oplus$, $\circ$, Sum, Prod and $\text{RPow}$ be operations with the signatures

\[
\begin{align*}
\boxdot & : \text{SumProd}(G) \times \mathbb{Z} \rightarrow \text{SumProd}(G) \\
\oplus & : \text{SumProd}(G) \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\circ & : \text{SumProd}(G) \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\text{Sum} & : \mathbb{N} \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\text{Prod} & : \mathbb{N} \times \text{SumProd}(G) \rightarrow \text{SumProd}(G) \\
\text{RPow} & : \mathcal{R} \rightarrow \text{SumProd}(G).
\end{align*}
\]

$\text{Prod}^*(G)$ = the smallest set that contains 1 with the following properties:

1. If $r \in \mathcal{R}$ then $\text{RPow}(r) \in \text{Prod}^*(G)$.
2. If $f \in G^*$ and $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\text{Prod}(l, f) \in \text{Prod}^*(G)$.
3. If $p, q \in \text{Prod}^*(G)$ then $p \circ q \in \text{Prod}^*(G)$.
4. If $p \in \text{Prod}^*(G)$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p \boxdot z \in \text{Prod}^*(G)$.

**Example:** In $G = \mathbb{Q}(x)$ we get

\[
P = (\text{Prod}(1, x)^\boxdot(-2)) \circ \text{RPow}(-1) \in \text{Prod}^*(G).
\]
\[ \mathcal{G} \rightarrow \text{SumProd}(\mathcal{G}) \text{ (nested sums over hypergeometric products)} \]

**SumProd**($\mathcal{G}$) = the smallest set containing $\mathcal{G} \cup \text{Prod}^*(\mathcal{G})$ with:

1. For all $f, g \in \text{SumProd}(\mathcal{G})$ we have $f \oplus g \in \text{SumProd}(\mathcal{G})$.
2. For all $f, g \in \text{SumProd}(\mathcal{G})$ we have $f \odot g \in \text{SumProd}(\mathcal{G})$.
3. For all $f \in \text{SumProd}(\mathcal{G})$ and $k \in \mathbb{Z}_{\geq 1}$ we have $f \wedge k \in \text{SumProd}(\mathcal{G})$.
4. For all $f \in \text{SumProd}(\mathcal{G})$ and $l \in \mathbb{N}$ we have $\text{Sum}(l, f) \in \text{SumProd}(\mathcal{G})$. 
Part 1: A term algebra for nested sums over hypergeometric products

\( G \rightarrow \text{SumProd}(G) \) (nested sums over hypergeometric products)

**SumProd**\( (G) = \) the smallest set containing \( G \cup \text{Prod}^*(G) \) with:

1. For all \( f, g \in \text{SumProd}(G) \) we have \( f \oplus g \in \text{SumProd}(G) \).
2. For all \( f, g \in \text{SumProd}(G) \) we have \( f \circ g \in \text{SumProd}(G) \).
3. For all \( f \in \text{SumProd}(G) \) and \( k \in \mathbb{Z}_{\geq 1} \) we have \( f \otimes k \in \text{SumProd}(G) \).
4. For all \( f \in \text{SumProd}(G) \) and \( l \in \mathbb{N} \) we have \( \text{Sum}(l, f) \in \text{SumProd}(G) \).

Furthermore, the **set of nested sums over hypergeometric products** is given by

\[ \Sigma(G) = \{ \text{Sum}(l, f) \mid l \in \mathbb{N} \text{ and } f \in \text{SumProd}(G) \} \]

and the **set of nested sums and hypergeometric products** is given by

\[ \Sigma\Pi(G) = \Sigma(G) \cup \Pi(G) \].
SumProd($G$) = the smallest set containing $G \cup \text{Prod}^*(G)$ with:

1. For all $f, g \in \text{SumProd}(G)$ we have $f \oplus g \in \text{SumProd}(G)$.
2. For all $f, g \in \text{SumProd}(G)$ we have $f \odot g \in \text{SumProd}(G)$.
3. For all $f \in \text{SumProd}(G)$ and $k \in \mathbb{Z}_{\geq 1}$ we have $f \wedge k \in \text{SumProd}(G)$.
4. For all $f \in \text{SumProd}(G)$ and $l \in \mathbb{N}$ we have $\text{Sum}(l, f) \in \text{SumProd}(G)$.

Example

With $G = \mathbb{K}(x)$ we get, e.g., the following expressions:

$$E_1 = \text{Sum}(1, \text{Prod}(1, x)) \in \Sigma(G) \subset \text{SumProd}(G),$$
$$E_2 = \text{Sum}(1, \frac{1}{x+1} \odot \text{Sum}(1, \frac{1}{x^3}) \odot \text{Sum}(1, \frac{1}{x})) \in \Sigma(G) \subset \text{SumProd}(G),$$
$$E_3 = (E_1 \oplus E_2) \odot E_1 \in \text{SumProd}(G).$$
ev : $G \times N \rightarrow K$ \quad \rightarrow \quad ev : $\text{SumProd}(G) \times N \rightarrow K$

Note: $\Pi(G)$ defines all hypergeometric products (which evaluate to sequences with non-zero entries).
ev : $\mathbb{G} \times \mathbb{N} \rightarrow \mathbb{K}$ $\rightarrow$ ev : $\text{SumProd}($ $\mathbb{G}) \times \mathbb{N} \rightarrow \mathbb{K}$

1. For $f, g \in \text{SumProd}($ $\mathbb{G})$, $k \in \mathbb{Z}\setminus\{0\}$ ($k > 0$ if $f \notin \text{Prod}^*$($\mathbb{G}$)) we set

\begin{align*}
\text{ev}(f \wedge k, n) &:= \text{ev}(f, n)^k, \\
\text{ev}(f \oplus g, n) &:= \text{ev}(f, n) + \text{ev}(g, n), \\
\text{ev}(f \odot g, n) &:= \text{ev}(f, n) \text{ev}(g, n);
\end{align*}

Note: $\prod($ $\mathbb{G})$ defines all hypergeometric products (which evaluate to sequences with non-zero entries).
1. For \( f, g \in \text{SumProd}(\mathbb{G}) \), \( k \in \mathbb{Z} \setminus \{0\} \) (\( k > 0 \) if \( f \notin \text{Prod}^*(\mathbb{G}) \)) we set
   \[
ev(f \otimes k, n) := \ev(f, n)^k,
   \]
   \[
ev(f \oplus g, n) := \ev(f, n) + \ev(g, n),
   \]
   \[
ev(f \odot g, n) := \ev(f, n) \ev(g, n);
   \]

2. For \( r \in \mathbb{R} \) and \( \text{Sum}(l, f), \text{Prod}(\lambda, g) \in \text{SumProd}(\mathbb{G}) \) we define
   \[
ev(RPow(r), n) := \prod_{i=1}^{n} r = r^n,
   \]
   \[
ev(\text{Sum}(l, f), n) := \sum_{i=l}^{n} \ev(f, i),
   \]
   \[
ev(\text{Prod}(\lambda, g), n) := \prod_{i=\lambda}^{n} \ev(g, i) = \prod_{i=\lambda}^{n} g(i).
   \]
ev : $G \times \mathbb{N} \rightarrow \mathbb{K}$ \quad \rightarrow \quad ev : SumProd($G$) \times \mathbb{N} \rightarrow \mathbb{K}

1. For $f, g \in \text{SumProd}(G)$, $k \in \mathbb{Z} \setminus \{0\}$ ($k > 0$ if $f \notin \text{Prod}^*(G)$) we set

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ev(f \otimes k, n) := ev(f, n)^k,$$
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ev(f \oplus g, n) := ev(f, n) + ev(g, n),$$
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ev(f \odot g, n) := ev(f, n) \cdot ev(g, n);$$

2. For $r \in \mathcal{R}$ and $\text{Sum}(l, f), \text{Prod}(\lambda, g) \in \text{SumProd}(G)$ we define

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ev(\text{RPow}(r), n) := \prod_{i=1}^{n} r = r^n,$$
$$
ev(\text{Sum}(l, f), n) := \sum_{i=l}^{n} ev(f, i),$$
$$
ev(\text{Prod}(\lambda, g), n) := \prod_{i=\lambda}^{n} ev(g, i) = \prod_{i=\lambda}^{n} g(i).$$

Note: $\Pi(G)$ defines all hypergeometric products (which evaluate to sequences with non-zero entries).
ev applied to \( f \in \text{SumProd}(G) \) represents a sequence.

\( f \) can be considered as a simple program and \( \text{ev}(f, n) \) with \( n \in \mathbb{N} \) executes it (like an interpreter/compiler) yielding the \( n \)th entry of the represented sequence.

**Definition**

For \( F \in \text{SumProd}(G) \) and \( n \in \mathbb{N} \) we write \( F(n) := \text{ev}(F, n) \).
ev applied to $f \in \text{SumProd}(G)$ represents a sequence. $f$ can be considered as a simple program and $\text{ev}(f, n)$ with $n \in \mathbb{N}$ executes it (like an interpreter/compiler) yielding the $n$th entry of the represented sequence.

**Definition**

For $F \in \text{SumProd}(G)$ and $n \in \mathbb{N}$ we write $F(n) := \text{ev}(F, n)$.

**Example**

For $E_i \in \text{SumProd}(\mathbb{K}(x))$ with $i = 1, 2, 3$ we get

$$E_1(n) = \text{ev}(E_1, n) = \text{ev}(\text{Sum}(1, \text{Prod}(1, x)), n) = \sum_{k=1}^{n} \prod_{i=1}^{k} i = \sum_{k=1}^{n} k!,$$
ev applied to $f \in \text{SumProd}(G)$ represents a sequence. $f$ can be considered as a simple program and $ev(f, n)$ with $n \in \mathbb{N}$ executes it (like an interpreter/compiler) yielding the $n$th entry of the represented sequence.

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For $F \in \text{SumProd}(G)$ and $n \in \mathbb{N}$ we write $F(n) := ev(F, n)$.

**Example**

For $E_i \in \text{SumProd}(K(x))$ with $i = 1, 2, 3$ we get

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$$E_2(n) = ev(\text{Sum}(1, \frac{1}{x+1} \circ \text{Sum}(1, \frac{1}{x^3}) \circ \text{Sum}(1, \frac{1}{x})), n)$$

$$= \sum_{k=1}^{n} \frac{1}{1+k} \left( \sum_{i=1}^{k} \frac{1}{i^3} \right) \sum_{i=1}^{k} \frac{1}{i}$$
ev applied to \( f \in \text{SumProd}(\mathbb{G}) \) represents a sequence. 
\( f \) can be considered as a simple program and \( \text{ev}(f, n) \) with \( n \in \mathbb{N} \) executes it (like an interpreter/compiler) yielding the \( n \)th entry of the represented sequence.

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For \( F \in \text{SumProd}(\mathbb{G}) \) and \( n \in \mathbb{N} \) we write \( F(n) := \text{ev}(F, n) \).

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\]

\[
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\]

\[
= \sum_{k=1}^{n} \frac{1}{1+k} \left( \sum_{i=1}^{k} \frac{1}{i^3} \right) \sum_{i=1}^{k} \frac{1}{i}
\]

\[
E_3(n) = (E_1(n) + E_2(n))E_1(n)
\]
Outline of the talk:
Definition

- An expression \( A \in \text{SumProd}(G) \) is in **reduced representation** if

\[
A = (f_1 \circ P_1) \oplus (f_2 \circ P_2) \oplus \cdots \oplus (f_r \circ P_r)
\]

(1)

with \( f_i \in G^* \) and

\[
P_i = (a_{i,1} \wedge z_{i,1}) \circ (a_{i,2} \wedge z_{i,2}) \circ \cdots \circ (a_{i,n_i} \wedge z_{i,n_i})
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Definition

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\]

for \( 1 \leq i \leq r \) where

- \( a_{i,j} = \text{Sum}(l_{i,j}, f_{i,j}) \) with \( l_{i,j} \in \mathbb{N}, f_{i,j} \in \text{SumProd}(G) \) and \( z_{i,j} \in \mathbb{Z}_{\geq 1} \),

- \( a_{i,j} = \text{Prod}(l_{i,j}, f_{i,j}) \) with \( l_{i,j} \in \mathbb{N}, f_{i,j} \in \text{Prod}^*(G) \) and \( z_{i,j} \in \mathbb{Z} \setminus \{0\} \),

- \( a_{i,j} = \text{RPow}(f_{i,j}) \) with \( f_{i,j} \in \mathcal{R} \) and \( 1 \leq z_{i,j} < \text{ord}(r_{i,j}) \)

such that the following properties hold:

1. for each \( 1 \leq i \leq r \) and \( 1 \leq j < j' < n_i \) we have \( a_{i,j} \neq a_{i,j'} \);

2. for each \( 1 \leq i < i' \leq r \) with \( n_i = n_{j'} \) there does not exist a \( \sigma \in S_{n_i} \) with \( P_{i'} = (a_{i,\sigma(1)} \land z_{i,\sigma(1)}) \circ (a_{i,\sigma(2)} \land z_{i,\sigma(2)}) \circ \cdots \circ (a_{i,\sigma(n_i)} \land z_{i,\sigma(n_i)}) \).
Definition

• An expression $A \in \text{SumProd}(\mathbb{G})$ is in **reduced representation** if

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\]

with $f_i \in \mathbb{G}^*$

• $H \in \text{SumProd}(\mathbb{G})$ is in **sum-product reduced representation** if
  
  ▶ it is in reduced representation;
Definition

• An expression $A \in \text{SumProd}(G)$ is in **reduced representation** if

$$A = (f_1 \circ P_1) \oplus (f_2 \circ P_2) \oplus \cdots \oplus (f_r \circ P_r)$$

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• $H \in \text{SumProd}(G)$ is in **sum-product reduced representation** if
  ▶ it is in reduced representation;
  ▶ for each Sum$(l, A)$ and Prod$(l, A)$ in $H$ the following holds:
    ▶ $A$ is in reduced representation as given in (1);
Definition

- An expression $A \in \text{SumProd}(\mathbb{G})$ is in **reduced representation** if

$$A = (f_1 \circ P_1) \oplus (f_2 \circ P_2) \oplus \cdots \oplus (f_r \circ P_r)$$

(1)

with $f_i \in \mathbb{G}^*$

- $H \in \text{SumProd}(\mathbb{G})$ is in **sum-product reduced representation** if
  - it is in reduced representation;
  - for each $\text{Sum}(l, A)$ and $\text{Prod}(l, A)$ in $H$ the following holds:
    - $A$ is in reduced representation as given in (1);
    - $l \geq \max(L(f_1), \ldots, L(f_r))$ (i.e., no poles occur);
The ground field (throughout this talk): \( \mathbb{G} = \mathbb{K}(x) \)

- For any element \( f = \frac{p}{q} \in \mathbb{G} \) with \( p, q \in \mathbb{K}[x] \) where \( q \neq 0 \) and \( p, q \) being coprime we define

\[
\text{ev}(f, k) = \begin{cases} 
0 & \text{if } q(k) = 0 \\
\frac{p(k)}{q(k)} & \text{if } q(k) \neq 0.
\end{cases}
\]

- We define \( L(f) \) to be the minimal value \( \delta \in \mathbb{N} \) such that \( q(k) \neq 0 \) holds for all \( k \geq \delta \); further,

\[
Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) \quad \text{if } f \neq 0.
\]
Definition

• An expression $A \in \text{SumProd}(\mathbb{G})$ is in **reduced representation** if

\begin{equation}
A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \cdots \oplus (f_r \odot P_r) \tag{1}
\end{equation}

with $f_i \in \mathbb{G}^*$

• $H \in \text{SumProd}(\mathbb{G})$ is in **sum-product reduced representation** if
  
  ▶ it is in reduced representation;
  
  ▶ for each $\text{Sum}(l, A)$ and $\text{Prod}(l, A)$ in $H$ the following holds:
    
    ▶ $A$ is in reduced representation as given in (1);
    
    ▶ $l \geq \max(L(f_1), \ldots, L(f_r))$ (i.e., no poles occur);
    
    ▶ the lower bound $l$ is greater than or equal to the lower bounds of the sums and products inside of $A$. 

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Definition

- An expression $A \in \text{SumProd}(\mathbb{G})$ is in **reduced representation** if

  $$A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \cdots \oplus (f_r \odot P_r)$$  \hspace{1cm} (1)

  with $f_i \in \mathbb{G}^*$

- $H \in \text{SumProd}(\mathbb{G})$ is in **sum-product reduced representation** if
  
  - it is in reduced representation;
  - for each $\text{Sum}(l, A)$ and $\text{Prod}(l, A)$ in $H$ the following holds:
    - $A$ is in reduced representation as given in (1);
    - $l \geq \max(L(f_1), \ldots, L(f_r))$ (i.e., no poles occur);
    - the lower bound $l$ is greater than or equal to the lower bounds of the sums and products inside of $A$.

Example

$E_3 = (E_1 \oplus E_2) \odot E_1$ is not in reduced representation
Definition

- An expression $A \in \text{SumProd}(G)$ is in **reduced representation** if

\[
A = (f_1 \circ P_1) \oplus (f_2 \circ P_2) \oplus \cdots \oplus (f_r \circ P_r)
\]  

(1)

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**Definition**

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**Lemma**

*For any $A \in \text{SumProd}(\mathbb{G})$, there is a $B \in \text{SumProd}(\mathbb{G})$ in sum-product reduced representation and $\lambda \in \mathbb{N}$ such that*

$$A(n) = B(n) \ \forall n \geq \lambda.$$
Key-Definitions: Let $W \subseteq \Sigma \Pi(G)$.

$\text{SumProd}(W, G) =$ the set of elements from $\text{SumProd}(G)$ which are in reduced representation and the arising sums/products are taken from $W$. 
**Key-Definitions:** Let $W \subseteq \Sigma \Pi(G)$.

**SumProd**$(W, G) =$ the set of elements from SumProd$(G)$ which are in reduced representation and the arising sums/products are taken from $W$.

- $W$ is called **shift-closed over** $G$ if for any $A \in \text{SumProd}(W, G), s \in \mathbb{Z}$ there are $B \in \text{SumProd}(W, G)$ and $\delta \in \mathbb{N}$ such that
  \[
  A(n + s) = B(n) \quad \forall n \geq \delta.
  \]
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- $W$ is called **shift-stable over** $G$ if for any product or sum in $W$ the multiplicand or summand is built by sums and products from $W$. 
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Example

$W = \{ \text{Sum}(1, \text{Sum}(1, \frac{1}{x}), \frac{1}{x}) \}$ is neither shift-closed nor shift-stable;
Part 1: A term algebra for nested sums over hypergeometric products

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Example

$W = \{\text{Sum}(1, \text{Sum}(1, \frac{1}{x}, \frac{1}{x}))\}$ is neither shift-closed nor shift-stable;

$W = \{\text{Sum}(1, \frac{1}{x}), \text{Sum}(1, \text{Sum}(1, \frac{1}{x}), \frac{1}{x})\}$ is shift-closed and shift-stable;
Key-Definitions: Let \( W \subseteq \Sigma \Pi(G) \).

\( \text{SumProd}(W, G) = \) the set of elements from \( \text{SumProd}(G) \) which are in reduced representation and the arising sums/products are taken from \( W \).

\( W \) is called **shift-closed over** \( G \) if for any \( A \in \text{SumProd}(W, G) \), \( s \in \mathbb{Z} \) there are \( B \in \text{SumProd}(W, G) \) and \( \delta \in \mathbb{N} \) such that

\[
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Example

\( W = \{\text{Sum}(1, \text{Sum}(1, \frac{1}{x}), \frac{1}{x})\} \) is neither shift-closed nor shift-stable;

\( W = \{\text{Sum}(1, \frac{1}{x}), \text{Sum}(1, \text{Sum}(1, \frac{1}{x}), \frac{1}{x})\} \) is shift-closed and shift-stable;

\[
W \text{ is shift-stable} \not\Rightarrow \ not \not\Rightarrow W \text{ is shift-closed}
\]
Key-Definitions: Let $W \subseteq \Sigma \Pi(G)$.

$\text{SumProd}(W, G) =$ the set of elements from $\text{SumProd}(G)$ which are in reduced representation and the arising sums/products are taken from $W$.

$\triangleright$ $W$ is called **shift-closed over** $G$ if for any $A \in \text{SumProd}(W, G)$, $s \in \mathbb{Z}$ there are $B \in \text{SumProd}(W, G)$ and $\delta \in \mathbb{N}$ such that

$$A(n + s) = B(n) \quad \forall n \geq \delta.$$

$\triangleright$ $W$ is called **shift-stable over** $G$ if for any product or sum in $W$ the multiplicand or summand is built by sums and products from $W$.

$\triangleright$ $W$ is called **canonical reduced over** $G$ if for any $A, B \in \text{SumProd}(W, G)$ with

$$A(n) = B(n) \quad \forall n \geq \delta$$

for some $\delta \in \mathbb{N}$ the following holds: $A$ and $B$ are the same up to permutations of the operands in $\oplus$ and $\odot$. 
Definition

$W \subseteq \Sigma\Pi(G)$ is called $\sigma$-reduced over $G$ if

1. the elements in $W$ are in sum-product reduced form,
2. $W$ is shift-stable (and thus shift-closed) and
3. $W$ is canonical reduced.

In particular, $A \in \text{SumProd}(W, G)$ is called $\sigma$-reduced (w.r.t. $W$) if $W$ is $\sigma$-reduced over $G$. 
Definition

$W \subseteq \Sigma\Pi(G)$ is called $\sigma$-reduced over $G$ if

1. the elements in $W$ are in sum-product reduced form,
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In particular, $A \in \text{SumProd}(W, G)$ is called $\sigma$-reduced (w.r.t. $W$) if $W$ is $\sigma$-reduced over $G$.

Problem SigmaReduce: Compute a $\sigma$-reduced representation

---

Given: $A_1, \ldots, A_u \in \text{SumProd}(G)$ with $G = \mathbb{K}(x)$.

Find: a $\sigma$-reduced set $W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(G)$,

$B_1 \ldots, B_u \in \text{SumProd}(W, G)$ and $\delta_1, \ldots, \delta_u \in \mathbb{N}$

such that for all $1 \leq i \leq r$ we get

$$A_i(n) = B_i(n) \quad n \geq \delta_i.$$
Application: Canonical representations in term algebras

\[ A_1 \quad \quad \quad \quad A_2 \quad \quad \quad \quad \text{in SumProd}(G) \]
Application: Canonical representations in term algebras

\[ \text{ev}(A_1, n) \quad \overset{?}{=} \quad \text{ev}(A_2, n) \quad \text{in SumProd}(G) \]
Application: Canonical representations in term algebras

\[
\begin{align*}
A_1 & \downarrow \quad \text{\(\sigma\)-reduced} \\
\downarrow & \quad \text{w.r.t. a \(W\)} \\
B_1 & \\
\forall n \in \mathbb{N} \quad \text{ev}(A_1, n) = \text{ev}(B_1, n) & \quad \text{ev}(B_2, n) = \text{ev}(A_2, n)
\end{align*}
\]
Application: Canonical representations in term algebras

\[ \forall n \in \mathbb{N} \quad \text{ev}(A_1, n) = \text{ev}(B_1, n) \quad \sigma\text{-reduced w.r.t. a } W \quad \text{ev}(B_2, n) = \text{ev}(A_2, n) \]

\[ \overset{?}{=} \]

\[ B_1 = B_2 \quad \overset{\updownarrow}{\text{canonical simplifier}} \]
Outline of the talk:
Outline of the talk:

- **user-level**
  - term algebra
  - \( \text{SumProd}(G) \)
  - \( \text{ev} \rightarrow \text{ev} \)
- **computer algebra-level**
  - \( \text{ring of sequences} \)
  - \( \text{ev} \)
  - \( \text{ev} \)
  - \( \text{ev} \)
- **formal difference rings**
- **user interface**

Part 1: A term algebra for nested sums over hypergeometric products
Outline of the talk:

Part 1: A term algebra for nested sums over hypergeometric products

Part 2: A canonical simplifier (justified by difference ring theory)

Part 3: Construction of appropriate difference rings
Part 2: A canonical simplifier (based on DR theory)

Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$
Represent \( H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{Q}) \) with
\[
H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.
\]

1. a formal ring \( \mathbb{A} = \underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}} [s] \)
   polynomial ring
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\text{ev'} : \mathbb{Q}(x) \times \mathbb{N} \rightarrow \mathbb{Q}$$

$$(\frac{p(x)}{q(x)}, n) \mapsto \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$
Part 2: A canonical simplifier (based on DR theory)

Represent \( H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G}) \) with

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\[
(p(x), q(x), n) \mapsto \begin{cases} 
\frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
ev : \mathbb{Q}(x)[s] \times \mathbb{N} \rightarrow \mathbb{Q}
\]

\[
ev(s, n) = H_n
\]
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

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$$(\frac{p(x)}{q(x)}, n) \mapsto \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{ev} : \mathbb{Q}(x)[s] \times \mathbb{N} \rightarrow \mathbb{Q}$$

$$\left( \sum_{i=0}^{d} f_i s^i, n \right) \mapsto \sum_{i=0}^{d} \text{ev}'(f_i, n) H_n^i$$

$$\text{ev}(s, n) = H_n$$

**Definition:** $(\mathbb{A}, \text{ev})$ is called an eval-ring
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(G)$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$ 

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\tau : \mathbb{A} \to \mathbb{Q}^{\mathbb{N}}$$

$$f \mapsto \langle \text{ev}(f, n) \rangle_{n \geq 0}$$

It is almost a ring homomorphism:

$$\tau(x) \tau\left(\frac{1}{x}\right) = \langle 0, 1, 2, 3, \ldots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$$
Part 2: A canonical simplifier (based on DR theory)

Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

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Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(G)$ with

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$$\parallel$$
$$\langle 0, 1, 1, 1, \ldots \rangle$$

$$\tau(x^{\frac{1}{x}}) = \tau(1) = \langle 1, 1, 1, 1, \ldots \rangle$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$

from a certain point on

It is a ring homomorphism:

$\tau(x)\tau(\frac{1}{x}) = \langle 0, 1, 2, 3, \ldots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$

$\parallel$

$\langle 0, 1, 1, 1, \ldots \rangle$

$\parallel$

$\tau(\frac{1}{x}) = \tau(1) = \langle 1, 1, 1, 1, \ldots \rangle$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(G)$ with

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1. a formal ring $A = \mathbb{Q}(x)[s]$
2. an evaluation function $ev : A \times \mathbb{N} \rightarrow \mathbb{Q}$

Consider the map

$$\tau : A \rightarrow \mathbb{Q}^\mathbb{N}/\sim$$

$$f \mapsto \langle ev(f, n) \rangle_{n \geq 0}$$

$(a_n) \sim (b_n)$ iff $a_n = b_n$
from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\tau(x)\tau(\frac{1}{x}) = \langle 0, 1, 2, 3, \ldots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$$
||
$$\langle 0, 1, 1, 1, \ldots \rangle$$
||

$$\tau(x \times \frac{1}{x}) = \tau(1) = \langle 1, 1, 1, 1, \ldots \rangle$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

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1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \rightarrow \mathbb{Q}$
3. a ring automorphism

$$\sigma' : \mathbb{Q}(x) \rightarrow \mathbb{Q}(x)$$

$$r(x) \leftrightarrow r(x + 1)$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$  

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3. a ring automorphism

$$\sigma' : \mathbb{Q}(x) \rightarrow \mathbb{Q}(x)$$

$$r(x) \mapsto r(x + 1)$$

$$\sigma : \mathbb{Q}(x)[s] \rightarrow \mathbb{Q}(x)[s]$$

$$s \mapsto s + \frac{1}{x+1}$$

$$H_{n+1} = H_n + \frac{1}{n+1}$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$ 

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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3. a ring automorphism

$$\sigma' : \mathbb{Q}(x) \rightarrow \mathbb{Q}(x)$$
$$r(x) \mapsto r(x + 1)$$

$$\sigma : \mathbb{Q}(x)[s] \rightarrow \mathbb{Q}(x)[s]$$
$$\sum_{i=0}^{d} f_i s^i \mapsto \sum_{i=0}^{d} \sigma'(f_i) \left(s + \frac{1}{x+1}\right)^i$$

$$H_{n+1} = H_n + \frac{1}{n+1}$$

**Definition:** $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism $\sigma$ is called a difference ring; the set of constants is

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$ 

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

$\text{ev}$ and $\sigma$ interact:

$$\text{ev}(\sigma(s), n) = \text{ev}(s + \frac{1}{x+1}, n) = H_n + \frac{1}{n+1} = \text{ev}(s, n + 1)$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(G)$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \rightarrow \mathbb{Q}$
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$$\text{ev}(\sigma(s), n) = \text{ev}(s + \frac{1}{x+1}, n) = H_n + \frac{1}{n+1} = \text{ev}(s, n + 1)$$

$\Updownarrow$

$$\tau(\sigma(s)) = \langle 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots \rangle = S(\langle 0, 1, 1 + \frac{1}{2}, \ldots \rangle) = S(\tau(s))$$

shift operator
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$ 

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$  
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$\uparrow$

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$\tau$ is an injective difference ring homomorphism:

$$\mathbb{K}(x)[s] \xrightarrow{\sigma} \mathbb{K}(x)[s]$$

$$\mathbb{K}^{\mathbb{N}} / \sim \xrightarrow{S} \mathbb{K}^{\mathbb{N}} / \sim$$
Represent $H = \text{Sum}(1, \frac{1}{x}) \in \text{SumProd}(G)$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

1. a formal ring $A = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : A \times \mathbb{N} \rightarrow \mathbb{Q}$
3. a ring automorphism $\sigma : A \rightarrow A$

$\text{ev}$ and $\sigma$ interact:

$$\text{ev}(\sigma(s), n) = \text{ev}(s + \frac{1}{x+1}, n) = H_n + \frac{1}{n+1} = \text{ev}(s, n + 1)$$

$\uparrow$

$$\tau(\sigma(s)) = \langle 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots \rangle = S(\langle 0, 1, 1 + \frac{1}{2}, \ldots \rangle) = S(\tau(s))$$

$\tau$ is an injective difference ring homomorphism:

$$\begin{align*}
\langle \mathbb{K}(x)[s], \sigma \rangle & \xrightarrow{\tau} \langle \tau(\mathbb{Q}(x))[\langle H_n \rangle_{n \geq 0}], S \rangle \\
\text{rat. seq.} & \leq (\mathbb{K}^\mathbb{N} / \sim, S)
\end{align*}$$
General construction

\[ H \in \text{SumProd}(G) \]

- a formal ring \( A \supseteq G \supseteq K \) with \( h \in A \);
- an evaluation function \( \text{ev} : A \times \mathbb{N} \rightarrow K \) with \( H(n) = \text{ev}(h, n) \);
- a ring automorphism \( \sigma : A \rightarrow A \) with \( H(n + 1) = \sigma(h) \).
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

\[ A := \mathbb{K}(x) \]

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

\[ \sigma(x) = x + 1 \]
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

\[
A := \mathbb{K}(x)
\]

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

\[
\sigma(x) = x + 1
\]

$Sk! = (k+1)k!$
A hypergeometric APS-extension of \((\mathbb{K}(x), \sigma)\) is

- a ring

\[
A := \mathbb{K}(x)[p_1, p_1^{-1}]
\]

- with an automorphism where \(\sigma(c) = c\) for all \(c \in \mathbb{K}\) and where

\[
\sigma(x) = x + 1
\]

\[
\text{Sk}! = (k+1)k! \quad \leftrightarrow \quad \sigma(p_1) = (x + 1)p_1
\]
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products $\iff \sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*$
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products $\leftrightarrow$ $\sigma(p_1) = a_1 p_1$ $a_1 \in \mathbb{K}(x)^*$

$$\sigma(p_2) = a_2 p_2$ $a_2 \in \mathbb{K}(x)^*$
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products

$$\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*$$

$$\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^*$$

$$\vdots$$

$$\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*$$
A hypergeometric \( APS \)-extension of \((\mathbb{K}(x), \sigma)\) is

- a ring

\[
\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z]
\]

- with an automorphism where \(\sigma(c) = c\) for all \(c \in \mathbb{K}\) and where

\[
\sigma(x) = x + 1
\]

\[
\begin{align*}
\sigma(p_1) &= a_1 p_1 & a_1 &\in \mathbb{K}(x)^* \\
\sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)^* \\
& \vdots \\
\sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)^*
\end{align*}
\]

\[
(\mathbf{-1})^k \leftrightarrow \sigma(z) = -z \quad z^2 = 1
\]
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

\[
\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z]
\]

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

\[
\sigma(x) = x + 1
\]

hypergeometric products \quad \leftrightarrow \quad \sigma(p_1) = a_1p_1 \quad a_1 \in \mathbb{K}(x)^* \\
\sigma(p_2) = a_2p_2 \quad a_2 \in \mathbb{K}(x)^* \\
... \\
\sigma(p_e) = a_ep_e \quad a_e \in \mathbb{K}(x)^*

\gamma \text{ is a primitive } \lambda \text{th root of unity} \quad \gamma^k \quad \leftrightarrow \quad \sigma(z) = \gamma z \quad z^\lambda = 1
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$A := \mathbb{K}(x)[p_1, p_1^{-1}] [p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][\tilde{z}][s_1]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products

$$\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*$$
$$\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^*$$
$$\vdots$$
$$\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*$$

$\gamma$ is a primitive $\lambda$th root of unity

$$\gamma^k \quad \leftrightarrow \quad \sigma(z) = \gamma z \quad z^\lambda = 1$$

$$H_{k+1} = H_k + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{x+1}$$
A hypergeometric \( APS \)-extension of \( (K(x), \sigma) \) is

- a ring

\[
A := K(x)[p_1, p_1^{-1}] [p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z][s_1]
\]

- with an automorphism where \( \sigma(c) = c \) for all \( c \in K \) and where

\[
\sigma(x) = x + 1
\]

\[
\text{hypergeometric products } \quad \leftrightarrow \quad \sigma(p_1) = a_1 p_1 \quad a_1 \in K(x)^*
\]
\[
\sigma(p_2) = a_2 p_2 \quad a_2 \in K(x)^*
\]
\[
\vdots
\]
\[
\sigma(p_e) = a_e p_e \quad a_e \in K(x)^*
\]

- \( \gamma \) is a primitive \( \lambda \)th root of unity

\[
\gamma^k \quad \leftrightarrow \quad \sigma(z) = \gamma z \quad z^\lambda = 1
\]

- \( (\text{nested}) \text{ sum} \) \( \leftrightarrow \) \( \sigma(s_1) = s_1 + f_1 \quad f_1 \in K(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z] \)
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products

$$\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*$$
$$\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^*$$
$$\vdots$$
$$\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*$$

$\gamma$ is a primitive $\lambda$th root of unity

$\gamma^k \leftrightarrow \sigma(z) = \gamma z \quad z^\lambda = 1$

(nested) sum

$$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z]$$
$$\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1]$$
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \cdots$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products

$$\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*$$

$$\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^*$$

$$\vdots$$

$$\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*$$

$\gamma$ is a primitive $\lambda$th root of unity

$$\gamma^k \quad \leftrightarrow \quad \sigma(z) = \gamma z \quad \Rightarrow \quad z^\lambda = 1$$

(nested) sum

$$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z]$$

$$\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1]$$

$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2]$$

$$\vdots$$
Definition (Evaluation function)

Take \((A, \sigma)\) with a subfield \(K\) of \(A\) with \(\sigma|_K = \text{id}\).

1. \(\text{ev} : A \times \mathbb{N} \rightarrow K\) is called \textbf{evaluation function} for \((A, \sigma)\) if for all \(f, g \in A, c \in K\) and \(l \in \mathbb{Z}\) there exists a \(\lambda \in \mathbb{N}\) with

\[
\forall n \geq \lambda : \text{ev}(c, n) = c, \quad (2)
\]
\[
\forall n \geq \lambda : \text{ev}(f + g, n) = \text{ev}(f, n) + \text{ev}(g, n), \quad (3)
\]
\[
\forall n \geq \lambda : \text{ev}(fg, n) = \text{ev}(f, n) \text{ev}(g, n), \quad (4)
\]
\[
\forall n \geq \lambda : \text{ev}(\sigma^l(f), n) = \text{ev}(f, n + l). \quad (5)
\]
Definition (Evaluation function)

Take \((A, \sigma)\) with a subfield \(\mathbb{K}\) of \(A\) with \(\sigma|_\mathbb{K} = \text{id}\).

1. \(\text{ev} : A \times \mathbb{N} \rightarrow \mathbb{K}\) is called **evaluation function** for \((A, \sigma)\) if for all \(f, g \in A, \ c \in \mathbb{K}\) and \(l \in \mathbb{Z}\) there exists a \(\lambda \in \mathbb{N}\) with

\[
\forall n \geq \lambda : \text{ev}(c, n) = c, \tag{2}
\]

\[
\forall n \geq \lambda : \text{ev}(f + g, n) = \text{ev}(f, n) + \text{ev}(g, n), \tag{3}
\]

\[
\forall n \geq \lambda : \text{ev}(fg, n) = \text{ev}(f, n) \text{ev}(g, n), \tag{4}
\]

\[
\forall n \geq \lambda : \text{ev}(\sigma^l(f), n) = \text{ev}(f, n + l). \tag{5}
\]

2. \(L : A \rightarrow \mathbb{N}\) is called **o-function** if for any \(f, g \in A\) with \(\lambda = \max(L(f), L(g))\) the properties (3) and (4) hold and for any \(f \in A\) and \(l \in \mathbb{Z}\) with \(\lambda = L(f) + \max(0, -l)\) property (5) holds.
Connection between SumProd($\mathcal{G}$) and hypergeometric $APS$-extension

$(\mathcal{E}, \sigma)$ with $\mathcal{E} = \mathcal{G}\langle t_1 \rangle \ldots \langle t_e \rangle$ a hypergeometric $APS$-extension of $(\mathcal{G}, \sigma)$

$\text{ev} : \mathcal{E} \times \mathbb{N} \rightarrow \mathbb{K}, \; L : \mathcal{E} \rightarrow \mathbb{N}$
Connection between SumProd($G$) and hypergeometric $APS$-extension

$(E, \sigma)$ with $E = G\langle t_1 \rangle \ldots \langle t_e \rangle$ a hypergeometric $APS$-extension of $(G, \sigma)$

$ev : E \times \mathbb{N} \to K$, $L : E \to \mathbb{N}$

\[
\forall n \geq L(t_i) : \quad ev(t_i, n) = T_i(n) \in \Sigma \Pi(G)
\]
Connection between SumProd($\mathbb{G}$) and hypergeometric $APS$-extension

($\mathbb{E}, \sigma$) with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \ldots \langle t_e \rangle$ a hypergeometric $APS$-extension of $(\mathbb{G}, \sigma)$

$ev : \mathbb{E} \times \mathbb{N} \rightarrow \mathbb{K}$, $L : \mathbb{E} \rightarrow \mathbb{N}$

$\forall n \geq L(t_i) :$
$ev(t_i, n) = T_i(n) \in \Sigma\Pi(\mathbb{G})$

$W = \{T_1, \ldots, T_e\} \subseteq \Sigma\Pi(\mathbb{G})$ is sum-product reduced and shift stable: sums/products in $T_i$ are from $\{T_1, \ldots, T_{i-1}\}$.
Connection between $\text{SumProd}(\mathbb{G})$ and hypergeometric $APS$-extension

$$(\mathbb{E}, \sigma) \text{ with } \mathbb{E} = \mathbb{G}\langle t_1 \rangle \ldots \langle t_e \rangle \text{ a hypergeometric } APS\text{-extension of } (\mathbb{G}, \sigma)$$

$\text{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}, \quad L : \mathbb{E} \to \mathbb{N}$

$$\forall n \geq L(t_i) : \quad \text{ev}(t_i, n) = T_i(n) \in \Sigma \Pi(\mathbb{G})$$

$W = \{T_1, \ldots, T_e\} \subseteq \Sigma \Pi(\mathbb{G})$ is sum-product reduced and shift stable: sums/products in $T_i$ are from $\{T_1, \ldots, T_{i-1}\}$.

In particular, if $f \in \mathbb{E} \setminus \{0\}$, then we can take the "unique" $0 \neq F \in \text{SumProd}(\{T_1, \ldots, T_e\}, \mathbb{G})$ with $F(n) = \text{ev}(f, n)$ for all $n \geq L(f)$. 
Connection between $\text{SumProd}(\mathbb{G})$ and hypergeometric $APS$-extension

\[(\mathbb{E}, \sigma) \text{ with } \mathbb{E} = \mathbb{G}\langle t_1 \rangle \ldots \langle t_e \rangle \text{ a hypergeometric } APS\text{-extension of } (\mathbb{G}, \sigma)\]
\[\text{ev} : \mathbb{E} \times \mathbb{N} \to K, \; L : \mathbb{E} \to \mathbb{N}\]

\[
\forall n \geq L(t_i) : \\
\text{ev}(t_i, n) = T_i(n) \in \Sigma\Pi(\mathbb{G})
\]

$W = \{T_1, \ldots, T_e\} \subseteq \Sigma\Pi(\mathbb{G})$ is sum-product reduced and shift stable: sums/products in $T_i$ are from $\{T_1, \ldots, T_{i-1}\}$.

In particular, if $f \in \mathbb{E} \setminus \{0\}$, then we can take the ”unique” $0 \neq F \in \text{SumProd}(\{T_1, \ldots, T_e\}, \mathbb{G})$ with $F(n) = \text{ev}(f, n)$ for all $n \geq L(f)$.

**Definition**

For $f \in \mathbb{E}$ we also write $\text{expr}(f) = F$ for this particular $F$. 
Connection between SumProd($\mathcal{G}$) and hypergeometric $APS$-extension

$(\mathcal{E}, \sigma)$ with $\mathcal{E} = \mathcal{G}\langle t_1 \rangle \ldots \langle t_e \rangle$ a hypergeometric $APS$-extension of $(\mathcal{G}, \sigma)$

$ev : \mathcal{E} \times \mathbb{N} \rightarrow \mathbb{K}, \ L : \mathcal{E} \rightarrow \mathbb{N}$

$\forall n \geq L(t_i) :$

$ev(t_i, n) = T_i(n) \in \Sigma \Pi(\mathcal{G})$

$W = \{T_1, \ldots, T_e\} \subseteq \Sigma \Pi(\mathcal{G})$ is sum-product reduced and

shift stable: sums/products in $T_i$ are from $\{T_1, \ldots, T_{i-1}\}$.

Example

For $f = x + \frac{x+1}{x} s^4 \in \mathbb{Q}(x)[s]$ we obtain

$\text{expr}(f) = F = x \oplus \left(\frac{x+1}{x} \odot (\text{Sum}(1, \frac{1}{x}) \odot 4) \right) \in \text{Sum}(\mathbb{Q}(x))$

with $F(n) = ev(f, n)$ for all $n \geq 1$. 
Connection between SumProd($\mathbb{G}$) and hypergeometric $APS$-extension

($\mathbb{E}, \sigma$) with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \ldots \langle t_e \rangle$ a hypergeometric $APS$-extension of $(\mathbb{G}, \sigma)$

$ev : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$, $L : \mathbb{E} \to \mathbb{N}$

$\forall n \geq L(t_i) :$

$ev(t_i, n) = T_i(n) \in \Sigma \Pi(\mathbb{G})$

$W = \{T_1, \ldots, T_e\} \subseteq \Sigma \Pi(\mathbb{G})$ is sum-product reduced and shift stable: sums/products in $T_i$ are from $\{T_1, \ldots, T_{i-1}\}$. 
Outline of the talk:

user-level

term algebra

SumProd(\mathcal{G})

user interface

formal difference rings

ring of sequences

computer algebra-level
Difference ring theory in action

Let \((E, \sigma)\) be a hypergeometric \(APS\)-extension of \((G, \sigma)\) with \(\text{ev} : E \times \mathbb{N} \to K\) and let \(\tau : E \to K^\mathbb{N}/\sim\) be the \(K\)-homomorphism given by

\[\tau(f) = (\text{ev}(f, n))_{n \geq 0}.\]
Difference ring theory in action

Let \((\mathbb{E}, \sigma)\) be a hypergeometric \(APS\)-extension of \((\mathbb{G}, \sigma)\) with \(\text{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}\) and let \(\tau : \mathbb{E} \to \mathbb{K}^{\mathbb{N}}/\sim\) be the \(\mathbb{K}\)-homomorphism given by

\[
\tau(f) = (\text{ev}(f, n))_{n \geq 0}.
\]

Lemma

Let \(W = \{T_1, \ldots, T_e\} \in \Sigma \Pi(\mathbb{G})\) with \(T_i = \text{expr}(t_i)\). Then:

\[
W \text{ is canonical reduced } \iff \tau \text{ is injective.}
\]
Difference ring theory in action

Let \((\mathcal{E}, \sigma)\) be a hypergeometric \(APS\)-extension of \((\mathcal{G}, \sigma)\) with \(\text{ev} : \mathcal{E} \times \mathbb{N} \rightarrow \mathbb{K}\) and let \(\tau : \mathcal{E} \rightarrow \mathbb{K}^\mathbb{N}/\sim\) be the \(\mathbb{K}\)-homomorphism given by

\[
\tau(f) = (\text{ev}(f, n))_{n \geq 0}.
\]

Lemma

Let \(W = \{T_1, \ldots, T_e\} \in \Sigma \Pi(\mathcal{G})\) with \(T_i = \text{expr}(t_i)\). Then:

\(W\) is canonical reduced \(\iff\) \(\tau\) is injective.

Using difference ring theory we get the following crucial property:

Theorem

\(\tau\) is injective \(\iff\) \(\text{const}_\sigma \mathcal{E} = \mathbb{K}\).
Example

For our difference field $\mathbb{G} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$ and $\text{const}_\sigma \mathbb{K} = \mathbb{K}$ we have $\text{const}_\sigma \mathbb{K}(x) = \mathbb{K}$. 
Example
For our difference field $G = \mathbb{K}(x)$ with $\sigma(x) = x + 1$ and $\text{const}_\sigma \mathbb{K} = \mathbb{K}$ we have $\text{const}_\sigma \mathbb{K}(x) = \mathbb{K}$.

Definition
A hypergeometric $APS$-extension $(E, \sigma)$ of $(G, \sigma)$ is called hypergeometric $R\Pi\Sigma$-extension if

$$\text{const}_\sigma E = \mathbb{K}.$$
Example
For our difference field $G = K(x)$ with $\sigma(x) = x + 1$ and $\text{const}_\sigma K = K$ we have $\text{const}_\sigma K(x) = K$.

Definition
A hypergeometric $APS$-extension $(E, \sigma)$ of $(G, \sigma)$ is called hypergeometric $R\Pi\Sigma$-extension if

$$\text{const}_\sigma E = K.$$ 

Theorem
Let $W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(G)$ be in sum-product reduced representation and shift-stable, i.e., for each $1 \leq i \leq e$ the arising sums and products in $T_i$ are contained in $\{T_1, \ldots, T_{i-1}\}$. Then the following is equivalent:

1. There is a hypergeometric $R\Pi\Sigma$-extension $(E, \sigma)$ of $(G, \sigma)$ with $E = G\langle t_1 \rangle \ldots \langle t_e \rangle$ equipped with an evaluation function $ev$ with $T_i = \text{expr}(t_i) \in \Sigma\Pi(G)$ for $1 \leq i \leq e$.
2. $W$ is $\sigma$-reduced over $G$. 
This yields a strategy (actually the only strategy for shift-stable sets).

**A strategy to solve Problem SigmaReduce**

---

**Given:**  
$A_1, \ldots, A_u \in \text{SumProd}(G)$ with $G = \mathbb{K}(x)$.

**Find:**  
a $\sigma$-reduced set $W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(G)$ with $B_1 \ldots, B_u \in \text{SumProd}(W, G)$ and $\delta_1, \ldots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$. 

---
This yields a strategy (actually the only strategy for shift-stable sets).

**A strategy to solve Problem SigmaReduce**

Given: \( A_1, \ldots, A_u \in \text{SumProd}(G) \) with \( G = \mathbb{K}(x) \).

Find: a \( \sigma \)-reduced set \( W = \{ T_1, \ldots, T_e \} \subset \Sigma\Pi(G) \) with \( B_1 \ldots, B_u \in \text{SumProd}(W, G) \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) such that \( A_i(n) = B_i(n) \) holds for all \( n \geq \delta_i \) and \( 1 \leq i \leq r \).

1. Construct \( \Pi\Sigma\Sigma\)-extension \((E, \sigma)\) of \((G, \sigma)\) with \( E = G \langle t_1 \rangle \ldots \langle t_e \rangle \) equipped with \( \text{ev} : E \times \mathbb{N} \rightarrow \mathbb{K} \) such that we get \( a_1, \ldots, a_u \in E \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) with

\[
A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}
\]
This yields a strategy (actually the only strategy for shift-stable sets).

**A strategy to solve Problem SigmaReduce**

Given: $A_1, \ldots, A_u \in \text{SumProd}(G)$ with $G = \mathbb{K}(x)$.

Find: a $\sigma$-reduced set $W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(G)$ with $B_1 \ldots, B_u \in \text{SumProd}(W, G)$ and $\delta_1, \ldots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $\Pi\Sigma\Sigma$-extension $(E, \sigma)$ of $(G, \sigma)$ with $E = G\langle t_1 \rangle \ldots \langle t_e \rangle$ equipped with $\text{ev} : E \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \ldots, a_u \in E$ and $\delta_1, \ldots, \delta_u \in \mathbb{N}$ with

$$A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \quad (9)$$

2. Set $W = \{T_1, \ldots, T_e\}$ with $T_i := \text{expr}(t_i) \in \Sigma\Pi(G)$ for $1 \leq i \leq e$. 
This yields a strategy (actually the only strategy for shift-stable sets).

A strategy to solve Problem SigmaReduce

Given: \( A_1, \ldots, A_u \in \text{SumProd}(G) \) with \( G = \mathbb{K}(x) \).
Find: a \( \sigma \)-reduced set \( W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(G) \) with \( B_1, \ldots, B_u \in \text{SumProd}(W, G) \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) such that \( A_i(n) = B_i(n) \) holds for all \( n \geq \delta_i \) and \( 1 \leq i \leq r \).

1. Construct \( R\Pi\Sigma \)-extension \( (E, \sigma) \) of \( (G, \sigma) \) with \( E = G\langle t_1 \rangle \ldots \langle t_e \rangle \) equipped with \( \text{ev} : E \times \mathbb{N} \rightarrow \mathbb{K} \) such that we get \( a_1, \ldots, a_u \in E \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) with

\[
A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \quad (9)
\]

2. Set \( W = \{T_1, \ldots, T_e\} \) with \( T_i := \text{expr}(t_i) \in \Sigma\Pi(G) \) for \( 1 \leq i \leq e \).
3. Set \( B_i := \text{expr}(a_i) \in \text{SumProd}(W, G) \) for \( 1 \leq i \leq u \).
This yields a strategy (actually the only strategy for shift-stable sets).

**A strategy to solve Problem SigmaReduce**

---

Given: $A_1, \ldots, A_u \in \text{SumProd}(G)$ with $G = \mathbb{K}(x)$.

Find: a $\sigma$-reduced set $W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(G)$ with $B_1 \ldots, B_u \in \text{SumProd}(W, G)$ and $\delta_1, \ldots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(E, \sigma)$ of $(G, \sigma)$ with $E = G\langle t_1 \rangle \ldots \langle t_e \rangle$ equipped with $ev : E \times \mathbb{N} \rightarrow \mathbb{K}$ such that we get $a_1, \ldots, a_u \in E$ and $\delta_1, \ldots, \delta_u \in \mathbb{N}$ with

   $$A_i(n) = ev(a_i, n) \quad \forall n \geq \delta_i.$$  

   \hspace{1cm} (9)

2. Set $W = \{T_1, \ldots, T_e\}$ with $T_i := \text{expr}(t_i) \in \Sigma\Pi(G)$ for $1 \leq i \leq e$.

3. Set $B_i := \text{expr}(a_i) \in \text{SumProd}(W, G)$ for $1 \leq i \leq u$.

4. Return $W$, $(B_1, \ldots, B_u)$ and $(\delta_1, \ldots, \delta_u)$.
Outline of the talk:

- user-level
- term algebra
- SumProd($\mathcal{G}$)
- user interface
- formal difference rings
- ring of sequences
- computer algebra-level
Outline of the talk:

Part 1: A term algebra for nested sums over hypergeometric products

Part 2: A canonical simplifier (justified by difference ring theory)

Part 3: Construction of appropriate difference rings
A hypergeometric $APS$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \cdots$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products

$$\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*$$
$$\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^*$$
$$\vdots$$
$$\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*$$

$\gamma$ is a primitive $\lambda$th root of unity

$$\gamma^k \quad \leftrightarrow \quad \sigma(z) = \gamma z \quad z^\lambda = 1$$

(nested) sum

$$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z]$$
$$\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1]$$
$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2]$$
$$\vdots$$
A hypergeometric $R\Pi\Sigma$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

\[
\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \cdots
\]

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

\[
\sigma(x) = x + 1
\]

for hypergeometric products

\[
\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^*
\]

\[
\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^*
\]

\[
\vdots
\]

\[
\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*
\]

\[
\gamma\text{ is a primitive } \lambda\text{th root of unity}
\]

\[
\gamma^k \quad \leftrightarrow \quad \sigma(z) = \gamma z \quad z^\lambda = 1
\]

for (nested) sum

\[
\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z]
\]

\[
\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1]
\]

\[
\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2]
\]

\[
\vdots
\]

such that $\text{const}_\sigma\mathcal{E} = \mathbb{K}$.
Represent sums (extension of Karr’s result, 1981)

Let \((\mathbb{A}, \sigma)\) be a difference ring with constant set

\[
\text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}.
\]

**Note 1:** \(\text{const}_\sigma \mathbb{A}\) is a ring that contains \(\mathbb{Q}\)

**Note 2:** We always take care that \(\text{const}_\sigma \mathbb{A}\) is a field
Represent sums (extension of Karr’s result, 1981)

Let \((\mathbb{A}, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}.
\]

Adjoin a new variable \(t\) to \(\mathbb{A}\) (i.e., \(\mathbb{A}[t]\) is a polynomial ring).
Represent sums (extension of Karr’s result, 1981)

Let \((A, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}.
\]

Adjoin a new variable \(t\) to \(A\) (i.e., \(A[t]\) is a polynomial ring).

Extend the shift operator s.t.

\[
\sigma(t) = t + f \quad \text{for some } f \in A.
\]
Represent sums (extension of Karr’s result, 1981)

Let \((A, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}.
\]

Adjoin a new variable \(t\) to \(A\) (i.e., \(A[t]\) is a polynomial ring).

Extend the shift operator s.t.

\[
\sigma(t) = t + f \quad \text{for some } f \in A.
\]

Then \(\text{const}_\sigma A[t] = \text{const}_\sigma A\) iff

\[
\not\exists g \in A : \sigma(g) = g + f
\]
Represent sums (extension of Karr’s result, 1981)

Let \((\mathbb{A}, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}.
\]

Adjoin a new variable \(t\) to \(\mathbb{A}\) (i.e., \(\mathbb{A}[t]\) is a polynomial ring).

Extend the shift operator s.t.

\[
\sigma(t) = t + f \quad \text{for some } f \in \mathbb{A}.
\]

Then \(\text{const}_\sigma \mathbb{A}[t] = \text{const}_\sigma \mathbb{A}\) iff

\[
\nexists g \in \mathbb{A} : \quad \sigma(g) = g + f
\]

Such a difference ring extension \((\mathbb{A}[t], \sigma)\) of \((\mathbb{A}, \sigma)\) is called \(\Sigma^*\)-extension.
Represent sums (extension of Karr’s result, 1981)

Let \((A, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}.
\]

Adjoin a new variable \(t\) to \(A\) (i.e., \(A[t]\) is a polynomial ring).

Extend the shift operator s.t.

\[
\sigma(t) = t + f \quad \text{for some } f \in A.
\]

Then \(\text{const}_\sigma A[t] = \text{const}_\sigma A\) iff

\[
\exists g \in A : \quad \sigma(g) = g + f
\]

There are 2 cases:

1. \(\exists g \in A : \sigma(g) = g + f\) : \((A[t], \sigma)\) is a \(\Sigma^*\)-extension of \((A, \sigma)\)
Represent sums \textit{(extension of Karr’s result, 1981)}

- Let \((A, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}.
\]

- Adjoin a new variable \(t\) to \(A\) (i.e., \(A[t]\) is a polynomial ring).

- Extend the shift operator s.t.

\[
\sigma(t) = t + f \quad \text{for some } f \in A.
\]

Then \(\text{const}_\sigma A[t] = \text{const}_\sigma A\) iff

\[
\forall g \in A : \sigma(g) = g + f
\]

There are 2 cases:

1. \(\exists g \in A : \sigma(g) = g + f\) : \((A[t], \sigma)\) is a \(\Sigma^*\)-extension of \((A, \sigma)\)

2. \(\exists g \in A : \sigma(g) = g + f\) : No need for a \(\Sigma^*\)-extension!
A hypergeometric $R\Pi\Sigma$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

\[
A := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \cdots
\]

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

\[
\sigma(x) = x + 1
\]

hypergeometric products

\[
\sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(x)^* \\
\sigma(p_2) = a_2 p_2 \quad a_2 \in \mathbb{K}(x)^* \\
\vdots
\]

\[
\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(x)^*
\]

$\gamma$ is a primitive $\lambda$th root of unity

$\gamma^k$ implies

\[
\sigma(z) = \gamma z \quad z^\lambda = 1
\]

(nested) sum

\[
\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z] \\
\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1] \\
\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \cdots [p_e, p_e^{-1}][z][s_1][s_2] \\
\vdots
\]

such that $\text{const}_\sigma \mathcal{E} = \mathbb{K}$
Represent products (extension of Karr's result, 1981)

Let \((\mathbb{A}, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} | \sigma(k) = k \}.
\]
Represent products (extension of Karr's result, 1981)

Let \((A, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma A := \{k \in A \mid \sigma(k) = k\}.
\]

Take the ring of Laurent polynomials \(A[t, \frac{1}{t}]\).
Represent products (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field
  
  \[ \text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}. \]

- Take the ring of Laurent polynomials $\mathbb{A}[t, \frac{1}{t}]$.

- Extend the shift operator s.t.
  \[ \sigma(t) = at \quad \text{for some } a \in \mathbb{A}^*. \]
Represent products (extension of Karr’s result, 1981)

- Let \((\mathbb{A}, \sigma)\) be a difference ring with constant field

\[
\text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}.
\]

- Take the ring of Laurent polynomials \(\mathbb{A}[t, \frac{1}{t}]\).

- Extend the shift operator s.t.

\[
\sigma(t) = at \quad \text{for some } a \in \mathbb{A}^*.
\]

Then \(\text{const}_\sigma \mathbb{A}[t, t^{-1}] = \text{const}_\sigma \mathbb{A}\) iff

\[
\forall g \in \mathbb{A} \setminus \{0\} : \quad \sigma(g) = ag
\]
Represent products (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field
  \[
  \text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}.
  \]
- Take the ring of Laurent polynomials $\mathbb{A}[t, \frac{1}{t}]$.
- Extend the shift operator s.t.
  \[
  \sigma(t) = a t \quad \text{for some } a \in \mathbb{A}^*.
  \]

Then $\text{const}_\sigma \mathbb{A}[t, t^{-1}] = \text{const}_\sigma \mathbb{A}$ iff

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Take the ring of Laurent polynomials \(\mathbb{A}[t, \frac{1}{t}]\).

Extend the shift operator s.t.

\[
\sigma(t) = a t \quad \text{for some } a \in \mathbb{A}^*.
\]

Then \(\text{const}_\sigma \mathbb{A}[t, t^{-1}] = \text{const}_\sigma \mathbb{A}\) iff

\[
\# g \in \mathbb{A} \setminus \{0\} \# n \in \mathbb{Z} \setminus \{0\} : \quad \sigma(g) = a^n g
\]
Represent products (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field
  \[ \text{const}_\sigma \mathbb{A} := \{ k \in \mathbb{A} \mid \sigma(k) = k \}. \]

- Take the ring of Laurent polynomials $\mathbb{A}[t, \frac{1}{t}]$.

- Extend the shift operator s.t.
  \[ \sigma(t) = a t \quad \text{for some} \; a \in \mathbb{A}^*. \]

Then $\text{const}_\sigma \mathbb{A}[t, t^{-1}] = \text{const}_\sigma \mathbb{A}$ iff

\[ \nexists g \in \mathbb{A} \setminus \{0\} \nexists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g \]

Such a difference ring extension $(\mathbb{A}[t, \frac{1}{t}], \sigma)$ of $(\mathbb{A}, \sigma)$ is called $\Pi$-extension.
Represent products (extension of Karr's result, 1981)

- Let \((A, \sigma)\) be a difference ring with constant field 
  \[ \text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}. \]
- Take the ring of Laurent polynomials \(A[t, \frac{1}{t}]\).
- Extend the shift operator s.t.
  \[ \sigma(t) = at \quad \text{for some } a \in A^*. \]

Then \(\text{const}_\sigma A[t, t^{-1}] = \text{const}_\sigma A\) iff 

\[ \exists g \in A \setminus \{0\} \exists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g \]

There are 3 cases:

1. \(\exists g \in A \setminus \{0\} \exists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g\) : \((A[t, \frac{1}{t}]), \sigma)\) is a \(\Pi\)-ext. of \((A, \sigma)\)
Represent products (extension of Karr's result, 1981)

- Let $(A, \sigma)$ be a difference ring with constant field

\[ \text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}. \]

- Take the ring of Laurent polynomials $A[t, \frac{1}{t}]$.

- Extend the shift operator s.t.

\[ \sigma(t) = a t \quad \text{for some } a \in A^*. \]

Then $\text{const}_\sigma A[t, t^{-1}] = \text{const}_\sigma A$ iff

\[ \nexists g \in A \setminus \{0\}, \exists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g \]

There are 3 cases:

1. \[ \nexists g \in A \setminus \{0\}, \exists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g \quad \text{: (} A[t, \frac{1}{t}], \sigma \text{) is a } \Pi\text{-ext. of } (A, \sigma) \]

2. \[ \exists g \in A \setminus \{0\} : \sigma(g) = a g \quad \text{: No need for a } \Pi\text{-extension!} \]
Represent products (extension of Karr’s result, 1981)

Let $(A, \sigma)$ be a difference ring with constant field

$$\text{const}_\sigma A := \{ k \in A \mid \sigma(k) = k \}. $$

Take the ring of Laurent polynomials $A[t, \frac{1}{t}]$.

Extend the shift operator s.t.

$$\sigma(t) = at \quad \text{for some } a \in A^*. $$

Then $\text{const}_\sigma A[t, t^{-1}] = \text{const}_\sigma A$ iff

$$\nexists g \in A \setminus \{0\} \nexists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g$$

There are 3 cases:

1. $\nexists g \in A \setminus \{0\} \nexists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n g$ : $(A[t, \frac{1}{t}], \sigma)$ is a $\Pi$-ext. of $(A, \sigma)$

2. $\exists g \in A \setminus \{0\} : \sigma(g) = ag$ : No need for a $\Pi$-extension!

3. $\exists g \in A \setminus \{0\} : \sigma(g) = a^n g$ only for $n \in \mathbb{Z} \setminus \{0, 1\}$ : 😞
The hypergeometric case

- Take the difference field \((\mathbb{K}(x), \sigma)\) with \(\sigma|_{\mathbb{K}} = \text{id}\) and \(\sigma(x) = x + 1\).
- Let \(\alpha_1, \ldots, \alpha_r \in \mathbb{K}(x)^*\)
The hypergeometric case

- Take the difference field \((\mathbb{K}(x), \sigma)\) with \(\sigma|_{\mathbb{K}} = \text{id}\) and \(\sigma(x) = x + 1\).
- Let \(\alpha_1, \ldots, \alpha_r \in \mathbb{K}(x)^*\)
- Then there is a difference ring

\[
\mathbb{E}
\]

such that for \(1 \leq i \leq r\) there are \(g_i \in \mathbb{E}^*\) with

\[
\sigma(g_i) = \alpha_i g_i
\]
The hypergeometric case

- Take the difference field \((\mathbb{K}(x), \sigma)\) with \(\sigma|_{\mathbb{K}} = \text{id}\) and \(\sigma(x) = x + 1\).
- Let \(\alpha_1, \ldots, \alpha_r \in \mathbb{K}(x)^*\)
- Then there is a difference ring

\[
\mathbb{E} = \mathbb{K}(x) [t_1, t_1^{-1}] \cdots [t_e, t_e^{-1}] [z]
\]

tower of \(\Pi\)-ext. \((-1)^k\) or \(\gamma^k\)

with

- \(\frac{\sigma(t_i)}{t_i} \in \mathbb{K}(x)^*\) for \(1 \leq i \leq e\)
- \(\sigma(z) = \gamma z\) and \(z^\lambda = 1\) for some primitive \(\lambda\)th root of unity \(\gamma \in \mathbb{K}^*\)
- \(\text{const}_{\sigma}\mathbb{E} = \mathbb{K}\)

such that for \(1 \leq i \leq r\) there are \(g_i \in \mathbb{E}^*\) with

\[
\sigma(g_i) = \alpha_i g_i
\]
The hypergeometric case

- Take the difference field \((\mathbb{K}(x), \sigma)\) with \(\sigma|_\mathbb{K} = \text{id}\) and \(\sigma(x) = x + 1\).
- Let \(\alpha_1, \ldots, \alpha_r \in \mathbb{K}(x)^*\)
- Then there is a difference ring

\[E = \mathbb{K}(x)[t_1, t_1^{-1}] \ldots [t_e, t_e^{-1}] \left(tower \ of \ \Pi\text{-ext.}\right) [z] \left((-1)^k \ or \ \gamma^k\right)\]

with
- \(\frac{\sigma(t_i)}{t_i} \in \mathbb{K}(x)^*\) for \(1 \leq i \leq e\)
- \(\sigma(z) = \gamma z\) and \(z^\lambda = 1\) for some primitive \(\lambda\)-th root of unity \(\gamma \in \mathbb{K}^*\)
- \(\text{const}_\sigma E = \mathbb{K}\)

such that for \(1 \leq i \leq r\) there are \(g_i \in E^*\) with

\[\sigma(g_i) = \alpha_i g_i\]

Note: There are similar results for the \(q\)-rational, multi-basic and mixed case.
A hypergeometric $R\Pi\Sigma$-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

\[
\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \ldots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \ldots
\]

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

\[
\sigma(x) = x + 1
\]

hypergeometric products

\[
\begin{align*}
\sigma(p_1) &= a_1 p_1 & a_1 &\in \mathbb{K}(x)^* \\
\sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)^* \\
&\cdots
\end{align*}
\]

\[
\sigma(p_e) = a_e p_e &\quad a_e \in \mathbb{K}(x)^*
\]

$\gamma$ is a primitive $\lambda$th root of unity

\[
\gamma^k \iff \sigma(z) = \gamma z &\quad z^\lambda = 1
\]

(nested) sum

\[
\begin{align*}
\sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \ldots [p_e, p_e^{-1}][z] \\
\sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \ldots [p_e, p_e^{-1}][z][s_1] \\
\sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \ldots [p_e, p_e^{-1}][z][s_1][s_2] \\
&\cdots
\end{align*}
\]

such that $\text{const}_\sigma \mathbb{E} = \mathbb{K}$
This yields a strategy (actually the only strategy for shift-stable sets).

A strategy to solve Problem SigmaReduce

Given: \( A_1, \ldots, A_u \in \text{SumProd}(G) \) with \( G = \mathbb{K}(x) \).
Find: a \( \sigma \)-reduced set \( W = \{T_1, \ldots, T_e\} \subset \Sigma \Pi(G) \) with \( B_1, \ldots, B_u \in \text{SumProd}(W, G) \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) such that \( A_i(n) = B_i(n) \) holds for all \( n \geq \delta_i \) and \( 1 \leq i \leq r \).

1. Construct \( R\Pi\Sigma \)-extension \((E, \sigma)\) of \((G, \sigma)\) with \( E = G\langle t_1 \rangle \ldots \langle t_e \rangle \) equipped with \( \text{ev} : E \times \mathbb{N} \rightarrow \mathbb{K} \) such that we get \( a_1, \ldots, a_u \in E \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) with

\[
A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}
\]

2. Set \( W = \{T_1, \ldots, T_e\} \) with \( T_i := \text{expr}(t_i) \in \Sigma \Pi(G) \) for \( 1 \leq i \leq e \).
3. Set \( B_i := \text{expr}(a_i) \in \text{SumProd}(W, G) \) for \( 1 \leq i \leq u \).
4. Return \( W, (B_1, \ldots, B_u) \) and \((\delta_1, \ldots, \delta_u)\).
This yields a strategy (actually the only strategy for shift-stable sets).

**An algorithm to solve Problem SigmaReduce**

Given: \( A_1, \ldots, A_u \in \text{SumProd}(G) \) with \( G = K(x) \).
Find: a \( \sigma \)-reduced set \( W = \{T_1, \ldots, T_e\} \subseteq \Sigma\Pi(G) \) with \( B_1, \ldots, B_u \in \text{SumProd}(W, G) \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) such that \( A_i(n) = B_i(n) \) holds for all \( n \geq \delta_i \) and \( 1 \leq i \leq r \).

1. Construct \( R\Pi\Sigma \)-extension \((E, \sigma)\) of \((G, \sigma)\) with \( E = G\langle t_1 \rangle \cdots \langle t_e \rangle \) equipped with \( \text{ev} : E \times \mathbb{N} \to K \) such that we get \( a_1, \ldots, a_u \in E \) and \( \delta_1, \ldots, \delta_u \in \mathbb{N} \) with

\[
A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}
\]

2. Set \( W = \{T_1, \ldots, T_e\} \) with \( T_i := \text{expr}(t_i) \in \Sigma\Pi(G) \) for \( 1 \leq i \leq e \).
3. Set \( B_i := \text{expr}(a_i) \in \text{SumProd}(W, G) \) for \( 1 \leq i \leq u \).
4. Return \( W, (B_1, \ldots, B_u) \) and \((\delta_1, \ldots, \delta_u)\).
Conclusion. All results can be generalized to the following setting:

- **the mixed multibasic hypergeometric case:**

  \[ \mathcal{G} := \mathbb{K}(x, x_1, \ldots, x_v) \]  
  \[ \text{with } \mathbb{K} = K(q_1, \ldots, q_v) \]  
  For \( f = \frac{p}{q} \in \mathcal{G} \) with \( p, q \in \mathbb{K}[x, x_1, \ldots, x_v] \) where \( q \neq 0 \) and \( p, q \) being coprime we define

  \[
  \text{ev}(f, k) = \begin{cases} 
  0 & \text{if } q(k, q_1^k, \ldots, q_v^k) = 0 \\
  \frac{p(k, q_1^k, \ldots, q_v^k)}{q(k, q_1^k, \ldots, q_v^k)} & \text{if } q(k, q_1^k, \ldots, q_v^k) \neq 0.
  \end{cases}
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  \end{cases}
  \]

- **simple products:** \( \text{Prod}^*(G) \) is the smallest set that contains 1 with:

  1. If \( r \in \mathcal{R} \) then \( \text{RPow}(r) \in \text{Prod}^*(G) \).
  2. If \( f \in G^*, l \in \mathbb{N} \) with \( l \geq Z(f) \) then \( \text{Prod}(l, f) \in \text{Prod}^*(G) \).
  3. If \( p, q \in \text{Prod}^*(G) \) then \( p \odot q \in \text{Prod}^*(G) \).
  4. If \( p \in \text{Prod}^*(G) \) and \( z \in \mathbb{Z} \setminus \{0\} \) then \( p \hat{\odot} z \in \text{Prod}^*(G) \).
Conclusion. All results can be generalized to the following setting:

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\[
evw(f, k) = \begin{cases} 0 & \text{if } q(k, q_1^k, \ldots, q_v^k) = 0 \\ \frac{p(k, q_1^k, \ldots, q_v^k)}{q(k, q_1^k, \ldots, q_v^k)} & \text{if } q(k, q_1^k, \ldots, q_v^k) \neq 0 \end{cases}\]

- **nested products:** \( \text{Prod}^*(\mathbb{G}) \) is the smallest set that contains 1 with:
  1. If \( r \in \mathcal{R} \) then \( \text{RPow}(r) \in \text{Prod}^*(\mathbb{G}) \).
  2. If \( p \in \text{Prod}^*(\mathbb{G}) \), \( f \in \mathbb{G}^* \), \( l \in \mathbb{N} \) with \( l \geq Z(f) \) then \( \text{Prod}(l, f \odot p) \in \text{Prod}^*(\mathbb{G}) \).
  3. If \( p, q \in \text{Prod}^*(\mathbb{G}) \) then \( p \odot q \in \text{Prod}^*(\mathbb{G}) \).
  4. If \( p \in \text{Prod}^*(\mathbb{G}) \) and \( z \in \mathbb{Z} \setminus \{0\} \) then \( p \odot z \in \text{Prod}^*(\mathbb{G}) \).
Conclusion. All results can be generalized to the following setting:

- **the mixed multibasic hypergeometric case:**
  \[ G := \mathbb{K}(x, x_1, \ldots, x_v) \quad \text{with} \quad \mathbb{K} = K(q_1, \ldots, q_v) \]
  For \( f = \frac{p}{q} \in G \) with \( p, q \in \mathbb{K}[x, x_1, \ldots, x_v] \) where \( q \neq 0 \) and \( p, q \) being coprime we define

  \[
  \text{ev}(f, k) = \begin{cases} 
  0 & \text{if } q(k, q_1^k, \ldots, q_v^k) = 0 \\
  p(k, q_1^k, \ldots, q_v^k) & \text{if } q(k, q_1^k, \ldots, q_v^k) \neq 0.
  \end{cases}
  \]

- **nested products:** \( \text{Prod}^*(G) \) is the smallest set that contains 1 with:
  1. If \( r \in \mathcal{R} \) then \( \text{RPow}(r) \in \text{Prod}^*(G) \).
  2. If \( p \in \text{Prod}^*(G), f \in G^*, l \in \mathbb{N} \) with \( l \geq Z(f) \) then \( \text{Prod}(l, f \odot p) \in \text{Prod}^*(G) \).
  3. If \( p, q \in \text{Prod}^*(G) \) then \( p \odot q \in \text{Prod}^*(G) \).
  4. If \( p \in \text{Prod}^*(G) \) and \( z \in \mathbb{Z} \setminus \{0\} \) then \( p^\odot z \in \text{Prod}^*(G) \).

For further details see