

A Loop Summit - new perturbative results and methods in precision physics

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Function spaces in QFT at higher loops and their algorithmic exploration

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Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N}_{f(\varepsilon, N, x, y, z)}$$

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The integrand is

- hypergeometric in N :

$$\frac{f(\varepsilon, N+1, x, y, z)}{f(\varepsilon, N, x, y, z)} \in \mathbb{Q}(\varepsilon, N, x, y, z)$$

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Ablinger's
MultiIntegrate.m \downarrow (9 hours)

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \dots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

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recurrence solver \downarrow

$F(\varepsilon, N)$ = expression in terms of special functions

Recurrence solving

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(N), \dots, a_\delta(N)$: polynomials in N

$h(N)$: expression in **indefinite nested sums**
defined over hypergeometric products.

$$a_0(N)F(N) + \dots + a_\delta(N)F(N + \delta) = h(N);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

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DECIDE constructively if $F(N)$ can be expressed in terms **indefinite nested sums** defined over hypergeometric products.

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Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(N) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j} \quad (\text{harmonic sums})$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];

J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^n \frac{2^k}{k} \sum_{i=1}^k \frac{2^{-i}}{i} \sum_{j=1}^i \frac{S_1(j)}{j} \quad (\text{generalized harmonic sums})$$

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^N \frac{1}{(1+2k)^2} \sum_{j=1}^k \frac{1}{j^2} \sum_{i=1}^j \frac{1}{1+2i} \quad (\text{cyclotomic harmonic sums})$$

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{j=1}^N \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \quad (\text{binomial sums})$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^N 2^{-2h} (1 - \eta)^h \binom{2h}{h} \sum_{k=1}^h \frac{2^{2k}}{k^2 \binom{2k}{k}} \quad (\text{generalized binomial sums})$$

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

Sigma.m is based on difference ring/field theory

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Ablinger's
MultiIntegrate.m

(9 hours)

↓

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \dots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

recurrence solver

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Ablinger's
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↓

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \dots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

Sigma

↓

$$F(\varepsilon, N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \dots$$

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(\varepsilon, N), \dots, a_\delta(\varepsilon, N)$: polynomials in ε, N
 $h_l(N), h_{l+1}(N), \dots, h_\lambda(N)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon, N)F(\varepsilon, N) + \dots + a_\delta(\varepsilon, N)F(\varepsilon, N + \delta) \\ = h_l(N)\varepsilon^l + h_{l+1}(N)\varepsilon^{l+1} + \dots h_\lambda(N)\varepsilon^r + O(\varepsilon^{r+1});$$

together with ε -expansions of $F(0), \dots, F(\delta - 1)$ up to a certain order.

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together with ε -expansions of $F(0), \dots, F(\delta - 1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(N)$ of

$$F(N) = F_l(N)\varepsilon^l + F_{l+1}(N)\varepsilon^{l+1} + \dots + F_\lambda(N)\varepsilon^r + O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]

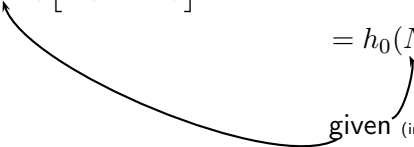
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, N) [F(N)] \\ & + a_1(\varepsilon, N) [F(N + 1)] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, N) [F(N + \delta)] \end{aligned}$$

$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$

given (in terms of indefinite nested sums and products)



Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, N) \left[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, N) \left[F(N + 1) \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, N) \left[F(N + \delta) \right] \end{aligned}$$

$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_\delta(\varepsilon, N) \left[F(N+\delta) \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

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 & \vdots \\
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 & + \\
 & \vdots \\
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 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, N)F_0(N) + a_1(0, N)F_0(N+1) + \dots + a_\delta(0, N)F_0(N+\delta) = h_0(N)$$

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 & + \\
 & \vdots \\
 & + a_\delta(\varepsilon, N) \left[F_0(N+\delta) + F_1(N+\delta)\varepsilon + F_2(N+\delta)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

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REC solver: Given the initial values $F_0(1), F_0(2), \dots, F_0(\delta)$,
decide if $F_0(N)$ can be written in terms of indefinite
 nested sums and products.

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
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 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, N)F_0(N) + a_1(0, N)F_0(N+1) + \dots + a_\delta(0, N)F_0(N+\delta) = h_0(N)$$

$$\begin{aligned} & a_0(\varepsilon, N) \left[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, N) \left[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, N) \left[F_1(N+\delta)\varepsilon + F_2(N+\delta)\varepsilon^2 + \dots \right] \\ & \qquad \qquad \qquad = h'_0(N) + h'_1(N)\varepsilon + h'_2(N)\varepsilon^2 + \dots \end{aligned}$$

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_\delta(\varepsilon, N) \left[F_1(N+\delta)\varepsilon + F_2(N+\delta)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = \underbrace{h'_0(N)}_{=0} + h'_1(N)\varepsilon + h'_2(N)\varepsilon^2 + \dots
 \end{aligned}$$

Devide by ε

$$\begin{aligned} & a_0(\varepsilon, N) \left[F_1(N) + F_2(N)\varepsilon + \dots \right] \\ & + a_1(\varepsilon, N) \left[F_1(N+1) + F_2(N+1)\varepsilon + \dots \right] \\ & + \\ & \vdots \\ & + a_\delta(\varepsilon, N) \left[F_1(N+\delta) + F_2(N+\delta)\varepsilon + \dots \right] = h'_1(N) + h'_2(N)\varepsilon + \dots \end{aligned}$$

Repeat to get $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

↓ (package MultiIntegrate.m)

$$a_0(\varepsilon, N)F(N) + a_1(\varepsilon, N)F(N+1) + \dots + a_5(\varepsilon, N)F(N+5) = 0$$

$$F(2) = \frac{20}{27\varepsilon^3} - \frac{40}{27\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{1393}{486} + \frac{5\zeta_2}{18} \right) + \dots$$

⋮

$$F(6) = \frac{22}{147\varepsilon^3} - \frac{535}{2058\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{630043}{1234800} + \frac{11\zeta_2}{196} \right) + \dots$$

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⋮

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↓ (summation package Sigma.m)

$$F(N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

We get

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

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$$F_{-2}(N) = -\frac{4(-1)^N(3N^3+18N^2+31N+18)}{3(N+1)^3(N+2)^2} - \frac{4(6N^3+32N^2+51N+26)}{3(N+1)^3(N+2)^2}$$

We get

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N(3N^3+18N^2+31N+18)}{3(N+1)^3(N+2)^2} - \frac{4(6N^3+32N^2+51N+26)}{3(N+1)^3(N+2)^2}$$

$$\begin{aligned} F_{-1}(N) &= (-1)^N \left(\frac{2(9N^5 + 81N^4 + 295N^3 + 533N^2 + 500N + 204)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)} \right) \\ &+ \frac{2(18N^5 + 150N^4 + 490N^3 + 755N^2 + 536N + 132)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)} \\ &+ \left(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)} \right) S_2 \\ &+ \left(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)} \right) S_{-2} \end{aligned}$$

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
 Almquist/Zeilberger
 (Jakob Ablinger)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n + d) = h(\varepsilon, n)$$

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 ε -recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(K. Wegschaider)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n + d) = h(\varepsilon, n)$$

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MultiSum Package
(K. Wegschaider)

Holonomic/difference ring approach

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n + d) = h(\varepsilon, n)$$

Solving coupled systems

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)

Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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\downarrow
 uncoupling algorithms
 (Zürcher, Abramov/Zima, Gauss, ...)

1. $\hat{I}_1(x)$ is a solution of

$$d_0(x)\hat{I}_1(x) + d_1(x)D_x\hat{I}_1(x) + \dots + d_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinCom}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinCom}(\dots, D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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↓
 uncoupling algorithms
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DE-solver

(see, e.g., [arXiv:1810.12261])

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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uncoupling algorithms
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DE-solver

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REC-solver

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A \hat{I}(x) + \hat{R}(x)$$

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A \hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$
$$\hat{I}_k(x) = \text{expr}_k(D_x^i \hat{I}_1(x)), k > 1$$

The DE-REC approach

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$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A \hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)
uncoupling algorithm

uncoupled DE system

$$\begin{aligned} \sum_i d_i(x) D_x^i \hat{I}_1(x) &= \hat{r}(x) \\ \hat{I}_k(x) &= \text{expr}_k(D_x^i \hat{I}_1(x)), k > 1 \end{aligned}$$

coeff. comparison w.r.t. x^N

linear recurrence

$$\sum_i a_i(N) I_1(N+i) = r(N)$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

The DE-REC approach

DE system

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coeff. comparison w.r.t. x^N

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

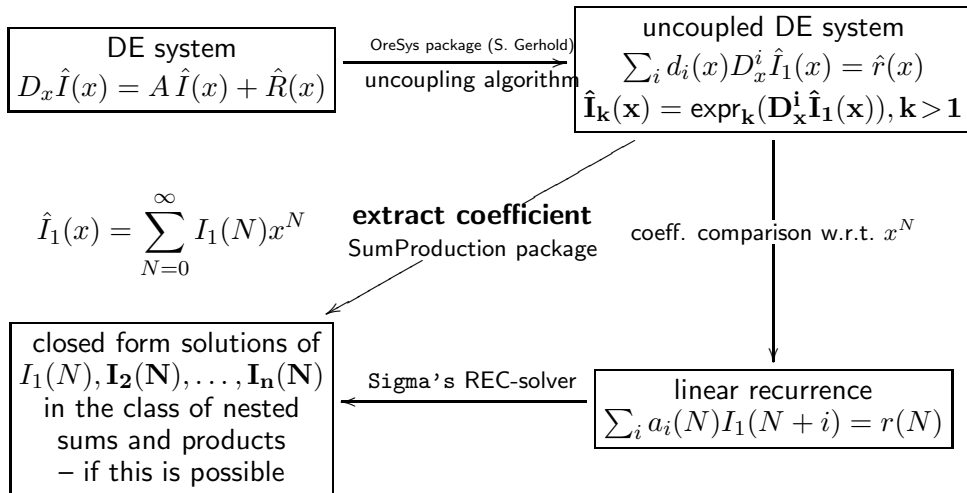
closed form solutions of $I_1(N)$
in the class of nested
sums and products
– if this is possible

Sigma's REC-solver

linear recurrence

$$\sum_i a_i(N) I_1(N+i) = r(N)$$

The DE-REC approach (SolveCoupledSystem package)



General strategy:

↓ IBP methods

- ▶ Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- ▶ $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

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↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$

$$D(N) = \varepsilon^{-3}D_{-3}(N) + \varepsilon^{-2}D_{-2}(N) + \varepsilon^{-1}D_{-1}(N) + \varepsilon^0D_0(N) + \dots$$

Concrete calculations:

- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, 2014. arXiv:1406.4654 [hep-ph].
- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, 2014. arXiv:1405.4259 [hep-ph].
- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, 2014. arXiv:1402.0359 [hep-ph].
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), 2015. arXiv:1508.01449 [hep-ph].
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, 2015. arXiv:1504.08217 [hep-ph].
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, 2015. arXiv:1409.1135.
- ▶ A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+ - W^-}(x, Q^2)$ and $F_2^{W^+ - W^-}(x, Q^2)$. Physical Review D 94(11), 2016. arXiv:1609.06255 [hep-ph].
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, 2016. arXiv:1509.08324 [hep-ph].
- ▶ J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit, 2018. arXiv:1804.07313 [hep-ph].
- ▶ J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D, 2018. arXiv:1712.09889.


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↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$ 

$$D(N) = \varepsilon^{-3}D_{-3}(N) + \varepsilon^{-2}D_{-2}(N) + \varepsilon^{-1}D_{-1}(N) + \varepsilon^0 D_0(N) + \dots$$


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↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$ 

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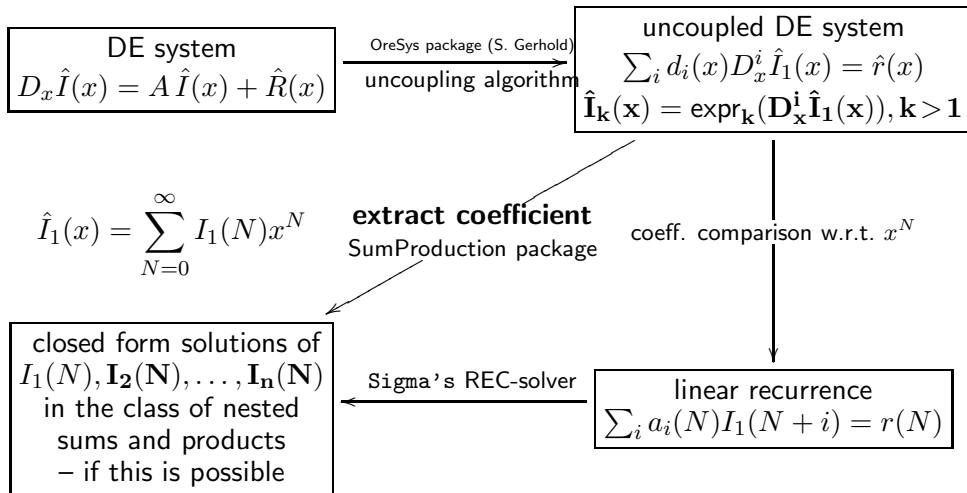
$$I_i(N) = \underbrace{\varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \varepsilon^0F_0(N) + \dots + \varepsilon^6F_6(N)}_{\text{often unknown functions}}$$

↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$ 

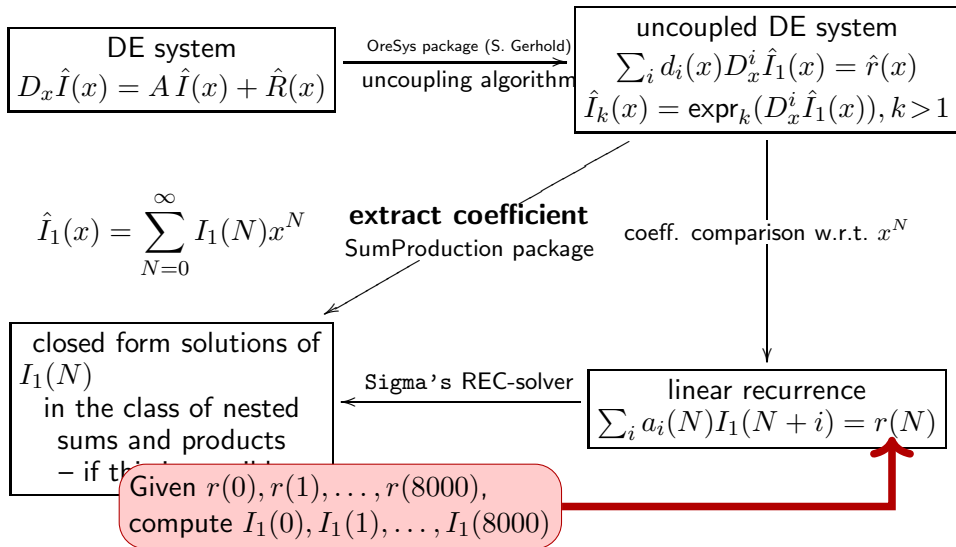
$$D(N) = \underbrace{\varepsilon^{-3}D_{-3}(N) + \varepsilon^{-2}D_{-2}(N) + \varepsilon^{-1}D_{-1}(N)}_{\text{often nice}} + \underbrace{\varepsilon^0D_0(N)}_{\text{partially nice}} + \dots$$

Computing large moments and guessing recurrences

The DE-REC approach (SolveCoupledSystem package)



The method of large moments (SolveCoupledSystem)



General strategy:

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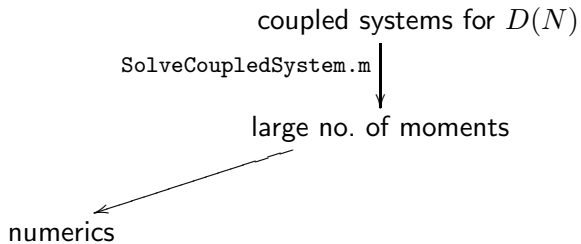
$N = 0, 1, \dots, 8000$

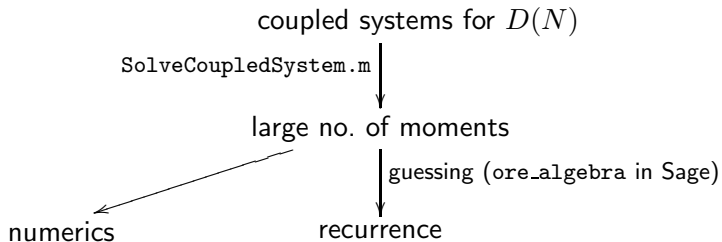
↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$

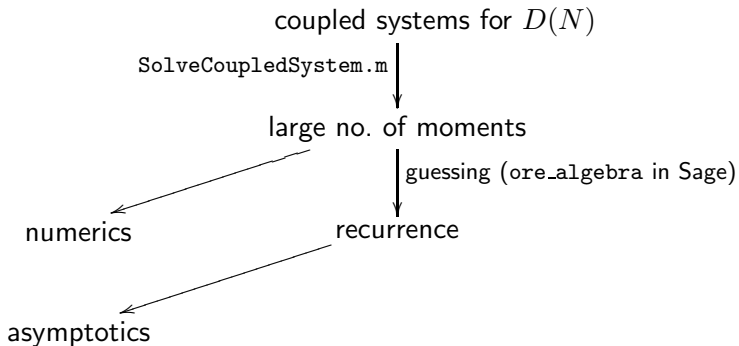
$$D(N) = \underbrace{\varepsilon^{-3}D_{-3}(N) + \varepsilon^{-2}D_{-2}(N) + \varepsilon^{-1}D_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^0D_0(N)}_{\text{numbers}} + \dots$$

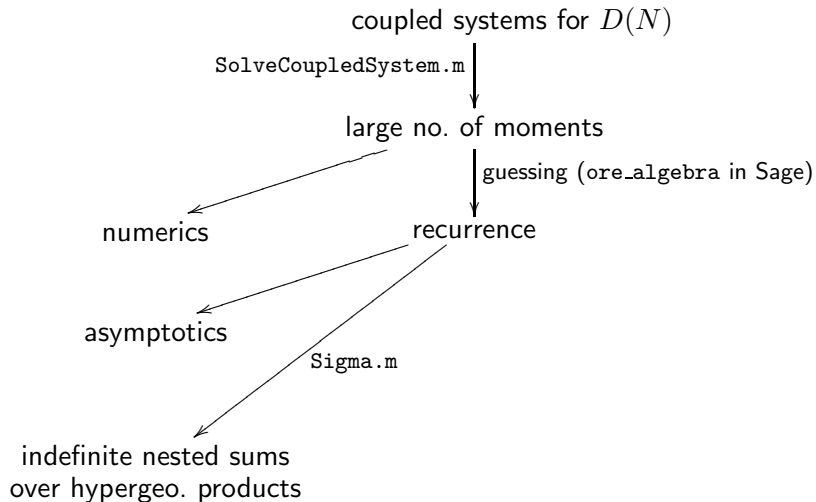
$N = 0, 1, \dots, 8000$

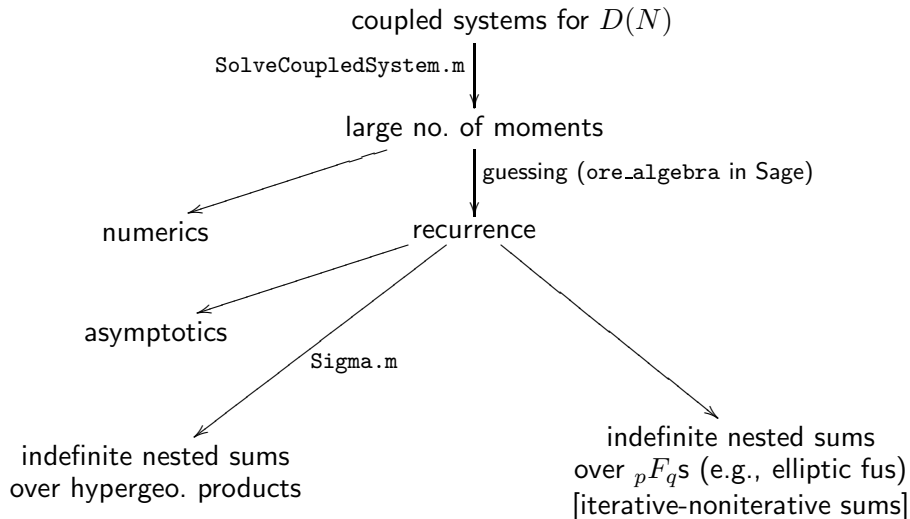
coupled systems for $D(N)$
`SolveCoupledSystem.m` ↓
large no. of moments

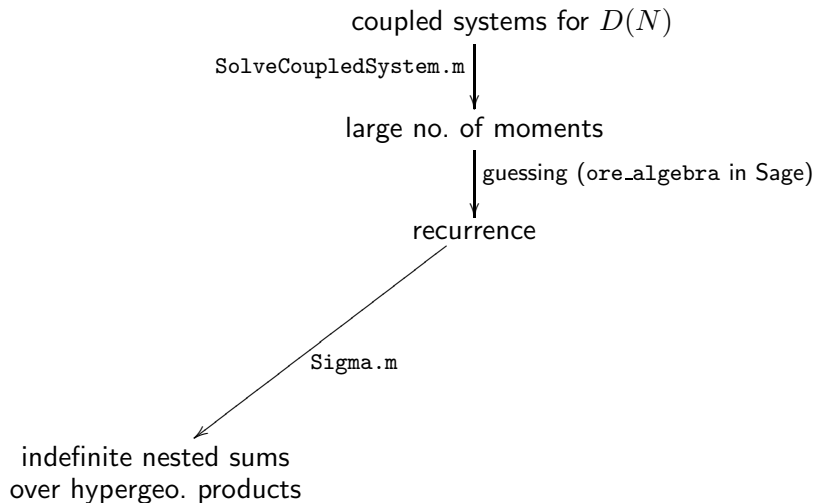












General strategy:

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- ▶ $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

↓ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)x^N$

$$I_i(N) = \varepsilon^{-3}F_{-3}(N) + \varepsilon^{-2}F_{-2}(N) + \varepsilon^{-1}F_{-1}(N) + \underbrace{\varepsilon^0F_0(N) + \dots + \varepsilon^6F_6(N)}_{\text{only numbers}}$$

$N = 0, 1, \dots, 8000$

↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$

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General strategy:

↓ IBP methods

- ▶ Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- ▶ $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

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↓ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N)x^N$ 

$$D(N) = \underbrace{\varepsilon^{-3}D_{-3}(N) + \varepsilon^{-2}D_{-2}(N) + \varepsilon^{-1}D_{-1}(N)}_{\text{nice}} + \underbrace{\varepsilon^0 D_0(N)}_{\text{partially nice}} + \dots$$

all N solution

Concrete calculations of large moments:

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 1. computed ~ 2400 moments
 2. guessed and solved all recurrences

[Nucl.Phys.B./2017]

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Mathematica is loaded...

```
In[4]:= << Sigma.m
```

Sigma - A summation package by Carsten Schneider © RISC-Linz

```
In[5]:= initial = << iFile16
```


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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[5]:= initial = << iFile16

Out[5]= { 37, 34577/1296, 7598833/151875, 13675395569/230496000,
 475840076183/7501410000, 1432950323678333/21965628762000,
 21648380901382517/328583783127600,
 52869784323778576751/802218994536960000,
 49422862094045523994231/753773992230616156800,
 33131879832907935920726113/509557943985299969760000,
 5209274721836755168448777/80949984111854180459136,
 56143711997344769021041145213/882589266383586456384353664,
 453500433353845628194790025124807/7217228048879468556886950000000,
 14061543374120479886110159898869387/226643167590350326435656036000000,
 715586522666491903324905785178619936571168370307700222807811495895030000000,
 16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,
 1428729642632302467951426905844691837805299/23940759575034122827861315961573673600000,
 498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000,
 555296381919643885816767597997123847103620469689/9546343089354642655682952088937477747472000000 };

In[6]:= **rec** ==<< **rFile16**

Out[6]= $(n + 1)^4(n + 2)^2(2n + 3)(2n + 5)(2n + 7)(2n + 9)(2n + 11) \left(309237645312n^{32} + 38256884318208n^{31} + 2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + 873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + 1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + 575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + 69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + 901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + 7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + 32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + 86710462147941775492301231896818n^9 + 112917328975807075881545543668548n^8 + 124873767581470867343743078943772n^7 + 115624836314544572769501784072647n^6 + 87938536330971046886456627610048n^5 + 53481897815980319933589323279298n^4 + 25000430622737750756669804052204n^3 + 8430930497463933665464836129855n^2 + 1825177817831282261293155379650n + 190428196025667395685609855000 \right) (2n + 1)^4 f[n]$

$$\begin{aligned}
& -(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11) \left(12369505812480n^{38} + 1613151061671936n^{37} + \right. \\
& 101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
& 2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
& 9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
& 7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
& 356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
& 9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
& 170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
& 1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
& 13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
& 59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
& 167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
& 284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
& 279679164311116651162116055961513301n^9 + 221781415386984655607595031093415136n^8 + \\
& 149214365004640710156345950062395186n^7 + 83882523964213110328265187672574356n^6 + \\
& 38609679702395410742361774562392789n^5 + 14149471988638475521561721269939086n^4 + \\
& 3963748138857399502678254252169734n^3 + 795659668131014454843348852372480n^2 + \\
& 101701393436276172443717692853400n + 6204709909986751913151675960000) f[n+1]
\end{aligned}$$

$$\begin{aligned}
& +2(n+3)^2(2n+5)^3(2n+9)(2n+11) \left(24739011624960n^{40} + 3317836466356224n^{39} + 215508170284466176n^{38} + 9032884062187945984n^{37} + \right. \\
& 274636134389959884800n^{36} + 6455501959255126179840n^{35} + 122094572934385260036096n^{34} + 1909387225793663151898624n^{33} + \\
& 25180108291969215434326016n^{32} + 284171960705270647479074816n^{31} + 2775794400720227034854326272n^{30} + \\
& 23677622163992853854566219776n^{29} + 177624312783583749157935120384n^{28} + 1178515602115604757944201871360n^{27} + \\
& 6947091965313419323781358354432n^{26} + 36515023100308314818702129258496n^{25} + 171621148571344894953594594017280n^{24} + \\
& 722837793013976317556258102507520n^{23} + 2732534027077907914497042720534528n^{22} + 9281028665970648470895368668485120n^{21} + \\
& 28337819215557708948254385336117248n^{20} + 77786125749274632150536464583130752n^{19} + 191877161455672780973502244537632256n^{18} + \\
& 424953221702140663089937921965135648n^{17} + 843818276409975584824720931649555264n^{16} + \\
& 1499359936674956711935311062995422344n^{15} + 2378007025570977662661938772843220240n^{14} + \\
& 3355671771434535852147325502571953770n^{13} + 4196375762867184563407432891655585484n^{12} + \\
& 4627675779563752366067861596232781096n^{11} + 4473175960511956000526499430851993603n^{10} + \\
& 3761696365025837909581516781307249585n^9 + 2726553473467254373993685951699145492n^8 + \\
& 1683383212304999468664293798012773485n^7 + 871926653651504419744271839781064837n^6 + \\
& 371307437598003570058538796122994147n^5 + 126427972742886389602285855482966072n^4 + 33048762330145623969058704448697313n^3 + \\
& 6217924746857741077419160100404560n^2 + 748298077423337427195946099994100n + 43181089548034246077698611794000 \Big) f[n+2]
\end{aligned}$$

$$\begin{aligned}
& -2(n+4)^2(2n+5)(2n+7)^3(2n+11) \left(24739011624960n^{40} + 3322784268681216n^{39} + 216160919414112256n^{38} + 9074528155284275200n^{37} + \right. \\
& 276348048819456311296n^{36} + 6506479077331107315712n^{35} + 123266585640616142569472n^{34} + 1931040885785102661976064n^{33} + \\
& 25510503383281445462081536n^{32} + 288418124175428279391485952n^{31} + 2822442799033603081019326464n^{30} + \\
& 24120717233320712351821332480n^{29} + 181295944719289040999116701696n^{28} + 1205246297785423925076555694080n^{27} + \\
& 7119049557560114436136213413888n^{26} + 37496933571993839665392189775872n^{25} + 176616172467048982234270428880896n^{24} + \\
& 745539218875020737621728364206080n^{23} + 2824909633156578132652259733712896n^{22} + 9618101958268071244680677589035520n^{21} + \\
& 29441860528446423517613263360742912n^{20} + 81033563306363873505877563416477312n^{19} + 200454769103641040142838133702338304n^{18} + \\
& 445286624972461749049425309485328992n^{17} + 887028447418790661018847407251573152n^{16} + \\
& 1581538101499869694224895701784875304n^{15} + 2517550244392724509968791166585362672n^{14} + \\
& 3566593026520465155504695877897282630n^{13} + 4479066125207404898722179511912639638n^{12} + \\
& 4962006990874351800791769650243464872n^{11} + 4819992643914265990647887896664485209n^{10} + \\
& 407489538669418224094153822230233221n^9 + 2970477229398746689186622534784613554n^8 + \\
& 1845274131994015990683957902602775337n^7 + 962091291302144537393228847830431614n^6 + \\
& 412595107814836563208757757032740146n^5 + 141540723940232563767779647013785485n^4 + 37292931812630561528276365992452010n^3 + \\
& 7074865777225416725452872895397100n^2 + 858794112392644074221312049837000n + 49997386738260112603615104780000) f[n+3]
\end{aligned}$$

$$\begin{aligned}
& + (n+5)^3(2n+5)(2n+7)(2n+9)^4 \left(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} + \right. \\
& 3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} + \\
& 38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} + \\
& 69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} + \\
& 32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} + \\
& 1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} + \\
& 20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} + \\
& 260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} + \\
& 2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} + \\
& 10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} + \\
& 34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} + \\
& 69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} + \\
& 82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^9 + \\
& 55935262205790259307904762197107653n^8 + 36322355479155199114489624391144238n^7 + \\
& 19756597118002557191991191826327042n^6 + 8822212911433711339358062994077203n^5 + \\
& 3145597282374650512689680780380605n^4 + 859907105684964990690798899478888n^3 + \\
& 168963309995629650025632011492580n^2 + 21205680751316222158938757272000n + \\
& 1274120732351744651125603886400) f[n+4]
\end{aligned}$$

$$\begin{aligned}
& -(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4 \left(309237645312n^{32} + 28361279668224n^{31} + \right. \\
& 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + \\
& 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + \\
& 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + \\
& 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + \\
& 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + \\
& 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + \\
& 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + \\
& 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^9 + 25770328899499991754425455738n^8 + \\
& 18817114309842270306167785140n^7 + 11424980760825630752861027739n^6 + 5656051955667821083952617134n^5 + \\
& 2221448212382554437709999491n^4 + 664859653803075491350122060n^3 + 142190920852333874895041748n^2 + \\
& 19313175036907229252501700n + 1248723341516324359641600) f[n+5] == 0
\end{aligned}$$

```
In[7]:= recSol = SolveRecurrence[rec, f[n]]
```


In[7]:= `recSol = SolveRecurrence[rec, f[n]]`

$$\begin{aligned}
 \text{Out[7]} = & \left\{ \left\{ 0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2} \right\} \right. \\
 & \left\{ 0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2} \right\} \\
 & \left\{ 0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n) \sum_{i=1}^n \frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} \right\} \\
 & \left\{ 0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n) \sum_{i=1}^n \frac{1}{i^2}}{(1+n)(1+2n)} + \left(- \right. \right. \\
 & \left. \frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} \right) \sum_{i=1}^n \frac{1}{i} + \\
 & \left. \frac{12(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^2}{(1+n)(1+2n)} + \frac{56(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \right. \\
 & \left. \frac{4(3+2n)(3+44n+48n^2) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2}{(1+n)(1+2n)} \right\}
 \end{aligned}$$

$$\begin{aligned}
& \{0, \frac{1}{16(1+n)^4(1+2n)^4} (72519 + 572343n + 1814716n^2 + 2918100n^3 + 2442240n^4 + 912896n^5 + 24576n^6 - \\
& 49152n^7) + \frac{16(3+2n) \sum_{i=1}^n \frac{1}{i^3}}{3(1+n)(1+2n)} + (-\frac{(3+2n)(29+307n+322n^2)}{4(1+n)^2(1+2n)^2} + \frac{44(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{i^2} + \\
& (\frac{(3+2n)(91+259n+974n^2+1784n^3+1024n^4)}{4(1+n)^3(1+2n)^3} + \frac{22(3+2n) \sum_{i=1}^n \frac{1}{i^2}}{(1+n)(1+2n)} + \frac{24(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \\
& \frac{4(3+2n)(-13-4n+16n^2) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{16(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^2}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{i} + (- \\
& \frac{(3+2n)(19+92n+80n^2)}{(1+n)^2(1+2n)^2} + \frac{40(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} (\sum_{i=1}^n \frac{1}{i})^2 + \frac{20(3+2n)(\sum_{i=1}^n \frac{1}{i})^3}{3(1+n)(1+2n)} + \\
& \frac{64(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)} - \frac{3(3+2n)(63+209n+150n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2} + \\
& (\frac{(3+2n)(347+1795n+4302n^2+4856n^3+2048n^4)}{2(1+n)^3(1+2n)^3} + \frac{48(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}) \sum_{i=1}^n \frac{1}{-1+2i} - \\
& \frac{4(3+2n)(19+92n+80n^2)(\sum_{i=1}^n \frac{1}{-1+2i})^2}{(1+n)^2(1+2n)^2} + \frac{32(3+2n)(\sum_{i=1}^n \frac{1}{-1+2i})^3}{3(1+n)(1+2n)} - \\
& \frac{8(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{i}}{(1+n)(1+2n)} - \frac{16(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j})^2}{-1+2i}}{(1+n)(1+2n)} \left(\sum_{j=1}^i \frac{1}{j} \right) \sum_{j=1}^i \frac{1}{-1+2j} \\
& - \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{j}) \sum_{j=1}^i \frac{1}{-1+2j}}{i}}{(1+n)(1+2n)} + \frac{64(3+2n) \sum_{i=1}^n \frac{1}{-1+2i}}{(1+n)(1+2n)} + \\
& \frac{32(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{i}}{(1+n)(1+2n)} + \frac{64(3+2n) \sum_{i=1}^n \frac{(\sum_{j=1}^i \frac{1}{-1+2j})^2}{-1+2i}}{(1+n)(1+2n)} \}, \{1, 0\} \}
\end{aligned}$$

```
In[8]:= sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]
```

In[8]:= sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]

$$\begin{aligned}
 \text{Out}[8]= & \frac{1}{3(1+n)^4(1+2n)^4} (111 + 1920n + 11765n^2 + 32545n^3 + 46476n^4 + 35376n^5 + 13440n^6 + 1968n^7) + \frac{32(3+2n) \sum_{i=1}^n \frac{1}{i^3}}{9(1+n)(1+2n)} - \\
 & \frac{(3+2n)(-3+101n+126n^2) \sum_{i=1}^n \frac{1}{i^2}}{(3+2n)(115+921n+1967n^2+1524n^3+340n^4) \sum_{i=1}^n \frac{1}{i}} + \\
 & \frac{3(1+n)^2(1+2n)^2}{44(3+2n) \left(\sum_{i=1}^n \frac{1}{i^2} \right) \sum_{i=1}^n \frac{1}{i}} - \frac{3(1+n)^3(1+2n)^3}{(3+2n)(23+139n+130n^2) \left(\sum_{i=1}^n \frac{1}{i} \right)^2} + \frac{40(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^3}{40(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^3} + \\
 & \frac{3(1+n)(1+2n)}{128(3+2n) \sum_{i=1}^n \frac{1}{(-1+2i)^3}} - \frac{3(1+n)^2(1+2n)^2}{4(3+2n)(77+261n+190n^2) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \frac{9(1+n)(1+2n)}{16(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right) \sum_{i=1}^n \frac{1}{(-1+2i)^2}} + \\
 & \frac{9(1+n)(1+2n)}{2(3+2n)(13-153n-303n^2+12n^3+172n^4) \sum_{i=1}^n \frac{1}{-1+2i}} + \frac{3(1+n)^2(1+2n)^2}{88(3+2n) \left(\sum_{i=1}^n \frac{1}{i^2} \right) \sum_{i=1}^n \frac{1}{-1+2i}} - \\
 & \frac{4(3+2n)(-41-53n+2n^2) \left(\sum_{i=1}^n \frac{1}{i} \right) \sum_{i=1}^n \frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3} + \frac{3(1+n)(1+2n)}{80(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right)^2 \sum_{i=1}^n \frac{1}{-1+2i}} + \\
 & \frac{32(3+2n) \left(\sum_{i=1}^n \frac{1}{(-1+2i)^2} \right) \sum_{i=1}^n \frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2} - \frac{3(1+n)(1+2n)}{4(3+2n)(23+139n+130n^2) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2} + \\
 & \frac{32(3+2n) \left(\sum_{i=1}^n \frac{1}{i} \right) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^2}{(1+n)(1+2n)} + \frac{64(3+2n) \left(\sum_{i=1}^n \frac{1}{-1+2i} \right)^3}{3(1+n)(1+2n)} - \frac{16(3+2n) \sum_{i=1}^n \left(\frac{\sum_{j=1}^i \frac{1}{j} \right)^2}{i}}{3(1+n)(1+2n)} - \\
 & \frac{32(3+2n) \sum_{i=1}^n \frac{\left(\sum_{j=1}^i \frac{1}{j} \right)^2}{-1+2i}}{64(3+2n) \sum_{i=1}^n \frac{\left(\sum_{j=1}^i \frac{1}{j} \right) \sum_{j=1}^i \frac{1}{-1+2j}}{i}} + \\
 & \frac{3(1+n)(1+2n)}{128(3+2n) \sum_{i=1}^n \frac{\left(\sum_{j=1}^i \frac{1}{j} \right) \sum_{j=1}^i \frac{1}{-1+2j}}{-1+2i}} - \frac{3(1+n)(1+2n)}{64(3+2n) \sum_{i=1}^n \frac{\left(\sum_{j=1}^i \frac{1}{-1+2j} \right)^2}{i}} + \\
 & \frac{128(3+2n) \sum_{i=1}^n \frac{\left(\sum_{j=1}^i \frac{1}{-1+2j} \right)^2}{-1+2i}}{3(1+n)(1+2n)} + \\
 & \frac{128(3+2n) \sum_{i=1}^n \frac{\left(\sum_{j=1}^i \frac{1}{-1+2j} \right)^2}{-1+2i}}{3(1+n)(1+2n)}
 \end{aligned}$$

```
In[9]:= << HarmonicSums.m
```

```
HarmonicSums by Jakob Ablinger © RISC-Linz
```

```
In[10]:= sol = TransformToSSums[sol];
```

```
In[11]:= sol = ReduceToBasis[MultipleSumLimit[sol,  
n, 2]//ToStandardForm, n]//CollectProdSum;
```

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In[10]:= sol = TransformToSSums[sol];

In[11]:= sol = ReduceToBasis[MultipleSumLimit[sol,
n, 2]//ToStandardForm, n]//CollectProdSum;

$$\begin{aligned} \text{Out[11]} = & \frac{1}{3(1+n)^4(1+2n)^4} (111 + 1920n + 11765n^2 + 32545n^3 + 46476n^4 + 35376n^5 + 13440n^6 + \\ & 1968n^7) + \frac{64(3+2n)^2 S[1, n]}{3(1+n)(1+2n)^2} + \frac{64(3+2n)(2+3n) S[1, n]^2}{3(1+n)(1+2n)^2} + (- \\ & \frac{2(3+2n)(147 + 985n + 1871n^2 + 1268n^3 + 212n^4)}{3(1+n)^3(1+2n)^3} + \frac{224(3+2n) S[2, 2n]}{3(1+n)(1+2n)} + \\ & \frac{128(3+2n) S[-2, 2n]}{3(1+n)(1+2n)}) S[1, 2n] - \frac{4(3+2n)(23 + 123n + 114n^2) S[1, 2n]^2}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[1, 2n]^3}{3(1+n)(1+2n)} + \frac{64(3+2n) S[2, n]}{3(1+n)(1+2n)} - \frac{4(3+2n)(53 + 229n + 190n^2) S[2, 2n]}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[3, 2n]}{3(1+n)(1+2n)} + (- \frac{64(3+2n)^2}{3(1+n)(1+2n)^2} - \frac{128(3+2n)(2+3n) S[1, 2n]}{3(1+n)(1+2n)^2}) S[-1, 2n] - \\ & \frac{64(3+2n)(2+3n) S[-1, 2n]^2}{3(1+n)(1+2n)^2} - \frac{32(3+2n)(1+8n+8n^2) S[-2, 2n]}{3(1+n)^2(1+2n)^2} + \\ & \frac{64(3+2n) S[-3, 2n]}{3(1+n)(1+2n)} - \frac{128(3+2n) S[-2, 1, 2n]}{3(1+n)(1+2n)} \end{aligned}$$

```
In[9]:= << HarmonicSums.m
```

```
HarmonicSums by Jakob Ablinger © RISC-Linz
```

```
In[10]:= sol = TransformToSSums[sol];
```

```
In[11]:= sol = ReduceToBasis[MultipleSumLimit[sol,
n, 2]//ToStandardForm, n]//CollectProdSum;
```

```
In[12]:= SExpansion[sol, n, 2]
```

$$\begin{aligned} \text{Out[12]} = & \ln^2 \left(\frac{64\text{LG}[n]}{n} + \frac{160}{3n^2} - \frac{44}{n} \right) + \\ & \ln 2 \left(\left(\frac{320}{3n^2} - \frac{88}{n} \right) \text{LG}(n) + \frac{64\text{LG}[n]^2}{n} - \frac{430}{3n^2} + \frac{160\zeta_2}{3n} - \frac{14}{n} \right) + \\ & \zeta_2 \left(\frac{160\text{LG}[n]}{3n} + \frac{40}{n^2} - \frac{84}{n} \right) + \left(\frac{160}{3n^2} - \frac{44}{n} \right) \text{LG}[n]^2 + \left(-\frac{430}{3n^2} - \frac{14}{n} \right) \text{LG}[n] + \frac{64\text{LG}[n]^3}{3n} + \\ & \frac{64\ln^2 3}{3n} + \frac{145}{2n^2} + \frac{32\zeta_3}{n} + \frac{41}{n} \end{aligned}$$

Special function algorithms

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remiddi, Blümlein, . . .)

$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

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$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx,$$

$$\zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n) \\ - \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O\left(\frac{\ln(n)}{n^6}\right).$$


limit computations
numerical evaluation

► Generalized algorithms for generalized harmonic sums

$$\begin{aligned}
 \sum_{k=1}^n \frac{2^k \sum_{i=1}^k \frac{2^{-i} \sum_{j=1}^i \frac{S_1(j)}{j}}{i}}{k} &= -\frac{21\zeta(2)^2}{20n} + \frac{1}{8n^2} + \frac{295}{216n^3} - \frac{1115}{96n^4} + O(n^{-5}) \\
 &+ \left(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\right)\zeta(2) \\
 &+ 2^n \left(\frac{3}{2n} + \frac{3}{2n^2} + \frac{9}{2n^3} + \frac{39}{2n^4} + O(n^{-5})\right)\zeta(3) \\
 &+ \left(\frac{1}{n} + \frac{3}{4n^2} - \frac{157}{36n^3} + \frac{19}{n^4} + O(n^{-5})\right)(\log(n) + \gamma) \\
 &+ \left(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\right)(\log(n) + \gamma)^2
 \end{aligned}$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for cyclotomic harmonic sums

$$\begin{aligned}
 \sum_{k=1}^n \frac{\sum_{j=1}^k \frac{1}{1+2i}}{j^2} &= \left(-3 + \frac{35\zeta(3)}{16}\right)\zeta(2) - \frac{31\zeta(5)}{8} \\
 &+ \frac{1}{n} - \frac{33}{32n^2} + \frac{17}{16n^3} - \frac{4795}{4608n^4} + O(n^{-5}) \\
 &+ \log(2)\left(6\zeta(2) - \frac{1}{n} + \frac{9}{8n^2} - \frac{7}{6n^3} + \frac{209}{192n^4} + O(n^{-5})\right) \\
 &+ \left(-\frac{7}{4} - \frac{7}{16n} + \frac{7}{16n^2} - \frac{77}{192n^3} + \frac{21}{64n^4} + O(n^{-5})\right)\zeta(3) \\
 &+ \left(\frac{1}{16n^2} - \frac{1}{8n^3} + \frac{65}{384n^4} + O(n^{-5})\right)(\log(n) + \gamma)
 \end{aligned}$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for nested binomial sums

$$\sum_{j=1}^n \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta(3) + \sqrt{\pi}\sqrt{n} \left\{ \left[-\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} \right. \right. \\ + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} \\ \left. \left. + O(n^{-10}) \right] \ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} \right. \\ \left. - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} \right. \\ \left. + O(n^{-10}) \right\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence solver (in the setting of difference rings)
3. coupled systems solver
4. the large moment method

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6. stable and efficient software packages

Main CA-packages

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **MultiIntegrate.m**

MultIntegrate by Jakob Ablinger © RISC-Linz

In[3]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[5]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[6]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

In[7]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

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Within the **RISC-DESY** cooperation

J. Ablinger, A. Behring (KIT), J. Blümlein, A. De Freitas, P. Marquard, M. Saragnese, CS, K. Schönwald (KIT)

I expect that we discover and explore many

new algorithms in CA and results in QFT!