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Project part 9: Computer Algebra for Nested Sums and Products

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Example

In the analysis of the simplex algorithm on the Klee-Minty cube (R. Pemantle) the following constant arose:

$$S = \sum_{k=1}^{\infty} \frac{H_{k+1} - 1}{k(k+1)} \boxed{\sum_{j=1}^{\infty} \frac{H_j}{j(j+k)}}$$

where $$H_k = \sum_{i=1}^{k} \frac{1}{i}.$$

Representation of summation objects in difference rings (I)

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{H_j}{j(j+k)}$$

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$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

Recurrence finder (II)

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Recurrence solver (III)

$$\mathbf{A}(k)=\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)} \in \left\{ c_1\frac{H_k}{k}+c_2\frac{1}{k}+\frac{kH_k^2-2H_k+kH_k^{(2)}}{2k^2} \mid c_1,c_2\in\mathbb{R}\right\}$$

Representation of summation objects in difference rings (I)

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Recurrence solver (III)

$$\mathbf{A}(k)=\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)} = \boxed{0\frac{H_k}{k}+\zeta(2)\frac{1}{k}+\frac{kH_k^2-2H_k+kH_k^{(2)}}{2k^2}}$$

where

$$\zeta(z)=\sum_{i=1}^{\infty}\frac{1}{i^z}\qquad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}$$

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$$= \frac{\zeta(2)}{k} + \frac{kH_k^2 - 2H_k + kH_k^{(2)}}{2k^2}$$

$$= -4\zeta(2) + (\zeta(2)-1)\sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2}$$

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$$= -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) = 0.999222\ldots$$

Exploring nested sums and related integrals (IV)

Project team:

- Jakob Ablinger
- Nikolai Fadeev
- Abilio De Freitas
- Johannes Middeke
- Evans Doe Ocansey
- Mark Round
- Ali Kemal Uncu
- `Sigma/HarmonicSums` etc.

Number of publications: 114

(I) An algorithmic difference ring theory.

(i) Simplification

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$A(\lambda) = B(\lambda)$ for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$

(δ can be computed explicitly)

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- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

- and such that

the arising sums and products in $B(k)$ (except γ^n with $(\gamma^n)^\lambda = 1$) are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

- **Indefinite summation over such representations:**

– **Sums with roots of unity products:**

$$\sum_{k=1}^{n}(-1)^{\binom{k+1}{2}}k^2\sum_{j=1}^{k}\frac{(-1)^j}{j}=\frac{1}{2}\sum_{j=1}^{n}\frac{(-1)^{\binom{j+1}{2}}}{j}-\frac{1}{4}(-1)^{\binom{n+1}{2}}\big(-1+(-1)^n+2n\big)$$
$$+(-1)^{\binom{n+1}{2}}\frac{1}{2}\big(n(n+2)+(-1)^n\big(n^2-1\big)\big)\sum_{j=1}^{n}\frac{(-1)^j}{j}$$

- **Indefinite summation over such representations:**

– **Sums with roots of unity products:**

$$\sum_{k=1}^{n}(-1)^{\binom{k+1}{2}}k^2\sum_{j=1}^{k}\frac{(-1)^j}{j} = \frac{1}{2}\sum_{j=1}^{n}\frac{(-1)^{\binom{j+1}{2}}}{j} - \frac{1}{4}(-1)^{\binom{n+1}{2}}\big(-1+(-1)^n+2n\big)$$
$$+(-1)^{\binom{n+1}{2}}\frac{1}{2}\Big(n(n+2)+(-1)^n\big(n^2-1\big)\Big)\sum_{j=1}^{n}\frac{(-1)^j}{j}$$

– **Sums with nested products** (using Ocansey's package NestedProducts):

$$\sum_{k=1}^{n}\Bigg(\Big(-1+(1+k)\,(2+k)^2\prod_{j=1}^{k}(1+j)^2\Big)\prod_{j=1}^{k}j\prod_{i=1}^{j}(1+i)^2$$
$$-\frac{4}{3}\Big(1+2\,(1+k)^2\,(3+k)\prod_{j=1}^{k}-j\,(2+j)\Big)\prod_{j=1}^{k}2\,j\prod_{i=1}^{j}-i\,(2+i)\Bigg)$$
$$=4-\frac{1}{3}\,(1+n)^5\,(2+n)^2\left(-3+(1+i)\left(-\mathbb{i}+(\mathbb{i}^n)^2\right)(3+n)\,\mathbb{i}^n\right)(n!)^5\left(\prod_{i=1}^{k}\prod_{j=1}^{i}j\right)^2;$$

here $\mathbb{i}$ denotes the imaginary unit, i.e., $\mathbb{i}^2=-1$.

- **Representation into "elementary" products**

Given

$$y_1 = \prod_{k=1}^{n} \frac{-13122k(1+k)}{(3+k)^3}, \qquad y_2 = \prod_{k=1}^{n} \frac{26244k^2(2+k)^2}{(3+k)^2},$$

$$y_3 = \prod_{k=1}^{n} \frac{\mathbb{i}k(2+k)^3}{729(5+k)}, \qquad y_4 = \prod_{k=1}^{n} -\frac{162k(2+k)}{5+k}$$

• Representation into "elementary" products

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$$y_3=\prod_{k=1}^{n}\frac{\mathbb{i}k(2+k)^3}{729(5+k)},\qquad y_4=\prod_{k=1}^{n}-\frac{162k(2+k)}{5+k}$$

we can compute

$$y_1=\frac{216\,(\mathbb{i}^n)^2\,2^n\,(3^n)^8}{(n+1)^2(n+2)^3(n+3)^3n!},\qquad y_2=\frac{9\,(2^n)^2\,(3^n)^8\,(n!)^2}{(n+3)^2},$$

$$y_3=\frac{15(n+1)^2(n+2)^2\mathbb{i}^n(n!)^3}{(n+3)(n+4)(n+5)\,(3^n)^6},\qquad y_4=\frac{60\,(\mathbb{i}^n)^2\,2^n\,(3^n)^4\,n!}{(n+3)(n+4)(n+5)}.$$

in terms of

$$n!,\quad 2^n,\quad 3^n\quad \text{and}\quad \mathbb{i}^n$$

• Representation into "optimal" products

Given

$$y_1 = \prod_{k=1}^{n} \frac{-13122k(1+k)}{(3+k)^3}, \qquad y_2 = \prod_{k=1}^{n} \frac{26244k^2(2+k)^2}{(3+k)^2},$$

$$y_3 = \prod_{k=1}^{n} \frac{\mathbb{i}k(2+k)^3}{729(5+k)}, \qquad y_4 = \prod_{k=1}^{n} -\frac{162k(2+k)}{5+k}$$

we can compute

$$y_1 = \frac{5(1+n)^2(2+n)^5(3+n)^8}{52488(4+n)(5+n)}(-1)^n\Phi_1\Phi_2^{-2}, \quad y_2 = \frac{(4+n)^2(5+n)^2}{400}\Phi_1^2,$$

$$y_3 = \frac{2754990144(4+n)^2(5+n)^2}{25(1+n)^4(2+n)^{10}(3+n)^{16}}\Phi_2^3, \quad y_4 = \Phi_1.$$

in terms of

$$\underbrace{\Phi_1 = \prod_{k=1}^{n} \frac{-162k(2+k)}{5+k}, \quad \Phi_2 = \prod_{k=1}^{n} \frac{-\mathbb{i}(3+k)^6}{9k(1+k)^2(2+k)(5+k)}}_{\text{minimal number of transcendental products}} \quad \text{and} \quad \underbrace{(-1)^n}_{\text{minimal order}}$$

(ii) Finding algebraic relations

Given

$$y_1 = \prod_{k=1}^{n} \tfrac{-13122k(1+k)}{(3+k)^3}, \qquad y_2 = \prod_{k=1}^{n} \tfrac{26244k^2(2+k)^2}{(3+k)^2},$$

$$y_3 = \prod_{k=1}^{n} \tfrac{\mathbb{i}k(2+k)^3}{729(5+k)}, \qquad y_4 = \prod_{k=1}^{n} -\frac{162k(2+k)}{5+k}$$

we can compute all its algebraic relations:

$$\Big\{ c_1 \left(\tfrac{y_2}{y_4^2} - \tfrac{(4+n)^2(5+n)^2}{400}\right) + c_2\left(\tfrac{y_2^2 y_4^2}{y_1^6 y_3^4} - \tfrac{(1+n)^4(2+n)^{10}(3+n)^{16}(4+n)^2(5+n)^2}{4199040^2}\right) \mid$$

$$c_1, c_2 \in \mathbb{Q}(\mathbb{i})(n)[y_1, y_1^{-1}][y_2, y_2^{-1}][y_3, y_3^{-1}][y_4, y_4^{-1}]\Big\}.$$

(iii) Summation theory for generic sequences (joint with P. Paule)

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^1$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^2$$

(iii) Summation theory for generic sequences (joint with P. Paule)

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^1 = (1+n)\sum_{i=0}^{n}X_i - \sum_{i=0}^{n}iX_i$$

see M. Kauers/CS in Discrete Math. 306(17), 2006.

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^2$$

(iii) Summation theory for generic sequences (joint with P. Paule)

- Case 1:

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- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^2 =(-c+n)\Big(\sum_{i=0}^{n}X_i\Big)^2+(-1-c)\sum_{i=0}^{n}X_i^2+\sum_{i=0}^{n}iX_i^2 - \sum_{i=0}^{n}X_{1+i}Z_i - X_0Z_{-1} + \Big(\sum_{i=0}^{n}X_i\Big)Z_n + X_{1+n}Z_n$$

for an arbitrary sequence Z_n satisfying

$$Z_{1+n}-Z_n=(c-1)2X_{1+n}$$

Sufficient and necessary condition for the class of indefinite nested sums

Specializations:

For $X_k = \binom{n}{k}$ we can compute $c = \frac{2-n}{2}$ and $Z_k = \binom{n}{k}(-k+n)$ s.t.

$$Z_{1+n} - Z_n = (c-1)2X_{1+n}$$

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$$Z_{1+n} - Z_n = (c-1)2X_{1+n}$$

This gives

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \binom{n}{a}(-a+n)\sum_{i=0}^{a}\binom{n}{i} + \frac{1}{2}(2+2a-n)\Big(\sum_{i=0}^{a}\binom{n}{i}\Big)^2 - \frac{1}{2}n\sum_{i=0}^{a}\binom{n}{i}^2$$

Specializations:

Similarly one can discover, e.g.,

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\frac{x^j}{\binom{n}{j}}\Big)^2=\frac{1+n+x}{x+1}\sum_{k=0}^{a}\frac{x^{2k}}{\binom{n}{k}^2}+\frac{x-1}{x+1}\sum_{k=0}^{a}\frac{kx^{2k}}{\binom{n}{k}^2}$$
$$+\frac{a-n+2x+ax}{x+1}\Big(\sum_{k=0}^{a}\frac{x^k}{\binom{n}{k}}\Big)^2-\frac{2(a+1)x^{a+1}}{(x+1)\binom{n}{a}}\sum_{k=0}^{a}\frac{x^k}{\binom{n}{k}},$$

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\frac{(-1)^j}{\binom{n}{j}}\Big)^2=\frac{(n+1)^2(4an^2+22an+30a+3n^2+23n+38)}{2(n+2)^2(n+3)(2n+5)}+\frac{2(-1)^a(a+1)(a+2)(n+1)}{(n+2)^2(n+3)}\frac{1}{\binom{n}{a}}$$
$$+\frac{(a+1)^2(6+2a+n)}{2(n+2)^2(2n+5)}\frac{1}{\binom{n}{a}^2}+\frac{n+2}{2(2n+5)}\sum_{k=0}^{a}\frac{1}{\binom{n}{k}^2}$$

for all $x\in\mathbb{K}\setminus\{-1\}$ and $a,n\in\mathbb{N}$ with $a\leq n$.

(II) Recurrence finding

(i) Complete creative telescoping algorithms for indefinite nested sums defined over (q–)hypergeometric products

Apéry sum

$$A(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(H_n^{(3)} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

$$\downarrow$$

$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0$$

(ii) A unified framework of the holonomic and DR approach

(with M. Round)

Example 1:

$$\sum_{m,n\geq 0} q^{\frac{3m^2+m}{2}+2mn+n^2} \begin{bmatrix} 2N-2m-2n+2 \\ m \end{bmatrix}_q \begin{bmatrix} N-m-n+1+\lfloor\frac{n}{2}\rfloor \\ \lfloor\frac{n}{2}\rfloor \end{bmatrix}_{q^4}$$

$$= \sum_{m,n\geq 0} q^{m^2+n^2} \begin{bmatrix} N-m+1 \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ m \end{bmatrix}_{q^2}$$

from A. Berkovich and A.K. Uncu. Refined q-trinomial coefficients and two infinite hierarchies of q-series identities.

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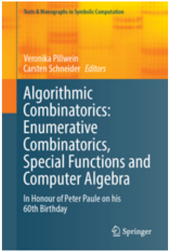

Example 2:

Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \, \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \, \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}\,.$$

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and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}}\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-m+j}{n}\frac{(t-j)_n}{n!}\,.$$

Then

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

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Theorem (CS, Sigma, Zudilin) For integers $n \geq m \geq 0$, define two rational functions

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and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}\,.$$

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Proof tactic: Both sides of

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satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

with

$$\begin{aligned}\alpha_0(n,m)&=(2n-m)^5,\\ \alpha_1(n,m)&=-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3\\ &\qquad+30n^2m-14nm^2+2m^3+8n^2-10nm+2m^2-4n+m),\\ \alpha_2(n,m)&=-(2n-m-1)^3(4n-m)(m+2).\end{aligned}$$

Proof tactic: Both sides of

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satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

$$
\begin{aligned}
S(n,m) = \sum_{j=0}^{n} \sum_{\nu=1}^{\infty} \Bigg(& \frac{\binom{n}{j}^2 \binom{j-m+2n}{n} (1+\nu)_{-m+2n} (1-j+\nu+n)_{-1+n}}{(1+\nu+n)_n (1+\nu+n)_{-m+2n} (\nu+n)^4 (\nu-m+2n)^3} \\
& \times \Bigg((\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) \\
& \quad + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
& \quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& - \nu(j-\nu-n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) \\
& \quad - S_1(\nu-m+3n) - S_1(-j+\nu+n) + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& + \nu(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) \\
& \quad - S_1(\nu-m+3n) - S_1(-j+\nu+n) + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& - (j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) \\
& \quad - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
& \quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& + \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{(j-\nu-2n)^2} - S_2(\nu) + 2S_2(\nu+n) \\
& \quad - S_2(\nu+2n) - S_2(\nu-m+3n) - S_2(-j+\nu+n) \\
& \quad + S_2(\nu-m+2n) + S_2(-j+\nu+2n)\Big)
\end{aligned}
$$

$$+ 4(j+n)(\nu+n) - 3(\nu+n)^2 + n(-m+n) - j(m+2n)\Big)$$

$$- 2(\nu+n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} - S_1(\nu)$$

$$+ 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$+ S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big)$$

$$+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big)$$

$$- 3(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} - S_1(\nu)$$

$$+ 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$+ S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big)$$

$$+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big)$$

$$- (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n}$$

$$- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$+ S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big)$$

$$+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big)$$

$$\times \big(- S_1(\nu+n) + S_1(\nu+2n)\big)$$

$$+ (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n}$$

$$- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)$$

$$
\begin{aligned}
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\times \big(- S_1(\nu) + S_1(\nu - m + 2n)\big) \\
&- (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} \\
&- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\times \big(- S_1(\nu+n) + S_1(\nu - m + 3n)\big) \\
&+ (\nu+n)(\nu-m+2n)\Big(- \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(- \frac{1}{-j+\nu+2n} \\
&- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 \\
&- (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\times \Big(- \frac{1}{-j+\nu+2n} - S_1(-j+\nu+n) + S_1(-j+\nu+2n)\Big)\Big)\Big)
\end{aligned}
$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+\,a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$

$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

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$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$\downarrow$ Holonomic-DR approach

$$(2n-m)^5S(n,m)$$
$$-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3$$
$$+30n^2m-14nm^2+2m^3+8n^2-10nm+2m^2-4n+m)S(n,m+1)$$
$$-(2n-m-1)^3(4n-m)(m+2)S(n,m+2)=R(n,m)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

SigmaReduce

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

Finally, check 2 initial values: another round of non-trivial summation...

(III) Solving difference and differential equations.

(i) New recurrence solver for $\Pi\Sigma$-fields

GIVEN a recurrence $a_0(n), \dots, a_d(n), h(n)$: expr. in terms of indefinite sums/products.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible with indefinite sums/produkts
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

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$$-F(2+n) + (1+n)\left(8+9n+2n^2\right)n!F(1+n) - 2(1+n)^3(3+n)n!^2F(n) = 0$$

$$\downarrow$$

$$\left\{c_1 \prod_{i=1}^{n} i! + c_2\left(-2^n n! \prod_{i=1}^{n} i! + \tfrac{3}{2} \prod_{i=1}^{n} i! \sum_{i=1}^{n} 2^i i!\right) \mid c_1, c_2 \in \mathbb{K}\right\}$$

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FIND all solutions expressible with indefinite sums/produkts
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$$\begin{aligned}&\big(1 + H_n + nH_n\big)^2\big(3 + 2n + 2H_n + 3nH_n + n^2 H_n\big)^2 F(n)\\ &\quad - (1+n)(3+2n)H_n\big(3 + 2n + 2H_n + 3nH_n + n^2 H_n\big)^2 F(n+1)\\ &\qquad + (1+n)^2(2+n)^3 H_n\big(1 + H_n + nH_n\big)F(n+2) = 0\end{aligned}$$

$$\downarrow$$

$$\Big\{c_1\, H_n \prod_{l=1}^{n} H_l + c_2 H_n^2 \prod_{l=1}^{n} H_l \mid c_1, c_2 \in \mathbb{K}\Big\}$$

(ii) Powerful differential equation solver

in Ablinger's HarmonicSums package (containing also Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0;$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x \tfrac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\int_0^{\tau_1}\tfrac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{(1+\tau_1)(1+\tau_2)}\,d\tau_2\,d\tau_1 + c_4\int_0^x\int_0^{\tau_1}\tfrac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{(1+\tau_1)(1+\tau_2)}\,d\tau_2\,d\tau_1 \mid c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}.$$

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(iii) New algorithms and software to solve coupled systems

- Tailored tools for particle physics based on uncoupling (with A. De Freitas, Nikolas Fadeev)
- Direct solver for q-case, for $\Pi\Sigma$-fields, for hypergeometric solutions (with J. Middeke)

(IV) Exploring nested sums and integrals

(J. Ablinger's HarmonicSums package)

(i) Calculation of (inverse) Mellin transforms

$$\sum_{i=1}^{n}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2}=\int_0^1((4\ x)^n-1)\Bigg(\frac{-4\pi\ \int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)}+\frac{2\left(\int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x}-\frac{2\left(\int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\Bigg)\,dx$$

(IV) Exploring nested sums and integrals

(J. Ablinger's HarmonicSums package)

(i) Calculation of (inverse) Mellin transforms, asymptotic expansions and numerics

$$\sum_{i=1}^{n}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2}=\int_0^1((4\ x)^n-1)\Bigg(\frac{-4\pi\ \int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)}+\frac{2\left(\int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x}-\frac{2\left(\int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\Bigg)dx$$

$$\downarrow$$

$$\sum_{i=1}^{n}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2}\sim-\frac{2\zeta(3)}{3}+2^{2n}\sqrt{n}\frac{1}{\sqrt{\pi}}\left(-\frac{5}{18n^4}-\frac{4}{3n^3}+\left(+\frac{121}{576n^4}+\frac{1}{12n^3}+\frac{2}{9n^2}\right)\pi^2+O\left(\frac{1}{n^5}\right)\right)$$

(ii) Evaluation of infinite sums.

$$\sum_{k=1}^{\infty} \frac{33H_k^{(5)} + 4/k^5}{k^2\binom{2k}{k}} = -\frac{45}{8}\zeta(7) + \frac{13}{3}\zeta(2)\zeta(5) + \frac{85}{6}\zeta(3)\zeta(4)$$

$$\sum_{k=1}^{\infty} \frac{2H_k^{(5)} - H_k^{(3)}/k^2}{k^2\binom{2k}{k}} = -\frac{\zeta(7)}{72} + \frac{8\pi^2\zeta(5)}{81} - \frac{17\pi^4\zeta(3)}{4860}$$

$$\sum_{k=1}^{\infty} \frac{3H_k^{(2)} - 1/k^2}{k^5\binom{2k}{k}} = -\frac{205\zeta(7)}{18} + \frac{5\pi^2\zeta(5)}{18} + \frac{\pi^4\zeta(3)}{18} - \frac{\pi^7}{486\sqrt{3}} + \frac{\sqrt{3}c\pi^3}{8}$$

$$\sum_{k=1}^{\infty} \frac{11H_k^{(3)} + 8H_k^{(2)}/k}{k^4\binom{2k}{k}} = \frac{7337\zeta(7)}{216} + \frac{11\pi^2\zeta(5)}{81} + \frac{1417\pi^4\zeta(3)}{4860} - \frac{4\pi^7}{729\sqrt{3}} + \frac{c\pi^3}{\sqrt{3}}$$

$$\sum_{k=1}^{\infty} \frac{2H_k^{(5)} - H_k^{(3)}/k^2}{k^2\binom{2k}{k}} = -\frac{\zeta(7)}{72} + \frac{8\pi^2\zeta(5)}{81} - \frac{17\pi^4\zeta(3)}{4860} \qquad c := \sum_{i=0}^{\infty} \frac{1}{(3i+1)^4}$$

$$\vdots$$

Some were conjectured, but most of them were discovered and proved for the first time by Jakob Ablinger!

Further highlights:

New and/or enhanced software packages:

- `EvaluateMultiSums.m`
- `HarmonicSums.m` (Ablinger)
- `MultiIntegrate.m` (Ablinger)
- `NestedProducts` (Ocansey)
- `RhoSum.m` (Round)
- `Sigma.m`
- `SolveCoupledSystem.m`
- `QObjects` (CS/Uncu)
- `QFunctions` (Ablinger/Uncu)

Cooperations within the SFB:

- **C. Krattenthaler**: multi-sums with absolute values
- **P. Paule**: identities for generic sequences
- **V. Pillwein**: recurrences for the Gillis–Reznick–Zeilberger conjecture
- **S.C. Radu**: elliptic integrals in QCD (joint with Bümlein/Hoeij/...)
- **R. Sulzgruber**: multi-sums for the Novelli-Pak-Stoyanovskii algorithm
- **A.K. Uncu**: $q-$multi-sums
- hopefully more will arise!