A comparison of methods for computing rational general solutions of algebraic ODEs

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In this paper, the two methods for finding rational general solutions of first-order algebraic ODEs introduced in Ngô and Winkler [15, 17, 16] and Vo, Grasegger and Winkler [22] are compared. Common to those methods is that both assign some algebraic set to an ODE. Provided the assigned algebraic sets are suitably parametrisable, the initial ODE can be reduced to a more fundamental (set of) differential equation(s). Both approaches possess a common rational parametrisation in certain situations, in which case the corresponding derived differential equation(s) are shown to coincide. Finally, a discussion on relations between certain classes of first-order algebraic ODEs with respect to their rational general solvability is provided.

1 Introduction

Recently, the algebro-geometric method for solving algebraic differential equations [24]—a novel method for finding exact solutions of such differential equations—has evolved. Starting with the work of Feng and Gao [7, 6], resulting in an algorithm for computing rational general solutions of autonomous first-order algebraic ODEs, several generalisations of their method have been proposed since. The idea for solving the autonomous case was to assign an algebraic curve to the ODE and derive a solution from a suitable parametrisation of this curve. An illustration of the general approach can be found in Winkler [24]. In this paper, two extensions of the method of Feng and Gao to non-autonomous algebraic ODEs of order one are discussed. The first method was proposed by Ngô and Winkler [15, 17, 16] and starts by assigning an algebraic surface to the differential equation. From a proper rational parametrisation of this surface, they derive an associated system of autonomous quasi-linear differential equations which is equal to the original ODE in terms of rational general solvability. A second method

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has been investigated by Vo, Grasegger and Winkler [22]. Their approach is to assign a curve over a rational function field to an algebraic ODE. Given a suitable rational parametrisation of this curve, they are able to reduce the original ODE to a single quasi-linear ODE. Again, this transformation preserves rational general solvability. It is natural to ask how these methods are connected and whether there are situations when the derived equations actually coincide.

The structure of this paper is as follows: Section 2 introduces necessary preliminaries of rational algebraic curves and surfaces and gives a precise definition of the term *rational general solution* for first-order algebraic ODEs. Afterwards, the two methods for constructing such solutions are outlined briefly in Section 3. Section 4 starts by proving that a suitable parametrisation of the associated curve yields a proper rational parametrisation of the surface associated to the same differential equation. The resulting surface parametrisation is of a special form, generating a simpler associated differential system. At the end, a discussion on relations between certain classes of algebraic ODEs of order one is provided. In particular, the classes of ODEs solvable by the two approaches are related to the class of first-order algebraic ODEs which have a rational general solution. Their relation is clarified for differential equations which can be solved by both approaches or none of them. Only one open question remains, cf. Section 5, which asks whether there are any algebraic ODEs solvable with the method via surface-parametrisation, but not with the other approach.

2 Preliminaries

Throughout this paper, $\mathcal{F}$ denotes an algebraically closed field of characteristic zero. The polynomial ring $\mathcal{F}[x]$ and its quotient field $\mathcal{F}(x)$ can be seen as a differential ring and a differential field, respectively, when endowed with the usual derivation ${}^{\mathrm{d}}/_{\mathrm{d}x}$. In both structures, $\mathcal{F}$ is the field of constants. Frequently, the derivation ${}^{\mathrm{d}}/_{\mathrm{d}x}$ is abbreviated by the symbol $'$. An algebraic ordinary differential equation (AODE) of order one is a differential equation of the form

$$A(x, y, y') = 0,$$

where $A \in \mathcal{F}[x, y, y'] \backslash \mathcal{F}[x, y]$ and y is a differential indeterminate over $\mathcal{F}(x)$. The polynomial A will be referred to as the defining polynomial of the AODE. For the purpose of constructing general solutions of such AODEs, the defining polynomial may be taken to be irreducible without loss of generality, which shall be assumed henceforth.

The notion of general solution of an AODE can be made precise in the language of differential algebra, cf. Ritt [18] and Kolchin [11] for an extensive treatise of the subject. Consider the defining polynomial A of a first-order AODE as an element of the differential polynomial algebra $\mathcal{F}(x)\{y\}$ over the differential field $(\mathcal{F}(x), {}^{\mathrm{d}}/_{\mathrm{d}x})$. Notice that the irreducibility of A is preserved when lifted to $\mathcal{F}(x)\{y\}$. The radical differential ideal generated by A, usually denoted by $\{A\}$, can be decomposed into

$$\{A\} = (\{A\} : S_A) \cap \{A, S_A\},$$

where $S_A = \partial A/\partial y'$ is the separant of A. It can be shown that the component $(\{A\} : S_A)$ is a prime differential ideal and as such has a generic zero[1], cf. Ritt [18, Chapter 2] or Kolchin [11, Chapter 4]. By a generic zero of a prime differential ideal $\mathfrak{p} \subsetneq \mathcal{F}(x)\{y\}$ one understands an element η in a universal differential extension field[2] of $\mathcal{F}(x)$ whose defining ideal in $\mathcal{F}(x)\{y\}$ is exactly $\mathfrak{p}$. In other words, a differential polynomial is in $\mathfrak{p}$ if and only if it vanishes at η. The prime component $(\{A\} : S_A)$ is characterised by the fact that it is the unique component which does not contain the separant. Literature refers to it as the general component of A.

Definition 1. Given a first-order AODE $A(x, y, y') = 0$ with A as its defining polynomial. A *general solution* of such an AODE is a generic zero of the general component of A. If in addition, a general solution $\hat{y}$ is of the form

$$\hat{y} = \frac{a_m\, x^m + a_{m-1}\, x^{m-1} + \cdots + a_0}{b_n\, x^n + b_{n-1}\, x^{n-1} + \cdots + b_0},$$

where $(a_i)_{i\in 0..m}, (b_j)_{j\in 0..n}$ are constants in a universal differential extension field of $\mathcal{F}(x)$ with $a_m, b_n \neq 0$, then $\hat{y}$ is called a *rational general solution*. ■

A general solution of an AODE serves as test point for the ideal membership problem of the general component. In order to verify whether a solution is indeed general one may use a reduction process for differential polynomials analogous to Euclidean division of algebraic polynomials. The precise reduction requires the introduction of differential rankings and autoreduced sets which is beyond the scope of this paper. Details may be found in Kolchin [11, Chapter 1, Section 8 and 9]. Consider the differential polynomial algebra $\mathcal{R} = \mathcal{F}(x)\{y\}$ with the orderly ranking[3] $y < y' < y'' < y''' < \cdots$ on the differential indeterminates over $\mathcal{F}(x)$. Let $A \in \mathcal{R} \setminus \mathcal{F}(x)$ be a differential polynomial of order o and denote by $y^{(o)}$ the o-th derivative of y. For every $F \in \mathcal{R}$ there exists a unique expression

$$I_A^{\iota}\, S_A^{\sigma}\, F \equiv R \pmod{[A]}, \tag{1}$$

where $\iota, \sigma \in \mathbb{N}$ and $R \in \mathcal{R}$ is reduced with respect to A. The expressions I_A and S_A denote the initial and separant of A, respectively, viz. I_A is the leading coefficient of A when considered as a univariate polynomial in the variable $y^{(o)}$ and $S_A = \partial A/\partial y^{(o)}$. Finally, $[A] \subseteq \mathcal{R}$ denotes the differential ideal generated by A. The statement of Equation (1) can be sharpened as follows: The difference $I_A^{\iota}\, S_A^{\sigma}\, F - R$ is expressible as an $\mathcal{R}$-linear combination of derivatives of A [11,

[1] Kolchin [11] requires such prime differential ideals to be $\mathcal{F}(x)$-separable in order to possess a generic zero. This is always the case for the general component [11, Chapter 4, Section 6].

[2] Such a differential field, denote it by $\mathcal{U}$, is characterised by the following property: Every prime differential ideal $\mathfrak{p} \subsetneq \mathcal{K}\{y_1, \ldots, y_n\}$ with $n > 0$ and $\mathcal{K}$ a finitely generated differential field extension of $\mathcal{F}(x)$ has a generic zero whose elements can be taken in $\mathcal{U}$. Alternatively, for every finitely generated differential field extension of $\mathcal{K}$ (not necessarily in $\mathcal{U}$) there exists a $\mathcal{K}$-isomorphism to a differential subfield of $\mathcal{U}$. Note that such a universal extension exists for every differential field, cf. Kolchin [12, Chapter 1, Section 5].

[3] In short, a differential ranking is a total order on the indeterminates and their derivatives which is compatible with the differential structure. Such a ranking is called *orderly* if derivatives of higher order are ranked higher. There exists precisely one orderly ranking in the case of the ordinary differential polynomial algebra $\mathcal{F}(x)\{y\}$, justifying the term “the orderly ranking”. In fact, it is also the only differential ranking of this algebra.

Proposition 1 in Chapter 1, Section 9]. Consequently, the congruence is of the subsequent form:

$$I_A^\iota\, S_A^\sigma\, F = \sum_{i\geq 0}\left(Q_i\,\frac{\mathrm{d}^i}{\mathrm{d}x^i}A\right) + R. \tag{2}$$

Note that the factors $Q_i \in \mathcal{R}$ are zero when the order of the i-th derivative of A exceeds the order of F. This identity motivates the following definition.

Definition 2. Given $F, A \in \mathcal{F}(x)\{y\}$ such that A is not an element of $\mathcal{F}(x)$. The differential polynomial R in Equation (2) is called the *differential pseudo-remainder* of F by A with respect to the orderly ranking and is denoted by $\operatorname{prem}(F, A)$. ■

Differential pseudo-remainders provide another tool for deciding the ideal membership of the general component of an AODE. Ritt [18, Chapter 2, Section 13] proved that a necessary and sufficient condition for a differential polynomial to be in the general component of an AODE is that the differential pseudo-remainder by the defining polynomial is zero. This allows to give the subsequent alternative characterisation of a general solution.

Lemma 1. *Let A be the defining polynomial of an AODE $A(x, y, y') = 0$. A solution $\hat{y}$ is a general solution if and only if*

$$\forall F \in \mathcal{F}(x)\{y\} : F(\hat{y}) = 0 \;\Leftrightarrow\; \operatorname{prem}(F, A) = 0.$$

The methods for constructing explicit solutions of AODEs discussed in this paper involve the association of certain algebraic sets to the initial differential equation. The following paragraphs outline necessary concepts of rational algebraic curves and surfaces. For a general overview, consult for instance Shafarevich [21], Walker [23] or Hartshorne [10, Chapters 1, 4, 5]. Details on rational curves and the computation of rational parametrisations thereof can be found in Sendra, Winkler and Pérez-Díaz [20]. For parametrisation methods of rational surfaces consult Schicho [19].

Consider now a first-order AODE $A(x, y, y') = 0$ with defining polynomial $A \in \mathcal{F}[x, y, y']$. Substitution of an arbitrary new variable, say z, for the differential variable y' turns the differential equation into a purely algebraic problem, viz. the differential polynomial into an algebraic polynomial. The latter shall be denoted by $A_{y'\to z}$. In this setting, the set of solutions of the algebraic problem constitutes a classical algebraic set. In fact, there are two natural choices on how to interpret the corresponding algebraic polynomial and, to that effect, in which space solutions are to be considered:

1. View $A_{y'\to z} \in \mathcal{F}[x, y, z]$, defining a surface in three-dimensional affine space over $\mathcal{F}$:

$$\mathcal{S}_A := \left\{ (x, y, z) \in \mathbb{A}^3(\mathcal{F}) \,\middle|\, A_{y'\to z}(x, y, z) = 0 \right\}. \tag{3}$$

2. View $A_{y'\to z} \in \mathcal{F}(x)[y, z]$, defining a curve in two-dimensional affine space over $\overline{\mathcal{F}(x)}$, the latter denoting the algebraic closure of $\mathcal{F}(x)$:

$$\mathcal{C}_A := \left\{ (y, z) \in \mathbb{A}^2\big(\overline{\mathcal{F}(x)}\big) \,\middle|\, A_{y'\to z}(y, z) = 0 \right\}. \tag{4}$$

Definition 3. Given a first-order AODE $A(x, y, y') = 0$ and let $\mathcal{S}_A/\mathcal{C}_A$ be the algebraic surface/curve from Equation (3)/(4), respectively. Then $\mathcal{S}_A$ is called the *associated surface* and $\mathcal{C}_A$ the *associated curve* of the AODE. ■

The antecedent algebraic sets are implicitly described by the vanishing set of a single polynomial. For the purpose of constructing solutions of the original AODE, however, an explicit characterisation is needed. From now on only irreducible algebraic sets are considered. Such sets are known as algebraic varieties.

Definition 4. Let $\mathcal{X} \subseteq \mathbb{A}^n(\mathcal{F})$ be an algebraic variety and $\mathcal{F}(t_1, \dots, t_d)$ be the field of rational functions in d variables over $\mathcal{F}$. A rational map $\mathcal{P}_\mathcal{X} : \mathbb{A}^d(\mathcal{F}) \to \mathcal{X}$ given by an n-tuple of rational functions $(\varphi_1, \dots, \varphi_n)$ with $(\varphi_i)_{i\in 1..n} \in \mathcal{F}(t_1, \dots, t_d)$ is called a *rational parametrisation* of $\mathcal{X}$ if $\operatorname{im}(\mathcal{P}_\mathcal{X})$ is Zariski-dense in $\mathcal{X}$. Furthermore, if $\mathcal{P}_\mathcal{X}$ has a rational inverse, i.e. a rational map $\mathcal{P}_\mathcal{X}^{-1} : \mathcal{X} \to \mathbb{A}^d(\mathcal{F})$ such that $\mathcal{P}_\mathcal{X}^{-1} \circ \mathcal{P}_\mathcal{X} = \mathrm{id}_{\mathbb{A}^d(\mathcal{F})}$ and $\mathcal{P}_\mathcal{X} \circ \mathcal{P}_\mathcal{X}^{-1} = \mathrm{id}_\mathcal{X}$, then the parametrisation is called *proper*[4]. ■

Irreducible algebraic curves and surfaces over an algebraically closed field of characteristic zero always have a proper rational parametrisation if they are rationally parametrisable. Should an algebraic variety possess a proper rational parametrisation then it is also called a *rational variety*. Notice that the associated surface of a first-order AODE is a variety since the defining polynomial is assumed to be irreducible. On the other hand, the associated curve might fail to be a variety and the defining polynomial of the curve factors over the algebraic closure of $\mathcal{F}(x)$. With the intention of parametrising this curve, the reducible case can be dismissed as only irreducible curves may possess a (proper) rational parametrisation [20, Theorem 4.4]. The following theorem provides a simple rationality criterion for the varieties considered in this paper.

Theorem 1. *An irreducible curve $\mathcal{C} \subseteq \mathbb{A}^2(\mathcal{F})$ is rational if and only if its genus is zero. Similarly, an irreducible surface $\mathcal{S} \subseteq \mathbb{A}^3(\mathcal{F})$ is rational if and only if both its arithmetic genus and second plurigenus are zero.*

A precise definition of those notions can be found in Hartshorne [10]. The subsequent theorem concludes this preliminary section by providing a well-known link between rational varieties and field extensions.

Theorem 2. *Consider an algebraic variety $\mathcal{X} \subseteq \mathbb{A}^n(\mathcal{F})$ and let $\mathcal{F}(\mathcal{X})$ denote its function field. Then the following conditions are equivalent:*

(i) There exists a proper rational parametrisation $\mathcal{P}_\mathcal{X} : \mathbb{A}^d(\mathcal{F}) \to \mathcal{X}$.

(ii) $\mathcal{F}(\mathcal{X})$ is isomorphic to $\mathcal{F}(t_1, \dots, t_d)$ over $\mathcal{F}$, where $\mathcal{F}(t_1, \dots, t_d)$ denotes the field of rational functions in d variables over $\mathcal{F}$.

[4] Equality of the maps should be understood in the sense that they are equal as rational maps, i.e. they agree on an open subset on which both sides of the equation are defined.

In fact, every proper rational parametrisation gives rise to a suitable isomorphism of the function fields via the induced pullback mapping on rational functions and every such isomorphism, constant on elements of the ground field $\mathcal{F}$, yields a proper rational parametrisation when applied to the coordinate functions of the variety. Observe that the transcendence degree d of the rational function field is one/two in the case of rational curves/surfaces, respectively.

3 Computational methods for parametrisable AODEs

The purpose of this section is to outline two recent methods for explicitly constructing rational general solutions of first-order AODEs whose associated surface/curve is rational. Over a suitable computable field—such as the algebraic numbers $\overline{\mathbb{Q}}$—these methods lead to algorithms which can be implemented in a computer algebra system.

Throughout this section, the algebraically closed field of characteristic zero $\mathcal{F}$ shall be fixed. Recall from the previous section that the defining polynomial of any first-order AODE is assumed to be irreducible and of positive degree in y'.

Definition 5. The class of all first-order AODEs is denoted by $\mathbf{A}_{ODE}$. Furthermore, $\mathbf{A}_{ODE}^{(RGS)}$ stands for the proper subclass of those AODEs which have a rational general solution. ■

In terms of solvability, any first-order ODE which is polynomial in y and y', but rational in the variable x can be transformed into a suitable AODE. More elaborate, let $F(x, y, y') = 0$ be an ODE such that the left-hand side is an irreducible element

$$F = \sum_{i=0}^{D}\sum_{j=0}^{i}\frac{n_{ij}}{d_{ij}}\, y^{i-j}(y')^{j} \;\in\; \mathcal{F}(x)[y, y'] \setminus \mathcal{F}(x)[y]$$

with $D \in \mathbb{N}$ and $n_{ij}, d_{ij} \in \mathcal{F}[x]$. Consider the polynomial

$$A_F = \mathrm{pp}_{\mathcal{F}[x]}\left(\prod_{\substack{0\le i\le D\\ 0\le j\le i}}(d_{ij})\cdot F\right) \;\in\; \mathcal{F}[x][y, y'], \tag{5}$$

where $\mathrm{pp}_{\mathcal{F}[x]}$ extracts the primitive part[5] over $\mathcal{F}[x]$. In other words, the argument, considered as a bivariate polynomial in $\mathcal{F}[x][y, y']$, is divided by the greatest common divisor of its coefficients. The differential polynomial A_F has the same general solution as F, because they differ by a mere unit as elements of $\mathcal{F}(x)\{y\}$. Furthermore, A_F is irreducible in $\mathcal{F}[x, y, y']$ and can be considered as the defining polynomial of an AODE.

[5]The primitive part operation is necessary, even when the fractions are normalised and the denominators are cleared by multiplying with their least common multiple. For example, $x(y' - y)$ is irreducible in $\mathcal{F}(x)[y, y']$, since x is a unit in this ring, but reducible as an element of $\mathcal{F}[x, y, y']$.

3.1 Surface-parametrisable AODEs

The content of this section is based on the work of Ngô and Winkler [15]. A pleasantly readable summary of this method, especially when it comes to solving the later introduced associated planar system, can be found in Ngô and Winkler [16].

Definition 6. Consider a first-order AODE with associated surface $\mathcal{S}_A$. Such an AODE is called *surface-parametrisable* if and only if $\mathcal{S}_A$ is a rational surface. The class of all surface-parametrisable AODEs is denoted by $\mathbf{A}_{ODE}^{(SP)}$. ■

Let $\mathcal{P}_{\mathcal{S}_A}(t_1, t_2) = (\varphi_1(t_1, t_2), \varphi_2(t_1, t_2), \varphi_3(t_1, t_2))$, where $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{F}(t_1, t_2)$, be a proper rational parametrisation of the surface $\mathcal{S}_A$ associated to the first-order AODE $A(x, y, y') = 0$. Furthermore, assume this AODE has a rational general solution $\hat{y}$. Consequently, such a solution must annihilate the defining polynomial, viz. $A(x, \hat{y}, \hat{y}') = 0$, and hence give rise to a family[6] of parametric curves $(x, \hat{y}(x), \hat{y}'(x))$ over $\mathcal{F}$ located on the associated surface $\mathcal{S}_A$. Now consider $(s(x), t(x)) = \mathcal{P}_{\mathcal{S}_A}^{-1}(x, \hat{y}(x), \hat{y}'(x))$, i.e. the image of $(x, \hat{y}(x), \hat{y}'(x))$ under the birational inverse of $\mathcal{P}_{\mathcal{S}_A}$. Application of $\mathcal{P}_{\mathcal{S}_A}$ on this image yields conditions on $s(x)$ and $t(x)$

$$\begin{cases} \varphi_1(s(x), t(x)) = x \\ \varphi_3(s(x), t(x)) = \varphi_2(s(x), t(x))', \end{cases}$$

since $\mathcal{P}_{\mathcal{S}_A}(s(x), t(x)) = (x, \hat{y}(x), \hat{y}'(x))$. Note that these maps are well-defined if $\hat{y}$ is a rational general solution. By using the chain rule on the second equation while differentiating the first equation results in a linear system in $s'(x)$ and $t'(x)$. A solution for the latter is of the form

$$\begin{cases} s' = \dfrac{\varphi_3(s,t)\frac{\partial \varphi_1(s,t)}{\partial t} - \frac{\partial \varphi_2(s,t)}{\partial t}}{\frac{\partial \varphi_1(s,t)}{\partial t}\frac{\partial \varphi_2(s,t)}{\partial s} - \frac{\partial \varphi_1(s,t)}{\partial s}\frac{\partial \varphi_2(s,t)}{\partial t}} \\[2ex] t' = \dfrac{\varphi_3(s,t)\frac{\partial \varphi_1(s,t)}{\partial s} - \frac{\partial \varphi_2(s,t)}{\partial s}}{\frac{\partial \varphi_1(s,t)}{\partial s}\frac{\partial \varphi_2(s,t)}{\partial t} - \frac{\partial \varphi_1(s,t)}{\partial t}\frac{\partial \varphi_2(s,t)}{\partial s}}, \end{cases} \tag{6}$$

where the parametric dependency on x is omitted for the sake of brevity. It should be mentioned that the denominators do not vanish for arbitrary variables s, t and neither when these are substituted by values $s(x), t(x)$ which correspond to a rational general solution of the original AODE [16].

Definition 7. Let $A(x, y, y') = 0$ be a surface-parametrisable first-order AODE and $\mathcal{P}_{\mathcal{S}_A}(s, t) = (\varphi_1(s, t), \varphi_2(s, t), \varphi_3(s, t))$ be a proper rational parametrisation of the associated surface with $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{F}(s, t)$. For this AODE, the autonomous system of differential equations (6) is called the *associated planar system* with respect to $\mathcal{P}_{\mathcal{S}_A}$. ■

[6]By necessity, any rational general solution contains a transcendental constant which may be specialised to a value of the ground field $\mathcal{F}$. Every such specialisation yields a particular rational solution of the AODE which in turn can be considered as a rational map depending on the parameter x.

This construction transforms a general (non-linear) first-order AODE into an autonomous system of quasi-linear ODEs of the same order. Furthermore, by the remark preceding Definition 7 and the results in Ngô and Winkler [15, Theorem 3.14 and Theorem 3.15] the subsequent theorem follows immediately.

Theorem 3. *If $A(x,y,y')=0$ is a surface-parametrisable AODE of order one and $\mathcal{P}_{S_A}$ a proper rational parametrisation of the associated surface, then there is a one-to-one correspondence between rational general solutions of this AODE and the rational general solutions of the associated planar system with respect to $\mathcal{P}_{S_A}$.*

Unsurprisingly, this one-to-one correspondence is realised via the rational parametrisation and its birational inverse. One direction is of special interest: Given a rational general solution $(\hat{s}(x),\hat{t}(x))$ of the associated planar system with respect to the parametrisation $\mathcal{P}_{S_A}(t_1,t_2)=(\varphi_1(t_1,t_2),\varphi_2(t_1,t_2),\varphi_3(t_1,t_2))$, then

$$\hat{y}=\varphi_2\big(\hat{s}(x-C),\hat{t}(x-C)\big), \tag{7}$$

where $C=\varphi_1(\hat{s}(x),\hat{t}(x))-x$, is a rational general solution of the original AODE [15, Theorem 3.15]. It is imperative to define what a general solution of a planar system is in order to make this correspondence precise. Consider the ordinary differential polynomial algebra $\mathcal{P}=\mathcal{F}(x)\{s,t\}$ over $(\mathcal{F}(x),{}^{\mathrm{d}}/_{\mathrm{d}x})$. Let $s^{(i)}$ and $t^{(j)}$ denote the i-th and j-th derivative of s and t, respectively, and define the orderly ranking [15, Definition 2.2]

$$\begin{cases} s^{(i)} < s^{(j)} & \text{if } i<j \\ t^{(i)} < t^{(j)} & \text{if } i<j \\ s^{(i)} < t^{(j)} & \text{if } i<j \\ t^{(i)} < s^{(j)} & \text{if } i\le j. \end{cases}$$

This ranking is a direct generalisation of the orderly ranking of a single differential indeterminate, cf. Section 2. From a normalised autonomous planar system

$$\begin{cases} s' = \dfrac{N_s}{D_s} \\ t' = \dfrac{N_t}{D_t}, \end{cases} \tag{8}$$

where $N_s,D_s,N_t,D_t\in\mathcal{F}[s,t]$ such that $D_s,D_t\neq 0$, one obtains two quasi-linear AODEs by clearing denominators. Let $A_1=D_s\,s'-N_s$, $A_2=D_t\,t'-N_t$ and notice that they constitute an autoreduced set $\{A_1,A_2\}$ with respect to the chosen ranking. Analogous to the approach in the preceding section, every differential polynomial $F\in\mathcal{P}$ has a unique representation of the form [15, Section 3]

$$D_s^{\sigma}\,D_t^{\tau}\,F=\sum_{i\ge 0}\left(Q_{1i}\,\frac{\mathrm{d}^i}{\mathrm{d}x^i}A_1\right)+\sum_{j\ge 0}\left(Q_{2j}\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}A_2\right)+R, \tag{9}$$

with $\sigma, \tau \in \mathbb{N}$, $Q_{1i}, Q_{2j} \in \mathcal{P}$ and R is reduced with respect to $\{A_1, A_2\}$. Once again, the differential polynomial R in Equation (9) is called the *differential pseudo-remainder* of F by $\{A_1, A_2\}$ and is denoted by $\operatorname{prem}(F, A_1, A_2)$. Now, a general solution of the planar system is defined as follows [15, Definition 3.9]. Notice that the differential polynomials with pseudo-remainder zero indeed form a prime ideal [15, Lemma 3.8].

Definition 8. Consider the planar system from Equation (8). A solution $(\hat{s}(x), \hat{t}(x))$ of this system is a *general solution* if and only if

$$\forall F \in \mathcal{F}(x)\{s,t\} : F\big(\hat{s}(x), \hat{t}(x)\big) = 0 \Leftrightarrow \operatorname{prem}(F, D_s\, s' - N_s, D_t\, t' - N_t) = 0,$$

where the differential pseudo-remainder is the polynomial R in Equation (9). Furthermore, if both $\hat{s}(x)$ and $\hat{t}(x)$ of a general solution are rational in x, then it is called a *rational general solution* of the planar system. ■

The significant benefit in reducing an AODE to a planar rational system lies in the existence of computational methods for finding rational general solutions of these systems. One such method has been investigated in Ngô and Winkler [17]. Here the authors show that rational general solutions of the associated planar system are intrinsically tied to a certain subclass of so-called rational first integrals of such systems [17, Theorem 5.6]. Rational first integrals of planar rational systems have received a lot of attention in the literature. A selection of recent publications with specific emphasis on the computation of such integrals may be found in [2, 5, 8]. The proposed algorithm of Ngô and Winkler computes a rational general solution—that is, if such an object exists—provided a degree bound for rational first integrals of the associated planar system is given. Upper bounds are known in the generic situation [3] and in this case their method is actually a decision algorithm. Finally, it is worth mentioning that the rational general solvability is independent of which proper rational parametrisation of the associated surface is used [16, Section 2.3].

3.2 Curve-parametrisable AODEs

The approach of this method shall be outlined up to the point—and at the same level of generality—as was done for the preceding method in Section 3.1. A complete derivation where all details are fleshed out is described in Vo, Grasegger and Winkler [22].

Definition 9. Given a first-order AODE with associated curve $\mathcal{C}_A$. Such an AODE is called *curve-parametrisable* if and only if $\mathcal{C}_A$ is a rational curve. The class of all curve-parametrisable AODEs is denoted by $\mathbf{A}_{ODE}^{(CP)}$. ■

Recall that the associated curve of a first-order AODE is defined over $\mathcal{F}(x)$, i.e. all coefficients of the defining polynomial of $\mathcal{C}_A$ are contained in this field. In general, one can not hope to find a rational parametrisation of a rational plane curve over the curve's field of definition [20, Corollary 5.9]. Of course, such a case can occur only if the field in question is not algebraically closed. The following proposition ensures that curve-parametrisable AODEs always possess proper rational parametrisations over $\mathcal{F}(x)$.

Proposition 1 ([22, Theorem 4.3]). *Every rational curve $\mathcal{C} \subseteq \mathbb{A}^2(\overline{\mathcal{F}(x)})$ defined over $\mathcal{F}(x)$ has a proper rational parametrisation $\mathcal{P}_{\mathcal{C}}(t) = (\psi_1(t), \psi_2(t))$ such that $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$.*

Following a similar reasoning as in Section 3.1, let $A(x, y, y') = 0$ be a curve-parametrisable AODE of order one with a rational general solution $\hat{y}$. Now, $(\hat{y}, \hat{y}')$ may be considered as a family of points over $\mathcal{F}(x)$. Given a proper rational parametrisation $\mathcal{P}_{\mathcal{C}_A}(t) = (\psi_1(t), \psi_2(t))$ such that $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$. By Proposition 1, such a parametrisation always exists and can be computed [22, Section 4]. Let $u = \mathcal{P}_{\mathcal{C}_A}^{-1}(\hat{y}, \hat{y}')$. From the identity $\mathcal{P}_{\mathcal{C}_A}(u) = (\hat{y}, \hat{y}')$ the following conditions for u must hold:

$$\begin{cases} \psi_1(u) = \hat{y} \\ \psi_2(u) = \hat{y}'. \end{cases} \tag{10}$$

This shows that it is sufficient to find a suitable family of points u such that $\psi_1(u)' = \psi_2(u)$. Expanding the left-hand side of this equation using the chain rule leads to the differential equation

$$u' = \frac{\psi_2(u) - \frac{\partial \psi_1(u)}{\partial x}}{\frac{\partial \psi_1}{\partial t}(u)}. \tag{11}$$

Again, the denominator is well-defined for a rational general solution [22, Section 5]. This approach transforms a general (non-linear) first-order AODE into a single quasi-linear ODE, whereas the method in Section 3.1 would give a system of two quasi-linear ODEs. However, the system of differential equations (6) is autonomous while the ODE of Equation (11) lacks this property in general.

Definition 10. Let $A(x, y, y') = 0$ be a curve-parametrisable first-order AODE and $\mathcal{P}_{\mathcal{C}_A}(t) = (\psi_1(t), \psi_2(t))$ be a proper rational parametrisation of the associated curve with $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$. For this AODE, the differential equation (11) is called the *associated quasi-linear equation* with respect to $\mathcal{P}_{\mathcal{C}_A}$. ■

The associated quasi-linear equation is, like the associated planar system, constructed in such a way that rational general solutions are preserved. The proof of the subsequent theorem is given in Vo, Grasegger and Winkler [22, Theorem 5.3].

Theorem 4. *Let $A(x, y, y') = 0$ be a curve-parametrisable AODE of order one and $\mathcal{P}_{\mathcal{C}_A}$ a proper rational parametrisation of the associated curve such that $\mathcal{P}_{\mathcal{C}_A}$ is defined over $\mathcal{F}(x)$. Then there is a one-to-one correspondence between rational general solutions of this AODE and the rational general solutions of the associated quasi-linear equation with respect to $\mathcal{P}_{\mathcal{C}_A}$.*

By construction, a rational general solution $\hat{u}$ of the associated quasi-linear equation with respect to $\mathcal{P}_{\mathcal{C}_A}(t) = (\psi_1(t), \psi_2(t))$ constitutes a rational general solution $\hat{y} = \psi_1(\hat{u})$ of the original AODE. Restricting the class of solutions to rational general solutions forces the associated quasi-linear equation to be of a special form. Behloul and Cheng [1] showed that a quasi-linear ODE which is neither linear nor a Riccati equation can have finitely many rational solutions at most. Since a rational general solution can be seen as an infinite family of rational solutions, a necessary condition that Equation (11) has a suitable solution is that it is of the form

$$u' = A_0 + A_1 u + A_2 u^2, \tag{12}$$

for some $A_0, A_1, A_2 \in \mathcal{F}(x)$. Thus, a characterisation for rational general solutions of the associated quasi-linear equation (11) is as follows: It is a solution $\hat{u}$ such that the differential equation is of the form (12) and $\hat{u}$ satisfies Lemma 1 for the corresponding AODE[7], viz. the AODE obtained by clearing the denominators of A_0, A_1 and A_2 followed by removing the content.

The notion of rational general solution is quite special for differential equations of the form (12). If such an object exists, then it can be chosen from a purely transcendental constant extension of $\mathcal{F}(x)$. Vo, Grasegger and Winkler [22] call such solutions strong rational general solutions.

Definition 11. Given a first-order AODE $A(x, y, y') = 0$ with a rational general solution $\hat{y}$. If $\hat{y} \in \mathcal{F}(x)(C)$, where C is a transcendental constant over $(\mathcal{F}(x), {}^{d}/_{dx})$, then $\hat{y}$ is called a *strong rational general solution*. Let $\mathbf{A}_{ODE}^{(SRGS)}$ denote the class of all first-order AODEs which possess a strong rational general solution. ■

In other words, strong rational general solutions are those rational general solutions where the constant coefficients appear rationally. This is not the case in general. Example 4 of Section 4 describes a first-order AODE with a rational general solution whose transcendental constant coefficients appear algebraically. The existence of (strong) rational general solutions of the associated quasi-linear equation can be decided and—in the positive case—computed effectively. For linear differential equations the solution is straightforward. In Kovacic [13, Section 3] a complete algorithm for computing all rational solutions of a Riccati equation is given. This algorithm can be modified, cf. Chen and Ma [4], to look for general solutions only. Notice that this makes the approach by curve parametrisation a full decision algorithm and every first-order AODE which has a strong rational general solution can be decided with this algorithm [22, Theorem 6.1].

4 Comparison of the methods

The aim of this section is to discuss connections between the surface and curve parametrisation method of Section 3 and to relate the classes $\mathbf{A}_{ODE}^{(SP)}$, $\mathbf{A}_{ODE}^{(CP)}$, $\mathbf{A}_{ODE}^{(SRGS)}$ and $\mathbf{A}_{ODE}^{(RGS)}$. A main tool for this investigation is the following lemma.

Lemma 2. *Let $\mathbb{F} := \mathcal{F}(x)$ and consider an irreducible polynomial $P \in \mathcal{F}[x, y, z]$ such that P is not an element of the base field in the polynomial ring $\mathbb{F}[y, z]$. Define the algebraic sets $\mathcal{S}_P \subseteq \mathbb{A}^3(\mathcal{F})$ and $\mathcal{C}_P \subseteq \mathbb{A}^2(\overline{\mathbb{F}})$ as follows:*

$$\mathcal{S}_P := \left\{ (x, y, z) \in \mathbb{A}^3(\mathcal{F}) \;\middle|\; P(x, y, z) = 0 \right\} \text{ and } \mathcal{C}_P := \left\{ (y, z) \in \mathbb{A}^2\left(\overline{\mathbb{F}}\right) \;\middle|\; P(y, z) = 0 \right\},$$

[7] Cf. Equation (5) at the beginning of Section 3.

where in the latter case, P is interpreted[8] *as an element of $\mathbb{F}[y,z]$. If $\mathcal{C}_P$ is rational, then $\mathcal{S}_P$ is rational and there exist $\chi_1, \chi_2 \in \mathbb{F}(t)$ such that $\mathcal{P}_{\mathcal{C}_P}(t) = (\chi_1(t), \chi_2(t))$ is a proper rational parametrisation of $\mathcal{C}_P$ and, by interpreting $\chi_1, \chi_2 \in \mathcal{F}(x,t)$, $\mathcal{P}_{\mathcal{S}_P}(x,t) = (x, \chi_1(x,t), \chi_2(x,t))$ is a proper rational parametrisation of $\mathcal{S}_P$.*

Proof. Assume that $\mathcal{C}_P$ is a rational curve. By Proposition 1 there exists a proper rational parametrisation $\mathcal{P}_{\mathcal{C}_P}(t) = (\chi_1(t), \chi_2(t))$ with $\chi_1, \chi_2 \in \mathbb{F}(t)$. Theorem 2 asserts that the function field $\overline{\mathbb{F}}(\mathcal{C}_P)$ is isomorphic to $\overline{\mathbb{F}}(t)$ over $\overline{\mathbb{F}}$. By the remark after the theorem this isomorphism is realised via the following pullback map:

$$\begin{aligned} \mathcal{P}^*_{\mathcal{C}_P} : \overline{\mathbb{F}}(\mathcal{C}_P) &\to \overline{\mathbb{F}}(t), \\ f &\mapsto f \circ \mathcal{P}_{\mathcal{C}_P}. \end{aligned}$$

Notice that the parametrisation can not be constant, i.e. not both $\chi_1, \chi_2 \in \mathbb{F}$, otherwise $\mathcal{P}^*_{\mathcal{C}_P}$ could not be an isomorphism. Consider now the function field $\mathbb{F}(\mathcal{C}_P)$ of the curve over the algebraically non-closed field $\mathbb{F}$. Obviously, this is a subfield of the domain of $\mathcal{P}^*_{\mathcal{C}_P}$ and by restricting the pullback to this subfield one obtains a map

$$\mathcal{P}^*_{\mathcal{C}_P}\big|_{\mathbb{F}(\mathcal{C}_P)} : \mathbb{F}(\mathcal{C}_P) \to \mathbb{F}(t).$$

The restriction of the codomain is only possible since the parametrisation $\mathcal{P}_{\mathcal{C}_P}$ is defined over $\mathbb{F}$. Evidently, the image of the restricted pullback map is an embedding of $\mathbb{F}(\mathcal{C}_P)$ in $\mathbb{F}(t)$ and—since the parametrisation is non-constant—the field $\mathbb{F}$ is a proper subfield of this image. By invoking Lüroth's theorem [23, Chapter 5, Theorem 7.2], the map $\mathcal{P}^*_{\mathcal{C}_P}\big|_{\mathbb{F}(\mathcal{C}_P)}$ establishes an isomorphism $\mathbb{F}(\mathcal{C}_P) \cong \mathbb{F}(t)$. If this would not be the case, i.e. the image of the restricted pullback map is merely an isomorphic proper subfield of $\mathbb{F}(t)$, then the embedding of $\mathbb{F}(\mathcal{C}_P)$ in $\mathbb{F}(t)$ is an extension of $\mathbb{F}$ generated by a necessarily non-linear rational function. Hence $\mathcal{P}_{\mathcal{C}_P}$ could be reparametrised by this generator, contradicting the properness of this parametrisation [20, Lemma 4.17].

Now recall the structure of the function field of an algebraic variety defined by an irreducible polynomial:

$$\mathbb{F}(\mathcal{C}_P) \cong \left\{ \frac{F}{G} \,\middle|\, F, G \in \mathbb{F}(y,z) \text{ and } P \nmid G \right\} \Big/ \sim, \tag{13}$$

where ${}^{F_1}/_{G_1} \sim {}^{F_2}/_{G_2} :\Leftrightarrow P \mid (F_1G_2 - F_2G_1)$. The denominators of the coefficients of F and G in Equation (13) can be cleared and a representative of each fraction ${}^{F}/_{G}$ can be chosen such that $F, G \in \mathcal{F}[x][y,z] \cong \mathcal{F}[x,y,z]$. Since $\mathcal{C}_P$ and $\mathcal{S}_P$ are defined by the very same polynomial $P \in \mathcal{F}[x,y,z]$ this shows that $\mathcal{F}(\mathcal{S}_P) \cong \mathbb{F}(\mathcal{C}_P)$ and hence

$$\mathcal{F}(\mathcal{S}_P) \cong \mathbb{F}(\mathcal{C}_P) \cong \mathbb{F}(t) \cong \mathcal{F}(x,t).$$

The first and last isomorphism is canonical, i.e. elements in one field have an interpretation as elements of the other, and the middle isomorphism is realised via the pullback map of the

[8] Actually, in both cases the polynomial is interpreted as a regular function on the corresponding affine space.

curve parametrisation. So clearly, $\mathcal{F}(\mathcal{S}_P)$ is isomorphic to $\mathcal{F}(x,t)$ over $\mathcal{F}$ and by Theorem 2 the surface $\mathcal{S}_P$ is rational.

To construct a proper rational parametrisation of $\mathcal{S}_P$ let

$$\mathcal{P}^*_{\mathcal{S}_P} : \mathcal{F}(\mathcal{S}_P) \xrightarrow{\text{interpret}} \mathbb{F}(\mathcal{C}_P) \xrightarrow{\mathcal{P}^*_{\mathcal{C}_P}|_{\mathbb{F}(\mathcal{C}_P)}} \mathbb{F}(t) \xrightarrow{\text{interpret}} \mathcal{F}(x,t)$$

be the previously described $\mathcal{F}$-isomorphism $\mathcal{F}(\mathcal{S}_P) \cong \mathcal{F}(x,t)$. By the remark after Theorem 2, the parametrisation may be obtained by applying $\mathcal{P}^*_{\mathcal{S}_P}$ on the surface's coordinate functions and interpreting the result as a function from $\mathbb{A}^2(\mathcal{F})$ to $\mathcal{S}_P$. By an abuse of notation, let $x, y, z \in \mathcal{F}(\mathcal{S}_P)$ also denote the coordinate functions[9] of $\mathcal{S}_P$. Then

$$\mathcal{P}_{\mathcal{S}_P}(x,t) = \left(\mathcal{P}^*_{\mathcal{S}_P}(x)(x,t), \mathcal{P}^*_{\mathcal{S}_P}(y)(x,t), \mathcal{P}^*_{\mathcal{S}_P}(z)(x,t)\right)$$

is a proper rational parametrisation of $\mathcal{S}_P$. The coordinate function x is a constant function as an element of $\mathbb{F}(\mathcal{C}_P)$ and hence is not changed by $\mathcal{P}^*_{\mathcal{C}_P}|_{\mathbb{F}(\mathcal{C}_P)}$. On the other hand, the coordinate functions y and z of the surface correspond to the coordinate functions of $\mathcal{C}_P$ in $\mathbb{F}(\mathcal{C}_P)$. By the definition of the parametrisation $y \circ \mathcal{P}_{\mathcal{C}_P} = \chi_1$ and $z \circ \mathcal{P}_{\mathcal{C}_P} = \chi_2$. Under the canonical identification of $\mathbb{F}(t)$ with $\mathcal{F}(x,t)$ and interpreting elements of the latter space as functions on $\mathbb{A}^2(\mathcal{F})$ it is found that $\mathcal{P}_{\mathcal{S}_P}(x,t) = (x(x,t), \chi_1(x,t), \chi_2(x,t))$. □

Remark 1. *Notice that the converse is not true. A rational surface defined by an irreducible polynomial $P \in \mathcal{F}[x,y,z] \setminus \mathcal{F}[x]$ does not imply that the corresponding curve is rational. A counterexample is readily given: Consider the polynomial $P = x - y^2 - z^3 \in \overline{\mathbb{Q}}[x,y,z]$. The surface $\mathcal{S}_P$ has the proper rational parametrisation*

$$\begin{aligned} \mathcal{P}_{\mathcal{S}_P} : \mathbb{A}^2\left(\overline{\mathbb{Q}}\right) &\to \mathcal{S}_P, \\ (s,t) &\mapsto \left(s^2 + t^3, s, t\right). \end{aligned}$$

On the other hand, the curve $\mathcal{C}_P$ has genus one and can not be rational by Theorem 1, thus.

Remark 2. *However, if the surface $\mathcal{S}_P$ is rational and admits a proper rational parametrisation of the special form $\mathcal{P}_{\mathcal{S}_P}(s,t) = (s, \chi_1(s,t), \chi_2(s,t))$, where $\chi_1, \chi_2 \in \mathcal{F}(s,t)$, then $\mathcal{C}_P$ is rational and $\mathcal{P}_{\mathcal{C}_P}(t) = (\chi_1(x,t), \chi_2(x,t))$ is a proper rational parametrisation of $\mathcal{C}_P$.*

In order to see this, consider the irreducible polynomial $P \in \mathcal{F}[x,y,z] \setminus \mathcal{F}[x]$ and assume for the moment that the curve $\mathcal{C}_P$ is irreducible. By renaming the parameter s of the surface parametrisation $\mathcal{P}_{\mathcal{S}_P}$ to x it follows that $P(x, \chi_1(x,t), \chi_2(x,t)) = 0$. At least one of the rational functions χ_1 or χ_2 must depend on the parameter t, otherwise $\mathcal{P}_{\mathcal{S}_P}$ could not parametrise a surface, hence $(\chi_1(x,t), \chi_2(x,t))$ is a pair of non-constant rational functions vanishing on the

[9] These are the projections on the first, second and third component of the points of $\mathcal{S}_P$. In other words, $x : \mathcal{S}_P \to \mathcal{F}, (x,y,z) \mapsto x$ and similarly for the other two coordinate functions. From a modern viewpoint, one may consider x, y, z as $\mathcal{F}$-algebra generators of the fraction field of the integral domain R, where R is the quotient ring of $\mathcal{F}[x,y,z]$ modulo the ideal generated by P.

defining polynomial of $\mathcal{C}_P$. By [20, Theorem 4.7] it follows that $\mathcal{P}_{\mathcal{C}_P}(t) = (\chi_1(x,t), \chi_2(x,t))$ is a rational parametrisation of $\mathcal{C}_P$. Furthermore, $\mathcal{P}_{\mathcal{C}_P}$ must be proper. The pullback mapping $\mathcal{P}^*_{\mathcal{S}_P} : \mathcal{F}(\mathcal{S}_P) \to \mathcal{F}(x,t)$ yields the identity function on the first parameter when applied to the coordinate function x of the surface $\mathcal{S}_P$. Consequently, $\mathcal{P}^*_{\mathcal{S}_P}$ defines an $\mathcal{F}(x)$-isomorphism to $\mathcal{F}(x)(t) \cong \mathcal{F}(x,t)$. By a similar reasoning as in the proof of Lemma 2, if $\mathcal{P}_{\mathcal{C}_P}$ was not proper, then there exists a reparametrisation by a non-linear rational function. In such a case, however, the pullback $\mathcal{P}^*_{\mathcal{S}_P}$ can not possibly give the full isomorphism which contradicts the assumed properness of $\mathcal{S}_P$.

Now to the irreducibility of $\mathcal{C}_P$. In Vo, Grasegger and Winkler [22, Theorem 3.1] is has been shown that an AODE is curve-parametrisable if it has a strong rational general solution $\hat{y} \in \mathcal{F}(x)(C)$, where C is a transcendental constant. In particular, they proved that in this case the defining polynomial A of the AODE is irreducible as an element of $\overline{\mathcal{F}(x)}[y, y']$. For this they constructed the ideal

$$I = \{F \in \overline{\mathcal{F}(x)}[y,z] \mid F(x, \hat{y}(x,C), \hat{y}'(x,C)) = 0\}$$

and proved that this is a principal prime ideal containing A as an irreducible element. By retracing the steps of their proof, it can be shown that P is irreducible as an element of the principal prime ideal

$$J = \{F \in \overline{\mathcal{F}(x)}[y,z] \mid F(x, \chi_1(x,t), \chi_2(x,t)) = 0\},$$

which completes the argument. In combination with Lemma 2, this last remark shows that curve-parametrisable AODEs are precisely those AODEs where the associated surface is a pencil of rational curves with a section along the x-axis.

4.1 Comparison of the associated differential equations

An interesting consequence of Lemma 2 is that it facilitates a direct comparison of the associated planar system and the associated quasi-linear equation with respect to a common parametrisation. Given a first-order AODE $A(x,y,y') = 0$ such that the associated curve $\mathcal{C}_A$ is rational. Let $\mathcal{P}_{\mathcal{C}_A}(t) = (\chi_1(t), \chi_2(t))$ be such that $\chi_1, \chi_2 \in \mathcal{F}(x)(t)$ is a proper rational parametrisation of $\mathcal{C}_A$. Recall that the associated quasi-linear equation with respect to $\mathcal{P}_{\mathcal{C}_A}$ is of the form

$$u' = \frac{\chi_2(u) - \frac{\partial \chi_1(u)}{\partial x}}{\frac{\partial \chi_1}{\partial t}(u)}. \tag{14}$$

By Lemma 2 the associated surface $\mathcal{S}_A$ is rational and possesses the proper rational parametrisation $\mathcal{P}_{\mathcal{S}_A}(x,t) = (x, \chi_1(x,t), \chi_2(x,t))$. In order to use the same notation as in Section 3.1, the parameter x is renamed to s. Due to the special form of this parametrisation the associated planar system with respect to $\mathcal{P}_{\hat{\mathcal{S}}_A}$ simplifies to

$$\begin{cases} s' = 1 \\ t' = \dfrac{\chi_2(s,t) - \frac{\partial \chi_1(s,t)}{\partial s}}{\frac{\partial \chi_1(s,t)}{\partial t}}. \end{cases} \tag{15}$$

Solutions of autonomous planar systems behave nicely under translation by constants. If $(\hat{s}(x), \hat{t}(x))$ is a rational general solution of the associated planar system, then $(\hat{s}(x+C), \hat{t}(x+C))$ is another expression for the same rational general solution [17]. The shift $x - C$ in Equation (7) is precisely such a shift by a constant. The general solution of the first equation of the system (15) obviously is $\hat{s}(x) = x + C$. Aiming for rational general solutions, if the planar system has such a solution $(\hat{s}(x), \hat{t}(x))$ with $\hat{s}(x) = x + C$, then $(\hat{s}(x-C), \hat{t}(x-C))$ is another expression for the same solution by the previous remark. In other words, by substitution of $\hat{s}(x-C) = x$ for s in the second equation of the system (15) one obtains a single quasi-linear ODE

$$t' = \frac{\chi_2(x,t) - \frac{\partial \chi_1(x,t)}{\partial x}}{\frac{\partial \chi_1(x,t)}{\partial t}}. \tag{16}$$

Note that the equivalence of the system (15) to the quasi-linear equation (16) can be shown with classical methods as well [14, Section 5.1]. Comparing Equation (14) and Equation (16) shows that they denote the very same quasi-linear ODE. This shows that the associated planar system reduces to the associated quasi-linear equation if a surface parametrisation obtained from a parametrisation of the associated curve is used. The observation that the associated planar system reduces to a single quasi-linear equation—given that the first component of the surface parametrisation coincides with the corresponding coordinate function—has been made in Ngô, Sendra and Winkler [14]. They also argued that a proper rational parametrisation of the associated curve defined over $\mathcal{F}(x)$ must translate to a parametrisation of the associated surface as in Lemma 2. AODEs with such a property are called *differential equations of pencil type* in their terminology [14, Section 5.1].

A rational general solution $\hat{u}$ of the quasi-linear equation (14) extends to a rational general solution $(\hat{s}(x) = x, \hat{t}(x) = \hat{u})$ of the planar system (15), cf. [15, Lemma 3.13]. By the remark after Theorem 4 in Section 3.2, the approach by curve parametrisation generates the rational general solution of the original AODE $\hat{y} = \chi_1(\hat{u})$ from the associated solution $\hat{u}$. Analogously, the surface parametrisation method constructs the general solution $\hat{y} = \chi_1(\hat{s}(x-C), \hat{t}(x-C)) = \chi_1(x, \hat{u})$, where $C = 0$ since the first component of the surface parametrisation is just x and $\hat{s}(x) = x$, cf. Equation (7) in Section 3.1. Therefore, both methods generate the same rational general solution of the original AODE from a common solution of the associated differential equation(s).

Example (Clairaut's equation)**.** *The subsequent ODE is known as* Clairaut's equation*:*

$$y(x) = x\,\frac{\mathrm{d}}{\mathrm{d}x}y(x) + f\Big(\frac{\mathrm{d}}{\mathrm{d}x}y(x)\Big),$$

where f is a continuously differentiable function. A general solution of this ODE can be found with the introduced methods if f is a polynomial function[10]. *Consider the defining polynomial*

$$A = y - x\,y' - f(y') \in \mathbb{C}[x, y, y'],$$

such that $f(y') \in \mathbb{C}[y']$. In this case, Clairaut's equation is curve-parametrisable and has the proper rational parametrisation $\mathcal{P}_{\mathcal{C}_A}(t) = (\chi_1(t), \chi_2(t))$ with $\chi_1 = x\,t + f(t)$ and $\chi_2 = t$. The associated quasi-linear equation has the form

$$u' = \frac{\chi_2(u) - \frac{\partial \chi_1(u)}{\partial x}}{\frac{\partial \chi_1}{\partial t}(u)} = \frac{u - u}{x + \frac{\partial f}{\partial t}(u)} = 0.$$

Obviously, the general solution of this equation is $\hat{u} = C$. This solution is transformed back to the general solution $\hat{y} = \chi_1(\hat{u}) = C\,x + f(C)$ of Clairaut's equation.

Alternatively, this AODE can be solved with the surface parametrisation approach. By Lemma 2, Clairaut's equation is surface-parametrisable and has—after renaming the parameter x to s—the proper rational parametrisation $\mathcal{P}_{\mathcal{S}_A}(s,t) = (\chi_0(s,t), \chi_1(s,t), \chi_2(s,t))$ with $\chi_0 = s$, $\chi_1 = s\,t + f(t)$ and $\chi_2 = t$. Since $\partial\chi_0/\partial s = 1$ and $\partial\chi_0/\partial t = 0$ the associated planar system is

$$\begin{cases} s' = \dfrac{-\frac{\partial \chi_1(s,t)}{\partial t}}{-\frac{\partial \chi_1(s,t)}{\partial t}} = 1 \\ t' = \dfrac{\chi_2(s,t) - \frac{\partial \chi_1(s,t)}{\partial s}}{\frac{\partial \chi_1(s,t)}{\partial t}} = \dfrac{t - t}{s + \frac{\partial f(t)}{\partial t}} = 0. \end{cases} \tag{17}$$

By the results of this section, $(\hat{s}(x) = x, \hat{t}(x) = C)$ is a rational general solution of this system from which the general solution $\hat{y} = \chi_1(x, \hat{u}) = C\,x + f(C)$ of Clairaut's equation is generated.

Remark 3. *If the system (17) is solved without the reduction to a single quasi-linear equation, then a rational general solution has the form $\hat{s}(x) = x + C_1$ and $\hat{t}(x) = C_2$, where C_1, C_2 are two transcendental constants. This associated solution is transformed back to the subsequent general solution, cf. Section 3.1:*

$$\hat{y} = \chi_1\big(\hat{s}(x - C_1), \hat{t}(x - C_1)\big) = \chi_1(x, C_2) = C_2\,x + f(C_2),$$

which is identical to the previous solution.

[10] Actually, f could be a rational function as well. In this case, one has to slightly modify the equation to constitute an AODE. However, the rational parametrisation will be the same.

4.2 Relations between subclasses of first-order AODEs

The following propositions relate the introduced classes of first-order AODEs. Recall that $\mathbf{A}_{ODE}$ denotes the class of all AODEs of order one, $\mathbf{A}_{ODE}^{(SP)}$ and $\mathbf{A}_{ODE}^{(CP)}$ stands for the subclass of surface-parametrisable and curve-parametrisable AODEs, respectively. Finally, $\mathbf{A}_{ODE}^{(RGS)}$ and $\mathbf{A}_{ODE}^{(SRGS)}$ refers to the subclass of AODEs with a rational general and strong rational general solution, respectively.

Proposition 2. *Curve-parametrisable AODEs of order one are surface-parametrisable and the inclusion is proper. In addition, not every first-order AODE with a rational general solution is surface-parametrisable. Stated in terms of the introduced notation:*

$$\mathbf{A}_{ODE}^{(CP)} \subsetneq \mathbf{A}_{ODE}^{(SP)} \text{ and } \mathbf{A}_{ODE}^{(RGS)} \setminus \mathbf{A}_{ODE}^{(SP)} \neq \emptyset.$$

Proof. This is a direct consequence of Lemma 2, Example 1 and Example 4. □

Proposition 3. *The class of first-order AODEs with a strong rational general solution are precisely those AODEs which have a rational general solution and are curve-parametrisable:*

$$\mathbf{A}_{ODE}^{(SRGS)} = \mathbf{A}_{ODE}^{(RGS)} \cap \mathbf{A}_{ODE}^{(CP)}.$$

Proof. This follows from Vo, Grasegger and Winkler [22, Theorem 5.4(i), Corollary 5.5]. □

The subsequent examples of first-order AODEs are used in a graphical depiction of the previous propositions given at Figure 1. Some of these examples are quite trivial, listed solely for the sake of completeness. Note that the extend of the regions in Figure 1 do not in any sense correspond to a mathematical measure of the sets. A statistical investigation for the class of curve-parametrisable AODEs is given in Grasegger, Vo and Winkler [9].

Example 1. *The AODE $y'^2 - y^3 - x = 0$ is surface-parametrisable, but does not have a rational general solution. Furthermore, its associated curve is not rational as the latter is of genus one.*

It is easy to see that the associated surface of the AODE in Example 1 has the proper rational parametrisation

$$\mathcal{P}_{\mathcal{S}_{Ex.1}}(s,t) = (t^2 - s^3, s, t)$$

which yields the following associated planar system

$$\begin{cases} s' = t \\ t' = \dfrac{3s^2 t + 1}{2t}. \end{cases}$$

From this special planar system one can derive the single quasi-linear differential equation [16, Section 4.3 (Equations solvable for x)]

$$t' = \frac{3x^2t + 1}{2t^2}$$

which is neither a linear nor a Riccati equation and hence can not have a rational general solution [22, Section 5].

Example 2. *The AODE $y' - y = 0$ is both surface-parametrisable and curve-parametrisable. A proper rational parametrisation is easily determined since the associated curve/surface is a line/plane, respectively. It should be clear that the only rational function which satisfies this autonomous AODE is the trivial solution. Consequently, such an AODE can not have a rational general solution.*

Example 3. *Consider the AODE $y' - y^2 = 0$ which is surface/curve-parametrisable and has a rational general solution. Proper rational parametrisations are found easily and a rational general solution is given by $\hat{y} = 1/(C-x)$.*

Example 4. *The subsequent AODE is neither curve-parametrisable nor surface-parametrisable. This follows from the fact that the arithmetic genus of the associated surface is negative and the genus of the associated curve is one:*

$$x^2y'^2 - 2xyy' - y'^3 + y^2 - 2 = 0.$$

However, this AODE has the rational general solution $\hat{y} = Cx + \sqrt{C^3 + 2}$.

Example 5. *Finally, consider the AODE $y'^2 + y^3 + 1 = 0$ which is not parametrisable as curve or as surface and does not have a rational general solution. Again, the arithmetic genus of the associated surface is negative and the genus of the associated curve is one. Since for autonomous first-order AODEs the notion of rational general solution and strong rational general solution coincides, yet this AODE is not curve-parametrisable, there can not exist a rational general solution [22]. Alternatively, this follows from the results in Feng and Gao [7] and the fact that the AODE is not parametrisable as a plane curve over the field of rational numbers.*

5 Conclusion and further questions

In this paper, two recent algobro-geometric methods for computing rational general solutions of first-order AODEs have been outlined and their associated (system of) differential equation(s) for a common rational parametrisation were compared. In addition, certain classes of first-order AODEs have been related. A graphical depiction of these relations is provided by Figure 1. One question which this diagram leaves open is whether there are surface-parametrisable AODEs which possess a rational general solution, but are not curve-parametrisable. In other words, is the following class

$$\left(\mathbf{A}_{ODE}^{(SP)} \cap \mathbf{A}_{ODE}^{(RGS)}\right) \setminus \mathbf{A}_{ODE}^{(CP)} = \left(\mathbf{A}_{ODE}^{(SP)} \setminus \mathbf{A}_{ODE}^{(CP)}\right) \cap \mathbf{A}_{ODE}^{(RGS)} \tag{18}$$

empty or not? By Proposition 3 it is clear that such AODEs can not have a strong rational general solution. Therefore, an element from the class (18) would have to be parametrisable as a rational surface, but the transcendental constants of any rational general solution do not form a purely transcendental extension of $\mathcal{F}(x)$, cf. Definition 11 in Section 3.2. An answer to this would settle the question whether the method via surface parametrisation is more general than the method by curve parametrisation.

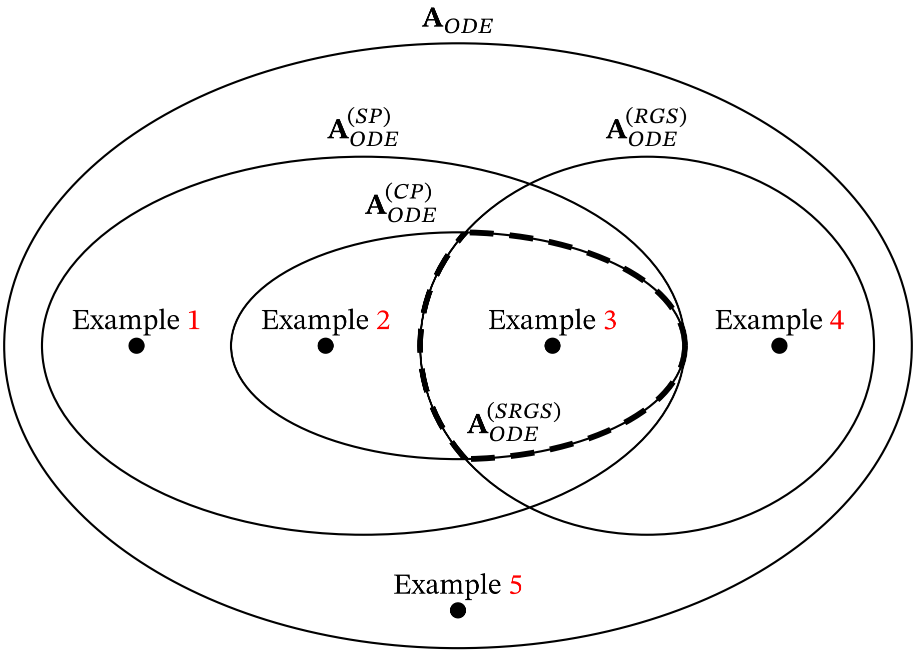

Figure 1: Relations between the subsequent classes of first-order AODEs:

$\mathbf{A}_{ODE}$	...	the class of first-order AODEs
$\mathbf{A}_{ODE}^{(RGS)}$	...	the class of first-order AODEs with a rational general solution
$\mathbf{A}_{ODE}^{(SRGS)}$	...	the class of first-order AODEs with a strong rational general solution
$\mathbf{A}_{ODE}^{(CP)}$	...	the class of first-order AODEs whose associated curve is parametrisable
$\mathbf{A}_{ODE}^{(SP)}$	...	the class of first-order AODEs whose associated surface is parametrisable

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