

# HOLONOMIC RELATIONS FOR MODULAR FUNCTIONS AND FORMS: FIRST GUESS, THEN PROVE

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*Dedicated to Bruce Berndt at the occasion of his 80th birthday.*

ABSTRACT. One major theme of this article concerns the expansion of modular forms and functions in terms of fractional (Puiseux) series. This theme is connected with another major theme, holonomic functions and sequences. With particular attention to algorithmic aspects, we study various connections between these two worlds. Applications concern partition congruences, Fricke-Klein relations, irrationality proofs à la Beukers, or approximations to  $\pi$  studied by Ramanujan and the Borweins. As a major ingredient to a “first guess, then prove” strategy, a new algorithm for proving differential equations for modular forms is introduced.

## 1. INTRODUCTION

The study of holonomic functions and sequences satisfying linear differential and difference equations, respectively, with polynomial coefficients has roots tracing back (at least) to the time of Gauß. More recently such objects have become fundamental ingredients for the modern theory of enumerative combinatorics; see, for example, the work of Stanley [34]. Concerning the promotion of the algorithmic relevance of these objects, a major role was played by Zeilberger’s pioneering “holonomic systems approach” to special functions identities [40].

The ubiquitous nature of holonomic functions and sequences is documented by the numerous examples given in the literature, for instance, in the magnum opus [34].<sup>1</sup> However, when browsing through Ramanujan’s Notebooks [8, 9, 10, 11, 12] and Lost Notebooks [3, 4, 5, 6, 7], one finds it surprisingly difficult to identify holonomic sequences or functions — apart from classical hypergeometric series having

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<sup>1</sup>In [34] holonomic sequences are called P-recursive (polynomially recursive), holonomic functions are called D-finite (differentiably finite).

hypergeometric summands which, by definition, satisfy a linear recurrence, with polynomial coefficients, of order 1. To some extent this phenomenon can be explained as follows: Many of Ramanujan’s entries are related to partition numbers  $p(n)$  or variations of such. As a matter of fact, proved in Proposition 3.2,  $(p(n))_{n \geq 0}$  is not holonomic — despite being one of the most prominent combinatorial and number theoretical sequences!

One objective of this paper is to cure Ramanujan’s work from this “non-holonomy syndrom.” This will be done by considering holonomic differential equations and algebraic relations, satisfied by modular functions and forms which are associated to the given non-holonomic sequences, resp. functions. Our study is driven by algorithmic consideration; applications concern results related to Ramanujan’s work, partition congruences, and special functions relations of Fricke-Klein type which, for instance, arise in approximations for  $\pi$  and in Beukers’ modular form version of Apèry’s proof of  $\zeta(3) \notin \mathbb{Q}$ . In all these examples the computer algebra package `GeneratingFunctions`, written by Christian Mallinger in Mathematica [26]<sup>2</sup>, is playing a crucial role.<sup>3</sup>

Another major aspect of this article concerns differential equations involving modular forms and functions. In general, any modular form satisfies a *non-linear* third order differential equation with constant coefficients; see, for instance, [39, Prop. 16]. But in many contexts relevant to Ramanujan’s work, modular form theory guarantees the existence of *holonomic* differential equations. Zagier [39, Prop. 21] introduces to this fact as follows: “... it is at the heart of the original discovery of modular forms by Gauss and of the later work of Fricke and Klein and others, and appears in modern literature as the theory of Picard-Fuchs differential equations or of the Gauss-Manin connection — but it is not nearly as well known as it ought to be.” The story connected to this quote is continued in Section 6. As a contribution to this topical area, in the spirit of the “first guess, then prove” paradigm<sup>4</sup>, in this article together with [31] we present an algorithm which automatically proves a claim stating that a given modular form of positive weight satisfies a differential equation which is linear and with coefficients being polynomials in a given modular function.

Our article is structured as follows. To make the exposition as self-contained as possible, in Section 2 we present the most important basic facts on modular functions and forms needed. Section 3 does the same for univariate holonomic functions and sequences; an illustrative example is taken from Ramanujan’s “Quarterly Reports”. Section 4 introduces to one of the major themes, the expansion of modular forms and functions in terms of fractional (Puiseux)

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<sup>2</sup>This package is freely available upon password request by email to the first named author.

<sup>3</sup>A much more powerful multivariate package `HolonomicFunctions` has been developed by Christoph Koutschan [23] also in the Mathematica system.

<sup>4</sup>This paradigm, e.g., was popularized by George Polya in many of his writings.

series. In Section 5 we present case studies dealing with relations of Fricke-Klein type, irrationality proofs based on modular forms a la Beukers, and approximations to  $\pi$  which were first discovered by Ramanujan and then extensively studied by Jonathan and Peter Borwein. In Section 6, again with focus on an algorithmic setting, we discuss aspects of a classical connection of modular forms with differential equations. This connection traces back to Gauß and was popularized prominently by Zagier. In addition, we present a brief account<sup>5</sup> of a new algorithm for proving differential equations for modular forms. Beginning with Section 7, we try make the following point: connecting Puiseux expansions for modular functions with holonomic differential equations is of interest for the theory of partition congruences. After an introductory example inspired by Ramanujan's  $11 \mid p(11n + 6)$  observation, in Section 8 we discuss algebraic relations between modular functions and corresponding issues of computational relevance. In Section 9 the focus is shifted from algebraic relations to differential equations involving modular functions. Choosing Andrews' 2-colored Frobenius partitions, Section 10 presents another case study to illustrate aspects treated in Sections 8 and 9. Linear differential equations with algebraic function coefficients can be transformed into holonomic differential equations. The Appendix Section 11 contains a proof of this fact<sup>6</sup>; it also comments very shortly on zero-recognition of meromorphic functions on Riemann surfaces.

## 2. CONVENTIONS AND BASIC FACTS: MODULAR FUNCTIONS AND FORMS

Conventions used throughout this paper:  $N$  denotes a positive integer,

$$\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}, \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}, \text{ and } \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}.$$

The ring of formal power series over the complex numbers, resp. over the integers  $\mathbb{Z}$ , is denoted by  $\mathbb{C}[[z]]$ , resp.  $\mathbb{Z}[[z]]$ . The ring of univariate polynomials with complex coefficients is denoted by  $\mathbb{C}[X]$ , and by  $\mathbb{C}[X, Y]$  in the bivariate case.

The group  $\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2} : ad - bc = 1 \right\}$  acts on elements  $\tau$  from the upper half complex plane  $\mathbb{H}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

Congruence subgroups  $\Gamma$  are subgroups of  $\text{SL}_2(\mathbb{Z})$  which are specified by congruence conditions on their entries.<sup>7</sup> The so-called principal congruence subgroup is

<sup>5</sup>Full details are given in [31].

<sup>6</sup>Proposition 6.2

<sup>7</sup>More precisely, such  $\Gamma$  is a discrete subgroup of  $\text{SL}_2(\mathbb{R})$  which is commensurable with  $\text{SL}_2(\mathbb{Z})$  (i.e.,  $\Gamma \cap \text{SL}_2(\mathbb{Z})$  has finite index in  $\Gamma$  and  $\text{SL}_2(\mathbb{Z})$ ) and  $\Gamma(N) \subseteq \Gamma$  for some  $N$ .

defined as

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

A congruence subgroup  $\Gamma$  is called to be of level  $N$ , if  $\Gamma(N) \subseteq \Gamma$  and  $N$  is minimal with this property. Besides  $\Gamma(N)$ , we need the congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \pmod{N} \right\}$$

and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

There are many refinements, for instance, for positive integers  $N, M, P$  such that  $M \mid NP$ :

$$\Gamma(N, M, P) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \right. \\ \left. a \equiv d \equiv 1 \pmod{M}, b \equiv 0 \pmod{P}, c \equiv 0 \pmod{N} \right\}.$$

In Section 6.3 we will need  $\Gamma(2, 4, 2)$  together with properties which we retrieve by using the computer algebra system Magma.<sup>8</sup> See also Example 2.6, Example 4.2, and Section 5.3.

**2.1. Modular Forms.** For a function  $f : \mathbb{H} \rightarrow \hat{\mathbb{C}}$ ,  $k \in \mathbb{Z}_{\geq 0}$ , and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  we define the weight  $k$  slash-action as usual by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau), \quad \tau \in \mathbb{H}.$$

Also as usual, we define (e.g., [28, Def. 1.8]): Let  $f : \mathbb{H} \rightarrow \hat{\mathbb{C}}$  be meromorphic,  $k \in \mathbb{Z}_{\geq 0}$ , and  $\Gamma$  a congruence subgroup. Then  $f$  is called a meromorphic *modular form* of weight  $k$  for  $\Gamma$ , if for all  $\gamma \in \Gamma$ ,

$$(1) \quad (f|_k\gamma)(\tau) = f(\tau), \quad \tau \in \mathbb{H},$$

and if for each  $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$  there exists  $N_0 = N_0(\gamma_0) \in \mathbb{Z}_{>0}$  and  $n_0 = n_0(\gamma_0) \in \mathbb{Z}$  together with a Fourier expansion,

$$(2) \quad (f|_k\gamma_0)(\tau) = \sum_{n \geq n_0} a_{\gamma_0}(n) q_{N_0}^n,$$

where  $q_{N_0} := e^{2\pi i\tau/N_0}$  and  $a_{\gamma_0}(n_0) \neq 0$ .

Modular forms with weight  $k = 0$  are called *modular functions*; in this case we write  $f|_{\gamma_0}$  instead of  $f|_0\gamma_0$ . Also if  $k = 0$ , the integer  $n_0 = n_0(\gamma_0)$  is called the order of  $f|_{\gamma_0}$  at infinity; notation:  $n_0 = \mathrm{ord}(f|_{\gamma_0})$ .

<sup>8</sup>In the Magma system, this group is specified as `CongruenceSubgroup([N,M,P])`; see <http://magma.maths.usyd.edu.au/magma/handbook/text/1589>

One can extend the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$  to an action on  $\hat{\mathbb{H}} := \mathbb{H} \cup \hat{\mathbb{Q}}$  by defining  $\gamma\infty = a/c$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . If  $\gamma_0\infty = \gamma_1\infty = a/c$  for  $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ , then  $n_0 = \mathrm{ord}(f|_{\gamma_0}) = \mathrm{ord}(f|_{\gamma_1})$ . If  $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , then we write in short  $\mathrm{ord} f := \mathrm{ord}(f|_{\gamma_0})$  for the order of  $f$  at infinity.

**2.2. Modular Functions.** For algorithmic zero recognition, modular functions to us are most important. For those, owing to  $k = 0$ , we have the invariance  $f(\gamma\tau) = f(\tau)$  for all  $\gamma \in \Gamma$ , and the expansions (2) then allow to extend  $f$  meromorphically to all the points  $a/c \in \hat{\mathbb{Q}}$  where  $a/0 := \infty$ . To this end, the first step is to extend the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$  to an action on  $\hat{\mathbb{H}} := \mathbb{H} \cup \hat{\mathbb{Q}}$  as already mentioned. Notation: for any congruence subgroup  $\Gamma$  the  $\Gamma$ -orbit of  $\tau \in \hat{\mathbb{H}}$  is written as  $[\tau]_{\Gamma} := \{\gamma\tau : \gamma \in \Gamma\}$ . We will write  $[\tau]$  instead of  $[\tau]_{\Gamma}$ , if the subgroup  $\Gamma$  is clear from the context.

After this first step, the meromorphic extension of a modular function  $f$  from  $\mathbb{H}$  to a function on  $\hat{\mathbb{H}}$  is done by choosing  $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma_0\infty = a/c$ , and one defines

$$f(a/c) := (f|_{\gamma_0})(\infty) := \begin{cases} \infty & \text{if } n_0 < 0, \\ a_{\gamma_0}(0) & \text{if } n_0 = 0. \\ 0 & \text{if } n_0 > 0 \end{cases}.$$

A straightforward verification shows that this definition is independent from the choice of  $\gamma_0$ . This means, if one chooses another  $\gamma_1 \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma_1\infty = a/c$ , one would obtain the same value for  $f(a/c)$ . Moreover,  $f(a/c) = f(\gamma\frac{a}{c})$  for any  $\gamma \in \Gamma$ . Together with (1), this means, a modular function is constant on all the  $\Gamma$ -orbits  $[\tau]_{\Gamma}$ . The set of all such orbits, denoted by  $X(\Gamma)$ , can be equipped with the structure of a compact Riemann surface. Hence a modular function  $f$  with respect to  $\Gamma$  can be interpreted as a function  $\hat{f} : X(\Gamma) \rightarrow \hat{\mathbb{C}}$ ; in fact, such  $\hat{f}$  will be meromorphic.

Orbits  $[\tau]_{\Gamma}$ , where  $\tau = a/c \in \hat{\mathbb{Q}}$ , are called cusps. Congruence subgroups have only a finite index in  $\mathrm{SL}_2(\mathbb{Z})$ , hence  $\hat{\mathbb{Q}} = \{\gamma\infty : \gamma \in \mathrm{SL}_2(\mathbb{Z})\} = \mathrm{SL}_2(\mathbb{Z})\infty = \cup_j \Gamma\gamma_j\infty = \cup_j [\gamma_j\infty]_{\Gamma}$  is a disjoint union of *finitely many* cusps. Fourier expansions as in (2) are called expansions of  $f$  at the cusp  $[a/c]$ , or simply at  $a/c$ .

Special attention is given to the case  $a/c = \infty$  (i.e.,  $c = 0$ ). In this case we can exploit the fact that each congruence subgroup contains a translation matrix  $\begin{pmatrix} \pm 1 & N_0 \\ 0 & \pm 1 \end{pmatrix}$  with  $N_0 \in \mathbb{Z}_{>0}$  minimal; this means, a modular function  $f$  satisfying (1) has minimal period  $N_0$ . Suppose for all  $\tau \in \mathbb{H}$  with  $\mathrm{Im}(\tau)$  sufficiently large the

Fourier expansion of  $f$  at infinity<sup>9</sup> is:

$$(3) \quad f(\tau) = \sum_{n \geq n_0} a(n)q^{n/N_0}.$$

In various contexts we need to put emphasis on representing  $f$  in the Fourier variable  $q$ ; in such cases we write

$$(4) \quad \tilde{f}(q) := \sum_{n \geq n_0} a(n)q^{n/N_0} = f(\tau) \text{ for } q = e^{2\pi i\tau} \text{ sufficiently small.}$$

*Remark 2.1.* As mentioned above,  $\hat{\mathbb{Q}}$  is a disjoint union of finitely many cusps. As a consequence, if  $f$  is a non-constant modular function holomorph on  $\mathbb{H}$  and with  $f(a/c) = \infty$  for some  $a/c \in \hat{\mathbb{Q}}$ , then  $f(r) = \infty$  for an infinite set of values  $r \in \hat{\mathbb{Q}}$ . In view of (4) this means,  $\tilde{f}(z) = \infty$  for infinitely many  $z$  on the complex unit circle  $|q| = 1$ .

**2.3. Zero recognition of modular functions.** For zero recognition of a modular function  $f$ , one exploits its extension to a meromorphic function  $\hat{f} : X(\Gamma) \rightarrow \hat{\mathbb{C}}$  on the compact Riemann surface  $X(\Gamma)$ . Namely, if such functions are non-constant, they have the property that

$$(5) \quad \text{number of poles of } \hat{f} = \text{number of zeros of } \hat{f},$$

counting multiplicities; for further details see Lemma 11.2 in the Appendix Section 11.2.

*Remark 2.2.* Particularly straightforward applications of this fact are with regard to modular functions having a possible pole only at infinity:

$$(6) \quad M^\infty(\Gamma) := \{f : \mathbb{H} \cup \hat{\mathbb{Q}} \rightarrow \hat{\mathbb{C}} : \hat{f} \text{ has poles only at } \infty, \text{ and} \\ f \text{ is a modular function for the congruence subgroup } \Gamma\}.$$

We note that  $M^\infty(\Gamma)$  is a  $\mathbb{C}$ -algebra; i.e., a ring with multiplication by scalars from  $\mathbb{C}$ . To prove equality  $f_1 = f_2$  of two modular functions in  $M^\infty(\Gamma)$ , it is sufficient to inspect the  $q$ -expansion at infinity,

$$f_1 - f_2 = \frac{c_{-m_0}}{q^{m_0/N_0}} + \cdots + \frac{c_{-1}}{q^{1/N_0}} + c_0 + c_1q^{1/N_0} + \cdots,$$

and to verify that

$$c_{-m_0} = \cdots = c_{-1} = c_0 = 0.$$

Notice that in this case  $f_1 - f_2 \in M^\infty(\Gamma)$ , resp.  $\hat{f}_1 - \hat{f}_2 : X(\Gamma) \rightarrow \hat{\mathbb{C}}$ , has no pole but a zero at infinity; hence according to (5) it must be a constant.

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<sup>9</sup>In other words, at the point  $\infty \in \hat{\mathbb{Q}}$ , resp. at the cusp  $[\infty]$ .

Finally, for the field of modular functions *meromorphic* on  $\mathbb{H}$  we write:

$$(7) \quad M(\Gamma) := \{f : \mathbb{H} \cup \hat{\mathbb{Q}} \rightarrow \hat{\mathbb{C}} : f \text{ is meromorphic on } \mathbb{H}, \text{ and} \\ f \text{ is a modular function for the congruence subgroup } \Gamma\};$$

for the  $\mathbb{C}$ -vector space of modular forms of weight  $k \geq 1$  for  $\Gamma$  we write  $M_k(\Gamma)$ .

#### 2.4. Examples of modular forms and functions.

*Example 2.3.* Consider the Dedekind eta function,

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{j=1}^{\infty} (1 - e^{2\pi i \tau j}) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j), \quad \tau \in \mathbb{H}.$$

For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  it transforms as

$$(8) \quad \eta\left(\frac{a\tau + b}{c\tau + d}\right) = v(\gamma)(c\tau + d)^{1/2} \eta(\tau),$$

where for the root one takes the principal branch<sup>10</sup> and with  $v(\gamma)$  being some 24th root of unity depending on  $\gamma$ . For details on  $\Delta$  and  $v(\gamma)$  see, e.g., [16, Thm. 5.8.1]. As a consequence of (8) the Delta function

$$\Delta(\tau) := \eta(\tau)^{24}, \quad \tau \in \mathbb{H},$$

is a modular form of weight 12 for  $\mathrm{SL}_2(\mathbb{Z})$ ; i.e.,  $\Delta \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ .

*Example 2.4.* In Subsection 5.1 we will need the normalized Eisenstein series

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n = 1 + 240q + 2160q^2 + \dots, \quad q = e^{2\pi i \tau} \text{ with } \tau \in \mathbb{H}.$$

As shown in [16, Sect. 5.2],  $E_4 \in M_4(\mathrm{SL}_2(\mathbb{Z}))$ .

*Example 2.5.* The most important modular function is the Klein  $j$  function, also called modular invariant,

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots, \quad q = e^{2\pi i \tau} \text{ with } \tau \in \mathbb{H}.$$

Being the quotient of two modular forms in  $M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ , one has that  $j \in M(\Gamma(1))$ . Since  $j$  has no pole in  $\mathbb{H}$ ,  $j \in M^\infty(\Gamma(1))$ ; see [16, Sect. 5.7].

*Example 2.6.* Consider the Jacobi theta functions with  $q = e^{2\pi i \tau}$ ,

$$(9) \quad \theta_2(\tau) = \sum_{n \in \mathbb{Z} + 1/2} q^{n^2/2} = 2 \frac{\eta(2\tau)^2}{\eta(\tau)}, \quad \tau \in \mathbb{H},$$

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<sup>10</sup>I.e., if  $z = re^{i\varphi}$ ,  $r \in \mathbb{R}_{>0}$  and  $-\pi < \varphi \leq \pi$ , then  $z^{1/2} = \sqrt{r}e^{i\varphi/2}$ .

and

$$(10) \quad \theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2/2} = \frac{\eta(\tau)^5}{\eta(\tau/2)^2 \eta(2\tau)^2}, \quad \tau \in \mathbb{H}.$$

One has, as a consequence of (8), that for all  $\gamma \in \Gamma(2, 4, 4)$ ,

$$\theta_2(\gamma\tau)^2 = (c\tau + d) \theta_2(\tau)^2$$

and for all  $\gamma \in \Gamma(2, 4, 2)$ ,

$$\theta_3(\gamma\tau)^2 = (c\tau + d) \theta_3(\tau)^2.$$

In other words,  $\theta_2^2$  and  $\theta_3^2$  are modular forms of weight 1 for  $\Gamma(2, 4, 4) \subseteq \Gamma(2, 4, 2)$ .<sup>11</sup>

### 3. UNIVARIATE HOLONOMIC FUNCTIONS AND SEQUENCES

An introduction to univariate holonomic theory and related algorithms is given in [22].<sup>12</sup> In this section we recall only the most fundamental holonomic notions. To illustrate the main concepts, we apply computer algebra to an example arising in the work of Ramanujan. Further details and proofs of the presented holonomic yoga, can be found in [22]; related theoretical extensions and applications in enumerative combinatorics can be found in [34].

A sequence  $(a(n))_{n \geq 0}$  is holonomic<sup>13</sup>  $:\Leftrightarrow$  there exist polynomials  $p_j(X)$ , not all zero, such that

$$(11) \quad p_d(n)a(n+d) + \cdots + p_0(n)a(n) = 0, \quad n \geq 0.$$

In the given context we restrict to complex sequences; as a consequence, the  $p_j(X)$  are from  $\mathbb{C}[X]$ , the ring of polynomials with coefficients in  $\mathbb{C}$ . The same restriction to  $\mathbb{C}$  applies to functions.

Let  $y(z) = \sum_{n=0}^{\infty} a(n)z^n$  be a function analytic at 0 or, alternatively, a formal power series with complex coefficients:  $y(x)$  is holonomic<sup>14</sup>  $:\Leftrightarrow$  there exist polynomials  $P_j(X) \in \mathbb{C}[X]$ , not all zero, such that

$$(12) \quad P_m(z)y^{(m)}(z) + \cdots + P_0(z)y(z) = 0.$$

The following fact, despite easy to prove, is of fundamental importance.

<sup>11</sup>Recalling definition (3), notice that  $\theta_2(\tau)$  and  $\theta_3(\tau)$  have period  $N_0 = 2$ . — In addition, we note that allowing multiplier systems in the definition of modular forms,  $\theta_2$  and  $\theta_3$  are modular forms of weight 1/2 for a bigger congruence subgroup than  $\Gamma(2, 4, 2)$ ; see, e.g., [16].

<sup>12</sup>[22] is intended as a computer algebra supplement to “Concrete Mathematics” [21].

<sup>13</sup>P-recursive (polynomially recursive) in [34].

<sup>14</sup>D-finite (differentiably finite) in [34].



**Proposition 3.1.** *If  $y(z)$  is a formal power series or a function analytic at 0:*

$$y(z) = \sum_{n=0}^{\infty} a(n)z^n \text{ is holonomic} \Leftrightarrow (a(n))_{n \geq 0} \text{ is holonomic.}$$

We will see many applications below. But first we apply Prop. 3.1 to prove

**Corollary 3.2.** *The sequence of partition numbers  $(p(n))_{n \geq 0}$  is not holonomic.*

*Proof.* Suppose it is holonomic. Then Prop. 3.1 implies that  $y(z) := \sum_{n=0}^{\infty} p(n)z^n$  is holonomic and would satisfy a differential equation as in (12) with  $P_m(z) \neq 0$ , say. As a polynomial,  $P_m(z)$  has only finitely many zeros; thus any analytic solution to (12) can have only finitely many singularities on its circle of convergence. This gives a contradiction to  $y(z) = \prod_{j=1}^{\infty} (1 - z^j)^{-1}$  having infinitely many singularities on the complex unit circle.  $\square$

For the same reason, modular forms and modular functions cannot be holonomic. We restrict to prove an important special case explicitly:

**Corollary 3.3.** *The  $q$ -expansion  $\hat{f}(q)$  of any non-constant function  $f \in M^\infty(\Gamma)$ ,  $\Gamma$  a congruence subgroup, is not holonomic.*

*Proof.* The statement follows from the fact that  $\tilde{f}(z) = \infty$  for infinitely many  $z = e^{2\pi i \tau}$  on the complex unit circle; see Remark 2.1.  $\square$

*Example 3.4.* For fixed  $n \in \mathbb{Z}_{\geq 1}$  consider the power series

$$(13) \quad y(z) := \sum_{k=0}^{\infty} a_n(k)z^k \text{ where } a_n(k) := n \frac{(n + 2k - 1)!}{(n + k)!k!} (-1)^k,$$

which arose in the work of Ramanujan. Before we point to references, we demonstrate how  $y(z)$  can be studied by following the “holonomic paradigm.” To this end, as announced in the Introduction, we use a RISC software package written in the Mathematica system. After putting the package in a directory where we open a Mathematica session, we read it in as follows<sup>15</sup>:

In[1]:= << RISC‘GeneratingFunctions‘

Package GeneratingFunctions version 0.8  
written by Christian Mallinger © RISC-JKU

As a hypergeometric sequence  $(a_n(k))_{k \geq 0}$  is holonomic. This means, the quotient of two consecutive terms is a rational function,

$$\frac{a_n(k + 1)}{a_n(k)} = - \frac{(2k + n)(2k + n + 1)}{(k + 1)(k + n + 1)};$$

<sup>15</sup>The package is freely available at <https://combinatorics.risc.jku.at/software> upon password request via email to the first named author.

in other words, it satisfies recurrence of order 1.

Hence, by Prop. 3.1,  $y(z)$  is holonomic and we use the package to pass from the order 1 recurrence as input, to a differential equation for  $y(z)$ :

```
In[2]:= de = RE2DE[{(k + 1)(k + n + 1)a[k + 1] + (2k + n)(2k + n + 1)a[k] == 0,
                  a[0] == 1}, a[k], y[z]]
Out[2]= {(n + n^2)y[z] + (1 + n + 6z + 4nz)y'[z] + (z + 4z^2)y''[z] = 0, y[0] = 1, y'[0] = -n}
```

We use a command from the Mathematica system to solve it:

```
In[3]:= DSolve[de, y[z], z, Assumptions -> (n ∈ Integers && n > 0)]
DSolve : For some branches of the general solution, the given boundary
conditions lead to an empty solution.
```

From the system's reply we extract that the built-in solver<sup>16</sup> cannot handle a generic positive integer  $n$  in this differential equation, so we ask the system to solve special instances of it:

```
In[4]:= {DSolve[de /. n -> 1, y[z], z], DSolve[de /. n -> 2, y[z], z]}
Out[4]= {{{y[z] -> (-1 + Sqrt[1 + 4z])/2z}}, {{y[z] -> (1 + 2z - Sqrt[1 + 4z])/2z^2}}}
```

We could continue with  $n = 3, 4$ , and so on, to collect more data for guessing a closed form for  $y(z)$ . However, in our context it is more instructive to consider an alternative way to proceed. The output expressions in `Out[4]` already suggest that  $y(x)$  might be an algebraic function (resp. power series); this means, a function (resp. power series) which satisfies a polynomial relation with polynomial coefficients. Connecting to holonomic objects, we have the

**Proposition 3.5.** *Any algebraic function (resp. power series)  $y(z)$  is holonomic. More concretely, if  $P(z, y(z)) = 0$  for a polynomial  $P(X, Y) \in \mathbb{C}[X][Y]$  of degree  $m = \deg_Y P(X, Y)$  in  $Y$ , then  $y(z)$  satisfies a homogeneous differential equation of order  $m$  as in (12), or an inhomogeneous differential equation of order  $m - 1$  of the form*

$$(14) \quad Q_{m-1}(z)y^{(m-1)}(z) + \cdots + Q_0(z)y(z) = Q(z)$$

with polynomials  $Q_j(X), Q(X) \in \mathbb{C}[X]$ .

*Proof.* The proof in [34, Thm. 6.4.6] works the same for our case where  $P(X, Y)$  is not necessarily irreducible.  $\square$

*Example 3.6* (Example 3.4 continued). To obtain a closed form for  $y(z)$ , our strategy is to apply *computer-supported guessing* successively for  $n = 1, 2$ , etc. The procedure to do so, `GuessAE`, is part of the `GeneratingFunctions` package. As input we take the first 15 coefficients  $a_n(0), \dots, a_n(14)$  successively fixing

<sup>16</sup>Using Mathematica V11.3.0.

$n = 1$ ,  $n = 2$ , and  $n = 3$ . As it turns out, these 15 initial values are sufficient to automatically guess an algebraic equation in each instance:

```

In[5]:= aList[n_] := Table[n  $\frac{(n + 2k - 1)!}{(n + k)!k!} (-1)^k$ , {k, 0, 14}];
In[6]:= GuessAE[aList[1], y[z]]
Out[6]= {{-1 + y[z] + zy[z]^2 = 0, y[0] = 1}, ogf}
In[7]:= GuessAE[aList[2], y[z]]
Out[7]= {{1 + (-1 - 2z)y[z] + z^2y[z]^2 = 0, y[0] = 1}, ogf}
In[8]:= GuessAE[aList[3], y[z]]
Out[8]= {{-1 + (1 + 3z)y[z] + z^3y[z]^2 = 0, y[0] = 1}, ogf}17

```

From this data, after solving the quadratic equations and fixing the sign of the solution, the following conjecture becomes evident:

$$(15) \quad y(z) = \left( \frac{2}{1 + \sqrt{4z + 1}} \right)^n, \quad n \geq 1.$$

*Remarks 3.7* (on Example 3.4 and Example 3.6). (i) Using computer-supported holonomic guessing, we derived the closed form representation (15) as a *conjecture*. On the other hand, by holonomic methods we computed (and thus proved!) the differential equation in `Out [2]` which is satisfied by  $y(z)$  for generic  $n \in \mathbb{Z}_{>0}$ .<sup>18</sup> Hence the task to prove (15) is trivialized: just take the differential equation and verify that it is satisfied by the right hand side of (15).

(ii) With the `GeneratingFunctions` package one can automatically obtain from any specific algebraic equation the corresponding differential equation; for example, starting with the relation in `Out [8]` gives,

```

In[9]:= AE2DE[{-1 + (1 + 3z)y[z] + z^3y[z]^2 == 0, y[0] == 1}, y[z]]
Out[9]= 3 - 3(1 + 4z + 2z^2)y[z] - (z + 5z^2 + 4z^3)y'[z] = 0

```

By differentiation we turn this inhomogeneous equation into a homogeneous one:

```

In[10]:= D[3 - 3(1 + 4z + 2z^2)y[z] - (z + 5z^2 + 4z^3)y'[z], z] //Simplify
Out[10]= -(1 + z)(12y[z] + 2(2 + 9z)y'[z] + z(1 + 4z)y''[z])

```

Up to the factor  $-(1 + z)$ , this is the case  $n = 3$  of `Out [2]`.

(iii) Identity (15) stems from the first of Ramanujan's "Quarterly Reports"; it is entry (1.12) in [8]. As explained in [8], this identity arose in the context of Ramanujan's "Master Theorem"; Bruce Berndt presents a wonderful account of this theorem, and also tells the story of Ramanujan's reports.

<sup>17</sup>"ogf" means, one has to take  $y(z)$  as an "ordinary generating function"  $y(z) = \sum a(k)x^k$ , in contrast to an interpretation as an "exponential generating function (egf)."

<sup>18</sup>In fact, this differential equation is satisfied by  $y(z)$  also if  $n$  is a non-zero real number. Thus, defining the  $a_n(k)$  for such  $n$  (using the Gamma function instead of factorials), the identity (15) extends to real  $n$ .

(iv) Consider the first 15 coefficients  $a_1(0), \dots, a_1(14)$  where we fix  $n = 1$ :

In[11]:= `aList[1]`

Out[11]= `{1, -1, 2, -5, 14, -42, 132, -429, 1430, -4862, 16796, -58786, 208012, -742900, 2674440}`

These are the first 15 Catalan numbers  $C_n = \frac{1}{n+1} \binom{2n}{n}$  with alternating signs. Indeed, consulting Sloane's OEIS at <http://oeis.org> (entry A00976, and also entries like A090749) one finds that Ramanujan's numbers for fixed  $n \geq 1$  are forming the  $n$ th diagonals of the table of the celebrated Ballot numbers  $B(\ell, k) := \frac{\ell-k}{\ell+k} \binom{\ell+k}{\ell}$ , where  $\ell > 0$  and  $0 \leq k \leq \ell$ :

$$|a_n(k)| = B(n+k, k), \quad n \geq 1, k \geq 0.$$

The Catalan numbers sit on the main diagonal  $(B(1+k, k))_{k \geq 0}$ .

#### 4. PUISEUX EXPANSIONS

In this section we present the basic mechanism of Puiseux expansion which will be used to connect holonomic functions and sequences with modular functions and forms.

As a major theme, we will study the following setting. Given a Laurent series,

$$g = \sum_{k=M}^{\infty} g(k)x^k, \quad M \in \mathbb{Z} \text{ fixed,}$$

and a power series,

$$h = x^m \underbrace{(1 + h_1x + h_2x^2 + \dots)}_{=: \psi = \psi(x)}, \quad m \in \mathbb{Z}_{\geq 1} \text{ fixed,}$$

find  $(c(k))_{k \geq M}$  such that

$$(16) \quad g = \sum_{k=M}^{\infty} c(k)h^{\frac{k}{m}},$$

where to define  $h^{\frac{k}{m}} := (h^{\frac{1}{m}})^k$  we choose the branch

$$h^{\frac{1}{m}} := x(1 + \psi)^{\frac{1}{m}} := \sum_{n=0}^{\infty} \binom{1/m}{n} \psi^n.$$

Expansions as in (16) are called Puiseux series or fractional series. Such expansions always exist, for formal power series as well as for complex functions  $g$  and  $h$  being meromorphic and holomorphic, respectively, at 0. The coefficients can be computed as follows.

**Lemma 4.1.** *Given  $g$  and  $h$  as above. Then there exists a sequence  $(c(k))_{k \geq M}$  of complex numbers such that*

$$g = \sum_{k=M}^{\infty} c(k)h^{\frac{k}{m}}.$$

Moreover, the  $c(k)$  are uniquely determined by

$$c(k) := \text{coefficient of } x^k \text{ in } \sum_{j=M}^{\infty} g(j)W(x)^j, \quad k \geq M,$$

where  $W$  is the compositorial inverse of  $U(x) := h^{\frac{1}{m}}(x) = x(1 + h_1x + \dots)^{\frac{1}{m}}$ ; i.e.,

$$W(U(x)) = U(W(x)) = x.$$

*Proof.*

$$\begin{aligned} \sum_{k=M}^{\infty} c(k)h^{\frac{k}{m}}(x) &= \sum_{k=M}^{\infty} (h^{\frac{1}{m}}(x))^k \left( \text{coeff. of } z^k \text{ in } \sum_{j=M}^{\infty} g(j)W(z)^j \right) \\ &= \sum_{j=M}^{\infty} g(j)W(h^{\frac{1}{m}}(x))^j = \sum_{j=M}^{\infty} g(j)x^j = g. \end{aligned}$$

□

To illustrate this kind of Puiseux expansion, we take a classical example from Zagier's classical exposition [39].

*Example 4.2.* Recall the modular forms  $\theta_2^2 \in M_1(\Gamma(2, 4, 4))$  and  $\theta_3^2 \in M_1(\Gamma(2, 4, 2))$  from Ex. 2.6. Let  $x := e^{i\pi\tau} = q^{1/2}$ . As in [39, p. 63] choose

$$g = \theta_3(\tau)^2 = 1 + 4x + 4x^2 + \dots \quad \text{and} \quad h = \frac{\theta_2(\tau)^4}{2^4\theta_3(\tau)^4} = x(1 - 8x + 44x^2 - \dots);$$

i.e.,  $m = 1$ . Here  $h(\tau) = \lambda(\tau)/16$  is the normalized modular lambda function. Applying (8) reveals that  $\lambda \in M(\Gamma(2, 4, 2))$ .<sup>19</sup> Using a computer algebra system like Mathematica, according to Lemma 4.1 the coefficients  $c(k)$  can be computed as follows:

```

In[12]:= g = 1 + 4x + 4x^2 + 4x^4 + 8x^5 + O[x]^6;
In[13]:= U = x(1 - 8x + 44x^2 - 192x^3 + 718x^4 + O[x]^5);
In[14]:= W = InverseSeries[U]
Out[14]= x + 8x^2 + 84x^3 + 992x^4 + O[x]^5
In[15]:= ComposeSeries[g, W]

```

<sup>19</sup>The product forms (9) and (10) of  $\theta_2^2$  and  $\theta_3^2$  imply that they are holomorphic in  $\mathbb{H}$ .

Out[15]=  $1 + 4x + 36x^2 + 400x^3 + 4900x^4 + 0[x]^5$

Interpreting the Mathematica output, we obtained

$$\theta_3(\tau)^2 = \sum_{k=0}^{\infty} c(k) \left( \frac{\lambda(\tau)}{16} \right)^k = 1 + 4 \frac{\lambda(\tau)}{16} + 36 \frac{\lambda(\tau)^2}{16^2} + 400 \frac{\lambda(\tau)^3}{16^3} + 4900 \frac{\lambda(\tau)^4}{16^4} + \dots,$$

With the same ease one can compute sufficiently many further values of the  $c(k)$  for additional information. For example, using the first 12 values, we can automatically guess a holonomic recurrence for the  $c(k)$ :

In[16]:= `cList = {1, 4, 36, 400, 4900, 63504, 853776, 11778624, 165636900, 2363904400, 34134779536, 497634306624};`

In[17]:= `GuessRE[cList, c[k]]`

Out[17]= `{{-4(1+2k)^2 c[k] + (1+k)^2 c[1+k] = 0, c[0] = 1}, ogf}`

The output recurrence tells that

$$(17) \quad c(k) = \frac{(1/2)_k (1/2)_k}{k!^2} 16^k = \binom{2k}{k}^2, \quad k \geq 0,$$

where  $(a)_k := a(a+1)\dots(a+k-1)$ . Summarizing, on the level of generating functions we have derived the conjecture that<sup>20</sup>

$$(18) \quad \theta_3(\tau)^2 = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left( \frac{\lambda(\tau)}{16} \right)^k = {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \lambda(\tau) \right).$$

*Remark 4.3.* In view of Prop. 3.1, an alternative way to describe the  $c(k)$  is to derive a holonomic differential equation for  $y(z) := \sum_{k=0}^{\infty} c(k)z^k$ . Using the `GeneratingFunctions` package, this can be done as follows:<sup>21</sup>

In[18]:= `GuessDE[cList, y[z]]`

Out[18]= `{{4y[z] + (-1 + 32z)y'[z] + (-z + 16z^2)y''[z] = 0, y[0] = 1, y'[0] = 4}, ogf}`

Comparing the guessed differential equation to the classical hypergeometric differential equation  $z(z-1)y''(z) + ((a+b+1)z-c)y'(z) + aby(z) = 0$ , which has  $y(z) = {}_2F_1(a, b; c; z)$  as a solution, again produces the generating function identity (18).

Using computer algebra, we derived relation (18) as a *conjecture*. In Section 6 we explain how such identities, once guessed, can be *proved* routinely in computer-assisted fashion. As a concrete example, in Section 6.3 we present a computer-supported proof of the equation in Out [18] which, relating back to  $h = h(\tau)$  and

<sup>20</sup>In [39, (73)] the binomial coefficient  $\binom{2n}{n}$  should be replaced by its square; i.e.,  $\binom{2n}{n}^2$ .

<sup>21</sup>Another option is to transform the recurrence in Out [17] to the desired differential equation by the command `RE2DE[Out [17] [[1]], c[k], y[x]]`.

$g = g(\tau) = y(h(\tau))$ , reads as,

$$(19) \quad (16h^2 - h) \frac{d^2 y(h)}{dh^2} + (32h - 1) \frac{dy(h)}{dh} + 4y(h) = 0 \text{ with } y(z) := \sum_{k=0}^{\infty} c(k)z^k.$$

*Remark 4.4.* Using the chain rule,

$$g'(\tau) = \left. \frac{dy(z)}{dz} \right|_{z=h(\tau)} h'(\tau) \text{ and } g''(\tau) = \left. \frac{d^2 y(z)}{dz^2} \right|_{z=h(\tau)} (h'(\tau))^2 + \left. \frac{dy(z)}{dz} \right|_{z=h(\tau)} h''(\tau),$$

the differential equation (19) translates into the differential equation:

$$\frac{h(-1 + 16h)}{(h')^2} g'' + \frac{hh'' - 16h^2 h'' - (h')^2 + 32h(h')^2}{(h')^3} g' + 4g = 0.$$

Despite being linear in the  $g^{(k)}$ , the coefficients are not modular forms anymore.<sup>22</sup> Nevertheless, as explained in Section 6, one can overcome this problem by a different processing of the holonomic version (19).

## 5. CASE STUDIES: PUISEUX EXPANSIONS OF MODULAR FORMS

In this section we present further examples to illustrate computational aspects.

**5.1. Fricke-Klein relations.** We begin by considering the second classical example given on page 63 of [39] where, taking  $q = e^{2\pi i\tau}$ , Zagier chooses

$$g = E_4(\tau) = 1 + 240q + 2160q^2 + \dots \text{ and } h = \frac{1}{j(\tau)} = q(1 - 744q + \dots).$$

In the setting of Lemma 4.1 this means,  $m = 1$  again. Here  $E_4(\tau) \in M_4(\text{SL}_2(\mathbb{Z}))$  is the normalized Eisenstein series from Ex. 2.4 and  $j \in M^\infty(\text{SL}_2(\mathbb{Z}))$  the modular invariant from Ex. 2.5. Again, we determine  $c(k)$  such that

$$E_4(\tau) = \sum_{k=0}^{\infty} c(k) \left( \frac{1}{j(\tau)} \right)^k.$$

As above, the coefficients can be computed as follows:

```
ln[19]:= g = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + O[q]^5;
ln[20]:= U = q(1 - 744q + 356652q^2 - 140361152q^3 + 49336682190q^4 + O[q]^5);
ln[21]:= W = InverseSeries[U]
Out[21]= q + 744q^2 + 750420q^3 + 872769632q^4 + 1102652742882q^5 + O[q]^6
ln[22]:= ComposeSeries[g, W]
```

<sup>22</sup>As the anonymous referee pointed out, they are quasi-modular forms.

Out[22]=  $1 + 240q + 180720q^2 + 183321600q^3 + 213917180400q^4 + 0[q]^5$

This means, we obtained

$$E_4(\tau) = 1 + 240 \frac{1}{j(\tau)} + 180720 \frac{1}{j(\tau)^2} + 183321600 \frac{1}{j(\tau)^3} + \dots$$

As in our first example, we compute further coefficients to produce a guess on their structure. To this end, we input the coefficients from  $c(0)$  to  $c(20)$  collected in the list `cList`:

In[23]:= `cList = {1, 240, 180720, 183321600, 213917180400, ...};`

In[24]:= `GuessRE[cList, c[k]]`

Out[24]= `{ {36864(1+k)(1+3k)(2+3k)(4+3k)(5+3k)c[k] - 24(3+2k)(4+3k)(5+3k)(21+24k+8k^2)c[1+k] + (2+k)^5 c[2+k] == 0, c[0] == 1, c[1] == 240}, ogf }`

Now, if we would proceed as in Ex. 4.2, we would need the first 56 coefficients  $c(k)$  to guess a differential equation. An algorithmic version of Prop. 3.1 allows us to directly transform the recurrence Out [24] into a differential equation:

In[25]:= `RE2DE[%24[[1]], c[k], f[x]]`

Out[25]= `{240(-1+6144x)f[x] + (1-30000x+81100800x^2)f'[x] + 15(x-9864x^2+17252352x^3)f''[x] + 5(5x^2-29616x^3+38486016x^4)f^{(3)}[x] + 10(x^3-4320x^4+4478976x^5)f^{(4)} + (x^4-3456x^5+2985984x^6)f^{(5)} = 0, f[0] = 1, f'[0] = 240, f''[0] = 361440, f^{(3)}[0] = 1099929600, f^{(3)}[0] = 5134012329600}`

Without giving this differential equation explicitly, Zagier makes the following comment on this example: “Since  $g := E_4$  is a modular form of weight 4, it should satisfy a fifth order linear differential equation with respect to  $h(\tau) := 1/j(\tau)$ , but by the third proof above<sup>23</sup> one should even have that the fourth root of  $E_4$  satisfies a second order differential equation, and indeed one finds

$$(20) \quad \sqrt[4]{E_4(\tau)} = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{12^3}{j(\tau)}\right) = 1 + \frac{1 \cdot 5}{1 \cdot 1} \frac{12}{j(\tau)} + \frac{1 \cdot 5 \cdot 13 \cdot 17}{1 \cdot 1 \cdot 2 \cdot 2} \frac{12^2}{j(\tau)^2} + \dots$$

a classical identity which can be found in the works of Fricke and Klein.”

Needless to say, that, as above, the corresponding differential equation in a computer-assisted way can be derived (as a guess) and proved (as the example Section 6.3) without any effort. Instead of going through this, we do a square root variation of the problem and determine  $c(k)$  such that

$$\sqrt[2]{E_4(\tau)} = \sum_{k=0}^{\infty} c(k) \left(\frac{1}{j(\tau)}\right)^k.$$

Again, the coefficients can be computed as follows:

In[26]:= `g = Series[1 + 240 Sum[DivisorSigma[3, n]q^n, {n, 1, 55}], {q, 0, 55}];`

<sup>23</sup>We come back to Zagier’s proofs in Section 6 in which we discuss proving.



```

In[27]:= gRoot2 = Series[g1/2, {q, 0, 55}];
          A test whether the square root is built correctly:
In[28]:= gRoot22 - g
Out[28]= 0[q]56
In[29]:= Del = Series[q Product[(1 - qn)24, {n, 1, 55}], q, 0, 55];
In[30]:= j = Series[g3/Del, {q, 0, 55}];
In[31]:= h = Series[1/j, {q, 0, 55}];
In[32]:= U = h; W = InverseSeries[U];
In[33]:= comp = ComposeSeries[gRoot2, W];
In[34]:= cRoot2List11 = CoefficientList[Normal[Series[comp, {q, 0, 11}]], q];
In[35]:= cRoot2List11
Out[35]= {1, 120, 83160, 81681600, 93699005400, 117386113965120, 155667030019300800,
          214804163196079142400, 305240072216678400087000, 443655767845074392936328000,
          656486312795713480715743268160, 985646873056680684690542988249600}
In[36]:= GuessRE[cRoot2List11, c[k]]
Out[36]= {{-24(1 + 2k)(1 + 6k)(5 + 6k)c[k] + (1 + k)3c[1 + k] = 0, c[0] = 1}, ogf}

```

The output recurrence `Out [36]` was guessed by using only the first 12 coefficients; it means that the  $c(k)$  form a hypergeometric sequence; more concretely:

$$(21) \quad \sqrt[2]{E_4(\tau)} = {}_3F_2\left(\begin{matrix} \frac{1}{6} & \frac{1}{2} & \frac{5}{6} \\ 1 & 1 \end{matrix}; \frac{12^3}{j(\tau)}\right) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(n!)^3} \frac{1}{j(\tau)^n}.$$

As hypergeometric series, identities (20) and (21) are related by Clausen's formula; for further details, including relevance of (21) to Calabi-Yau varieties and string theory, see [25].

**5.2. Relations connected to irrationality proofs.** The next example is taken from Beukers' modular-form-based proof [14] for the irrationality of  $\zeta(3)$ . Following Beukers, resp. Zagier [39, pp. 63–64], choose

$$g = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^7(1 - q^{3n})^7}{(1 - q^n)^5(1 - q^{6n})^5} = 1 + 5q + 13q^3 + 23q^3 + 29q^4 + 30q^5 + 31q^6 + \dots$$

and

$$h = q \prod_{n=1}^{\infty} \frac{(1 - q^n)^{12}(1 - q^{6n})^{12}}{(1 - q^{2n})^{12}(1 - q^{3n})^{12}} = q(1 - 12q + 66q^2 - 220q^3 + 495q^4 - 804q^5 + \dots);$$

again  $q = e^{2\pi i\tau}$ . Relating to the setting of Lemma 4.1,  $m = 1$ , and we determine  $c(k)$  such that

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^7(1 - q^{3n})^7}{(1 - q^n)^5(1 - q^{6n})^5} = \sum_{k=0}^{\infty} c(k)q^k \prod_{n=1}^{\infty} \frac{(1 - q^n)^{12k}(1 - q^{6n})^{12k}}{(1 - q^{2n})^{12k}(1 - q^{3n})^{12k}}.$$

As above, the coefficients  $c(k)$  can be computed as follows:

$$\text{In[37]}:= \mathbf{g} = \mathbf{q} \prod_{n=1}^{25} \frac{(1 - \mathbf{q}^{2n})^7 (1 - \mathbf{q}^{3n})^7}{1 - \mathbf{q}^n)^5 (1 - \mathbf{q}^{6n})^5}; \quad \mathbf{h} = \mathbf{q} \prod_{n=1}^{25} \frac{(1 - \mathbf{q}^n)^{12} (1 - \mathbf{q}^{6n})^{12}}{1 - \mathbf{q}^{2n})^{12} (1 - \mathbf{q}^{3n})^{12}};$$

$$\text{In[38]}:= \mathbf{U} = \mathbf{h}; \mathbf{W} = \text{InverseSeries}[\mathbf{U}];$$

$$\text{In[39]}:= \text{comp} = \text{ComposeSeries}[\mathbf{g}, \mathbf{W}]; \text{Series}[\text{comp}, \mathbf{q}, \mathbf{0}, \mathbf{6}]$$

$$\text{Out[39]}= 1 + 5\mathbf{q} + 73\mathbf{q}^2 + 1445\mathbf{q}^3 + 33001\mathbf{q}^4 + 819005\mathbf{q}^5 + 21460825\mathbf{q}^6 + 0[\mathbf{q}]^7$$

This means,

$$g = \sum_{k=0}^{\infty} c(k)h^k = 1 + 5h + 73h^2 + 1445h^3 + 33001h^4 + 819005h^5 + \dots$$

To gain further insight into the structure of the  $c(k)$ , we proceed as above:

$$\text{In[40]}:= \text{cList20} = \text{CoefficientList}[\text{Normal}[\text{Series}[\text{comp}, \{\mathbf{q}, \mathbf{0}, \mathbf{20}\}], \mathbf{q}];$$

$$\text{In[41]}:= \text{GuessRE}[\text{cList20}, \text{c}[\mathbf{k}]]$$

$$\text{Out[41]}= \{\{(1+\mathbf{k})^3 \text{c}[\mathbf{k}] - (3+2\mathbf{k})(39+51\mathbf{k}+17\mathbf{k}^2) \text{c}[1+\mathbf{k}] + (2+\mathbf{k})^3 \text{c}[2+\mathbf{k}] == 0, \text{c}[0] == 1, \text{c}[1] == 5\}, \text{ogf}\}$$

This guess, recurrence `Out[41]`, was computed by using only the first 21 coefficients. The sequence  $(c(k))_{k \geq 0}$  can be defined as a definite hypergeometric sum,

$$c(k) = \sum_{\ell=0}^k \binom{k}{\ell}^2 \binom{k+\ell}{\ell}^2, \quad k \geq 0,$$

which satisfies the same recurrence with the same initial conditions, and which can be proven automatically by using implementations of Zeilberger's holonomic systems approach [40]. This sequence is the celebrated Apéry sequence which, as beautifully described in [35], has played a key role in Apéry's proof of  $\zeta(3) \notin \mathbb{Q}$ .

According to Prop. 3.1 the recurrence in `Out[41]` translates into a differential equation for the generating function  $y(x) = \sum_{k \geq 0} c(k)x^k$ . E.g., with the `GeneratingFunctions` package, one computes:

$$\text{In[42]}:= \text{RE2DE}[\%41[[1]], \text{c}[\mathbf{k}], \mathbf{y}[\mathbf{x}]]$$

$$\text{Out[42]}= \{(-5 + \mathbf{x})\mathbf{y}[\mathbf{x}] + (1 - 112\mathbf{x} + 7\mathbf{x}^2)\mathbf{y}'[\mathbf{x}] + 3(\mathbf{x} - 51\mathbf{x}^2 + 2\mathbf{x}^3)\mathbf{y}''[\mathbf{x}] + (\mathbf{x}^2 - 34\mathbf{x}^3 + \mathbf{x}^4)\mathbf{y}^{(3)} = 0, \\ \mathbf{y}[0] = 1, \mathbf{y}'[0] == 5, \mathbf{y}''[0] == 146\}$$

Owing to Prop. 6.1, discussed in the next section, the order 3 of the differential equation is no surprise: Using the modular transformation property of  $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ , one can show that  $g$  is a modular form of weight 2 for  $\Gamma_1(6)$ ; moreover,  $h$  is a Hauptmodul<sup>24</sup> for  $\Gamma_1(6)$ , and  $X(\Gamma_1(6))$  has genus 0. This means, when taking  $\sqrt{g}$  instead of  $g$ , one obtains a differential equation of order 2:

$$\text{In[43]}:= \mathbf{g} = \text{Series}[\mathbf{g}^{1/2}, \mathbf{q}, \mathbf{0}, \mathbf{25}]; \text{Series}[\mathbf{g}, \mathbf{q}, \mathbf{0}, \mathbf{6}]$$

$$\text{Out[43]}= 1 + \frac{5\mathbf{q}}{2} + \frac{27\mathbf{q}^2}{8} + \frac{49\mathbf{q}^3}{16} + \frac{147\mathbf{q}^4}{128} + \frac{459\mathbf{q}^5}{256} + \frac{2511\mathbf{q}^6}{1024} + 0[\mathbf{q}]^7$$

$$\text{In[44]}:= \mathbf{U} = \mathbf{h}; \mathbf{W} = \text{InverseSeries}[\mathbf{U}]; \text{comp} = \text{ComposeSeries}[\mathbf{g}, \mathbf{W}];$$

<sup>24</sup>I.e., each function in  $M(\Gamma_1(6))$  can be represented as a rational function in  $h$ .

```

In[45]:= Series[comp, q, 0, 6]
Out[45]= 1 +  $\frac{5q}{2}$  +  $\frac{267q^2}{8}$  +  $\frac{10225q^3}{16}$  +  $\frac{1836275q^4}{128}$  +  $\frac{90191115q^5}{256}$  +  $\frac{9386644575q^6}{1024}$  + 0[q]^7
In[46]:= cList11 = CoefficientList[Normal[Series[comp, q, 0, 11]], q]; GuessRE[cList11, c[k]]
Out[46]= {{(1 + 2k)^2 c[k] - 2(107 + 170k + 68k^2) c[1 + k] + 4(2 + k)^2 c[2 + k] = 0, c[0] = 1, c[1] = 5/2}, ogf}
In[47]:= RE2DE[%[[1]], c[k], y[x]]
Out[47]= {(-10 + x)y[x] + 4(1 - 51x + 2x^2)y'[x] + 4(x - 34x^2 + x^3)y''[x] = 0, y[0] = 1, y'[0] = 5/2}

```

In [13] Beukers gives further details about how the differential equations in Out [42] and Out [47] are related.

**5.3. Relations for approximations to  $\pi$ .** Relation (21) can also be found as entry (c) of Thm. 5.7 in the famous monograph [15] by Jon and Peter Borwein. This entry is listed in the company of 11 identities [15, Thm. 5.6, Thm. 5.7] of the same flavor. Besides other applications, the Borweins used these identities to derive and explain identities which Ramanujan [33] gave (without too many details) to establish formulas to approximate  $\pi$ , respectively  $1/\pi$ .

The 12 identities listed in [15, Thm. 5.6, Thm. 5.7] can be written in our format where  $g$  is a modular form of some weight and  $h$  is a Hauptmodul. For this translation one only needs the formulas,

$$K(k(\tau)) = \frac{\pi}{2} \theta_3(\tau)^2 \text{ where } k(\tau)^2 = \lambda(\tau),$$

for the complete elliptic integral of the first kind  $K(k(\tau))$  with modulus  $k(\tau)$ . In [15] the 12 identities were discovered (and proved) by using tools from hypergeometric functions like Kummer's identity or Clausen's product formula, and by "piecing together quadratic and cubic transformations given in Erdély et al. [18, Sect. 2.1]."

Knowing that  $\theta_3^2$  is a modular form in  $M_1(\Gamma(2, 4, 2))$  and  $\lambda \in M(\Gamma(2, 4, 2))$ , in our setting the proofs of all these 12 identities become routine and can be carried out with full computer-support. We illustrate this by choosing as a concrete example [15, Thm. 5.6(ai)],

$$(22) \quad \frac{2}{\pi} K = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; 4k^2(1 - k^2)\right).$$

(i) *Computer-supported discovery of (22).* In our setting this relation can be discovered as follows: for

$$h := 4\lambda(1 - \lambda)/64 = x - 24x^2 + \dots \text{ with } x = e^{\pi i \tau},$$

we determine  $c(k)$  such that

$$g := \theta_3(\tau)^2 = \sum_{k=0}^{\infty} c(k)h(\tau)^k.$$

```

In[48]:= g = Series[EllipticTheta[3, 0, x]^2, {x, 0, 9}]
Out[48]= 1 + 4x + 4x^2 + 4x^4 + 8x^5 + 4x^8 + 4x^9 + 0[x]^10
In[49]:= λ = Series[EllipticTheta[2, 0, x]^4 / EllipticTheta[3, 0, x]^4, {x, 0, 9}];
In[50]:= h = Series[1/64 4λ(1 - λ), {x, 0, 8}]
Out[50]= x - 24x^2 + 300x^3 - 2624x^4 + 18126x^5 - 105504x^6 + 538296x^7 - 2471424x^8 + 0[x]^9
In[51]:= U = h; W = InverseSeries[U]; comp = ComposeSeries[g, W];
In[52]:= cList = CoefficientList[Normal[Series[comp, {x, 0, 8}]], x]
Out[52]= {1, 4, 100, 3600, 152100, 7033104, 344622096, 17582760000, 924193822500}
In[53]:= GuessRE[cList, c[k]]
Out[53]= {{-4(1 + 4k)^2 c[k] + (1 + k)^2 c[1 + k] == 0, c[0] == 1}, ogf}

```

This means, the program guesses that

$$c(k) = \frac{(1/4)_k (1/4)_k 4^{3k}}{(1)_k k!}, \quad k \geq 0,$$

which confirms (22).

(ii) *Computer-supported proof of (22)*. The first step is to *guess* a holonomic differential equation satisfied by  $y(z) = \sum_{k=0}^{\infty} c(k)z^k$ . For various purposes, including order estimation of the desired differential equations, modular function/form knowledge is useful. In our case,  $X(\Gamma(2, 4, 2))$  has genus 0. In addition,  $g$  is a modular form of weight 1 not only for  $\Gamma(2, 4, 2)$ , but also for the larger group

$$(23) \quad \Gamma := \Gamma(2, 4, 2) \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma(2, 4, 2);$$

moreover,  $h$  is a Hauptmodul for this group.<sup>25</sup> As a consequence, according to Prop. 6.1 there exists a differential equation of order 2:

```

In[54]:= RE2DE[{-4(1 + 4k)^2 c[k] + (1 + k)^2 c[1 + k] == 0, c[0] == 1}, c[k], y[z]]
Out[54]= {-4y[z] - (-1 + 96z)y'[z] - (-z + 64z^2)y''[z] = 0, y[0] = 1, y'[0] = 4}

```

As a consequence, in order to prove (22) we have to prove that

$$(24) \quad 4y(h) + (-1 + 96h) \frac{dy(h)}{dh} + (-h + 64h^2) \frac{d^2y(h)}{dh^2} = 0 \quad \text{with } y(z) := \sum_{k=0}^{\infty} c(k)z^k,$$

and

$$(25) \quad y(h)|_{h=0} = 1 \quad \text{and} \quad \left( \frac{d}{dh} y \right) (h)|_{h=0} = 4,$$

where, as above,  $g = y(h) = 1 + 4h + 100h^2 + 3600h^3 + 152100h^4 + \dots$ , and  $h = x - 24x^2 + 300x^3 - 2624x^4 + 18126x^5 + \dots$  with  $x = e^{\pi i \tau}$ .

<sup>25</sup>We are grateful to the anonymous referee for pointing to this larger group.

The verification of (25) is immediate from the series expansion of  $y$  in powers of  $h$ . The “computer-proof” of (24) works analogously to that of (19) given in Section 6.3.

## 6. PROVING HOLONOMIC DIFFERENTIAL EQUATIONS FOR MODULAR FORMS

Using computer algebra, we derived relations like (22) as *conjectures*. In this section we explain how such identities, once guessed, can be *proved* routinely in computer-assisted fashion. As pointed out in Section 5.3, such proofs carry out the equivalent task to show that the given modular form  $g$  and the given modular function  $h$  satisfy an associated holonomic differential equation; for example, the differential equation (24) is associated to (22).

**6.1. Holonomic DEs for modular forms.** In several examples we showed how these associated holonomic differential equations can be derived, as a *guess*, in computer-supported fashion. Modular form theory guarantees the existence of such differential equations. Zagier [39, Prop. 21] introduces to this fact as follows: “... it is at the heart of the original discovery of modular forms by Gauss and of the later work of Fricke and Klein and others, and appears in modern literature as the theory of Picard-Fuchs differential equations or of the Gauss-Manin connection — but it is not nearly as well known as it ought to be. Here is a precise statement:”

**Proposition 6.1.** *Let  $g(\tau)$  be a modular form of weight  $k > 0$  and  $h(\tau)$  a modular function, both with respect to the congruence subgroup  $\Gamma$ . Express  $g(\tau)$  locally as  $y(h(\tau))$ . Then the function  $y(h)$  satisfies a linear differential equation of order  $k + 1$  with algebraic coefficients, or with polynomial coefficients if the compact Riemann surface  $X(\Gamma)$  has genus 0 and  $\text{ord}(h(\tau)) = 1$ .<sup>26</sup>*

After stating this proposition, Zagier [39, p. 21] continues: “This proposition is perhaps the single most important source of applications of modular forms in other branches of mathematics, so with no apology we sketch three different proofs, ...”

Zagier’s third proof is constructive; i.e., given  $g$  and  $h$ , it constructs the associated differential equation. Along this line, the following observation is highly relevant for applying holonomic proving strategies:<sup>27</sup>

**Proposition 6.2.** *In the setting of Prop. 6.1, the function  $y(h)$  satisfies a linear differential equation with rational coefficients also when the genus of  $X(\Gamma)$  is non-zero or when  $\text{ord}(h(\tau)) > 1$ . — In these cases, the order of the differential equation in general will be larger than  $k + 1$ .*

<sup>26</sup>In view of Lemma 4.1,  $\text{ord } h = 1$  means  $m = 1$ .

<sup>27</sup>We were not able to find this proposition in the literature.

*Proof.* See the Appendix Section 11.1. □

Another major aspect we want to stress is that, as an alternative to constructing the associated holonomic differential equation, one can follow the “first guess, then prove” strategy. This means, using software like the `GeneratingFunctions` package, one first guesses the holonomic equation, and then proves it by using the following algorithm.

**6.2. An algorithm to prove holonomic DEs for modular forms.** We describe an algorithm, `ModFormDE`, which solves the following problem.

GIVEN a modular form  $g \in M_k(\Gamma)$  with weight  $k \in \mathbb{Z}_{\geq 1}$  for the congruence subgroup  $\Gamma$ , and a modular function  $h \in M(\Gamma)$ ; moreover, suppose  $g$  has a local expansion of the form

$$g(\tau) = y(h(\tau)) \text{ where } y(z) := \sum_{k=0}^{\infty} c(k)z^k.$$

PROVE that  $y(h)$  satisfies a holonomic differential equation of the form

$$(26) \quad P_m(h)y^{(m)}(h) + P_{m-1}(h)y^{(m-1)}(h) + \cdots + P_0(h)y(h) = 0,$$

where the  $P_j(X)$  are given polynomials in  $\mathbb{C}[X]$  with  $P_m(X) \neq 0$ .

Note. Notice that  $y^{(k)}(h) := \frac{dy}{dz}(z)|_{z=h}$ .

Our algorithm `ModFormDE` is based on work of Yifan Yang [37]. So, before describing the steps of `ModFormDE`, we recall notation and notions used in [37].

First, we recall the differential operators used by Yang:

- for functions  $\varphi(\tau)$  defined on  $\mathbb{H}$  having  $q$ -expansions<sup>28</sup>  $\tilde{\varphi}(q) = \sum_{k \geq n_0} a(k)q^k$ ,

$$D_q \varphi = D_q \varphi(\tau) := \frac{1}{2\pi i} \frac{d\varphi}{d\tau}(\tau) = \frac{1}{2\pi i} \varphi'(\tau) = q \tilde{\varphi}'(q);$$

- for functions  $\psi = \psi(z)$  being analytic in  $z$ ,

$$D_h \psi := h \frac{d\psi}{dz}(h).$$

Note. Owing to  $q = e^{2\pi i \tau}$ ,  $D_q = \frac{1}{2\pi i} \frac{d}{d\tau}$ ; also,  $D_h y = h y'(h)$ .

As in [37], define the fundamental functions

$$(27) \quad G_1 := \frac{D_q h}{h} = \frac{1}{2\pi i} \frac{h'}{h} \text{ and } G_2 := \frac{D_q g}{g} = \frac{1}{2\pi i} \frac{g'}{g};$$

---

<sup>28</sup>As in [37], in this general description we assume  $q = e^{2\pi i \tau}$ . But the setting without any difficulties generalizes from  $q$  to  $x = q^{1/N_0} = e^{2\pi i \tau / N_0}$ ; see, e.g., Section 6.3 or [31].

notice that  $h' = h'(\tau) = \frac{dh(\tau)}{d\tau}$  and  $g' = g'(\tau) = \frac{dg(\tau)}{d\tau}$ .

Because of

$$g'(\tau) = \frac{d}{d\tau}y(h(\tau)) = \frac{dy}{dz}(h(\tau))\frac{dh}{d\tau}(\tau) = y'(h(\tau))h'(\tau),$$

we have

$$(28) \quad D_h y(h) = h y'(h) = h \frac{g'}{h'} = g \frac{G_2}{G_1}.$$

Similarly, one has

$$(29) \quad h^2 y''(h) = D_h^2 y - D_h y.$$

Yang [37, p. 9] computed,

$$(30) \quad D_h^2 y = (-p_2) \cdot g + (-p_1) \cdot g \frac{G_2}{G_1} + \left(1 - \frac{1}{k}\right) \cdot g \frac{G_2^2}{G_1^2},$$

where the  $p_j \in M(\Gamma)$  are modular functions<sup>29</sup> defined as

$$p_1 = p_1(h) := \frac{D_q G_1 - 2G_1 G_2/k}{G_1^2} \quad \text{and} \quad p_2 = p_2(h) := -\frac{D_q G_2 - G_2^2/k}{G_1^2}.$$

Yang [37, p. 10] also computed that

$$D_h^3 y = \alpha_3 \cdot g \frac{G_2^3}{G_1^3} + \alpha_2 \cdot g \frac{G_2^2}{G_1^2} + \alpha_1 \cdot g \frac{G_2}{G_1} + \alpha_0 \cdot g,$$

with  $\alpha_j$  being polynomials in  $h, p_1, p_2, p_1',$  and  $p_2'$ ; and this pattern continues. Define the polynomial ring

$$R := \mathbb{C} \left[ h, p_1, p_2, \frac{dp_1}{dh}, \frac{dp_2}{dh}, \dots, \frac{d^m p_1}{dh^m}, \frac{d^m p_2}{dh^m}, \dots \right].$$

Notice that the elements of this ring are modular functions in  $M(\Gamma)$ .

Yang showed that any expression of the form,

$$(31) \quad Y := Q_m(h) D_h^m y + Q_{m-1}(h) D_h^{m-1} y + \dots + Q_0(h) y,$$

with polynomials  $Q_j(X) \in \mathbb{C}[X]$ , can be written into ‘‘Yang form’’ as

$$(32) \quad Y = \alpha_m \cdot g \frac{G_2^m}{G_1^m} + \alpha_{m-1} \cdot g \frac{G_2^{m-1}}{G_1^{m-1}} + \dots + \alpha_0 \cdot g \quad \text{with } \alpha_j \in R.$$

In [31] we show that these  $\alpha_j$  are uniquely determined.

Our algorithm ModFormDE consists of the following steps:

Step 0: Rewrite the left side of (26) into the form (31).

<sup>29</sup>This is proved in [37, Lemma 1]. Hence the  $p_j$  are also algebraic functions in  $h \in M(\Gamma)$ ; see the remark after Thm. 8.1.

Step 1: Transform the resulting expression into Yang form (32).

Step 2: Owing to the uniqueness of the coefficients in (32), the proof of (26) finally is reduced to prove that

$$(33) \quad \alpha_m = 0, \alpha_{m-1} = 0, \dots, \alpha_0 = 0.$$

Since the  $\alpha_j$  are modular functions, this task, owing to (5), reduces to determine upper bounds for the number of poles of each  $\alpha_j$ . Such bounds are given explicitly in [31]; because of  $\alpha_j \in R$ , it is sufficient to provide such bounds for  $h$  and for the  $\frac{d^n p_j}{dh^n}$ ,  $n \geq 0$ . The zero test is completed by expanding the  $q$ -series expansions of the  $\alpha_j$  at infinity<sup>30</sup> to sufficiently high powers of  $q$  such that the number of possible zeros exceeds the corresponding bounds for the poles.

**6.3. The ModFormDE algorithm on an example.** To illustrate the behavior of the ModFormDE algorithm, we choose the task to prove the validity of (19). For a better matching to Yang [37, Ex. 1], we choose  $h := \lambda$ , instead of  $h = \lambda/16$  as in the setting of Ex. 4.2. As a consequence, the differential equation (19) changes into the equivalent form,

$$(34) \quad h(h-1) \frac{d^2 y}{dz^2}(h) + (2h-1) \frac{dy}{dh}(h) + \frac{1}{4} y(h) = 0$$

with  $y(z) := \sum_{k=0}^{\infty} \frac{c(k)}{16^k} z^k$ , and where  $g = y(h)$ .

Step 0: Use (28) and (29) to rewrite the left side of the differential equation in (34) into the form (31):

$$(35) \quad Y := (h-1)D_h^2 y + hD_h y + \frac{h}{4}y.$$

Step 1: By using (28) and (30) we transform (35) into Yang form (32):<sup>31</sup>

$$(36) \quad Y = \alpha_1 \cdot g \frac{G_2}{G_1} + \alpha_0 \cdot g \text{ with } \alpha_1 := -p_1 h + p_1 + h \in M(\Gamma(2, 4, 2)),$$

$$\alpha_0 := -p_2 h + p_2 + \frac{h}{4} \in M(\Gamma(2, 4, 2)).$$

Step 2: The proof of (34) is reduced to zero recognition of the modular functions  $\alpha_1$  and  $\alpha_0$ .

<sup>30</sup>As mentioned, if required we can work also with  $x$ -expansions at infinity where  $x = q^{1/N_0}$  for fixed  $N_0 \in \mathbb{Z}_{\geq 1}$ .

<sup>31</sup>It is important to note that in this application  $q$  is replaced by  $x := q^{1/2}$ ; compare the remark given in the first paragraph of Ex. 4.2.



Step 2a:  $\alpha_1$  is a modular function for  $\Gamma(2, 4, 2)$ . Hence the task to prove  $\alpha_1 = 0$ , owing to (5), reduces to determine upper bounds for the total number poles of  $\alpha_1$ , resp.  $\hat{\alpha}_1$ . Obviously,

$$(37) \quad \begin{aligned} \text{no. of poles}(\alpha_1) &= \text{no. of poles}(p_1(1 - h) + h) \\ &\leq \text{no. of poles}(h) + \text{no. of poles}(p_1) + \text{no. of poles}(h). \end{aligned}$$

To proceed, one needs further information about  $\Gamma(2, 4, 2)$ , resp.  $X(\Gamma(2, 4, 2))$ . For instance, it has three cusps at 0, 1, and  $\infty$ . To obtain such information, one can run systems like Magma; for example:

```
> G:=CongruenceSubgroup([2,4,2]);
> Cusps(G);
[
  oo,
  0,
  1
]
> Widths(G);
[ 2, 2, 2 ]
```

Inspecting the  $x$ -expansions ( $x = e^{\pi i \tau}$ ) at these cusps, one sees that  $h$ , resp.  $\hat{h}$ , has its only pole at 1 with multiplicity 1. To bound the number of poles of  $p_1$ , in view of  $g \in M_1(\Gamma(2, 4, 2))$ , i.e., weight  $k = 1$ , we apply Theorem 5.3 from [31]:

$$\text{no. of poles}(p_1) \leq 8 \text{ no. of poles}(h) + 3 \text{NofPoles}(g^2) + (2k + 4)(g_{\Gamma(2,4,2)} - 1).$$

Here  $g_{\Gamma(2,4,2)} = 0$ , the genus of  $X(\Gamma(2, 4, 2))$ ; in addition, according to the definition given in [31], one has  $\text{NofPoles}(g^2) = 2$ .

*Remark 6.3.* Our definition [31, Def. 6.12] of  $\text{NofPoles}(g^2)$  uses a notion of “order of modular forms” for even weight; see Def. 6.9 in [31]. This order is coinciding with the order of a differential on a compact Riemann surface, although a priori there is no direct connection between these two notions.

Hence we obtain,

$$\text{no. of poles}(p_1) \leq 8 + 3 \cdot 2 + 6(-1) = 8;$$

and, turning back to (37),

$$\text{no. of poles}(\alpha_1) \leq 1 + 8 + 1 = 10.$$

This means, to prove  $\alpha_1 = 0$ , it is sufficient to verify that the first 10 coefficients in the  $x$ -expansion of  $\alpha_1$  at  $\infty$  are zero.<sup>32</sup>

---

<sup>32</sup>Recall that  $x = q^{1/2}$ .

Step 2b: The task to prove  $\alpha_0 = 0$  works analogously to Step 2a. In this case, Theorem 5.3 from [31] gives,

$$\text{no. of poles}(p_2) \leq 22.$$

This means, to prove  $\alpha_0 = 0$ , it is sufficient to verify that the first 24 coefficients in the  $x$ -expansion of  $\alpha_0$  at  $\infty$  are zero.<sup>32</sup>

Note. This algorithmic proving strategy carries over to the general case (26). One can establish explicit formulas for the bounds on the number of poles of  $p_1$  and  $p_2$ , and of the derivatives involved. A full account of all these details is given in [31]. In addition, in a forthcoming version of this paper we present an improved degree estimation. For example, in this setting, for  $p_1$  and  $p_2$  as above,

$$(38) \quad \text{no. of poles}(p_1) \leq 2 \text{ and no. of poles}(p_2) \leq 12.$$

## 7. PUISEUX EXPANSIONS OF MODULAR FUNCTIONS

Given a modular function  $h$ , in the previous sections we discussed the case when its “Puisseux mate”  $g$  is a *modular form* with positive integer weight. From this section on, we consider what happens when the weight of  $g$  is zero; i.e., when both  $g$  and  $h$  are modular functions. Again we will follow a “first guess, then prove” strategy and we shall see that holonomic guessing will work essentially the same. On the other hand, in contrast to the case of modular forms, for modular *functions*  $g$ , proving will be a more elementary routine step, owing to classical theory.

All the modular functions in this section and in the following sections are for congruence subgroups  $\Gamma_0(N)$ ; consequently, the abbreviations

$$M(N) := M(\Gamma_0(N)) \text{ and } M^\infty(N) := M^\infty(\Gamma_0(N))$$

will be convenient.

To illustrate the basic features and the potential for applications in the field of partition congruences, we again consider a concrete example.

Define,

$$f_2(\tau) := \frac{\eta(\tau)\eta(2\tau)^3}{\eta(11\tau)^3\eta(22\tau)} = q^{-2} \prod_{k=1}^{\infty} \frac{(1-q^k)(1-q^{2k})^3}{(1-q^{11k})^3(1-q^{22k})} \in M(22),$$

$$f_3(\tau) := \frac{\eta(\tau)^3\eta(2\tau)}{\eta(11\tau)\eta(22\tau)^3} = q^{-3} \prod_{k=1}^{\infty} \frac{(1-q^k)^3(1-q^{2k})}{(1-q^{11k})(1-q^{22k})^3} \in M(22).$$

Owing to Newman's Lemma [28, Thm. 1.64] it is straightforward to show that these eta-quotients are in  $M(22)$ . In [30] we derived and proved the relation

$$(39) \quad \frac{1}{q^4} \prod_{j=1}^{\infty} \frac{(1-q^j)^{12}}{(1-q^{11j})^{11}} \sum_{n=0}^{\infty} p(11n+6)q^n = 11F_2^2 + 11^2F_3 + 11^3F_2 + 11^4,$$

for the generating function of the partition numbers  $p(11n+6)$  with

$$F_2(\tau) := f_2(\tau) - (U_2f_3)(\tau) = q^{-2} + 2q^{-1} - 12 + 5q + 8q^2 + \dots \in M^\infty(11),$$

$$F_3(\tau) := f_3(\tau) - 4(U_2f_2)(\tau) = q^{-3} - 3q^{-2} - 5q^{-1} + 24 - \dots \in M^\infty(11),$$

where

$$U_2 \sum_{n=N}^{\infty} a(n)q^n := \sum_{2n \geq N} a(2n)q^n.$$

As the main problem in this section, we consider Puiseux expansion of the left side of (39). Concretely, we consider the following task: Given

$$(40) \quad g(\tau) := \frac{1}{q^4} \prod_{j=1}^{\infty} \frac{(1-q^j)^{12}}{(1-q^{11j})^{11}} \sum_{n=0}^{\infty} p(11n+6)q^n \in M^\infty(11)$$

$$= \frac{11}{q^4} + \frac{165}{q^3} + \frac{748}{q^2} + \frac{1639}{q} + 3553 + 4136q + O(q^2)$$

and

$$(41) \quad f(\tau) := F_2(\tau) = f_2(\tau) - (U_2f_3)(\tau) \in M^\infty(11)$$

$$= \frac{1}{q^2} + \frac{2}{q} - 12 + 5q + 8q^2 + O(q^3),$$

determine  $c(k)$  such that

$$(42) \quad g = \sum_{k=-4}^{\infty} c(k) \left(\frac{1}{f}\right)^{\frac{k}{2}} = \sum_{k=-4}^{\infty} c(k) \left(\frac{1}{F_2}\right)^{\frac{k}{2}}.$$

Using the `GeneratingFunctions` package, according to Lemma 4.1 with  $h := (\frac{1}{f})^{1/2}$  and  $m = 2$ , the coefficients  $c(k)$  can be computed as follows:

$$\text{In}[55]:= \mathbf{g} = \frac{11}{\mathbf{q}^4} + \frac{165}{\mathbf{q}^3} + \frac{748}{\mathbf{q}^2} + \frac{1639}{\mathbf{q}} + 3553 + 4136\mathbf{q} + 6347\mathbf{q}^2 + 3586\mathbf{q}^3 + 7414\mathbf{q}^4 + O[\mathbf{q}]^5;$$

The first eight coefficients of  $g$ :

$$\text{In}[56]:= \mathbf{gg}[\mathbf{n}_.] := \mathbf{Coefficient}[\mathbf{Normal}[\mathbf{g}], \mathbf{q}, \mathbf{n}]$$

$$\text{In}[57]:= \mathbf{Table}[\mathbf{gg}[\mathbf{k}], \mathbf{k}, -4, 4]$$

$$\text{Out}[57]= \{11, 165, 748, 1639, 3553, 4136, 6347, 3586, 7414\}$$

$$\text{In}[58]:= \mathbf{f} = \frac{1}{\mathbf{q}^2} + \frac{2}{\mathbf{q}} - 12 + 5\mathbf{q} + 8\mathbf{q}^2 + \mathbf{q}^3 + 7\mathbf{q}^4 - 11\mathbf{q}^5 + 10\mathbf{q}^6 + O[\mathbf{q}]^7;$$

$$\text{In}[59]:= \mathbf{h} = \mathbf{Series}[\mathbf{q} \left(\frac{1}{\mathbf{q}^2 \mathbf{f}}\right)^{1/2}, \{\mathbf{q}, 0, 6\}]$$

```

Out[59]= q - q^2 + 15q^3/2 - 23q^4 + 855q^5/8 - 3481q^6/8 + 0[q]^7
In[60]:= U = h; W = InverseSeries[U]; ComposeSeries[g, W]
Out[60]= 11/q^4 + 121/q^3 + 605/q^2 + 5687/2q + 14641/2 + 84095q/8 + 0[q]^2

```

Alternatively, we can express the answer in powers of  $h$ :

```

In[61]:= Clear[h]
In[62]:= Normal[Sum[gg[k]W^k, {k, -4, 3}]] + O[h]^2 /. q -> h
Out[62]= 11/h^4 + 121/h^3 + 605/h^2 + 5687/2h + 14641/2 + 84095h/8 + 0[h]^2

```

This means, we obtained,

$$g = \sum_{k=-4}^{\infty} c(k) \left(\frac{1}{f}\right)^{\frac{k}{2}} = \sum_{k=-4}^{\infty} c(k) h^k = \frac{11}{h^4} + \frac{121}{h^3} + \frac{605}{h^2} + \frac{5687}{2h} + \frac{14641}{2} + \dots$$

Using additional terms in the  $q$ -expansions of  $f$  and  $g$ , one computes in the same fashion that for  $h := (\frac{1}{f})^{1/2}$ :

$$(43) \quad h^4 g = 11 + 11^2 h + 11^2 \cdot 5 h^2 + \frac{11^2 \cdot 47}{2} h^3 + \frac{11^4}{2} h^4 + \frac{11^2 \cdot 5 \cdot 139}{8} h^5 \\ - \frac{11^2 \cdot 17 \cdot 199}{16} h^7 + \frac{11^2 \cdot 3 \cdot 50993}{128} h^9 - \frac{11^2 \cdot 67 \cdot 107 \cdot 347}{256} h^{11} + \dots$$

We note explicitly that all the coefficients are quotients with integers as numerators, and denominators being powers of 2. This is owing to

$$(1 + \psi)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{1/2}{n} \psi^n \quad \text{and} \quad \binom{1/2}{n+1} = \frac{1}{2} \frac{(-1)^n}{4^n} \frac{1}{n+1} \binom{2n}{n},$$

together with the following lemma for formal power series which is folklore.

**Lemma 7.1.** *Suppose  $\psi = 1 + f_1 x + f_2 x^2 + \dots \in \mathbb{Z}[[x]]$ . Then*

$$\frac{1}{\psi} = 1 - f_1 x + (f_1^2 - f_2) x^2 + \dots \in \mathbb{Z}[[x]]$$

and

$$\phi = x - f_1 x^2 + (2f_1^2 - f_2) x^3 + \dots \in \mathbb{Z}[[x]],$$

where  $\phi$  is the compositorial inverse of  $x\psi$ ; i.e.,  $\phi(x\psi(x)) = \phi(x)\psi(\phi(x)) = x$ .

In the next step we take the first 24 coefficients in the expansion (43) of  $y(h) := h^4 g$  and use the `GeneratingFunctions` package to guess an algebraic equation<sup>33</sup>:

```

In[63]:= cList = {11, 121, 605, 5687/2, 14641/2, 84095/8, 0, -(409343/16), 0, 18510459/128,
                0, -(301004803/256), 0, 12660067739/1024, 0, -(323203962851/2048),

```

<sup>33</sup>The addition of 2 and 6 in the procedure call `In[64]` asks the program to guess an algebraic relation of degree 2 with polynomial coefficients of degree maximally 6.

**0, 75351596221803/32768, 0, -(2379607141538539/65536), 0,  
157195485892282497/262144, 0, -(5317572127702768593/524288)}**

In[64]:= **Clear[h]; GuessAE[cList, y[h], 2, 6]**

Out[64]= **{{y[h]<sup>2</sup> - 11(1331h<sup>4</sup> + 110h<sup>2</sup> + 2)y[h] - 121(14641h<sup>6</sup> + 1331h<sup>4</sup> + 11h<sup>2</sup> - 1) = 0,  
y[0] = 11}, ogf}**

With  $y(h) := h^4 g = \frac{1}{f^2} g$ , the algebraic relation in `Out[64]` translates into an algebraic relation involving  $f$  and  $g$  which, owing to its importance, we state as a separate proposition.

**Proposition 7.2.** *The modular functions  $f$  and  $g$  in  $M^\infty(11)$  as given in (41) and (40) satisfy*

$$(44) \quad g^2 - 11(11^3 + 11 \cdot 10f + 2f^2)g - 11^2(11^4 f + 11^3 f^2 + 11f^3 - f^4) = 0.$$

*Proof.* Knowing from [30] that  $f, g \in M^\infty(11)$ , according to Remark 2.2, to prove (44) it suffices to show that the principal part and the constant term in the  $q$ -expansion of the left side of (44) are equal to zero.  $\square$

*Remark 7.3.* To verify this by actual computation, it turns out that the precision we entered for  $f$  and  $g$  in our Mathematica session suffices:

In[65]:= **Y<sup>2</sup> - 11(11<sup>3</sup> + 110X + 2X<sup>2</sup>)Y - 11<sup>2</sup>(11<sup>4</sup>X + 11<sup>3</sup>X<sup>2</sup> + 11X<sup>3</sup> - X<sup>4</sup>) /.{Y -> g, X -> f}**  
 Out[65]= **0[q]<sup>1</sup>**

## 8. EXISTENCE OF ALGEBRAIC RELATIONS

Holonomic guessing can be used to set up algebraic relations like (44). As explained in Section 7, *proving* such relations between two modular *functions* in  $M^\infty(N)$  amounts to straightforward computational verification. In this section we discuss why such algebraic relations exist, state some facts of computational relevance, and connect to Puiseux's theorem for solving algebraic equations.

In [29, Thm. 7.1] we proved the following theorem:

**Theorem 8.1.** *Given a compact Riemann surface  $S$  together with non-constant functions  $f$  and  $g$  being meromorphic on  $S$ . Suppose  $f$  has  $m$  poles, counted with their multiplicity. If for any  $p \in S$  the condition,*

$$(45) \quad p \text{ a pole of } g \Rightarrow p \text{ a pole of } f,$$

*holds, then there exist POLYNOMIALS  $c_1, \dots, c_m \in \mathbb{C}[X]$  such that*

$$(46) \quad g^m + c_1(f)g^{m-1} + \dots + c_m(f) = 0.$$

Without the pole condition (45), this theorem holds with the  $c_j$  being *rational functions* in  $\mathbb{C}(X)$ . In this version the statement is folklore; see for instance [20, §8.2 and §8.3]. Our version, where condition (45) forces the  $c_j$  to be *polynomials*, is algorithmically relevant, but it seems to be less known. A special instance of Theorem 8.1 where  $f$  is allowed to have poles only at one point of  $S$  is used in [38].

In the classical context, i.e., without the pole condition (45), another useful result is proved in [38, Lemma 1]:

**Lemma 8.2.** *Let  $f$  and  $g$  be as in Theorem 8.1. Suppose  $f$  and  $g$  have  $m$  and  $M$  poles, respectively, counted with their multiplicity. If  $\gcd(m, M) = 1$  then there exists a polynomial,*

$$(47) \quad c(X, Y) = Y^m + c_1(X)Y^{m-1} + \cdots + c_m(X) \in \mathbb{C}(X)[Y],$$

*which is irreducible over  $\mathbb{C}(X)$ , and where the  $c_j(X) \in \mathbb{C}(X)$  are rational functions such that  $c(f, g) = 0$ .*

Now we are ready to state and prove a general theorem which explains the existence of algebraic relations such as (44).

**Theorem 8.3.** *Let  $f$  and  $g$  be modular functions from  $M^\infty(N)$  with  $\text{ord } f = -m \leq -1$  and  $\text{ord } g = -M \leq -1$ . Then there exist polynomials  $c_j(X) \in \mathbb{C}[X]$  such that*

$$g^m + c_1(f)g^{m-1} + \cdots + c_m(f) = 0.$$

*Moreover, if*

$$(48) \quad c(X, Y) := Y^m + c_1(X)Y^{m-1} + \cdots + c_m(X) \in \mathbb{C}[X][Y] = \mathbb{C}[X, Y]$$

*is irreducible with  $\deg c_m(X) = M$ , then*

$$(49) \quad \deg c_j(X) \leq M - 1, \quad j = 1, \dots, m - 1.$$

*The condition,*

$$(50) \quad \gcd(m, M) = 1,$$

*is sufficient, but not necessary, to imply that  $c(X, Y)$  is irreducible with degree  $\deg c_m(X) = M$ .*

*Proof.* The existence of  $c(X, Y) \in \mathbb{C}[X, Y]$  such that  $c(f, g) = 0$  is implied by Theorem 8.1 with  $S = \mathbb{C} \cup \{\infty\}$ , the Riemann sphere. Notice that both  $f$  and  $g$  have their only pole at  $\infty$  with multiplicities  $m$  and  $M$ , respectively.

To prove the second part of the theorem, consider the polynomial

$$d(X, Y) := X^M + d_1(Y)X^{M-1} + \cdots + d_M(Y) \in \mathbb{C}[Y][X] = \mathbb{C}[X, Y]$$

with  $d(f, g) = 0$ ;  $d(X, Y)$  exists in this form by applying the same argument as for  $c(X, Y)$ . The polynomials  $c(X, Y), d(X, Y) \in \mathbb{C}[X, Y]$  cannot be relatively

prime. Suppose they are, then owing to  $c(f, g) = 0 = d(f, g)$  there are infinitely many common roots which contradicts the fact that the set of common roots of two bivariate polynomials has to be finite; see, e.g., [1, Exercise 1.3.8]. As a consequence, the irreducible  $c(X, Y)$  has to be a factor of  $d(X, Y)$ ; i.e.,

$$d(X, Y) = (a_0(Y)X^L + a_1(Y)X^{L-1} + \dots + a_L(Y))c(X, Y)$$

for some polynomials  $a_j \in \mathbb{C}[Y]$ . Now the assumption  $\deg c_m(X) = M$  implies  $L = 0$  and  $a_0(Y) = a_0 \in \mathbb{C}$ , and comparing the coefficients of  $X^i Y^j$  of both sides of  $d(X, Y) = a_0 \cdot c(X, Y)$  proves (49).

The third part of the theorem is an immediate consequence of Lemma 8.2 which implies that both  $c(X, Y)$  and  $d(X, Y)$  are irreducible; hence, as in part two,

$$d(X, Y) = a_0 \cdot c(X, Y) \text{ for some } a_0 \in \mathbb{C}.$$

Applying coefficient comparison gives  $\deg c_m(X) = M$ .

Finally, in Example 8.4 we show that the condition (50) is not necessary for  $c(X, Y)$  to be irreducible with  $\deg c_m(X) = M$ .  $\square$

*Example 8.4.* Consider the relation (44) with modular functions  $f, g \in M^\infty(11)$  as in (41) and (40), where the orders  $\text{ord } f = -2$  and  $\text{ord } g = -4$  are not relatively prime. Nevertheless, using computer algebra one can check that the corresponding polynomial

$$\begin{aligned} c(X, Y) &= Y^2 + c_1(X)Y + c_2(X) \\ &= Y^2 - 11(11^3 + 11 \cdot 10X + 2X^2)Y - 11^2(11^4X + 11^3X^2 + 11X^3 - X^4), \end{aligned}$$

such that  $c(f, g) = 0$  as in (44), is irreducible in  $\mathbb{C}[X, Y]$  with  $\deg c_2(X) = 4 = M$ .

Recall that, in view of the Laurent series

$$(51) \quad g = \sum_{k=-4}^{\infty} c(k) \left(\frac{1}{f}\right)^{\frac{k}{2}} = \sum_{k=-4}^{\infty} c(k)h^k \text{ with } h := \left(\frac{1}{f}\right)^{1/2},$$

we set  $y(h) := h^4g$  to obtain a power series expansion

$$y(h) = \sum_{k=0}^{\infty} c(k-4)h^k = 11 + 11^2h + 11^2 \cdot 5h^2 + \frac{11^2 \cdot 47}{2}h^3 + \dots,$$

where the coefficients  $c(k)$  form a holonomic sequence with coefficients as in (43). Using the `GeneratingFunctions` package, we algorithmically *guessed* the algebraic relation `Out [64]`,

$$(52) \quad y(h)^2 - 11(11^3h^4 + 11 \cdot 10h^2 + 2)y(h) - 11^2(11^4h^6 + 11^3h^4 + 11h^2 - 1) = 0.$$

Noting that  $y(h) := h^4g$  with  $h := (\frac{1}{f})^{1/2}$ , this relation is equivalent to (44),

$$g^2 - 11(11^3 + 11 \cdot 10f + 2f^2)g - 11^2(11^4f + 11^3f^2 + 11f^3 - f^4) = 0.$$

We want to stress that this equivalence is no surprise in view of ‘‘Puiseux’s theorem’’; see, e.g., [34, Thm. 6.1.5, Prop. 6.1.6, Cor. 6.1.7].

**Theorem 8.5** (Puiseux’s Theorem). *Let  $P(X, Y) \in \mathbb{C}[X][Y]$  be an irreducible polynomial in  $Y$  with  $\deg_Y P(X, Y) = m$ . Then there exist positive integers  $d_1, \dots, d_r$  such that  $d_1 + \dots + d_r = m$ , and for each  $j \in \{1, \dots, r\}$  there exist Laurent series*

$$y_{j,\ell}(X) = \sum_{k=M_j}^{\infty} a_{j,\ell}(k) e^{\frac{2\pi i k \ell}{d_j}} X^k \quad \text{with } a_{j,\ell}(k) \in \mathbb{C} \text{ such that } P(X^{d_j}, y_{j,\ell}(X)) = 0,$$

for  $\ell \in \{0, \dots, d_j - 1\}$ .

After the substitution  $X \rightarrow X^{1/d_j}$ , the Laurent series  $y_{j,\ell}(X)$  in the theorem become objects of the form  $y_{j,\ell}(X^{1/d_j}) = \sum_{k=M_j}^{\infty} \tilde{a}_{j,\ell}(k) X^{k/d_j}$ ; i.e., Puiseux series (also called fractional series) which satisfy  $P(X, y_{j,\ell}(X^{1/d_j})) = 0$ .

*Remark 8.6.* The classic version of ‘‘Puiseux’s theorem’’ traces back to Newton. For proofs in the setting of complex analysis see, for example, [19, Ch. 7] or [24, Ch. 6.3]. For computational aspects see, for instance, [22, Thm. 6.2] or [36, Ch. 4.3].

Also in our context Puiseux’s theorem can be useful for computational reasons: it implies lower bounds supplementing the general degree bound given in Thm. 8.3(49). We restrict to state a special case:

**Corollary 8.7.** *Let  $P(X, Y)$  be as in Puiseux’s Theorem 8.5 with  $r = 1$  (i.e.,  $d_1 = m$ ) and with coefficient polynomials  $P_n(X) \in \mathbb{C}[X]$  such that*

$$P(X, Y) = P_m(X)Y^m + P_{m-1}(X)Y^{m-1} + \dots + P_0(X).$$

*Suppose  $d = \deg P_m(X)$ , then for  $\ell = 1, \dots, m$  and  $M_1$  as in Theorem 8.5:*

$$(53) \quad \text{lowest power of } X \text{ in } P_{m-\ell}(X) \geq \max\left\{d + \ell \frac{M_1}{m}, 0\right\}.$$

*Proof.* The statement follows directly by comparing coefficients of  $Y^k$  in

$$\begin{aligned} P(X, Y) &= P_m(X) \left( Y - y_1(X^{1/m}) \right) \dots \left( Y - y_m(X^{1/m}) \right) \\ &= P_m(X) Y^m - P_m(X) \left( y_1(X^{1/m}) + \dots + y_m(X^{1/m}) \right) Y^{m-1} + \dots \\ &\quad + (-1)^m P_m(X) y_1(X^{1/m}) \dots y_m(X^{1/m}), \end{aligned}$$

where the  $y_\ell(X)$  are the  $y_{1,\ell}(X)$  in Theorem 8.5. To this end, notice that the fractional series formed by the elementary symmetric functions have to be polynomials.  $\square$



*Example 8.8.* Consider

$$(54) \quad P(X, Y) := X^4 Y^2 - 11X^2(2 + 11 \cdot 10X + 11^3 X^2)Y \\ + 11^2(1 - 11X - 11^3 X^2 - 11^4 X^3);$$

i.e.,  $P(1/f, g) = 0$  with  $f$  as in (41) and  $g$  as in (40) with expansion (42). In other words, the relation  $P(1/f, g) = 0$  is nothing but (44), produced from the equivalent relation `Out [64]` by setting  $y(h) := h^4 g$  with  $h^2 = 1/f$ . In the light of of Puiseux's Thm. 8.5 and (51), this means,

$$P(X, Y) = X^4 \left( Y - y_1(X^{1/2}) \right) \left( Y - y_2(X^{1/2}) \right) \\ = X^4 Y^2 + P_1(X)Y + P_0(X),$$

with

$$y_1(X^{1/2}) := \sum_{k=-4}^{\infty} c(k)X^{k/2} \quad \text{and} \quad y_2(X^{1/2}) := \sum_{k=-4}^{\infty} (-1)^k c(k)X^{k/2}.$$

The coefficient polynomials  $P_0(X)$  and  $P_1(X)$  can be recovered from truncated versions  $y_j$  of the  $y_j(X^{1/2})$  as follows. For instance, set

$$y_1 := \frac{11}{X^2} + \frac{121}{X^{3/2}} + \frac{605}{X} + \frac{5687}{2X^{1/2}} + \frac{14641}{2} + \frac{84095X^{1/2}}{8} - \frac{409343X^{3/2}}{16} + \frac{18510459X^{5/2}}{128}$$

and

$$y_2 := \frac{11}{X^2} - \frac{121}{X^{3/2}} + \frac{605}{X} - \frac{5687}{2X^{1/2}} + \frac{14641}{2} - \frac{84095X^{1/2}}{8} + \frac{409343X^{3/2}}{16} - \frac{18510459X^{5/2}}{128}.$$

Then

$$y_1 + y_2 = \frac{11}{X^2}(2 + 110X + 1331X^2)$$

and

$$y_1 y_2 = \frac{11^2}{X^4}(1 - 11X - 1331X^2 - 14641X^3) \\ - \frac{36421581163X^2}{128} - \frac{1891760432903X^3}{512} + \dots$$

Finally, in view of Prop. 8.7, we note that with  $P_2(X) := X^4$ :

$$d = \deg P_2(X) = 4,$$

and one has for the lowest powers of  $X$  in  $P_1(X)$ , resp.  $P_0(X)$ :

$$2 = \text{lowest power of } X \text{ in } P_1(X) \geq 4 + \frac{-4}{2} = 2$$

and

$$0 = \text{lowest power of } X \text{ in } P_0(X) \geq 4 - 4 = 0;$$

this confirms the estimates given by (53).

## 9. HOLONOMIC DIFFERENTIAL EQUATIONS FOR MODULAR FUNCTIONS

In this section we continue to study the situation where  $g$  is a modular *function* as  $h$ . But, in contrast to algebraic relations as considered in Section 8, we now put our focus on *differential equations*. Again we see that the underlying mathematics is somewhat simpler than that for modular *forms*  $g$ . Nevertheless, applications of this setting still seem to be of some interest. In this section and in Section 10 we illustrate this aspect by examples related to congruences for partition numbers.

Suppose we are interested in  $\sum_{n=0}^{\infty} p(11n+6)q^n$  with coefficients being the numbers of partitions of  $11n+6$ . Using Smoot's implementation [?] of Radu's Ramanujan-Kolberg algorithm, we compute a  $q$ -product,  $q = e^{2\pi i\tau}$ , such that when multiplied to this generating function,

$$(55) \quad g(\tau) := \frac{1}{q^4} \prod_{j=1}^{\infty} \frac{(1-q^j)^{12}}{(1-q^{11j})^{11}} \sum_{n=0}^{\infty} p(11n+6)q^n \in M^{\infty}(11).$$

Given  $f = \frac{1}{q^2} + \frac{2}{q} - 12 + 5q + 8q^2 + \dots \in M^{\infty}(11)$  as in (41), in Section 7 we considered the task to determine  $c(k)$  such that

$$g = \sum_{k=-4}^{\infty} c(k) \left(\frac{1}{f}\right)^{\frac{k}{2}}.$$

Using holonomic guessing and modular function proving, we derived and proved the algebraic relation (44) between  $f$  and  $g$ ; for convenience we display (44) again:

$$(44) \quad g^2 = 11(11^3 + 11 \cdot 10f + 2f^2)g + 11^2(11^4f + 11^3f^2 + 11f^3 - f^4).$$

Connecting this fact with the  $q$ -expansion (55), one obtains a proof of a classical congruence of Ramanujan [32]:

**Theorem 9.1.** *For non-negative integer  $n$ ,*

$$(56) \quad p(11n+6) \equiv 0 \pmod{11}.$$

*Proof.* By (44) each coefficient of  $g^2$  has 11 as a factor. Hence 11 divides each coefficient of  $g$ , and thus  $11 \mid p(11n+6)$ .  $\square$

For computational purposes it is convenient to set  $y(h) := h^4g$  with  $h := (\frac{1}{f})^{1/2}$ , and to rewrite (44) into the equivalent form (52), which we repeat to display for the reader's convenience:

$$(52) \quad y(h)^2 - 11(11^3h^4 + 11 \cdot 10h^2 + 2)y(h) - 11^2(11^4h^6 + 11^3h^4 + 11h^2 - 1) = 0.$$

This means,

$$(57) \quad y(h) = \sum_{k=0}^{\infty} c(k-4)h^k = 11 + 11^2h + 11^2 \cdot 5h^2 + \frac{11^2 \cdot 47}{2}h^3 + \dots$$

is an algebraic power series and, according to Prop. 3.5, the coefficients  $c(k)$  constitute a holonomic sequence.

After these preliminary remarks we finally turn to the aspect of *differential equations*. To this end, using the `GeneratingFunctions` package, we transform (52) into a holonomic differential equation:

```
In[66]:= AE2DE[y[h]^2 - 11(11^3h^4 + 110h^2 + 2)y[h] - 121(11^4h^6 + 11^3h^4 + 11h^2 - 1) == 0, y[h]]
Out[66]= -44 - 1716h^2 - 7986h^4 + 351384h^6 + 3543122h^8 + 4(1 + 94h^2 + 2178h^4 + 14641h^6)y[h]
          - (4h + 188h^3 + 2904h^5 + 14641h^7)y'[h] = 0
```

Notice that the differential equation `Out [66]` is valid for  $y(h)$  as in (57), where  $h$  can be an indeterminate or a complex number sufficiently close to 0, depending on a formal power series or analytic function context.

We also note that, according to Prop. 3.5, the existence of `Out [66]` in the form of an *inhomogeneous* differential equation of order 1 is no surprise, owing to the degree 2 of the algebraic relation.

In order to learn more about the  $c(k)$ , we set for convenience,

$$(58) \quad d(k) := c(k - 4), \quad k \geq 0,$$

and use the `GeneratingFunctions` package to transform (52), via the differential equation `Out [66]`, into a recurrence:

```
In[67]:= DE2RE[%66, y[h], d[k]]
Out[67]= {-2357947691(-4 + k)(-2 + k) d[k] - 701538156(-2 + k)k d[2 + k] - 161051(-176 + 666k +
          443k^2) d[4 + k] - 2662(-9104 - 1218k + 545k^2) d[6 + k] + 242(60032 + 17226k + 1075k^2) d[8 + k] +
          12(151726 + 33975k + 1827k^2) d[10 + k] + 8(10 + k)(1087 + 86k) d[12 + k] + 8(13 + k)(14 + k) d[14 + k] =
          0, d[0] = 11, d[2] = 605, d[3] = (47d[1])/2, d[4] = 14641/2, d[5] = (695d[1])/8, d[6] = 0, d[7] =
          -((3383d[1])/16), d[8] = 0, d[9] = (152979d[1])/128, d[10] = 0, d[11] = -((2487643d[1])/256),
          d[12] = 0}
```

The output also contains initial conditions; one observes that  $c[1]$  can be freely chosen and the recurrence is still valid. In order to match the initial conditions with the corresponding values of  $c(k)$ , respectively with  $y(0) = c(-4) = 11$  and  $y'(0) = 11^2 = c(-3)$ , we choose  $d[1] := c(-3) = 11^2$ .

An alternative way to produce a recurrence for the  $d(k)$  is by computer-supported *guessing*. Again we use the `GeneratingFunctions` package; as input we take the first 35 coefficients  $d(0) := c(-4), \dots, d(34) := c(30)$ :

```
In[68]:= dList35 = {11, 121, 605, 5687/2, 14641/2, 84095/8, 0, -409343/16, 0, 18510459/128,
          0, -301004803/256, 0, 12660067739/1024, 0, -323203962851/2048, 0,
          75351596221803/32768, 0, -2379607141538539/65536, 0,
          157195485892282497/262144, 0, -5317572127702768593/524288, 0,
          728392438513303248223/4194304, 0, -25092691314419458322263/8388608, 0,
```

1733298832317545489307443/33554432, 0,  
 – 59906702629110941247099115/67108864, 0,  
 33116914468409338915089124427/2147483648, 0}

In[69]:= **GuessRE**[cList35, d[k], 6, 3]

Out[69]= {{14641(-4 + k)(-2 + k)k d[k] + 2904(-2 + k)(-1 + k)k d[2 + k] +  
 188(-2 + k)k(2 + k) d[4 + k] + 4(-2 + k)k(5 + k) d[6 + k] = 0,  
 d[0] = 11, d[1] = 121, d[2] = 605, d[3] = 5687/2, d[4] = 14641/2,  
 d[5] = 84095/8, d[6] = 0, d[7] = -(409343/16), d[8] = 0}, ogf}

With the procedure call in In[69] we asked the program to guess a recurrence of order 6 with polynomial coefficients of degree maximally 3; the output Out[69] shows that such a recurrence indeed exists.

The following remark is elementary, but important for practical computations: As we have seen in Out[69], when using the *inhomogeneous* differential equation Out[66], one obtains a recurrence which overshoots the order significantly, 14 instead of 6. The explanation for this increase of the order is caused by the presence of the powers  $h^2$  up to  $h^8$  in the inhomogeneous polynomial part. Using as input the *homogeneous* version of Out[66], one also obtains a valid recurrence which is produced by comparing coefficients of  $h^k$ , and this comparison is correct from  $k = 9$  on:

In[70]:= **DE2RE**[-4(1+94x<sup>2</sup>+2178x<sup>4</sup>+14641x<sup>6</sup>)y[x]+(4x+188x<sup>3</sup>+2904x<sup>5</sup>+14641x<sup>7</sup>)y'[x] ==  
 0, y[x], d[k]]

Out[70]= 14641(-4 + k)d[k] + 2904(-1 + k)d[2 + k] + 188(2 + k)d[4 + k] + 4(5 + k)d[6 + k] = 0

This is, up to a common multiplicative factor, the guessed recurrence Out[66]; but this time we *proved* this relation by an algorithmic version of the fundamental holonomic conversion, Prop. 3.1.

To summarize, we rewrite Out[70] as

$$d(k) = \frac{-1}{4(k-1)} \left( 11^4(k-10)d(k-6) + 2^3 \cdot 3 \cdot 11^2(k-7)d(k-4) + 2^2 \cdot 47(k-4)d(k-2) \right),$$

and apply the relabeling (58):

**Proposition 9.2.** *Given  $f$  as in (41) and  $g$  as in (55). Then the uniquely determined coefficients  $c(k)$  such that*

$$(59) \quad g = \sum_{k=-4}^{\infty} c(k) \left( \frac{1}{f} \right)^{\frac{k}{2}},$$

for  $k \geq 5$  satisfy the recurrence

$$(60) \quad c(k) = \frac{-1}{4(k+3)} \left( 11^4(k-6)c(k-6) + 2^3 \cdot 3 \cdot 11^2(k-3)c(k-4) + 2^2 \cdot 47k c(k-2) \right)$$

with initial values

$$c(-4) = 11, c(-3) = 121, c(-2) = 605, c(-1) = 5687/2, c(0) = 14641/2, \\ c(1) = 84095/8, c(2) = 0, c(3) = -(409343/16), c(4) = 0.$$

Proposition 6.1 is fundamental when working with modular forms  $g$  of positive weight.<sup>34</sup> We conclude the present section with a version for the case when the weight is zero; i.e., when  $g$  is a modular function.

**Proposition 9.3.** *Let  $f$  and  $g$  be modular functions from  $M^\infty(N)$  with  $\text{ord } f = -m \leq -1$ ,  $\text{ord } g = -M \leq -1$ , and  $m \leq M$ . For  $h := (\frac{1}{f})^{1/m}$  express  $g(\tau)$  locally as  $y(h(\tau))$ ; i.e.,*

$$(61) \quad g = y(h) = \sum_{k=-M}^{\infty} c(k)h^k.$$

Then  $y(h)$  satisfies a homogeneous holonomic differential equation of order  $m$ ,

$$P_m(h) \left( \frac{dy}{dh} \right)^m + \dots + P_0(h)y = 0,$$

or an inhomogeneous differential equation of order  $m - 1$  of the form

$$(62) \quad Q_{m-1}(h) \left( \frac{dy}{dh} \right)^{m-1} + \dots + Q_0(h)y = Q(h),$$

where the  $P_i(X)$ ,  $Q_j(X)$ , and  $Q(X)$  are polynomials in  $\mathbb{C}[X]$ .

*Proof.* By Thm. 8.3 there exists a polynomial

$$P(X, Y) := Y^m + c_1(X)Y^{m-1} + \dots + c_m(X) \in \mathbb{C}[X][Y] = \mathbb{C}[X, Y]$$

such that  $P(f, g) = 0$ . Equivalently we have,

$$0 = P(1/h^m, g) = P(1/h^m, y(h)) = y(h)^m + c_1(1/h^m)y(h)^{m-1} + \dots + c_m(1/h^m).$$

This means,  $y(h)$  is an algebraic function satisfying a relation  $0 = Q(h, y(h)) := P(1/h^m, y(h))$  where  $Q(X, Y) \in \mathbb{C}(X)[Y]$  is a polynomial with degree  $m = \text{deg}_Y Q(X, Y)$  in  $Y$ . The rest follows from Prop. 3.5.  $\square$

*Remark 9.4.* As noted in connection with Lemma 7.1, all the coefficients  $c(k)$  in (59) are quotients with integers as numerators, and denominators being powers of 2; i.e., for each fixed  $k$  they have the form

$$(63) \quad c(k) = \frac{\text{integer}}{\text{power of } 2}.$$

In addition, the  $c(k)$  inherit property (56) of the  $p(11n+6)$  in the following sense: each numerator in the quotient representation (63) is divisible by 11. We proved

<sup>34</sup>See the Zagier quote in Section 6.1.

this fact as a consequence of the algebraic relation (44), resp. (52). In the  $q$ -series interpretation<sup>35</sup>, relation (61) reads as

$$(64) \quad \tilde{g}(q) = \frac{1}{q^4} \prod_{j=1}^{\infty} \frac{(1-q^j)^{12}}{(1-q^{11j})^{11}} \sum_{n=0}^{\infty} p(11n+6)q^n = \sum_{k=-4}^{\infty} c(k)\tilde{h}(q)^k.$$

Consequently, the 11 divisibility of the  $c(k)$ , by coefficient comparison of  $q$ -powers, implies  $11 \mid p(11n+6)$ ,  $n \geq 0$ . In other words, Ramanujan's congruence (56) involving a *non-holonomic* sequence  $p(11n+6)$  is implied by the analogous divisibility property of a *holonomic* sequence  $c(k)!$

## 10. CASE STUDY: A PARTITION CONGRUENCE BY ANDREWS

In this section we present another example to illustrate the relevance of holonomic differential equations to partition congruences.

Consider the number  $c\phi_2(n)$  of 2-colored Frobenius partitions of  $n$  introduced by Andrews [2]; their generating function is

$$(65) \quad \sum_{n=0}^{\infty} c\phi_2(n)q^n = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^5}{(1-q^j)^4(1-q^{4j})^2}.$$

Picking the subsequence  $(c\phi_2(5n+3))_{n \geq 0}$ , as for (55) we use Smoot's package [?] to compute a  $q$ -product  $G(\tau)$ ,  $q = e^{2\pi i\tau}$ , which, when multiplied with the generating function, gives a modular function,

$$(66) \quad g(\tau) := G(\tau) \sum_{n=0}^{\infty} c\phi_2(5n+3)q^n \in M^{\infty}(20).$$

Matching the product representation in (65) with modular function facts for eta-quotients,  $N = 20$  for the ambient space  $M^{\infty}(N)$  is a natural choice. Using the command

```
RKDelta[20, 20, {-4, 5, -2, 0, 0, 0}, 5, 3],
```

Smoot's package returns,

$$G(\tau) = \frac{1}{q^9} \prod_{j=1}^{\infty} \frac{(1-q^j)^{14}(1-q^{4j})^7(1-q^{10j})^{10}}{(1-q^{2j})^{11}(1-q^{5j})^2(1-q^{20j})^{17}}.$$

The genus of  $X(\Gamma_0(20))$  is 1, hence by Weierstraß' gap theorem<sup>36</sup> there exists a modular function  $F \in M^{\infty}(20)$  such that  $\text{ord } f = -2$ . Such  $h$  can also be

<sup>35</sup>For the notation  $\tilde{g}$  see (4)

<sup>36</sup>See, for instance, [30, Thm. 12.2].

computed using Smoot's package:

$$(67) \quad f = \frac{1}{q^2} \prod_{j=1}^{\infty} \frac{(1 - q^{2j})^3 (1 - q^{10j})}{(1 - q^{4j})(1 - q^{20j})^3} \in M^{\infty}(20).$$

*Remark 10.1.* In general, such  $q$ -product (resp. eta-quotient) generators of modular function spaces  $M^{\infty}(N)$ —provided they exist for a particular choice of  $N$ —are not uniquely determined. Algorithmically, they are members of the solution space of a system of linear Diophantine equations and inequalities. Another element of the solution space for the given problem, for instance, is

$$f^* = \frac{1}{q^2} \prod_{j=1}^{\infty} \frac{(1 - q^{4j})^4 (1 - q^{10j})^2}{(1 - q^{2j})^2 (1 - q^{20j})^4} \in M^{\infty}(20).$$

These two modular functions are related,

$$(68) \quad f = f^* + 5,$$

which owing to the fact that  $f, f^* \in M^{\infty}(20)$  can be easily verified by comparing the coefficients of  $q^{-2}$ ,  $q^{-1}$ , and  $q^0$  in their  $q$ -expansions. After the replacement  $q^2 \rightarrow q$ , relation (68) is Thm. 1.6.1(ii),

$$\begin{aligned} \psi^2(q) - 5 q \psi^2(q^5) &= (q; q)_{\infty}^2 \frac{\chi(-q)}{\chi(-q^5)} \\ &= q^{-1/4} \frac{\eta(\tau)^3 \eta(10\tau)}{\eta(2\tau) \eta(5\tau)}, \end{aligned}$$

in Andrews and Berndt [3]; see also (10.7) in Cooper's monograph [17].

Again we use the `GeneratingFunctions` package to determine  $c(k)$  such that

$$g = \sum_{k=-9}^{\infty} c(k) \left(\frac{1}{f}\right)^{\frac{k}{2}};$$

i.e., according to Lemma 4.1 with  $h := (\frac{1}{f})^{1/2}$  and  $m = 2$ , the coefficients  $c(k)$  can be computed as follows:

$$\text{In[71]:= } g = \frac{20}{q^9} + \frac{165}{q^8} + \frac{270}{q^7} + \frac{1190}{q^6} + \frac{1100}{q^5} + \frac{3680}{q^4} + \frac{2990}{q^3} + \frac{8220}{q^2} + \frac{6520}{q} + 16825 + O[q]^1;$$

The first eight coefficients of  $g$ :

`In[72]:= gg[n_] := Coefficient[Normal[g], q, n]`

`In[73]:= Table[gg[k], {k, -9, 0}]`

`Out[73]= {20, 165, 270, 1190, 1100, 3680, 2990, 8220, 6520, 16825}`

`In[74]:= f = \frac{1}{q^2} - 3 + q^2 + 2q^4 + 2q^6 - 2q^8 - q^{10} - 4q^{14} - 2q^{16} + 5q^{18} + O[q]^{19};`

`In[75]:= h = Series[q \left(\frac{1}{q^2 f}\right)^{1/2}, {q, 0, 14}]`

```

Out[75]= q +  $\frac{3q^3}{2}$  +  $\frac{23q^5}{8}$  +  $\frac{83q^7}{16}$  +  $\frac{1099q^9}{128}$  +  $\frac{3637q^{11}}{256}$  +  $\frac{23091q^{13}}{1024}$  + 0[q]14
In[76]:= U = h; W = InverseSeries[U]; ComposeSeries[g, W]
Out[76]=  $\frac{20}{q^9}$  +  $\frac{165}{q^8}$  +  $\frac{540}{q^7}$  +  $\frac{3170}{q^6}$  +  $\frac{10525}{2q^5}$  +  $\frac{22640}{q^4}$  +  $\frac{23615}{q^3}$  +  $\frac{71400}{q^2}$  +  $\frac{1508395}{32q}$  + 84000 + 0[q]1

```

To match our problem setting, we express the answer in powers of  $h$ :

```

In[77]:= Clear[h]
In[78]:= Normal[Sum[gg[k]Wk, {k, -9, 0}]] + O[h]1 /. q -> h
Out[78]=  $\frac{20}{h^9}$  +  $\frac{165}{h^8}$  +  $\frac{540}{h^7}$  +  $\frac{3170}{h^6}$  +  $\frac{10525}{2h^5}$  +  $\frac{22640}{h^4}$  +  $\frac{23615}{h^3}$  +  $\frac{71400}{h^2}$  +  $\frac{1508395}{32h}$  + 84000 + 0[h]1

```

This means, we obtained,

$$g = \sum_{k=-4}^{\infty} c(k) \left(\frac{1}{f}\right)^{\frac{k}{2}} = \sum_{k=-9}^{\infty} c(k) h^k = \frac{20}{h^9} + \frac{165}{h^8} + \frac{540}{h^7} + \frac{3170}{h^6} + \frac{10525}{2h^5} + \dots$$

In the next step we guess an algebraic relation between  $f$  and  $g$ . To this end, we put the first 60 coefficients  $c(-9), c(-10), \dots, c(50)$  into a list:

```

In[79]:= cList = {20, 165, 540, 3170, 10525/2, 22640, 23615, 71400, 1508395/32, 84000,
                792795/32, 0, -(2775035/256), 0, 856325/64, 0, ...,
                -(49862066867855050577211121507905/2251799813685248), 0,
                68861170372827393869036145946955/2251799813685248, 0}
In[80]:= AlgRel = GuessAE[cList, Y[h], 2, 18][[1, 1]]
Out[80]= -25(-16+225h2+21760h4+435680h6+4480640h8+27700736h10+107417600h12+257024000h14+
          348160000h16+204800000h18) + 10(33h+634h3+4528h5+14280h7+16800h9)Y[h] - Y[h]2 = 0

```

The program has guessed the algebraic relation for

$$(69) \quad Y(h) := \sum_{k=0}^{\infty} c(k-9)h^k = h^9 \sum_{k=-9}^{\infty} c(k)h^k = h^9 g,$$

hence to relate to  $g$ , we substitute accordingly:

```

In[81]:= AlgRel /. Y[h] -> h9g
Out[81]= -g2h18+10gh9(33h+634h3+4528h5+14280h7+16800h9)-25(-16+225h2+21760h4+435680h6+
          4480640h8+27700736h10+107417600h12+257024000h14+348160000h16+204800000h18) = 0

```

Finally, in Out [81] we substitute  $h = (\frac{1}{f})^{1/2}$  to obtain, after factoring the coefficient polynomials, the following result.

**Proposition 10.2.** *The modular functions  $f$  and  $g$  in  $M^\infty(20)$  as given in (67) and (66) satisfy*

$$(70) \quad g^2 - 10(f+4)(4200 + 2520f + 502f^2 + 33f^3)g + 25(f+4)^4(800000 + 560000f + 144000f^2 + 15600f^3 + 481f^4 - 16f^5) = 0.$$



*Proof.* Knowing that  $f, g \in M^\infty(20)$ , to prove (70) it suffices to show that the principal part and the constant term in the  $q$ -expansion of the left side of (70) are equal to zero:

$$\begin{aligned} \text{In[82]}:= f &= \frac{1}{q^2} - 3 + q^2 + 2q^4 + 2q^6 - 2q^8 - q^{10} - 4q^{14} - 2q^{16} + 5q^{18} + \mathcal{O}[q]^{19}; \\ \text{In[83]}:= g &= \frac{20}{q^9} + \frac{165}{q^8} + \frac{270}{q^7} + \frac{1190}{q^6} + \frac{1100}{q^5} + \frac{3680}{q^4} + \frac{2990}{q^3} + \frac{8220}{q^2} + \frac{6520}{q} + 16825 + 11500q + \\ &\quad 28760q^2 + 17340q^3 + 36820q^4 + 22050q^5 + 40760q^6 + 19680q^7 + 36345q^8 + 9000q^9 + \\ &\quad \mathcal{O}[q]^{10}; \\ \text{In[84]}:= g^2 - 10(f+4)(4200+2520f+502f^2+33f^3)g + 25(f+4)^4(800000+560000f+144000f^2+ \\ &\quad 15600f^3 + 481f^4 - 16f^5) \\ \text{Out[84]}= & \mathcal{O}[q]^1 \end{aligned}$$

□

Relation (70) implies a result which has been proven already by Andrews [2, Cor. 10.1]:

**Corollary 10.3.** *For non-negative integer  $n$ ,*

$$(71) \quad c\phi_2(5n + 3) \equiv 0 \pmod{5}.$$

*Proof.* By (70) each coefficient of  $g^2$  has 5 as a factor. Hence 5 divides each coefficient of  $g$ , and thus  $5 \mid c\phi_2(5n + 3)$ . □

*Remark 10.4.* Another direct consequence of relation (70) is  $g^2 \equiv f^8 \pmod{2}$ ; hence  $g \equiv f^4 \pmod{2}$  or equivalently,

$$g \equiv \frac{1}{q^8} \prod_{j=1}^{\infty} \frac{(1 - q^{2j})^{12}(1 - q^{10j})^4}{(1 - q^{4j})^4(1 - q^{20j})^{12}} \pmod{2}.$$

Taking the the  $q$ -product factor  $G(\tau)$  of  $g(\tau)$  from (66) into account, one can reduce this further. By applying freshman's dream,  $(1 + x)^p \equiv 1 + x^p \pmod{p}$  with  $p$  a prime, one, for instance, obtains

$$(72) \quad \sum_{n=0}^{\infty} c\phi_2(5n + 3)q^n \equiv q \prod_{j=1}^{\infty} \frac{(1 - q^{10j})^5}{(1 - q^{4j})^3} \pmod{2}.$$

On other words, the  $q$ -series expansion of the  $q$ -product on the right side describes the parity sequence of the partition numbers  $c\phi_2(5n + 3)$ .

Finally, we return to the aspect of differential and recurrence equations. First, using the `GeneratingFunctions` package, we convert relation `AlgRel` in `Out [80]`, which is equivalent to (70), into a holonomic differential equation:

$$\text{In[85]}:= \text{AE2DE}[\text{AlgRel}[[1]], \text{Y}[h]]$$

Notice that here  $h$  is taken as an indeterminate.

$$\begin{aligned} \text{Out}[85]= & 330+12255h^2+205730h^4+2029080h^6+12774880h^8+52211200h^{10}+134438400h^{12}+198400000h^{14}+ \\ & 128000000h^{16} + (108h + 2807h^3 + 28810h^5 + 147660h^7 + 380600h^9 + 396000h^{11})Y[h] - (2 + 67h^2 + \\ & 895h^4 + 6246h^6 + 24340h^8 + 50600h^{10} + 44000h^{12})Y'[h] = 0 \end{aligned}$$

To obtain a recurrence for the coefficients  $d(n) := c(n - 9)$  of  $Y(h)$ , we input, analogously to Out [70], the *homogeneous* version of the differential equation in Out [85]:

$$\begin{aligned} \text{In}[86]:= & \text{DE2RE}[(108h + 2807h^3 + 28810h^5 + 147660h^7 + 380600h^9 + 396000h^{11})Y[h] \\ & - (2 + 67h^2 + 895h^4 + 6246h^6 + 24340h^8 + 50600h^{10} + 44000h^{12})Y'[h] = 0, d[n]] \\ \text{Out}[86]= & -44000(-9 + n)d[n] - 2200(-127 + 23n)d[2 + n] - 20(-2515 + 1217n)d[4 + n] - 2(4333 + \\ & 3123n)d[6 + n] + (-4353 - 895n)d[8 + n] + (-562 - 67n)d[10 + n] - 2(12 + n)d[12 + n] = 0 \end{aligned}$$

As with Out [70], the way we derived this recursion also *proves* it. Analogously to Prop. 9.2 we summarize what one obtains after relabeling and using  $d(n) = c(n - 9)$ :

**Proposition 10.5.** *Given  $f$  as in (67) and  $g$  as in (66). Then the uniquely determined coefficients  $c(k)$  such that*

$$(73) \quad g = \sum_{n=-9}^{\infty} c(n) \left(\frac{1}{f}\right)^{\frac{n}{2}},$$

for  $n \geq 9$  satisfy the holonomic recurrence

$$(74) \quad \begin{aligned} c(n) = & \frac{-1}{2(n+9)} \left( 2^5 5^3 \cdot 11(n-12) c(n-12) + 2^3 5^2 \cdot 11(23n-196) c(n-10) \right. \\ & + 2^2 \cdot 5(1217n-6166) c(n-8) + 2(3123n-5036) c(n-6) \\ & \left. + (895n+1668) c(n-4) + (67n+361) c(n-2) \right), \end{aligned}$$

with initial values

$$\begin{aligned} c(-9) &= 20, c(-8) = 165, c(-7) = 540, c(-6) = 3170, c(-5) = 10525/2, \\ c(-4) &= 22640, c(-3) = 23615, c(-2) = 71400, c(-1) = 1508395/32, \\ c(0) &= 84000, c(1) = 792795/32, c(2) = 0, c(3) = -(2775035/256), c(4) = 0, \\ c(5) &= 856325/64, c(6) = 0, c(7) = -(246729945/8192), \text{ and } c(8) = 0. \end{aligned}$$

## 11. APPENDIX: PROOF OF PROPOSITION 6.2 AND ZERO-RECOGNITION OF MEROMORPHIC FUNCTIONS ON RIEMANN SURFACES

**11.1. Proof of Proposition 6.2.** Proposition 6.2 is an immediate consequence of the following, more general Proposition 11.1 which we state and prove in the setting of the ring of formal power series over the complex numbers, denoted by  $\mathbb{C}[[z]]$ . The ring of polynomials with complex coefficients is denoted by  $\mathbb{C}[z]$ ;

its quotient field, the rational functions, by  $\mathbb{C}(z)$ . This notation is extended to multivariate polynomials,  $\mathbb{C}[z_1, \dots, z_n]$ , and to rational functions,  $\mathbb{C}(z_1, \dots, z_n)$ .

**Proposition 11.1.** *Let  $y(z) = \sum_{k=0}^{\infty} c(k)z^k \in \mathbb{C}[[z]]$  be such that*

$$(75) \quad A_\ell(z)y^{(\ell)}(z) + \dots + A_0(z) = 0,$$

where the  $A_j(z) \in \mathbb{C}[[z]]$  are algebraic Puiseux series, not all zero. Then  $y(z)$  is holonomic series; i.e., there exist polynomials  $P_k(z) \in \mathbb{C}[z]$ , not all zero, such that

$$(76) \quad P_d(z)y^{(d)}(z) + \dots + P_0(z) = 0.$$

*Proof.* By definition, each  $A_j(z)$  satisfies an algebraic equation  $P_j(z, A_j(z)) = 0$  with  $P_j(X, Y) \in \mathbb{C}(X)[Y]$ . As a consequence, for each  $j$  the derivative  $A'_j(z)$  is a rational function in  $z$  and  $A_j(z)$ ; i.e.,

$$A'_j(z) = \frac{dA_j(z)}{dz} \in \mathbb{C}(z, A_j(z)).$$

This property extends to the higher derivatives  $A_j^{(i)}(z)$ . Hence the rational function field,

$$K := \mathbb{C}(z)(A_0(z), \dots, A_\ell(z), A'_0(z), \dots, A'_\ell(z), A''_0(z), \dots, A''_\ell(z), \dots),$$

generated by  $z$  and all possible derivatives  $A_j^{(i)}(z)$ , is nothing but the field generated by  $z$  and the  $A_j(z)$ ; i.e.,

$$(77) \quad K = \mathbb{C}(z)(A_0(z), \dots, A_\ell(z)).$$

Again using the fact that the  $A_j(z)$  are algebraic, one sees that  $K$  is finitely generated as a  $\mathbb{C}(z)$ -module,

$$(78) \quad K = \{\beta_1(z)b_1(z) + \dots + \beta_r(z)b_r(z) : \beta_j(z) \in \mathbb{C}(z)\}.$$

As a concrete choice of the linear generators  $b_j(z)$ , for instance, one can take all the power products,

$$A_0(z)^{a_0} \dots A_\ell(z)^{a_\ell} \text{ with } a_0 < \deg_Y P_0(X, Y), \dots, a_\ell < \deg_Y P_\ell(X, Y).$$

The next observation is that for  $k \geq \ell$ :

$$(79) \quad y^{(k)}(z) = c_{k,1}(z)y^{(\ell-1)}(z) + \dots + c_{k,\ell}(z) \text{ for some } c_{k,1}(z), \dots, c_{k,\ell}(z) \in K.$$

This can be seen by the following steps which can be applied successively to prove the statement by mathematical induction. Assuming  $A_\ell(z) \neq 0$ , rewrite (75) as

$$(80) \quad y^{(\ell)}(z) = -\frac{A_{\ell-1}(z)}{A_\ell(z)}y^{(\ell-1)}(z) - \dots - \frac{A_0(z)}{A_\ell(z)},$$

which by (77) is of the required form. Taking the derivative, again by (77), gives

$$y^{(\ell+1)}(z) = d_\ell(z)y^{(\ell)}(z) + \dots + d_0(z) \text{ for some } d_j(z) \in K.$$

Now reduce with (80) to arrive at the required form.

Fact (79) prepares us to define another set of linear generators by

$$(81) \quad \{B_1, \dots, B_{r\ell}\} := \{b_i(z) y^{(j)}(z) : 1 \leq i \leq r, 0 \leq j \leq \ell - 1\}.$$

Using this set and setting  $d := r\ell$ , fact (79) translates into a matrix relation,

$$(82) \quad \begin{bmatrix} y(z) \\ y'(z) \\ \dots \\ y^{(d)}(z) \end{bmatrix} = \begin{bmatrix} c_{0,1}(z) & c_{0,2}(z) & \dots & c_{0,d}(z) \\ c_{1,1}(z) & c_{1,2}(z) & \dots & c_{1,d}(z) \\ \dots & \dots & \dots & \dots \\ c_{d,1}(z) & c_{d,2}(z) & \dots & c_{d,d}(z) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \dots \\ B_d \end{bmatrix},$$

where the  $c_{i,j}$  are rational functions in  $\mathbb{C}(z)$ . Denote the  $(d+1) \times d$  matrix by  $C$ . Its transpose  $C^t$  is a  $d \times (d+1)$  matrix, hence there exists a non-zero vector  $x = (x_0(z), \dots, x_d(z)) \in \mathbb{C}(z)^{d+1}$  such that  $C^t x^t = 0$ . Equivalently,  $x C = 0$ , and as a consequence of applying this to both sides of (82),

$$x_0(z)y(z) + x_1(z)y'(z) + \dots + x_d(z)y^{(d)}(z) = 0,$$

where the rational functions  $x_j(z) \in \mathbb{C}(z)$  are not all zero. Multiplying the  $x_j(z)$  with a common denominator completes the proof of Prop. 11.1.  $\square$

**11.2. Zero recognition of meromorphic functions.** The basic fundamental fact we use for zero recognition of meromorphic functions on Riemann surfaces is:

**Lemma 11.2.** *Let  $f$  be a non-constant meromorphic function on a compact Riemann surface  $X$ . Then*

$$\sum_{x \in X} \text{ord}_x f = 0.$$

*Proof.* See, for instance, [27, Prop. 4.12].  $\square$

In other words,

$$\text{number of poles of } f = \text{number of zeros of } f,$$

counting multiplicities.

Here  $\text{ord}_{x_0} f$  is defined as follows: Suppose  $f(x) = \sum_{n \geq m} c_n (\varphi(x) - \varphi(x_0))^n$ ,  $c_m \neq 0$ , is the local Laurent expansion of  $g$  at  $x_0$  using the local coordinate chart  $\varphi : U_0 \rightarrow \mathbb{C}$  which homeomorphically maps a neighborhood  $U_0$  of  $x_0 \in X$  to an open set  $V_0 \subseteq \mathbb{C}$ . Then  $\text{ord}_{x_0} f := m$ .

## 12. CONCLUSION

In this article we attempt to connect two worlds which, at the first glance, look very different: the “web of modularity”, i.e., modular forms and functions, with the universe of holonomic functions and sequences. It is our hope to inspire subsequent investigations in this direction.

To point to one of the more general aspects: in the given context, we see quite some application potential of the “first guess, then prove” strategy. For instance, it led us to develop, on the “shoulders” of Yang [37], a new algorithm, ModFormDE, to verify holonomic differential equations involving modular forms; see Section 6 and, for full details, [31]. Concerning the computational complexity of this approach, we remark that the holonomic procedure to *guess* the differential equation only requires to solve a system of linear equations. The *proving* step by algorithm ModFormDE boils down to zero-testing of the coefficients  $\alpha_j$  in (32). Each of these tests needs linear time in the number of coefficients of the corresponding  $q$ -expansions, resp.  $x$ -expansions with  $x = q^{1/N_0}$ , at infinity of the particular  $\alpha_j$ .

We restrict to mention only one more concrete aspect for further exploration. Choosing modular functions  $g$  and  $h$ , as explained, one can relate a non-holonomic sequence with a holonomic sequence; for example,

$$(p(11n + 6))_{n \geq 0} \longleftrightarrow (c(k))_{k \geq 0},$$

where  $c(k)$  is defined by the holonomic recurrence (60). As remarked at the end of Section 9, Ramanujan’s observation  $11 \mid p(11n + 6)$  is implied by the corresponding divisibility property of the holonomic sequence  $(c(k))_{k \geq 0}$ . Nevertheless, in the examples we considered (e.g., the sequence defined in (74) is another such instance) we found it a non-trivial task to prove the corresponding arithmetic property of the  $c(k)$  directly from the defining holonomic recurrence.

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