AMS Sectional Meeting, Gainesville, Florida, November 3

Special Session on Experimental Mathematics in Number Theory and Combinatorics, IV

The summation package Sigma and $(q-)$applications

Carsten Schneider

Research Institute for Symbolic Computation
Johannes Kepler University Linz

JOHANNES KEPLER UNIVERSITY LINZ

Der Wissenschaftsfonds

Indefinite summation

Simplify

$$\sum_{k=0}^{a} \big(1 + (n-2k)\, S_1(k)\big) \binom{n}{k} = \text{?},$$

where $S_1(k) := \sum_{i=1}^{k} \frac{1}{i} \;\; (= H_k).$

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

My summation package Sigma computes

$$g(k) = (k\,S_1(k) - 1)\binom{n}{k}$$

GIVEN $f(k) = \big(1 + (n-2k)\, S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Summing the telescoping equation over k from 0 to a gives

$$\boxed{\sum_{k=0}^{a} \big(1 + (n-2k)\, S_1(k)\big)\binom{n}{k}} = \boxed{g(a+1) - g(0)}$$

$$= 1 + (n-a) S_1(a) \binom{n}{a}.$$

GIVEN $f(k) = \left(1 + (n-2k)\,S_1(k)\right)\binom{n}{k}$

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

Summing the telescoping equation over k from 0 to a gives

$$\sum_{k=0}^{a}\left(1 + (n-2k)\,S_1(k)\right)\binom{n}{k} = g(a+1) - g(0)$$

$$= 1 + (n-a)S_1(a)\binom{n}{a}.$$

based difference ring algorithms

M. Karr. *J. ACM*, 28:305–350, 1981.
M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
P. A. Hendriks and M. F. Singer. *J. Symbolic Comput.*, 27(3):239–259, 1999.
M. Bronstein. *J. Symbolic Comput.*, 29(6):841–877, 2000.
CS. Symbolic summation in difference fields. J. Kepler University, May 2001. PhD Thesis.
CS. *An. Univ. Timişoara Ser. Mat.-Inform.*, 42(2):163–179, 2004.
CS. *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
CS. *Appl. Algebra Engrg. Comm. Comput.*, 16(1):1–32, 2005.
CS. *Ann. Comb.*, 9(1):75–99, 2005.
CS. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
CS. In *Proc. ISSAC'05*, pages 285–292. ACM Press, 2005.
CS. *J. Algebra Appl.*, 6(3):415–441, 2007.
CS. *J. Symbolic Comput.*, 43(9):611–644, 2008.
S.A. Abramov, M. Petkovšek. *J. Symbolic Comput.*, 45(6): 684–708, 2010.
CS. In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, pages 285–308. 2010.
CS. *Ann. Comb.*, 14(4):533–552, 2010.
CS. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
CS. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
CS. *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014.
CS. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. J. Symb. Comput. 80(3), pp. 616-664. 2017.
E.D. Ocansey, CS. In: Advances in Computer Algebra. WWCA 2016., C. Schneider, E. Zima (ed.), pp. 175-213. 2018.
S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n} \big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\boxed{\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?}$$

$\alpha = 1$:

$$\sum_{k=0}^{a}\big(1+(n-2k)S_1(k)\big)\binom{n}{k} = 1+(n-a)S_1(a)\binom{n}{a}$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?$$

$\alpha = 1$:

$$\sum_{k=0}^{n}\big(1+(n-2k)S_1(k)\big)\binom{n}{k} = 1$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha = 1$:

$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:

$$\sum_{k=0}^{a}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = \frac{(a-n)^2(1+2nS_1(a))}{n^2}\binom{n}{a}$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\boxed{\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?}$$

$\alpha = 1$:

$$\sum_{k=0}^{n}\big(1+(n-2k)S_1(k)\big)\binom{n}{k} = 1$$

$\alpha = 2$:

$$\sum_{k=0}^{n}\big(1+2(n-2k)S_1(k)\big)\binom{n}{k}^{2} = 0$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha = 1$:

$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:

$$\sum_{k=0}^{n}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = 0$$

$\alpha = 3$:

$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3} =?$$

$\alpha = 4$:

$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4} =?$$

$\alpha = 5$:

$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5} =?$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n} \left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

$\alpha = 1$:
$$\sum_{k=0}^{n} \left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:
$$\sum_{k=0}^{n} \left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = 0$$

$\alpha = 3$:
$$\sum_{k=0}^{n} \left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3} = (-1)^n$$

$\alpha = 4$:
$$\sum_{k=0}^{n} \left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4} = (-1)^n\binom{2\,n}{n}$$

$\alpha = 5$:
$$\sum_{k=0}^{n} \left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5} =?$$

Telescoping

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1+5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{\qquad f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1 + 5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1+5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$, $c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)\ , c_2(n), c_3(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\cdots+c_3(n)\,f(n+3,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1+5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)\ , c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Sigma *computes*: $c_0(n),\dots,c_4(n) \in \mathbb{Q}[n]$

$$g(n,k) := \binom{n}{k}^5 \frac{p_1(k,n,S_1(k))}{(k-n-4)^5(k-n-3)^5(k-n-2)^5(k-n-1)^5},$$

$$g(n,k+1) := \binom{n}{k}^5 \frac{p_2(k,n,S_1(k))}{(k-n-3)^5(k-n-2)^5(k-n-1)^5}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1 + 5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)\ , c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \cdots + c_4(n)\, f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1) - g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n) + \\ c_1(n)\,\big[\mathsf{SUM}(n+1) - f(n+1,n+1)\big] \\ \vdots \\ c_4(n)\,\big[\mathsf{SUM}(n+4) - f(n+4,n+1) - f(n+4,n+2) - \cdots - f(n+4,n+4)\big]. \end{array}}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$, $c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 0 to n gives: Sigma

$$\begin{aligned}
&(1+n)^3(2+n)(41752+59264n+31245n^2+7250n^3+625n^4)\mathsf{SUM}(n)\\
&-(2+n)(3007560+10401664n+15087509n^2+11895816n^3+5506508n^4+1496890n^5+221375n^6+13750n^7)\mathsf{SUM}(1+n)\\
&+(66648040+240325672n+372720670n^2+325025288n^3+174496185n^4+59121186n^5+12356530n^6+1457750n^7+74375n^8)\mathsf{SUM}(2+n)\\
&+(3+n)(6783960+21058536n+27279834n^2+19134404n^3+7861553n^4+1895640n^5+248875n^6+13750n^7)\mathsf{SUM}(3+n)\\
&+(3+n)(4+n)^3(7108+16024n+13245n^2+4750n^3+625n^4)\mathsf{SUM}(4+n)=0
\end{aligned}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$, $c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\cdots+c_4(n)\,f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$(1+n)^2\mathsf{SUM}(n) - \left(25+33n+11n^2\right)\mathsf{SUM}(1+n) - (2+n)^2\mathsf{SUM}(2+n) = 0$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

extended → Krattenthaler/Rivoal 07

$\alpha = 1$:
$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:
$$\sum_{k=0}^{n}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = 0$$

$\alpha = 3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3} = (-1)^n$$

$\alpha = 4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4} = (-1)^n\binom{2\,n}{n}$$

$\alpha = 5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5} = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^{2}\binom{n+j}{j}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?}$$

$\alpha=-1$:
$$\sum_{k=0}^{a}\big(1-(n-2k)S_1(k)\big)\binom{n}{k}^{-1}=\frac{(a+1)S_1(a)+1}{\binom{n}{a}}$$

The other direction:

$$\boxed{\sum_{k=0}^{n} \left(1 + \alpha(n-2k)S_1(k)\right) \binom{n}{k}^{\alpha} \stackrel{?}{=}}$$

$\alpha = -1$:

$$\sum_{k=0}^{n} \left(1 - (n-2k)S_1(k)\right) \binom{n}{k}^{-1} = (n+1)S_1(n) + 1$$

The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} = ?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1} = (n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{a}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{(a+1)(-a+2n+2(a+1)(n+2)S_1(a)+3)}{(n+2)^2\binom{n}{a}^{-2}}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:
$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:
$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$
$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

$\alpha = -1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1} = (n+1)S_1(n)+1$$

$\alpha = -2$:

$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2} = \frac{(n+1)^2}{(n+2)^2} + \frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

$\alpha = -3$:

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3} = ?$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=0}^{n} \big(1 - 3(n - 2k)S_1(k)\big)\binom{n}{k}^{-3};$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=0}^{n} \left(1 - 3(n-2k)S_1(k)\right)\binom{n}{k}^{-3};$$

Compute a recurrence

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $$(2+n)^4(3+n)^2\mathrm{SUM}[n] + (1+n)^3(3+n)^2(5+2n)\mathrm{SUM}[1+n] + (1+n)^3(2+n)^3\mathrm{SUM}[2+n] = (1+n)^2(2+n)\left(229+311n+138n^2+20n^3\right) + 6(1+n)^3(2+n)^2(3+n)(5+2n)S_1(n)$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=0}^{n} \left(1 - 3(n-2k)S_1(k)\right)\binom{n}{k}^{-3};$$

Compute a recurrence

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $$(2+n)^4(3+n)^2\mathrm{SUM}[n] + (1+n)^3(3+n)^2(5+2n)\mathrm{SUM}[1+n] + (1+n)^3(2+n)^3\mathrm{SUM}[2+n] = (1+n)^2(2+n)\left(229+311n+138n^2+20n^3\right) + 6(1+n)^3(2+n)^2(3+n)(5+2n)S_1(n)$$

Solve a recurrence

In[4]:= **recSol = SolveRecurrence[rec, SUM[n]]**

Out[4]= $$\{\{0, (1+n)^3(-1)^{1+n}\},$$
$$\{0, \left((1+n)^2 + (1+n)^3 S_1(n)\right)(-1)^n\},$$
$$\{1, 1 + \left(5(1+n)^3 \sum_{i=1}^{n} \frac{(-1)^i}{i^3} - 6(1+n)^3 \sum_{i=1}^{n} \frac{(-1)^i S_1[i]}{i^2}\right)(-1)^n + 6(1+n)S_1[n]\}\}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=0}^{n} \left(1 - 3(n-2k)S_1(k)\right)\binom{n}{k}^{-3};$

Compute a recurrence

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $(2+n)^4(3+n)^2\mathtt{SUM}[n] + (1+n)^3(3+n)^2(5+2n)\mathtt{SUM}[1+n] + (1+n)^3(2+n)^3\mathtt{SUM}[2+n] = (1+n)^2(2+n)\left(229+311n+138n^2+20n^3\right) + 6(1+n)^3(2+n)^2(3+n)(5+2n)S_1(n)$

Solve a recurrence

In[4]:= **recSol = SolveRecurrence[rec, SUM[n]]**

Out[4]= $\{\{0, (1+n)^3(-1)^{1+n}\},$
$\{0, \left((1+n)^2 + (1+n)^3 S_1(n)\right)(-1)^n\},$
$\{1, 1 + \left(5(1+n)^3 \sum_{i=1}^{n} \frac{(-1)^i}{i^3} - 6(1+n)^3 \sum_{i=1}^{n} \frac{(-1)^i S_1[i]}{i^2}\right)(-1)^n + 6(1+n)S_1[n]\}\}$

Combine the solutions

In[5]:= **FindLinearCombination[recSol, mySum, n, 2]**

Out[5]= $1 + 6(1+n)S_1(n) + 5(1+n)^3(-1)^n \sum_{i=1}^{n} \frac{(-1)^i}{i^3} - 6(1+n)^3(-1)^n \sum_{i=1}^{n} \frac{(-1)^i S_1(i)}{i^2}$

The other direction:

$$\sum_{k=0}^{n} \left(1 + \alpha(n-2k)S_1(k)\right) \binom{n}{k}^{\alpha} = ?$$

$\alpha = -1$:

$$\sum_{k=0}^{n} \left(1 - (n-2k)S_1(k)\right) \binom{n}{k}^{-1} = (n+1)S_1(n) + 1$$

$\alpha = -2$:

$$\sum_{k=0}^{n} \left(1 - 2(n-2k)S_1(k)\right) \binom{n}{k}^{-2}$$

$$= \frac{(n+1)^2}{(n+2)^2} + \frac{\left(n + 2\left(n^2+3n+2\right)S_1(n) + 3\right)(n+1)}{(n+2)^2}$$

$\alpha = -3$:

$$\sum_{k=0}^{n} \left(1 - 3(n-2k)S_1(k)\right) \binom{n}{k}^{-3} = 1 + 6(1+n)S_1(n)$$

$$+ 5(1+n)^3(-1)^n \sum_{i=1}^{n} \frac{(-1)^i}{i^3} - 6(1+n)^3(-1)^n \sum_{i=1}^{n} \frac{(-1)^i S_1(i)}{i^2}$$

The other direction:

$$\sum_{k=0}^{n} \left(1 + \alpha(n-2k)S_1(k)\right) \binom{n}{k}^{\alpha} =?$$

$\alpha = -4$:

$$\sum_{k=0}^{n} \left(1 - 4(n-2k)S_1(k)\right) \binom{n}{k}^{-4} = \frac{(10(n+1)S_1(n)+3)(n+1)}{2n+3}$$

$$+ \frac{(-1)^n \binom{2n}{n}^{-1} (n+1)^5}{(4n(n+2)+3)} \left(\frac{7}{2} \sum_{i=1}^{n} \frac{(-1)^i \binom{2i}{i}}{i^3} - 5 \sum_{i=1}^{n} \frac{(-1)^i \binom{2i}{i} S_1(i)}{i^2} \right)$$

Example 1: CA for partition theory conjectures

(joint with Ali Uncu and Jakob Ablinger)

Kanade–Russel Conjectures coming from partition theory

Conjecture (Kanade-Russel 2018) - one of 21

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k \geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k \geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Guess

$$H_1(x) + (-1 - qx - q^2x + q^3x)H_1(q^2x)$$
$$-q^3x(1-q^2x+q^3x+q^4x)H_1(q^4x)+q^8x^2(-1+q^4x)H_1(q^6x) = 0$$

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Guess

$$H_1(x) + (-1 - qx - q^2x + q^3x)H_1(q^2x)$$
$$-q^3x(1-q^2x+q^3x+q^4x)H_1(q^4x)+q^8x^2(-1+q^4x)H_1(q^6x) = 0$$

2. Solve the recurrence and show that the triple sum equals the solution
$\rightarrow$ many steps of formal, but skillful/artistic manipulations

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Guess

$$H_1(x) + (-1 - qx - q^2x + q^3x) H_1(q^2x)$$
$$-q^3x(1-q^2x+q^3x+q^4x)H_1(q^4x)+q^8x^2(-1+q^4x)H_1(q^6x) = 0$$

2. Solve the recurrence and show that the triple sum equals the solution $\rightarrow$ many steps of formal, but skillful/artistic manipulations

3. Tracing back the manipulations with extra transformations shows (*)

In[6]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= << **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= << **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= $\mathbf{summand} = \frac{(-1)^{k} q^{(i+2j+3k)(i+2j+3k-1)+3k^{2}+i+6j+6k}}{(q;q)_{i}(q^{4};q^{4})_{j}(q;q)_{k}} x^{i+2j+3k};$

In[10]:= $\mathbf{GuessQShiftEquation}[\mathbf{summand}, \{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$

In[6]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= $<<$ **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= $<<$ **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= $\mathbf{summand} = \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k};$

In[10]:= **GuessQShiftEquation[summand,**

$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$

Out[10]= $H_1[x] - (1+qx+q^2x-q^3x)H_1[q^2x] - q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x] - q^8x^2(1-q^4x)H_1[q^6x]$

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k \geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Guess

$$H_1(x) + (-1 - qx - q^2x + q^3x) H_1(q^2x)$$
$$- q^3x(1 - q^2x + q^3x + q^4x) H_1(q^4x) + q^8x^2(-1 + q^4x) H_1(q^6x) = 0$$

2. Solve the recurrence and show that the triple sum equals the solution
   $\to$ many steps of formal, but skillful/artistic manipulations

3. Tracing back the manipulations with extra transformations shows (*)

In[6]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= $<<$ **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= $<<$ **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= $\mathbf{summand} = \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k};$

In[10]:= $\mathbf{GuessQShiftEquation[summand,}$

$$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$$

Out[10]= $H_1[x] - (1+qx+q^2x-q^3x)H_1[q^2x] - q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x] - q^8x^2(1-q^4x)H_1[q^6x]$

$$\Updownarrow\ H_1[x] = \sum_{m=0}^{\infty} x^m h_1[m]$$

In[11]:= $\mathbf{QSEToQRE[H1qShift[[1]], H_1[x], h_1[m]]}$

Out[11]= $q^{(12+6m)}h_1[m] - q^(9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$

$$-q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m] - (-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m]$$

In[6]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= $<<$ **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= $<<$ **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= $\mathbf{summand} = \frac{(-1)^{k} q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k};$

In[10]:= **GuessQShiftEquation[summand,**

$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$

Out[10]= $H_1[x] - (1+qx+q^2x-q^3x)H_1[q^2x] - q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x] - q^8x^2(1-q^4x)H_1[q^6x]$

$$\Updownarrow \quad H_1[x] = \sum_{m=0}^{\infty} x^m \underbrace{\sum_{k,j\geq 0} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{=h_1[m]}$$

In[11]:= **QSEToQRE[H1qShift[[1]], H_1[x], h_1[m]]**

Out[11]= $q^{(12+6m)}h_1[m] - q^{(}9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
$- q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m] - (-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m]$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j (q;q)_k}}_{F[k]}$$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[12]:= $\mathbf{innerSum} = \sum_{\mathbf{j=0}}^{\infty} \frac{(-1)^{\mathbf{k}}\mathbf{q}^{\mathbf{4j+3k+3k^2+m^2}}}{(\mathbf{q;q})_{\mathbf{m-2j-3k}}(\mathbf{q^4;q^4})_{\mathbf{j}}(\mathbf{q;q})_{\mathbf{k}}};$ $\quad (= \boldsymbol{F[k]})$

In[13]:= $\mathbf{recK} = \mathbf{GenerateRecurrence[innerSum, k]/.SUM \to F}$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[12]:= $\mathbf{innerSum} = \sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k};$ $\quad (= \boldsymbol{F[k]})$

In[13]:= $\mathbf{recK = GenerateRecurrence[innerSum, k]/.SUM \to F}$

Out[13]= $a_0[m,k]F[k] + a_1[m,k]F[i+k] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3] + a_4[m,k]F[k+4] == 0$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[12]:= $\mathbf{innerSum} = \sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k};$ $\quad (= \boldsymbol{F[k]})$

In[13]:= **recK = GenerateRecurrence[innerSum, k]/.SUM → F**

Out[13]= $a_0[m,k]F[k] + a_1[m,k]F[i+k] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3] + a_4[m,k]F[k+4] == 0$

In[14]:= **recKM = GenerateRecurrence[innerSum, k, OneShiftIn → m]/.SUM → F**

Out[14]= $F[m+1,k] == a_0[m,k]F[k] + a_1[m,k]F[k+1] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3]$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[12]:= $\mathbf{innerSum} = \sum_{\mathbf{j=0}}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k};$ $\quad (= \boldsymbol{F[k]})$

In[13]:= **recK = GenerateRecurrence[innerSum, k]/.SUM → F**

Out[13]= $a_0[m,k]F[k] + a_1[m,k]F[i+k] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3] + a_4[m,k]F[k+4] == 0$

In[14]:= **recKM = GenerateRecurrence[innerSum, k, OneShiftIn → m]/.SUM → F**

Out[14]= $F[m+1,k] == a_0[m,k]F[k] + a_1[m,k]F[k+1] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3]$

2. A recurrence for $h_1[m]$:

In[15]:= $\mathbf{doubleSum} = \sum_{k=0}^{\infty} \mathbf{F[k]};$ $\quad (= \boldsymbol{h_1[m]})$

In[16]:= **GenerateRecurrence[doubleSum, m, recK, F[k], recKM]/.SUM →** $\mathbf{h_1}$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[12]:= $\mathbf{innerSum} = \sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k};$ $\qquad (= F[k])$

In[13]:= **recK = GenerateRecurrence[innerSum, k]/.SUM → F**

Out[13]= $a_0[m,k]F[k] + a_1[m,k]F[i+k] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3] + a_4[m,k]F[k+4] == 0$

In[14]:= **recKM = GenerateRecurrence[innerSum, k, OneShiftIn → m]/.SUM → F**

Out[14]= $F[m+1,k] == a_0[m,k]F[k] + a_1[m,k]F[k+1] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3]$

2. A recurrence for $h_1[m]$:

In[15]:= $\mathbf{doubleSum} = \sum_{k=0}^{\infty} F[k];$ $\qquad (= h_1[m])$

In[16]:= **GenerateRecurrence[doubleSum, m, recK, F[k], recKM]/.SUM → h_1**

Out[16]= $q^{(12+6m)} h_1[m] - q^{(}9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
$\qquad - q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m] - (-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m] == 0$

In[6]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= $<<$ **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= $<<$ **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= $\mathbf{summand} = \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i(q^4;q^4)_j(q;q)_k} x^{i+2j+3k};$

In[10]:= **GuessQShiftEquation[summand,**

$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\},H_1[x],6,\{3,12\},50]$

Out[10]= $H_1[x]-(1+qx+q^2x-q^3x)H_1[q^2x]-q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x]-q^8x^2(1-q^4x)H_1[q^6x]$

$$\Big\Updownarrow \quad H_1[x]=\sum_{m=0}^{\infty}x^m\underbrace{\sum_{k,j\geq 0}\frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{=h_1[m]}$$

In[11]:= $\mathbf{QSEToQRE[H1qShift[[1]],H_1[x],h_1[m]]}$

Out[11]= $q^{(12+6m)}h_1[m]-q^{(9+4m)}(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
$-q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m]-(-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m]$

In[6]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= **<< qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= **<< qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= $\mathbf{summand} = \frac{(-1)^{k} q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k};$

In[10]:= $\mathbf{GuessQShiftEquation[summand,}$

$$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$$

Out[10]= $H_1[x] - (1+qx+q^2x-q^3x)H_1[q^2x] - q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x] - q^8x^2(1-q^4x)H_1[q^6x]$

$$\Updownarrow \quad H_1[x] = \sum_{m=0}^{\infty} x^m \underbrace{\sum_{k,j\geq 0} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{=h_1[m]}$$

In[11]:= $\mathbf{QSEToQRE[H1qShift[[1]], H_1[x], h_1[m]]}$

Out[11]= $q^{(12+6m)}h_1[m] - q^{(9+4m)}(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
$\qquad - q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m] - (-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m]$

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Find and prove in one stroke

$$H_1(x) + (-1 - qx - q^2x + q^3x) H_1(q^2x)$$
$$- q^3x(1 - q^2x + q^3x + q^4x) H_1(q^4x) + q^8x^2(-1 + q^4x) H_1(q^6x) = 0$$

2. Solve the recurrence and show that the triple sum equals the solution
   $\to$ many steps of formal, but skillful/artistic manipulations

3. Tracing back the manipulations with extra transformations shows (*)

In[6]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= $<<$ **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[8]:= $<<$ **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[9]:= **summand** = $\frac{(-1)^k q^{(i+2j+\dots}}{}$ ... $^{i+6k}$ ... $^{i+2j+3k}$;

In[10]:= **GuessQS**...

Remark 1: The qFunctions package contains many other interesting features to support the discovery of identities in the context of partition theory

see arXiv:1910.12410 [cs.SC]

Out[10]= $\mathrm{H_1[x}$ ... $\mathrm{q^4x)H_1[q^6x]}$

... m^2

... $q^4)_j(q;q)_k$

$= h_1[m]$

In[11]:= **QSEToQRE[H1qShift[[1]],** ...

Out[11]= $\mathrm{q^{(12+6m)}h_1[m]-q^{(9+4m)}(-1+q+q^2+q^{(5+2m)})h_1[1+m]}$

$\mathrm{-q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m]-(-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m]}$

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[12]:= $\mathbf{innerSum} = \sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}$; $\quad (= F[k])$

In[13]:= **recK = Gene**

Out[13]= $a_0[m,k]F[k$ … $k]F[k+4] == 0$

In[14]:= **recKM** … $\mathbf{SUM} \to \mathbf{F}$

Out[14]= $F[m+$ … $3]$

Remark 2: These Σ-tools have been used/developed to obtain the first computer-assisted proof of Stembridge's TSPP Theorem

(joint with G.E. Andrews and P. Paule, 2005)

2. A recur

In[15]:= $\mathbf{doubleSum} = \sum_{k=0}$

In[16]:= **GenerateRecurrence[doubleSum, m, recK, F[k], recKM]/.SUM → h_1**

Out[16]= $q^{(12+6m)}h_1[m] - q^{(}9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
$- q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m] - (-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m] == 0$

Example 2: Exploring the Calkin–identities

G.E. Andrews, P. Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums, From the Andrews Festschrift, Springer, Berlin (2001), pp. 189-208.
N.J. Calkin. A curious binomial identities Discrete Math., 131 (1994), pp. 335-337.
M. Hirschhorn Calkin's binomial identity Discrete Math., 159 (1996), pp. 273-278.
C. Schneider. C. Schneider Symbolic Summation Assists Combinatorics. Sem. Lothar. Combin. 56, pp. 1-36. 2007.
J. Wang, Z.Z. Zhang. On extensions of Calkin's binomial identities Discrete Math., 274 (2004), pp. 331-342.
Z.Z. Zhang. A kind of curious binomial identity Discrete Math., 306 (2006), pp. 2740-2754.

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1} = (2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}=(2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}=(2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1} = (2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2} = (2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3} = (2+n)2^{-1+3n}-3n2^{-2+n}\binom{2n}{n}$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1=(-1)^n2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1 = (-1)^n 2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2 = 2^{-1+4n}$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2 = -2^{1+4n}+(-1)^{1+n}\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1=(-1)^n2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2=2^{-1+4n}$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2=-2^{1+4n}+(-1)^{1+n}\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3=-2^{2+6n}-3(-1)^n2^{2n}\binom{2n}{n}$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3=2^{-1+6n}+\frac{(-1)^{1+n}4^{-2+3n}\sum_{i=0}^{-1+n}64^{-i}(3+11i)\binom{2i}{i}^2\binom{3i}{i}}{n\binom{2n}{n}}$$

Case 1:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \text{?}$$

Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

Case 1:

- $x \neq 1$
$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$

$$\sum_{k=0}^{a} \sum_{i=0}^{k} y^i \binom{n}{i} = -\frac{(a-n)y^{1+a}\binom{n}{a}}{1+y} + \frac{1+a+y+ay-ny}{1+y} \sum_{i=0}^{a} y^i \binom{n}{i}$$

Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$

$$\sum_{k=0}^{a} \sum_{i=0}^{k} y^i \binom{n}{i} = -\frac{(a-n) y^{1+a} \binom{n}{a}}{1+y} + \frac{1+a+y+ay-ny}{1+y} \sum_{i=0}^{a} y^i \binom{n}{i}$$

  - $y \neq -1$

$$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = (1+n+y)(1+y)^{-1+n}$$

Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$

$$\sum_{k=0}^{a} \sum_{i=0}^{k} y^i \binom{n}{i} = -\frac{(a-n)y^{1+a}\binom{n}{a}}{1+y} + \frac{1+a+y+ay-ny}{1+y} \sum_{i=0}^{a} y^i \binom{n}{i}$$

  - $y \neq -1$

$$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = (1+n+y)(1+y)^{-1+n}$$

  - $y = -1$

$$\sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^i \binom{n}{i} = 0$$

Case 1:

- $x \neq 1$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$
  - $y \neq -1$

    $$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = (1+n+y)(1+y)^{-1+n}$$

  - $y = -1$

    $$\sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^i \binom{n}{i} = 0$$

q-Case 1:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

q-Case 1:

- $x \neq 1$:
$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

q-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

q-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

- $x = 1$:

$$\sum_{k=0}^{a} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = (1+a) \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} i q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

q-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\big(-\frac{xy}{q};q\big)_{1+n}}{(-1+x)(q+xy)} + \frac{q\big(-\frac{y}{q};q\big)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

- $x = 1$:

$$\sum_{k=0}^{a} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = (1+a) \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} i q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

  - $y \neq -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{q\big(-\frac{y}{q};q\big)_{1+n}}{q+y} + \frac{q^2\big(-\frac{y}{q};q\big)_{1+n}}{q+y} \sum_{i=1}^{n} \frac{1}{q+q^i y}$$

q-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

- $x = 1$:

$$\sum_{k=0}^{a} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = (1+a) \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} i q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

  - $y \neq -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{q+y} + \frac{q^2\left(-\frac{y}{q};q\right)_{1+n}}{q+y} \sum_{i=1}^{n} \frac{1}{q+q^i y}$$

  - $y = -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} (-1)^i q^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{(q;q)_{1+n}}{1-q^{1+n}} - \frac{(q;q)_{1+n}}{1-q^{1+n}} \sum_{i=1}^{n} \frac{1}{-1+q^i}$$

q-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\big(-\frac{xy}{q};q\big)_{1+n}}{(-1+x)(q+xy)} + \frac{q\big(-\frac{y}{q};q\big)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

  - $y \neq -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{q\big(-\frac{y}{q};q\big)_{1+n}}{q+y} + \frac{q^2\big(-\frac{y}{q};q\big)_{1+n}}{q+y} \sum_{i=1}^{n} \frac{1}{q+q^i y}$$

  - $y = -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} (-1)^i q^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{(q;q)_{1+n}}{1-q^{1+n}} - \frac{(q;q)_{1+n}}{1-q^{1+n}} \sum_{i=1}^{n} \frac{1}{-1+q^i}$$

Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i \binom{n}{i} \right)^2 = ?$$

Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = \text{?}$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}, a = n$:

$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i}\binom{n}{i}\right)^2 = \frac{-2+\bar{x}^{1+n}+\bar{x}^{2+n}}{-1+\bar{x}}\bar{x}^{1-n}(1+\bar{x})^{-2+2n} + \frac{\binom{2n}{n}}{(1+\bar{x})^2} - \bar{x}^{-n}(1+\bar{x})^{-2+2n}\sum_{i=1}^{n}\bar{x}^i(1+\bar{x})^{-2i}\binom{2i}{i}$$

Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = \,?$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}, a = n$:

$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} \binom{n}{i}\right)^2 = \frac{-2 + \bar{x}^{1+n} + \bar{x}^{2+n}}{-1 + \bar{x}} \bar{x}^{1-n} (1 + \bar{x})^{-2+2n} + \frac{\binom{2n}{n}}{(1 + \bar{x})^2} - \bar{x}^{-n} (1 + \bar{x})^{-2+2n} \sum_{i=1}^{n} \bar{x}^i (1 + \bar{x})^{-2i} \binom{2i}{i}$$

- $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}, a = n$:

$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left(\sum_{i=0}^{k} \frac{(-1)^i}{\bar{x}^i} \binom{n}{i}\right)^2 = \frac{\bar{x}^2 (-1+\bar{x})^{-1+2n}}{1+\bar{x}} - \frac{2(-1)^n (-1+\bar{x})^{-2+2n} \bar{x}^{1-n}}{1+\bar{x}} + \frac{\binom{2n}{n}}{(-1 + \bar{x})^2} + \frac{(-1)^{1+n} (-1+\bar{x})^{-2+2n}}{\bar{x}^n} \sum_{i=1}^{n} \frac{(-1)^i \bar{x}^i \binom{2i}{i}}{(-1 + \bar{x})^{2i}}$$

- $x=1$
  - $y\neq -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}y^i\binom{n}{i}\right)^2=-\frac{2(a-n)y^{1+a}\binom{n}{a}\sum_{i=0}^{a}y^i\binom{n}{i}}{1+y}-\frac{ny\sum_{i=0}^{a}y^{2i}\binom{n}{i}^2}{1+y}+\frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a}y^i\binom{n}{i}\Big)^2+\frac{(-1+y)\sum_{i=0}^{a}iy^{2i}\binom{n}{i}^2}{1+y}$$

- $x = 1$
  - $y \neq -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = -\frac{2(a-n)y^{1+a}\binom{n}{a}\sum_{i=0}^{a} y^i\binom{n}{i}}{1+y} - \frac{ny\sum_{i=0}^{a} y^{2i}\binom{n}{i}^2}{1+y} + \frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a} y^i\binom{n}{i}\Big)^2 + \frac{(-1+y)\sum_{i=0}^{a} i y^{2i}\binom{n}{i}^2}{1+y}$$

$$\downarrow \quad a = n,\ y = 1$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n\binom{2n}{n}$$

- $x = 1$
  - $y \neq -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = -\frac{2(a-n)y^{1+a}\binom{n}{a}\sum_{i=0}^{a} y^i\binom{n}{i}}{1+y} - \frac{ny\sum_{i=0}^{a} y^{2i}\binom{n}{i}^2}{1+y} + \frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a} y^i\binom{n}{i}\Big)^2 + \frac{(-1+y)\sum_{i=0}^{a} iy^{2i}\binom{n}{i}^2}{1+y}$$

$$\downarrow \quad a = n,\ y = 1$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n\binom{2n}{n}$$

  - $y = -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{-2a^2-2a^3+4an+7a^2n-2n^2-8an^2+3n^3}{2n^2(-1+2n)}\binom{n}{a}^2 + \frac{n\sum_{i=0}^{a}\binom{n}{i}^2}{2(-1+2n)}$$

- $x = 1$
  - $y \neq -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = -\frac{2(a-n)y^{1+a}\binom{n}{a}\sum_{i=0}^{a} y^i\binom{n}{i}}{1+y} - \frac{ny\sum_{i=0}^{a} y^{2i}\binom{n}{i}^2}{1+y} + \frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a} y^i\binom{n}{i}\Big)^2 + \frac{(-1+y)\sum_{i=0}^{a} i y^{2i}\binom{n}{i}^2}{1+y}$$

$$\downarrow \quad a = n,\ y = 1$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n\binom{2n}{n}$$

  - $y = -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{-2a^2-2a^3+4an+7a^2n-2n^2-8an^2+3n^3}{2n^2(-1+2n)}\binom{n}{a}^2 + \frac{n\sum_{i=0}^{a}\binom{n}{i}^2}{2(-1+2n)}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{n\binom{2n}{n}}{2(-1+2n)}$$

- $x = 1$
  - $y \neq -1$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n\binom{2n}{n}$$

  - $y = -1$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{n\binom{2n}{n}}{2(-1+2n)}$$

- $x = -1$
  - $y = -1$

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{(-a+n)^2(-1)^a\binom{n}{a}^2}{2n^2} + \frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2 - \frac{\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2}{n}$$

- $x=-1$
  - $y=-1$

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{(-a+n)^2(-1)^a\binom{n}{a}^2}{2n^2} + \frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2 - \frac{\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2}{n}$$

$$\downarrow \quad a=n,\ y=1$$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

- $x = -1$
  - $y = -1$

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{(-a+n)^2(-1)^a\binom{n}{a}^2}{2n^2} + \frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2 - \frac{\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2}{n}$$

$$\downarrow \quad a = n,\, y = 1$$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

  - $y = 1$

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \Big(\frac{(-a+n)\binom{n}{a}\sum_{i=0}^{a}\binom{n}{i}}{n} + \frac{1}{2}\Big(\sum_{i=0}^{a}\binom{n}{i}\Big)^2\Big)(-1)^a - \frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2 + \frac{1}{n}\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2$$

- $x = -1$
  - $y = -1$

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{(-a+n)^2(-1)^a\binom{n}{a}^2}{2n^2} + \frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2 - \frac{\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2}{n}$$

$$\downarrow \quad a = n,\ y = 1$$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

  - $y = 1$

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \Big(\frac{(-a+n)\binom{n}{a}\sum_{i=0}^{a}\binom{n}{i}}{n} + \frac{1}{2}\big(\sum_{i=0}^{a}\binom{n}{i}\big)^2\Big)(-1)^a - \frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2 + \frac{1}{n}\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2$$

$$\downarrow \quad a = n,\ y = 1$$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \begin{cases} 2^{-1+2n} & n \text{ even} \\ -2^{2n-1} + (-1)^{\frac{n+1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

- $x = -1$
  - $y = -1$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases}0 & n \text{ even}\\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd}\end{cases}$$

  - $y = 1$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \begin{cases}2^{-1+2n} & n \text{ even}\\ -2^{2n-1}+(-1)^{\frac{n+1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd}\end{cases}$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\begin{aligned}
\sum_{k=0}^{n} (\bar{x}^2)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 &= \frac{q^2 \bar{x}^3 (1+q^2-2q\bar{x}^2) \left(-\frac{1}{q\bar{x}};q\right)_{1+n} \left(-\frac{\bar{x}}{q};q\right)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})(-q+\bar{x}^2)(-1+q\bar{x}^2)} \\
&+ \frac{\left(-\frac{1}{q};q\right)_{1+n} \left(\frac{1}{q};q^2\right)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q;q)_{1+n}} (-q^3 - q^3\bar{x} - q^4\bar{x} - q^3\bar{x}^2) \\
&+ \frac{\left(-\frac{1}{q};q\right)_{1+n} \left(\frac{1}{q};q^2\right)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q;q)_{1+n}} (q^4 + q^3\bar{x} + q^4\bar{x} + q^4\bar{x}^2) q^n \\
&+ \frac{q^2 \left(\left(-\frac{1}{q\bar{x}};q\right)_{1+n}\right)^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2} \bar{x}^{4+2n} \\
&+ \frac{q^2(1+q^2)\bar{x} \left(-\frac{1}{q\bar{x}};q\right)_{1+n} \left(-\frac{\bar{x}}{q};q\right)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^{n} \frac{\left(-\frac{1}{q};q\right)_{1+i} \left(\frac{1}{q};q^2\right)_{1+i}}{(q;q)_{1+i} \left(-\frac{1}{q\bar{x}};q\right)_{1+i} \left(-\frac{\bar{x}}{q};q\right)_{1+i}} \\
&- \frac{q^3(1+q^2)\bar{x} \left(-\frac{1}{q\bar{x}};q\right)_{1+n} \left(-\frac{\bar{x}}{q};q\right)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^{n} \frac{\left(-\frac{1}{q};q\right)_{1+i} \left(\frac{1}{q};q^2\right)_{1+i}}{(q+q^i)(q;q)_{1+i} \left(-\frac{1}{q\bar{x}};q\right)_{1+i} \left(-\frac{\bar{x}}{q};q\right)_{1+i}}
\end{aligned}$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 =?$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\begin{aligned}
\sum_{k=0}^{n} (\bar{x}^2)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 &= \frac{q^2\bar{x}^3(1+q^2-2q\bar{x}^2)(-\frac{1}{q\bar{x}};q)_{1+n}(-\frac{\bar{x}}{q};q)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})(-q+\bar{x}^2)(-1+q\bar{x}^2)} \\
&+ \frac{(-\frac{1}{q};q)_{1+n}(\frac{1}{q};q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q;q)_{1+n}}(-q^3-q^3\bar{x}-q^4\bar{x}-q^3\bar{x}^2) \\
&+ \frac{(-\frac{1}{q};q)_{1+n}(\frac{1}{q};q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q;q)_{1+n}}(q^4+q^3\bar{x}+q^4\bar{x}+q^4\bar{x}^2)q^n \\
&+ \frac{q^2((-\frac{1}{q\bar{x}};q)_{1+n})^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2}\bar{x}^{4+2n} \\
&+ \frac{q^2(1+q^2)\bar{x}(-\frac{1}{q\bar{x}};q)_{1+n}(-\frac{\bar{x}}{q};q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^{n} \frac{(-\frac{1}{q};q)_{1+i}(\frac{1}{q};q^2)_{1+i}}{(q;q)_{1+i}(-\frac{1}{q\bar{x}};q)_{1+i}(-\frac{\bar{x}}{q};q)_{1+i}} \\
&- \frac{q^3(1+q^2)\bar{x}(-\frac{1}{q\bar{x}};q)_{1+n}(-\frac{\bar{x}}{q};q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^{n} \frac{(-\frac{1}{q};q)_{1+i}(\frac{1}{q};q^2)_{1+i}}{(q+q^i)(q;q)_{1+i}(-\frac{1}{q\bar{x}};q)_{1+i}(-\frac{\bar{x}}{q};q)_{1+i}}
\end{aligned}$$

- $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}$: similar

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r \; [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r\ [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

E.g., $x = q$, $y = 1$

$$\sum_{k=0}^{a} q^k \left(\sum_{i=0}^{k} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = \frac{-1 + 2q^{1+a} - q^n}{2(-1+q)} \Big(\big(\sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \big)^2$$

$$- \big(-1 + q^n \big) \sum_{i=0}^{a} q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$

$$+ 2\big(-q^a + q^n \big) q^{\frac{1}{2}(-1+a)a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix} \Big)$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r \; [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

E.g., $x = q$, $y = 1$

$$\sum_{k=0}^{a} q^k \left(\sum_{i=0}^{k} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = \frac{-1+2q^{1+a}-q^n}{2(-1+q)} \Big(\big(\sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}\big)^2$$

$$- \big(-1+q^n\big) \sum_{i=0}^{a} q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$

$$+ 2\big(-q^a+q^n\big) q^{\frac{1}{2}(-1+a)a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix} \Big)$$

$$\downarrow \quad a = n$$

$$\Bigg(q^2\big(-\frac{1}{q};q\big)_{1+n} \big((1+q)\big(-1+q^n\big)\big(-1+q^{1+n}\big)\big(\frac{1}{q};q^2\big)_{1+n}$$
$$+(-1+q)\big(-1+q^n(-1+2q)\big)\big(-\frac{1}{q};q\big)_{1+n}(q;q)_{1+n}\big)\Bigg) / (2(-1+q)^2(1+q)^2(q;q)_{1+n})$$

q-Case 2: $$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r\ [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

E.g., $x = q$, $y = q$

$$\sum_{k=0}^{a} q^k \left(\sum_{i=0}^{k} q^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \Bigg(\big(-1 + (1+q) q^{1+a} - q^{1+n} \big) \left(\sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2$$
$$+ \left(1 - q^{1+n}\right) \sum_{i=0}^{a} q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1+q) \sum_{i=0}^{a} q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$
$$+ 2\big(-q^a + q^n \big) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \Bigg) \Big/ (-1+q)(1+q)$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r\ [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

E.g., $x = q$, $y = q$

$$\sum_{k=0}^{a} q^k \left(\sum_{i=0}^{k} q^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \Bigg(\left(-1 + (1+q)q^{1+a} - q^{1+n} \right) \left(\sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2$$

$$+ \left(1 - q^{1+n}\right) \sum_{i=0}^{a} q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1+q) \sum_{i=0}^{a} q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$

$$+ 2\left(-q^a + q^n\right) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \Bigg) \Big/ (-1+q)(1+q)$$

$$\downarrow \quad a = n$$

$$\Bigg((-1+q)^2 \left(-1 + q^{2+n}\right) \left((-1;q)_{1+n}\right)^2 - \frac{2(-1+q)^2 \left(1+q^{2+2n}\right) (-1;q)_{1+n} (q;q^2)_{1+n}}{\left(-1+q^{1+n}\right)(q;q)_{1+n}}$$

$$+ \frac{4(1+q^2) + \left((q;q)_{1+n}\right)^2}{-1+q^{1+n}} + \frac{4(-1+q)^2 q \left((q;q)_{1+n}\right)^2}{-1+q^{1+n}} \sum_{i=1}^{n} \frac{q^i (-1;q)_{1+i} (q;q^2)_{1+i}}{\left((q;q)_{1+i}\right)^3} \Bigg) \Big/ \left(4(-1+q)^3(1+q)\right)$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1} = (2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2} = (2+n)2^{-1+2n} - \frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3} = (2+n)2^{-1+3n} - 3n2^{-2+n}\binom{2n}{n}$$

Sigma