Kolchin Seminar in Differential Algebra

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An Algorithmic Difference Ring Theory for Symbolic Summation

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Der Wissenschaftsfonds.

Outline of the talk:

Part 1: A warm-up example

Part 2: The underlying framework: difference ring theory

Part 3: The simplification of Feynman integrals

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \quad (=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

↑ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

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Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[1]:= $<<$ **Sigma.m**

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In[2]:=
$$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]=
$$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= **SigmaLimit[res, {n}, a]**

Out[4]= $$\frac{1}{n!}\frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Telescoping

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

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no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n, k)$

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\,f(n,k) + \sum_{k=1}^{a} c_1(n)\,f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

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$$||\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} \in \boxed{\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $$\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

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In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k]+S[1,n]-S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\}\}$

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

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Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

> Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

> HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

> EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}]}$

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In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

Sigma.m is based on difference ring/field theory

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The underlying framework: difference ring theory

[a gentle introduction]

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

Simplify

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}}[s]}_{\text{polynomial ring}}$

Simplify
$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{array}{rcl} \mathrm{ev}' : \quad \mathbb{Q}(x) \times \mathbb{N} & \to & \mathbb{Q} \\ (\frac{p(x)}{q(x)}, k) & \mapsto & \begin{cases} \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

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\end{aligned}$$

$$\begin{aligned}
\mathrm{ev} : \quad & \mathbb{A} \times \mathbb{N} && \to \quad \mathbb{Q} \\
& \left(\sum_{i=0}^{d} f_i\, s^i, k\right) && \mapsto \quad \sum_{i=0}^{d} \mathrm{ev}'(f_i, k) S_1(k)^i \qquad \mathrm{ev}(s,k) = S_1(k)
\end{aligned}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f,k) \rangle_{k \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \qquad = \quad \langle 0,1,2,3,\dots \rangle \langle 0,1,\tfrac{1}{2},\tfrac{1}{3},\dots \rangle$$

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$$||$$
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$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \\ & & \| \\ & & \langle 0, 1, 1, 1, \dots \rangle \\ & & \nparallel \\ \tau(x \, \frac{1}{x}) = \tau(1) & = & \langle 1, 1, 1, 1, \dots \rangle \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0,1,2,3,\dots\rangle\langle 0,1,\frac{1}{2},\frac{1}{3},\dots\rangle \\ & & || \\ & & \langle 0,1,1,1,\dots\rangle \\ & & || \\ \tau(x\,\frac{1}{x}) = \tau(1) & = & \langle 1,1,1,1,\dots\rangle \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{llcl} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

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$$\sigma : \quad \mathbb{Q}(x)[s] \to \mathbb{Q}(x)[s]$$

$$s \mapsto s + \frac{1}{x+1}$$

$$S_1(k+1) = S_1(k) + \frac{1}{k+1}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

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$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\Big(s + \frac{1}{x+1}\Big)^i \end{array}$$

$$s \mapsto s + \tfrac{1}{x+1}$$

$$S_1(k+1) = S_1(k) + \tfrac{1}{k+1}$$

Definition: $(\mathbb{A}, \sigma)$ is a difference ring with constant set

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

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$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\Big(s + \frac{1}{x+1}\Big)^i \end{array}$$

$s \mapsto s + \frac{1}{x+1}$

$S_1(k+1) = S_1(k) + \frac{1}{k+1}$

In this example: $(\mathbb{A}, \sigma)$ is a difference ring with constant set

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), k) = \mathrm{ev}(s + \tfrac{1}{x+1}, k) = S_1(k) + \tfrac{1}{k+1} = \mathrm{ev}(s, k+1)$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify

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τ is an injective difference ring homomorphism:

$$\forall f \in \mathbb{A} \quad \tau(\sigma(f)) = S(\tau(f))$$

Simplify

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τ is an injective difference ring homomorphism:

$$\forall f \in \mathbb{A} \quad \tau(\sigma(f)) = S(\tau(f))$$

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \simeq \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(k)\rangle_{k\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

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Given: $F(k) = S_1(k)$

Find: $G = \langle G(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$G(k+1) - G(k) = S_1(k)$$

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CONSTRUCT a difference ring $(\mathbb{A}, \sigma)$ with ev for $F(k)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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CONSTRUCT a difference ring $(\mathbb{A}, \sigma)$ with ev for $F(k)$:

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Sk!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}]}$$

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$$\begin{aligned}
\sigma(p_1) &= a_1\,p_1 && a_1 \in \mathbb{K}(x)^*\\
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\end{aligned}$$

$$\mathbf{(-1)^k} \quad \leftrightarrow \quad \sigma(\mathbf{z}) = -\mathbf{z} \qquad \mathbf{z^2 = 1}$$

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γ is a primitive λth root of unity $\quad \gamma^{\mathbf{k}} \leftrightarrow \boldsymbol{\sigma(z) = \gamma\, z} \qquad \mathbf{z^{\lambda} = 1}$

$$\mathcal{S} S_1(k) = S_1(k) + \tfrac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \tfrac{1}{x+1}$$

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CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with ev for $F(k)$: (Karr81,CS16,CS17)

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such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

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(nested) su

$e, p_e^{-1}][z]$

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GIVEN $f \in \mathbb{A}$ with $\text{ev}(f, k) = F(k)$;
FIND, in case of existence, a $g \in \mathbb{A}$ such that

$$\sigma(g) - g = f.$$

Represent sums

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant set

$$\text{const}_\sigma \mathbb{A} := \{k \in \mathbb{A} | \sigma(k) = k\}.$$

Note 1: $\text{const}_\sigma \mathbb{A}$ is a ring that contains $\mathbb{Q}$

Note 2: We always take care that $\text{const}_\sigma \mathbb{A}$ is a field

Represent sums

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Such a difference ring extension $(\mathbb{A}[t], \sigma)$ of $(\mathbb{A}, \sigma)$ is called Σ^*-extension

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There are 2 cases:

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There are 2 cases:

1. $\boxed{\nexists g \in \mathbb{A} : \ \sigma(g) = g + f}$: $(\mathbb{A}[t], \sigma)$ is a Σ^*-extension of $(\mathbb{A}, \sigma)$
2. $\boxed{\exists g \in \mathbb{A} : \sigma(g) = g + f}$: No need for a Σ^*-extension!

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$$\begin{aligned}\sigma(p_1) &= a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2) &= a_2p_2 && a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*\\ &\vdots\\ \sigma(p_e) &= a_ep_e && a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*\end{aligned}$$

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Then $\text{const}_\sigma\mathbb{A}[t, t^{-1}] = \text{const}_\sigma\mathbb{A}$ iff

$$\nexists g \in \mathbb{A} \setminus \{0\}\, \nexists n \in \mathbb{Z} \setminus \{0\} : \quad \boxed{\sigma(g) = a^n\, g}$$

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Such a difference ring extension $(\mathbb{A}[t, t^{-1}], \sigma)$ of $(\mathbb{A}, \sigma)$ is called Π-extension

Represent products

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field
$$\text{const}_\sigma\mathbb{A} := \{k \in \mathbb{A} | \sigma(k) = k\}.$$
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The hypergeometric case

- Take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma|_{\mathbb{K}} = \text{id}$ and $\sigma(x) = x + 1$.
- Let $\alpha_1, \dots, \alpha_r \in \mathbb{K}(x)^*$

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  - $\frac{\sigma(t_i)}{t_i} \in \mathbb{K}(x)^*$ for $1 \leq i \leq e$
  - $\sigma(z) = \gamma\, z$ and $z^\lambda = 1$ for some primitive λth root of unity $\gamma \in \mathbb{K}^*$
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CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$ with ev for $F(k)$: (Karr81,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

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GIVEN $f\in\mathbb{A}$ with $\mathrm{ev}(f,k)=F(k)$;
FIND, in case of existence, a $g\in\mathbb{A}$ such that

$$\sigma(g)-g=f.$$

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda)$$

for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
(δ can be computed explicitly)

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- such that

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- and such that

the arising sums and products in $B(k)$ (except γ^n with $(\gamma^n)^\lambda = 1$)
are **algebraically independent**
(i.e., they do not satisfy any polynomial relation)

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Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\text{ev}(\sigma(s), k) = \text{ev}(s + \tfrac{1}{x+1}, k) = S_1(k) + \tfrac{1}{k+1} = \text{ev}(s, k+1)$$

$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\forall f \in \mathbb{A} \quad \tau(\sigma(f)) = S(\tau(f))$$

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \simeq \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(k)\rangle_{k\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

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$$\mathbb{A} = e_1\, \mathbb{E}_1 \oplus \dots \oplus e_\lambda \mathbb{E}_\lambda.$$

CS, A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS, Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

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Note: Similar results have been worked out in the Galois theory of difference equations (van der Put/Singer, 1997)

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We get $a \in \mathbb{A}$ plus
an embedding τ from $(\mathbb{A}, \sigma)$ into the ring of sequences s.t.

$$\tau(a) = \langle A(\nu) \rangle_{\nu \geq 0}$$

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Reinterpreting a in terms of these nested sums and products yields $B(k)$

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$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2]$$

$\vdots$

such that $\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with ev for $A(k)$: (Karr81,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

Reinterpreting a in terms of these nested sums and products yields $B(k)$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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$$\bigcap$$

$$\tau(\mathbb{A}) = \underbrace{\tau(\mathbb{K}(x))}_{\text{rational seq.}} [\langle\gamma^k\rangle_{k\geq 0}] \underbrace{[\tau(p_1), \tau(p_1^{-1})] \dots [\tau(p_e), \tau(p_e^{-1})]}_{\text{nested products}} \underbrace{[\tau(s_1)] \dots [\tau(s_r)]}_{\text{nested sums}}$$

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Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda)$$

for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
(δ can be computed explicitly)

- and such that

the arising sums and products in $B(k)$ (except γ^n with $(\gamma^n)^\lambda = 1$)
are **algebraically independent**
(i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

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Application 1: the expression $B(k)$ is usually much smaller

Simplification of nested product-sum expressions

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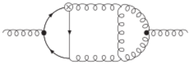
Application 1: the expression $B(k)$ is usually much smaller

Application 2: We solve the zero-recognition problem.

$$A(k) \text{ evaluates to 0 from a certain point on} \quad \Leftrightarrow \quad B(k) = 0$$

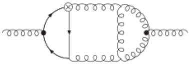
Application: The simplification of Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles

$$\longrightarrow$$

$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

$$\int_0^1 x\,dx = \text{?}$$

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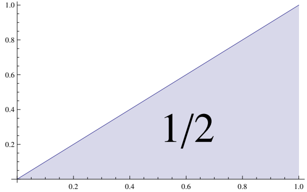
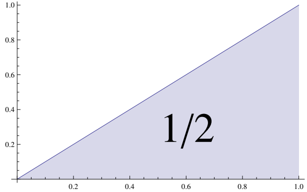
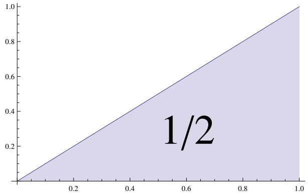


$$\int_0^1 x^1\,dx =$$



$$\int_0^1 x^2\,dx = \text{?}$$

$$\int_0^1 x^1 \, dx =$$



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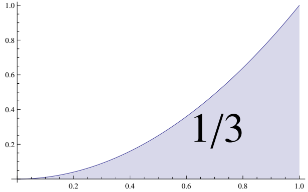

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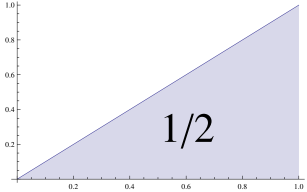

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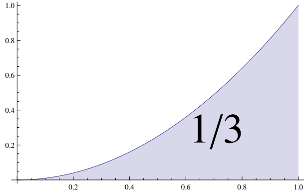

$\int_0^1 x^3 \, dx =$?

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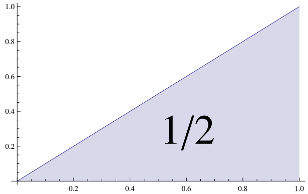

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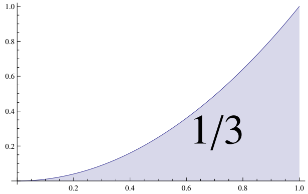
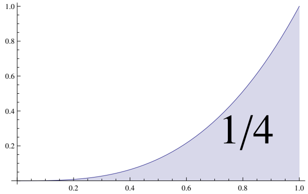


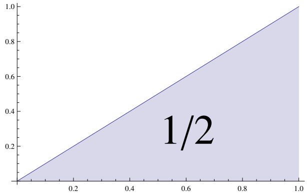
$$\int_0^1 x^3\,dx =$$



$$\boxed{\int_0^1 x^N \, dx = \frac{1}{N+1}}$$

für $N = 1, 2, 3, 4, \ldots$

$$\int_0^1 x^1 \, dx =$$



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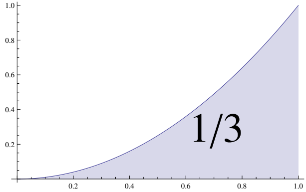

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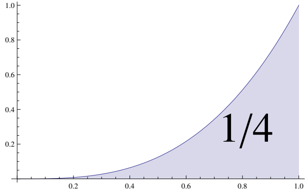


Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1\, dx_2\, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

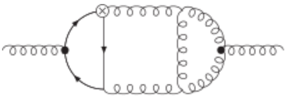
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$||$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

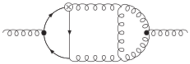
$$\Bigg[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Bigg]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times\left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

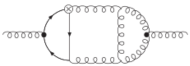
Evaluation of Feynman Integrals



Behavior of particles $\longrightarrow$ $$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



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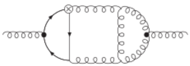
Feynman integrals

DESY
(J. Blümlein)

$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



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(J. Blümlein)

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RISC

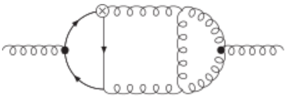
(Sigma-package)

expression in special functions

Example 1:

massive 3-loop ladder integrals

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$\|$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

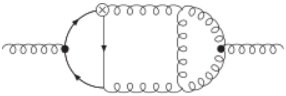
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

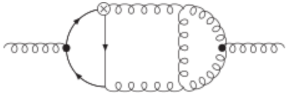
$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k \Big]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times [x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+ \Big(- \frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+ \Big(- \frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

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$$+ \big(- \ldots 1)^N(2N+1) \over N(N+1)} - \frac{13}{N}\big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

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$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \ldots\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\ldots\frac{(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(\ldots 1)^N(2N+1) \ldots N(N+1) - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \Big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \ldots} \ldots 3 \ldots)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \ldots)^N \ldots 1\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)} \ldots$$

$$+ \frac{4(3N \ldots}{N(N+ \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N \ldots}{N(\ldots} \ldots (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(- \ldots}{\ldots} \ldots\Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N \ldots}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

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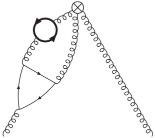
$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

Example 2:

2-mass 3-loop Feynman integrals

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

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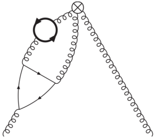

Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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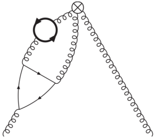

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Typical triple sum:

$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^{k}\times$$

$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

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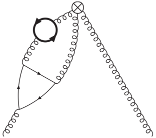

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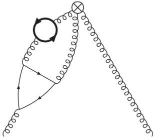
6 hours for this sum

$\sim$ 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

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- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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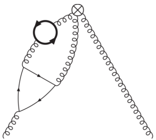

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$\downarrow$ EvaluateMultiSums.m

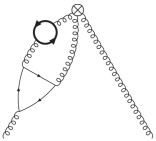
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation		number of indef. sums
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s	(2.1 days)	1188
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s	(11.4 days)	747
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s	(6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s	(8.0 days)	44
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s	(3.5 days)	1933
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s	(8.2 days)	621
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s	(5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s	(2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies ⟶

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

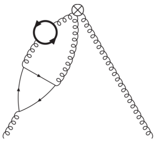
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left(\sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies →

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

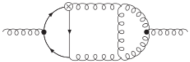
↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

← Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

DESY
(J. Blümlein)

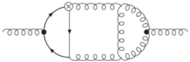
$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY

(J. Blümlein)

LHC at CERN

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expression in special functions

applicable