A UNIFIED ALGORITHMIC FRAMEWORK FOR RAMANUJAN'S CONGRUENCES MODULO POWERS OF 5, 7, AND 11

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Abstract. In 1919 Ramanujan conjectured three infinite families of congruences for the partition function modulo powers of 5, 7, and 11. In 1938 Watson proved the 5-case and a corrected version of the 7-case. In 1967 Atkin proved the remaining 11-family using a method significantly different from Watson's. Finally, in 1983 Gordon found a way to adapt Watson's method for the 11-family. We present a new proof for the 11-family by generalizing Watson's method in a direction different from Gordon's. Until now the case 11 has been viewed as substantially different from 5 and 7. We show that this is not the case by proving the families for $5, 7$ and 11 with the same new algorithmic framework. In addition, we eliminate elements needed in the original proof of Watson. Focusing on eta-quotient representations of modular functions in the new setting, we derived new compact representations of Atkin's basis functions. This, for instance, gives a new simple witness identity for the divisibility of the partition numbers $p(11n + 6)$ by 11.

1. Introduction

We start by quoting A.O.L. Atkin, a pioneer in the use of computers in number theory [6, p. 14]: "Watson's method of modular equations, while theoretically available for the case $p = 11$, does not seem to be so in practice even with the help of present-day computers."

Atkin's statement refers to infinite families of congruences for $p(n)$, the number of partitions of n, which were first considered by Ramanujan [28] in 1919 when he conjectured that for $\ell \in \{5, 7, 11\}$ and $\alpha \geq 1$:

$$p(\ell^\alpha n + \mu_{\alpha,\ell}) \equiv 0 \pmod{\ell^\alpha}, \quad n \geq 0; \tag{1}$$

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here $\mu_{\alpha,\ell}$ is defined to be the smallest positive integer such that $24\mu_{\alpha,\ell} \equiv 1 \pmod{\ell^\alpha}$. Chowla [9] observed using a table by Gupta [16] that the conjecture failed for $\ell = 7$ and $\alpha = 3$. In 1938 Watson [29] proved the conjecture for $\ell = 5$ and all α, and a suitably modified form for $\ell = 7$ and all α. However, it should be noted that according to [7] the task of assigning credit for the proofs for $\ell = 5, 7$ poses an interesting historical challenge concerning the contribution of Ramanujan. Regarding the case $\ell = 11$ Watson states, "Da die Untersuchung der Aussage über 11^α recht langweilig ist, verschiebe ich den Beweis dieses Falles auf eine spätere Abhandlung."

In 1967 Atkin [6] proved (1) for $\ell = 11$. Inspired by Lehner's work [21, 20] he uses an approach significantly different from Watson's. The method has many pieces linked together which makes the proof rather technical. Atkin ends with the comment "We may observe finally that, in comparison with $\ell = 5$ and $\ell = 7$, this proof is indeed "langweilig" as Watson suggested.". In 1983 Gordon [14, p. 108, Thm. 2] proved a result which in a special case implies:

$$p(11^\alpha n + \mu_{\alpha,11}) \equiv 0 \pmod{11^{\alpha+\epsilon}},$$

where ϵ is some integer. By following carefully the proof of Gordon one easily can show that $\epsilon = 0$. Consequently, Gordon in [14] provides another proof of (1) for $\ell = 11$. He concludes by saying that although his method contains a lot of ideas from Atkin [6], it is more a generalization in the spirit of Watson's method. On pages 116 and 117 of [14] Gordon uses some recurrences which are too big to be listed, but which are essential in the proof. Furthermore, in [14] and also in the paper with Hughes [15, p. 112] he needs to compute the structure constants of a certain algebra.

Restricting to modules with polynomial coefficient rings our method provides a much simpler conceptual framework which avoids to deal in any way with structure constants related to multiplication. This feature comes close to Atkin's simplification of Watson's proof for the powers of 5 and 7; Atkin made his unpublished work available to Knopp who published it in Chapter 8 of his book [18]. Another streamlining of Watson's proof was done by Andrews [2]. Besides [7] another rich source concerning the history of Ramanujan's congruences is [8]. With regard to proofs not using modular function machinery in explicit ways, Berndt [8, p. 374] remarks, "A more classically oriented proof for powers of 5 was found by M.D. Hirschhorn and D.C. Hunt [17], while a proof in the same vein for powers of 7 was given by F.G. Garvan [13]."

With respect to modular functions, an important new feature of our approach concerns the use of different modular equations in comparison to those used classically for 5 and 7, and also for 11. Our modular equation gives the action of the U-operator on the basis elements in each of these cases—as in the other proofs. But using our equation this information is obtained in much more direct fashion.

Despite its conceptual simplicity our setting has a strong algorithmic backbone which computationally is somewhat demanding. But the arising complexity, in view of the Atkin quote at the beginning of this introduction, is easily manageable for present-day computers. To prove the 11-case, our proof uses a modular equation of degree 55 and needs 550 fundamental relations to set up the basis for the induction. These 550 relations can be computed in about three days on a standard workstation used in academic environment.

Exactly the same method works for $\ell = 5$ and $\ell = 7$: the modular equation is of degree 5 in the 5-case and of degree 7 in the 7-case; for the induction basis we need 10 fundamental relations for $\ell = 5$ and 14 fundamental relations for $\ell = 7$. These relations can be computed in a couple of seconds; for further details see Section 12.2.

At this point we want to stress two further aspects: (i) all these relations can be (and were) obtained algorithmically; (ii) all the families of Ramanujan congruences modulo powers of $\ell = 5, 7$, and 11 in our setting are proved using *one uniform approach*, which we feel is desirable, for instance, in view of the following statement by Ahlgren and Ono [1, p. 981], "By contrast, the case of powers of 11 is much more difficult."

To underline the uniform nature of our setting, we mention our Conjecture 8.1 which extrapolates the common structure of the modular equations for $\ell = 5, 7,$ and 11 to *all* primes $\ell \geq 5$. More concretely, we conjecture lower bounds for the ℓ-adic valuation of the coefficients of the modular equation. For $\ell = 5, 7,$ and 11 this, in essence, is all what is needed to prove Ramanujan's congruences.

Another uniform aspect concerns the module description of the algebra of modular functions for congruence subgroup $\Gamma_0(\ell)$. Namely, all the subalgebras needed here can be represented as free modules $\langle 1, f_1, \ldots, f_{n_\ell - 1}\rangle_{\mathbb{Z}[z_\ell]}$ over the ring $\mathbb{Z}[z_\ell]$ with fixed modular functions f_i and

$$z_\ell = \left(\frac{\eta(\ell\tau)}{\eta(\tau)}\right)^{\frac{24}{\gcd(\ell-1,24)}},$$

and where $(n_5, n_7, n_{11}) = (1, 1, 5)$.

Our article is structured as follows. In Section 2 and Section 3 we introduce basic notions and provide the necessary modular functions background. In Section 4 we state the Main Theorem, Theorem 4.15, of our paper. It describes the action of the U-operator on quotients of eta function products being crucial for proving the Ramanujan congruences modulo powers of $\ell = 5, 7, 11$; see Corollary 4.16.

The rest of the main part of the paper deals with proving the Main Theorem. In Section 5 we state and prove the Fundamental Lemma, Lemma 5.8, which will play a key role in the proof of the Main Theorem presented in Section 6. In Section 7 we prove Theorem 7.1 which implies the existence of the fundamental polynomials $F_\ell(X, Y)$ stated in Theorem 5.2. Section 8 concludes the main part of the paper; here we state a conjecture on lower bounds for the ℓ-adic valuation of the coefficients of the modular equation for all primes $\ell \geq 5$.

Algorithmic aspects were a major driving force for the development of the material of the main part of this paper. Nevertheless, for better readability we put various constructions and results of algorithmic relevance in a separate part, the Appendix. In fact, the main part up to Section 7 in its essence is independent from the material in the Appendix with the only exception of Section 12 (Appendix 4), where we describe the derivation and proof of the fundamental relations needed to prove the crucial Lemma 6.5.

In Section 9, Appendix 1, we give a detailed description of the derivation of our new representations of Atkin's basis functions g_i using an approach which is close to an algorithm. To this end, we focus on properties of module generators, in particular, the concept of an Atkin basis. In order to work with compact representations of the modular functions involved, we make use of special instances of a trace operator formula. Sections 10 and 11 (Appendix 2 and 3) are included for the sake of completeness. Section 10 presents proofs of formulas for the modular functions F_i, resp. J_i, arising in Section 9. In Section 11 we give an elementary proof of the non-existence of a principal modular function ("Hauptmodul") on $\Gamma_0(11)$.

2. Basic Notions

Notational conventions used throughout are: For $x \in \mathbb{R}$ the symbol $\lceil x \rceil$ ("ceiling" of x) denotes the smallest integer greater or equal to x.

$\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ and $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. As usual, η denotes the Dedekind eta function defined for $\tau \in \mathbb{H}$,

$$\eta(\tau) = q(\tau/24) \prod_{n=1}^{\infty} (1 - q(\tau)^n) \text{ where } q(\tau) = \exp(2\pi i \tau). \tag{2}$$

Frequently we write $q = q(\tau)$. In general, we often suppress writing the argument τ of functions $f(\tau)$ defined on the upper half plane. With regard to q we do this also in view of the fact that, depending on the context, many infinite q-products and q-series can be interpreted as formal power (resp. Laurent) series taking q as an indeterminate.

For $k \in \mathbb{Z}_{>0}$ the function η_k is defined by

$$\eta_k(\tau) := \eta(k\tau), \quad \tau \in \mathbb{H}. \tag{3}$$

Let $f = \sum_{n=N}^{\infty} a_n q^n$ with $N \in \mathbb{Z}$ such that $a_N \neq 0$. Then $N =: \operatorname{ord} f$ is the order of f. More generally, if $t = \sum_{n\geq 1}^{\infty} b_n q^{n/w}, w \in \mathbb{Z}_{>0}$, and $F = f \circ t := \sum_{n=N}^{\infty} a_n t^n$, then $N =: \operatorname{ord}_t f$ is the t-order of f. For example, if $N = \operatorname{ord} f = -1$ and $t = q^2$, then $\operatorname{ord}_t F = -1$ but $\operatorname{ord} F = -2$; if $N = \operatorname{ord} f = -2$ and $t = q^{1/2}$, then $\operatorname{ord}_t F = -2$ but $\operatorname{ord} F = -1$.

The modular group $\mathrm{SL}_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} : ad - bc = 1\}$ acts on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau := \frac{a\tau+b}{c\tau+d}$; this action is inherited by the congruence subgroups

$$\Gamma_0(\ell) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \ell | c\right\},$$

where $\ell \in \mathbb{Z}_{>0}$. Note that $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. These subgroups have finite index in $\mathrm{SL}_2(\mathbb{Z})$:

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(\ell)] = \ell \prod_{\text{prime } p|\ell} \left(1 + \frac{1}{p}\right), \quad \ell \geq 2; \tag{4}$$

see the standard literature on modular forms like [11] or [10]. Particularly related to our context are [18] and [26].

The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ extends to an action on meromorphic functions $f : \mathbb{H} \to \hat{\mathbb{C}}$. With regard to the action of the $W = \left(\begin{smallmatrix} 0 & -1 \\ 11 & 0 \end{smallmatrix}\right)$ operator, we define the slightly more general action of the general linear group $\mathrm{GL}_2(\mathbb{Z})$: for all $\tau \in \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$,

$$(f \mid \gamma)(\tau) := f(\gamma\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right).$$

3. Modular Functions

To make this article as much self-contained as possible, in this section we recall some facts about modular functions.

In this article a *modular function for* $\Gamma_0(\ell)$ is: (i) a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that (ii) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\ell)$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau), \quad \tau \in \mathbb{H},$$

and (iii) if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ then $f\left(\frac{a\tau+b}{c\tau+d}\right)$ admits a Laurent series expansion in powers of $q^{\gcd(c^2,\ell)/\ell}$ with finite principal part. We will use the notation $w_\ell(c) := \ell/\gcd(c^2,\ell)$, and $M(\ell)$ for the set of modular functions for $\Gamma_0(\ell)$.

By (iii) with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, any $f \in M(\ell)$ admits a Laurent series expansion in powers of q with finite principal part; i.e.,

$$f(\tau) = \sum_{n=-N}^{\infty} f_n q^n. \tag{5}$$

Hence one can extend f to $\mathbb{H} \cup \{\infty\}$ by defining $f(\infty) := \infty$, if $N > 0$, and $f(\infty) := f_0$, otherwise. Subsequently, a Laurent expansion of f as in (5) will be also called *q-expansion of f at infinity.*

Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $f \in M(\ell)$, consider the Laurent series expansion of $f(\gamma\tau)$ in powers of $q^{1/w_\ell(c)}$,

$$f(\gamma\tau) = \sum_{n=-M}^{\infty} g_n q^{n/w_\ell(c)}. \tag{6}$$

In view of $\gamma\infty = \lim_{\mathrm{Im}(\tau)\to\infty} \gamma\tau = a/c$, we say that (6) is the *q-expansion of f at a/c.* Understanding that $a/0 = \infty$, this also covers the definition of q-expansions at ∞. Concerning uniqueness of such expansions, let $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma'\infty = \gamma\infty = a/c$. Then the q-expansion of $f(\gamma'\tau)$ differs from that of $f(\gamma\tau)$ only by a root-of-unity factor in the coefficients. Namely, then $\gamma' = \gamma \left(\begin{smallmatrix} \pm 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$, which implies

$$f(\gamma'\tau) = \sum_{n=-M}^{\infty} g_n \left(e^{2\pi i m/w_\ell(c)}\right)^n q^{n/w_\ell(c)}.$$

As a consequence, one can extend f from $\mathbb{H}$ to $\hat{\mathbb{H}} := \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ by defining $f(a/c) := (f \circ \gamma)(\infty)$ where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is chosen such that $\gamma\infty = a/c$. Another consequence is that the q-expansions of f at ∞ are uniquely determined owing to

$$\gamma\infty = \infty \Leftrightarrow \gamma = \left(\begin{smallmatrix} \pm 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right) \text{ and } w_\ell(0) = 1. \tag{7}$$

In the next step, the action of $\mathrm{SL}_2(\mathbb{Z})$, and thus of $\Gamma_0(\ell)$, is extended in an obvious way to an action on $\hat{\mathbb{H}}$. The orbits of the $\Gamma_0(\ell)$ action are denoted by

$$[\tau]_\ell := \{\gamma\tau : \gamma \in \Gamma_0(\ell)\}, \quad \tau \in \hat{\mathbb{H}}.$$

In many cases, ℓ will be clear from the context and we will write $[\tau]$ instead of $[\tau]_\ell$. The set of all such orbits is denoted by

$$X_0(\ell) := \{[\tau]_\ell : \tau \in \hat{\mathbb{H}}\}.$$

The $\Gamma_0(\ell)$ action maps $\mathbb{Q} \cup \{\infty\}$ to itself, and owing to (4) each $\Gamma_0(\ell)$ produces only finitely many orbits $[\tau]_\ell$ with $\tau \in \mathbb{Q} \cup \{\infty\}$; such orbits are called *cusps* of $X_0(\ell)$.

Lemma 3.1. *For any prime p:*

(1) $X_0(p)$ *has two cusps,* $[\infty]_p$ *and* $[0]_p$;

(2) $X_0(p^2)$ *has* $p+1$ *cusps,* $[\infty]_{p^2}, [0]_{p^2}$, *and* $[k/p]_{p^2},\ k = 1, \dots, p-1$.

Proof. This fact can be found in many sources; a detailed description of how to construct a set of representatives for the cusps of $\Gamma_0(\ell)$, for instance, is given in [27, Lemma 5.3]. □

Suppose the domain of $f \in M(\ell)$ is extended from $\mathbb{H}$ to $\hat{\mathbb{H}}$ as described above. Then the resulting function f (using the same notation for the extended function) gives rise to a function $f^* : X_0(\ell) \to \hat{\mathbb{C}}$, which is defined as follows:

$$f^*([\tau]_\ell) := f(\tau), \quad \tau \in \hat{\mathbb{H}}.$$

The fact that f^* is well-defined follows from our previous discussion. We say that f^* is induced by f.

As described in detail in [11], $X_0(\ell)$ can be equipped with the structure of a compact Riemann surface. This analytic structure turns the induced functions f^* into meromorphic functions on $X_0(\ell)$ having possible poles only at the cusps of $X_0(\ell)$. The following classical lemma [22, Thm. 1.37], a Riemann surface version of Liouville's theorem, is crucial for zero recognition of modular functions:

Lemma 3.2. *Let X be a compact Riemann surface. Suppose that $g : X \to \mathbb{C}$ is an analytic function on all of X. Then g is a constant function.*

As already mentioned, possible poles of f^* can only sit at cusps. More precisely, owing to the definition of induced functions f^*, all possible poles of f^* can be spotted by checking whether $f^*([a/c]) = f(a/c) = \infty$ for $a/c \in \mathbb{Q} \cup \{\infty\}$. Because of (4), $\mathbb{Q} \cup \{\infty\}$ splits only into a *finite* number of cusps,

$$\mathbb{Q} \cup \{\infty\} = [a_1/c_1]_\ell \cup \dots \cup [a_k/c_k]_\ell.$$

Hence knowing all the cusps $[a_j/c_j]$ reduces the task of finding all possible poles to the inspection of q-expansions of f at a_j/c_j; i.e., of q-expansions of $f(\gamma_j \tau)$ as in (6) with $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_j \infty = a_j/c_j$. We call these expansions also local q-expansions of f^* at the cusps $[a_j/c_j]_\ell$; $w_\ell(c_j)$ is called the width of the cusp $[a_j/c_j]_\ell$. It is straightforward to show that it is independent of the choice of the respresentative a_j/c_j of the cusp $[a_j/c_j]_\ell$, and that $w_\ell(c_j) = \ell / \gcd(c_j^2, \ell)$ for relatively prime a_j and c_j. Note that $[\infty]_\ell = [1/0]_\ell$.

The order $\mathrm{ord}_{[a/c]_\ell} f^*$ of f^* at a cusp $[a/c]_\ell$ is defined to be the $q^{1/w_\ell(c)}$-order of a local q-expansion of f^* at $[a/c]_\ell$; i.e.,

$$\mathrm{ord}_{[a/c]_\ell} f^* := \mathrm{ord}_{q^{1/w_\ell(c)}}(f \mid \gamma) \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

It is straightforward to verify that $\mathrm{ord}_{[a/c]} f^*$ is well-defined.

Example 3.3. [18, Ch. 7, Thm. 1] Consider

$$z_5(\tau) := \Big(\frac{\eta(5\tau)}{\eta(\tau)}\Big)^6 = q\prod_{j=1}^{\infty}\Big(\frac{1-q^{5j}}{1-q^j}\Big)^6 = q + 6q^2 + 27q^3 + 98q^4 + \cdots . \tag{8}$$

Observing that $5 \cdot \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\tau = \left(\begin{smallmatrix} a & 5b \\ c/5 & d\end{smallmatrix}\right)(5\tau)$ together with a straightforward application of the η transformation formula [3, Thm. 3.4] shows that $z_5(\gamma\tau) = z_5(\tau)$ for all $\gamma \in \Gamma_0(5)$ and

$$z_5(T\tau) = z_5\Big(-\frac{1}{\tau}\Big) = \frac{5^{-3}}{z_5(\tau/5)} = \frac{1}{5^3}\Big(\frac{1}{q^{1/5}} - 6 + 9q^{1/5} + 10q^{2/5} - \dots\Big), \tag{9}$$

where $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. For p a prime, $X_0(p)$ has exactly two cusps $[\infty]_p$ and $[0]_p$ with widths 1 and 5, respectively; see [18, Ch. 2, Sect. 2]. Hence (8) and (9) are the local q-expansions of z_5^* at these cusps with

$$\operatorname{ord}_{[\infty]_5} z_5^* = \operatorname{ord}_q z_5 = 1 \quad \text{and} \quad \operatorname{ord}_{[0]_5} z_5^* = \operatorname{ord}_{q^{1/5}} z_5(-1/\tau) = -1. \tag{10}$$

Obviously, z_5 has no zero in $\mathbb{H}$. Consequently, the orders in (10) tell that z_5^* has its only zero of order 1 at $[\infty]_5$ and its only pole, also of order 1, at $[0]_5$. This is also in accordance with Lemma 3.4 which says that, with regard to counting orders, induced functions f^* have exactly as many zeroes as poles:

Lemma 3.4. *For any $f \in M(\ell)$ the induced function $f^* : X_0(\ell) \to \hat{\mathbb{C}}$ is surjective and*

$$\sum_{[\tau]_\ell} \operatorname{ord}_{[\tau]_\ell} f^* = 0,$$

where the sum runs over all cusps of $X_0(\ell)$.

The lemma is implied by another fundamental fact from compact Riemann surfaces; see, for instance, [22, Prop. 4.12].

The functions z_5 are embedded in a general class of modular functions treated in detail in [18, Ch. 7, Thm. 1]. In our setting we need a subclass defined as

$$z_\ell(\tau) := \Big(\frac{\eta(\ell\tau)}{\eta(\tau)}\Big)^{\frac{24}{\gcd(\ell-1,12)}}, \quad \ell \text{ a prime } \geq 5. \tag{11}$$

Lemma 3.5. *Let $\ell \geq 5$ be a prime,*

(1) $$z_\ell \in M(\ell);$$

(2) $$\operatorname{ord}_{[\infty]_\ell} z_\ell^* = \frac{\ell-1}{\gcd(\ell-1,12)} = -\operatorname{ord}_{[0]_\ell} z_\ell^*;$$

(3) $$z_\ell\Big(-\frac{1}{\tau}\Big) z_\ell\Big(\frac{\tau}{\ell}\Big) = \ell^{-\frac{12}{\gcd(\ell-1,12)}}.$$

Proof. [18, Ch. 7, Thm. 1]. $\square$

We note that

$$\mathrm{ord}_{[\infty]_5} z_5^* = \mathrm{ord}_{[\infty]_7} z_7^* = 1, \quad \text{whereas} \quad \mathrm{ord}_{[\infty]_{11}} z_{11}^* = 5.$$

Obviously, the modular functions in $M(\ell)$ form a $\mathbb{C}$-algebra; i.e., a commutative ring with 1 which is also a vector space over $\mathbb{C}$. In our context, the subalgebras $M_{\mathbb{Z}}(\ell)$, resp. $M_{\mathbb{Q}}(\ell)$, of modular functions in $M(\ell)$ with *integer*, resp. *rational*, coefficients in their q-expansions at ∞, which by (7) are uniquely determined, will be relevant too. Other important subrings are those having a pole only at 0,

$$M^0(\ell) := \{f \in M(\ell) : \mathrm{ord}_{[\infty]_\ell} f^* \geq 0\},$$

and those having a pole only at ∞,

$$M^\infty(\ell) := \{f \in M(\ell) : \mathrm{ord}_{[0]_\ell} f^* \geq 0\}.$$

When we want to focus on subalgebras with functions having q-expansions at ∞ with coefficients from $R := \mathbb{Z}$ or $R := \mathbb{Q}$ we use the notation

$$M_R^0(\ell) := M^0(\ell) \cap M_R(\ell), \text{ resp. } M_R^\infty(\ell) := M^\infty(\ell) \cap M_R(\ell).$$

Example 3.6. By Lemma 3.5: $z_\ell \in M_{\mathbb{Z}}^0(\ell)$ for ℓ a prime.

Despite the naturally given ring structure of these function domains, inspired by Atkin [6], Gordon [14], and Newman [24] and [25] we will make fundamental use of representing these rings as freely generated modules.

Example 3.7. For $\ell = 5$ and $\ell = 7$ the module structure for $M_{\mathbb{Z}}^0(\ell)$ boils down to the naturally given ring structure. Namely, because of $\mathrm{ord}_{[0]_5} z_5^* = \mathrm{ord}_{[0]_7} z_7^* = -1$, together with the fact that z_5 and z_7 are analytic in $\mathbb{H}$, it is easily verified that

$$M_{\mathbb{Z}}^0(5) = \{p(z_5) : p(z_5) \in \mathbb{Z}[z_5]\} = \mathbb{Z}[z_5] \text{ and } M_{\mathbb{Z}}^0(7) = \mathbb{Z}[z_7].$$

This means, $M_{\mathbb{Z}}^0(\ell)$ for $\ell = 5$ and $\ell = 7$ is a module over the ring $\mathbb{Z}[z_\ell]$ with one module generator, the constant function 1. In other words, $M_{\mathbb{Z}}^0(5)$ and $M_{\mathbb{Z}}^0(7)$ are polynomials rings in z_5 and z_7, respectively, with integer coefficients.

Example 3.8. Let $\mathbb{Z}[z_{11}]$ denote the ring of polynomials in z_{11} with integer coefficients. Atkin [6, Lem. 3] proved that $M_{\mathbb{Z}}^0(11)$ can be represented as a $\mathbb{Z}[z_{11}]$-module which is freely generated by modular functions g_2, g_3, g_4, and $g_6 \in M_{\mathbb{Z}}^0(11)$ where the g_i are as in [6], resp. (100) below. Notationally,

$$\begin{aligned} M_{\mathbb{Z}}^0(11) = \big\{&p_0(z_{11}) + p_2(z_{11})g_2 + p_3(z_{11})g_3 \\ &+ p_4(z_{11})g_4 + p_6(z_{11})g_6 : p_i(z_{11}) \in \mathbb{Z}[z_{11}]\big\} \\ =:\ &\langle 1, g_2, g_3, g_4, g_6\rangle_{\mathbb{Z}[z_{11}]}. \end{aligned} \tag{12}$$

Atkin [6] defined the functions g_i in a skillful manner by following Newman [23]. It turns out that using "summing the even part",

$$U_2 \sum_{k=N}^{\infty} a(k)q^k := \sum_{k=\lceil N/2\rceil}^{\infty} a(2k)q^k,$$

the special case $m = 2$ of the U-operator in Definition 4.2, the g_i can be represented in a simple and compact manner:

Theorem 3.9. *Let*

$$f_2(\tau) := q^{-2} \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{2n})^3}{(1-q^{11n})^3(1-q^{22n})} \tag{13}$$

and

$$f_3(\tau) := q^{-3} \prod_{n=1}^{\infty} \frac{(1-q^n)^3(1-q^{2n})}{(1-q^{11n})(1-q^{22n})^3}. \tag{14}$$

Then

$$g_2 = U_2\frac{1}{f_2} - \frac{1}{f_3}, \quad g_3 = \frac{4}{11}U_2\frac{1}{f_3} - \frac{1}{11}\frac{1}{f_2}, \quad g_4 = -\frac{1}{2}g_3 + \frac{1}{2}U_2\frac{1}{f_2^2} + \frac{1}{f_3^2}, \tag{15}$$

and

$$g_6 = z_{11} - \frac{8}{11^2}g_4 + \frac{8}{11^2}U_2\frac{1}{f_3^2} + \frac{1}{11^2}\frac{1}{f_2^2}. \tag{16}$$

In the Appendix we prove this theorem; see Theorem 9.8 together with (100). More generally, in Section 9 of the Appendix we describe how to derive these representations using an approach which is close to an algorithm. This derivation among other things also explains that in the choice of (16) there is some freedom. For instance, one could omit the summand z_{11} there.

4. The Main Theorem

To state the main theorem of this paper, Theorem 4.15, we need some preparations.

Notation. It will be convenient to use the following shorthand (as in Lemma 9.6) for the multiplicative inverse of functions:

$$\bar{f} := \frac{1}{f}.$$

For a prime $\ell \geq 5$ one of the key players will be the following modular function and its multiplicative inverse,

$$u_\ell(\tau) := \frac{\eta(\tau)}{\eta(\ell^2\tau)}. \tag{17}$$

Lemma 4.1. *For any prime $\ell \geq 5$,*

$$u_\ell \in M_{\mathbb{Z}}^{\infty}(\ell^2) \ \ \textit{and} \ \bar{u}_\ell \in M_{\mathbb{Z}}^{0}(\ell^2); \tag{1}$$

$$\begin{aligned} &\operatorname{ord}_{[\infty]_{\ell^2}} u_\ell^* = -\frac{\ell^2-1}{24} = -\operatorname{ord}_{[0]_{\ell^2}} u_\ell^* \quad \textit{and} \\ &\operatorname{ord}_{[k/\ell]_{\ell^2}} u_\ell^* = 0,\ k = 1, \dots, \ell-1. \end{aligned} \tag{2}$$

Proof. The statements are implied by [26, Thms. 1.64 and 1.65]. □

We will use the U-operator in its usual definition:

Definition 4.2. *For $f : \mathbb{H} \to \mathbb{C}$ and $m \in \mathbb{Z}_{>0}$ we define $U_m(f) : \mathbb{H} \to \mathbb{C}$ by*

$$U_m(f)(\tau) := \frac{1}{m} \sum_{\lambda=0}^{m-1} f\Big(\frac{\tau+\lambda}{m}\Big), \quad \tau \in \mathbb{H}.$$

If f has period 1 and a q-expansion with principal part involving only finitely many q-powers, then one has:

$$f(\tau) = \sum_{k=N}^{\infty} a(k) q^k \ \Rightarrow \ (U_m f)(\tau) = \sum_{k=\lceil N/m \rceil}^{\infty} a(mk) q^k, \tag{18}$$

and for any $g : \mathbb{H} \to \mathbb{C}$,

$$U_m(f(m\tau)g(\tau)) = f(\tau) U_m(g(\tau)). \tag{19}$$

Example 4.3.

$$\begin{aligned} U_{11}(\bar{u}_{11}) &= U_{11} \frac{\eta(11^2\tau)}{\eta(\tau)} = \prod_{j=1}^{\infty}(1-q^{11j}) U_{11}\Big(\sum_{k=0}^{\infty} p(k) q^{k+5}\Big) \\ &= \prod_{j=1}^{\infty}(1-q^{11j}) U_{11}\Big(\sum_{k\geq 5}^{\infty} p(k-5) q^k\Big) = \prod_{j=1}^{\infty}(1-q^{11j}) \sum_{k=\lceil 5/11 \rceil}^{\infty} p(11k-5) q^k \\ &= q \prod_{j=1}^{\infty}(1-q^{11j}) \sum_{k=0}^{\infty} p(11k+6) q^k. \end{aligned}$$

Obviously U_m is linear (over $\mathbb{C}$); in addition, it is easy to verify that

$$U_{mn} = U_m \circ U_n = U_n \circ U_m, \quad m, n \in \mathbb{Z}_{>0}. \tag{20}$$

The U-operator turns modular functions for the full modular group into functions from $M(p)$, p a prime; but more is true:

Lemma 4.4. *For any prime p:*

$$f \in M(1) \Rightarrow U_p(f) \in M(p). \tag{1}$$

If $p \mid N \in \mathbb{Z}_{>0}$:

$$f \in M(N) \Rightarrow U_p(f) \in M(N). \tag{2}$$

If $p^2 \mid N \in \mathbb{Z}_{>0}$:

$$f \in M(N) \Rightarrow U_p(f) \in M\Big(\frac{N}{p}\Big). \tag{3}$$

Proof. For instance, [4, p. 138, Lem. 7]. $\square$

Example 4.5. By Lemma 4.4(3), $U_{11}(\bar{u}_{11}) \in M^0_{\mathbb{Z}}(11)$; by Lemma 4.4(2), $U^2_{11}(\bar{u}_{11}) \in M^0_{\mathbb{Z}}(11)$. By (20) and (19),

$$\begin{aligned} U^2_{11}(\bar{u}_{11}) = U_{11^2}\frac{\eta(11^2\tau)}{\eta(\tau)} &= \prod_{j=1}^{\infty}(1-q^j)\sum_{k=\lceil 5/11^2\rceil}^{\infty} p(11^2k-5)q^k \\ &= q\prod_{j=1}^{\infty}(1-q^j)\sum_{k=0}^{\infty} p(11^2k+116)q^k. \end{aligned}$$

In view of the congruences (1) the following explicit expressions for the numbers $\mu_{\alpha,\ell}$ are easily verified.

Lemma 4.6. *For $\ell \in \{5,7,11\}$ and $\beta \in \mathbb{Z}_{>0}$:*

$$\mu_{2\beta-1,\ell} = \frac{1+(24-\ell)\cdot \ell^{2\beta-1}}{24} \quad \textit{and} \quad \mu_{2\beta,\ell} = \frac{1+23\cdot \ell^{2\beta}}{24}.$$

Definition 4.7. *Let $\ell \in \{5,7,11\}$ and u_ℓ as in* (17)*. For $f : \mathbb{H} \to \mathbb{C}$ we define $U^{(1)}_\ell(f), U^{(2)}_\ell(f) : \mathbb{H} \to \mathbb{C}$ by*

$$U^{(1)}_\ell(f) := U_\ell(\bar{u}_\ell f) \;\; \textit{and} \;\; U^{(2)}_\ell(f) := U_\ell(f).$$

Definition 4.8. *For $\ell \in \{5,7,11\}$ we define $L_{0,\ell} := 1$ and for all $\beta \in \mathbb{Z}_{>0}$:*

$$L_{2\beta-1,\ell} := U^{(1)}_\ell(L_{2\beta-2,\ell}) \;\; \textit{and} \;\; L_{2\beta,\ell} := U^{(2)}_\ell(L_{2\beta-1,\ell}).$$

In view of the Examples 4.3 and 4.5, which give the base cases $L_{1,11}$ and $L_{2,11}$, the proof of the following lemma is an easy induction exercise.

Lemma 4.9. *For $\ell \in \{5,7,11\}$ and $\beta \in \mathbb{Z}_{>0}$ we have:*

$$L_{2\beta-1,\ell} = q\prod_{n=1}^{\infty}(1-q^{\ell n})\sum_{n=0}^{\infty} p(\ell^{2\beta-1}n+\mu_{2\beta-1,\ell})q^n \in M^0_{\mathbb{Z}}(\ell) \tag{21}$$

and

$$L_{2\beta,\ell} = q\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty} p(\ell^{2\beta}n + \mu_{2\beta,\ell})q^n \in M^0_{\mathbb{Z}}(\ell). \tag{22}$$

Example 4.10. Example 4.3 gives the q-expansion of $L_{11} = U_{11}(\bar{u}_{11}) \in M^0_{\mathbb{Z}}(11)$. Atkin's functions g_i are module generators which satisfy (12). Hence one can apply the reduction strategy as described in Section 9.3 and finds, as Atkin [6, p. 26], that

$$L_{1,11} = 11^4 z_{11} + 11 g_2 + 2\cdot 11^2 g_3 + 11^3 g_4. \tag{23}$$

By Theorem 3.9 this turns into

$$L_{11} = 11^3\Big(\frac{1}{f_3^2} + \frac{1}{2}U_2\frac{1}{f_2^2}\Big) + 11\cdot\frac{7}{2}\Big(\frac{1}{f_2} - 4U_2\frac{1}{f_3}\Big) - 11\Big(\frac{1}{f_3} - U_2\frac{1}{f_2}\Big) + 11^4 z_{11}. \tag{24}$$

This is Atkin's (23) in the clothes of a new witness identity for $11 \mid p(11n+6)$ involving only eta quotients and the U_2-operator ("summing the even part") acting on eta quotients.

To state the Main Theorem 4.15 of the paper, we introduce convenient shorthand notation.

Definition 4.11. *A map $a : \mathbb{Z} \to \mathbb{Z}$ is called a discrete function if it has finite support. A map $a : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is called discrete array if for each $i \in \mathbb{Z}$ the map $a(i,-) : \mathbb{Z} \to \mathbb{Z}$, $j \mapsto a(i,j)$ is a discrete function.*

Definition 4.12. *For $\ell \in \{5,7,11\}$ and $s \in \{1,2\}$ define numbers $\xi_i^{(s,\ell)}$ via maps $\xi^{(s,\ell)} : \{0,\dots,n_\ell - 1\} \to \mathbb{Z}, i \mapsto \xi_i^{(s,\ell)}$ as follows:*

$$(\xi_0^{(1,11)},\dots,\xi_4^{(1,11)}) := (-5,-1,1,2,6) \textit{ and } \xi_j^{(2,11)} := \xi_j^{(1,11)} + 1,\ j \in \{0,\dots,4\};$$
$$\xi_0^{(1,7)} := -7 \textit{ and } \xi_0^{(2,7)} := -10;$$
$$\xi_0^{(1,5)} := -6 \textit{ and } \xi_0^{(2,5)} := -5.$$

Definition 4.13. *For the sake of uniform treatment we define*

$$J_{0,5} = J_{0,7} = J_{0,11} := 1, \textit{ and } J_{i,11} := g_i \textit{ for } i = 2,3,4,6, \tag{25}$$

where the g_i are the Atkin generators from (100).

With the help of these numbers together with

$$A_\ell := \frac{12}{\gcd(\ell-1,12)}\frac{\ell}{\ell+1}, \tag{26}$$

we define specific sets of modular functions which by Examples 3.8 and 3.7 are in $M^0_{\mathbb{Z}}(\ell)$:

Definition 4.14. *With $J_{i,\ell}$ as in Def. 4.13 and $\xi_i^{(s,\ell)}$ as in Def. 4.12:*

$$X^{(s,\ell)} := \Big\{ \sum_{i=0}^{n_\ell-1} J_{i,\ell} \sum_{j=0}^{\infty} \ell^{\left\lceil \frac{A_\ell}{\ell}\left(\ell j+\xi_i^{(s,\ell)}\right)\right\rceil} a_i(j) z_\ell^j : \text{the } a_i \text{ are discrete functions with } a_0(0)=0 \Big\} \subseteq M_{\mathbb{Z}}^0(\ell).$$

Theorem 4.15 ("Main Theorem")**.** *For $\ell \in \{5,7,11\}$ and $\beta \in \mathbb{Z}_{>0}$ there exist $f_{2\beta-1} \in X^{(1,\ell)}$ and $f_{2\beta} \in X^{(2,\ell)}$ such that*

$$L_{2\beta-1,\ell} = \ell^{p_\ell(2\beta-1)} f_{2\beta-1} \tag{27}$$

and

$$L_{2\beta,\ell} = \ell^{p_\ell(2\beta)} f_{2\beta}, \tag{28}$$

where for $n \in \mathbb{Z}_{>0}$,

$$p_\ell(n) := \begin{cases} n, & \text{if } \ell \in \{5,11\}, \\ \left\lceil \frac{n+1}{2} \right\rceil, & \text{if } \ell = 7. \end{cases}$$

The next two sections are devoted to proving the Main Theorem by mathematical induction on β. To this end we need to investigate the algebra underlying the induction step.

We conclude this section by proving the Ramanujan congruences modulo powers of ℓ for $\ell \in \{5,7,11\}$.

Corollary 4.16. *Let $\ell \in \{5,7,11\}$, then for $\alpha \in \mathbb{Z}_{>0}$ and $\mu_{\alpha,\ell}$ as in Lemma 4.6:*

$$p(\ell^\alpha n + \mu_{\alpha,\ell}) \equiv 0 \pmod{\ell^{p_\ell(\alpha)}}, \quad n \geq 0.$$

Proof. The statement is immediate by applying Lemma 4.9 to (27) and (28). □

5. The Fundamental Lemma

In this section we state and prove the Fundamental Lemma, Lemma 5.8, which will play a key role in the proof of the Main Theorem presented in Section 6. Besides the $z_\ell \in M_{\mathbb{Z}}^0(\ell)$ and the $u_\ell \in M_{\mathbb{Z}}^\infty(\ell^2)$, resp. $\bar{u}_\ell \in M_{\mathbb{Z}}^0(\ell^2)$, further crucial functions, $\ell \geq 5$ a prime, are

$$t_\ell(\tau) := \left(\frac{\eta(\tau)}{\eta(\ell\tau)}\right)^{\nu_\ell} = \bar{z}_\ell(\tau) \in M_{\mathbb{Z}}^\infty(\ell) \text{ and}$$

$$T_\ell(\tau) := t_\ell(\ell\tau) = \left(\frac{\eta(\ell\tau)}{\eta(\ell^2\tau)}\right)^{\nu_\ell} = u_\ell(\tau)^{\nu_\ell} z_\ell(\tau) \in M_{\mathbb{Z}}^\infty(\ell^2),$$

where

$$\nu_\ell = \frac{24 d_\ell}{(\ell-1)\ell} \text{ with } d_\ell := \frac{(\ell-1)\ell}{\gcd(\ell-1,12)}. \tag{29}$$

Lemma 5.1. *For any prime $\ell \geq 5$,*

(1) $t_\ell, \bar{t}_\ell \in M_{\mathbb{Z}}(\ell)$ *(and thus in $M_{\mathbb{Z}}(\ell^2)$), and $T_\ell, \bar{T}_\ell \in M_{\mathbb{Z}}(\ell^2)$;*

(2) $$\operatorname{ord}_{[\infty]_{\ell^2}} \bar{t}^*_\ell = \frac{d_\ell}{\ell},\ \operatorname{ord}_{[0]_{\ell^2}} \bar{t}^*_\ell = -d_\ell, \text{ and}$$
$$\operatorname{ord}_{[k/\ell]_{\ell^2}} \bar{t}^*_\ell = \frac{d_\ell}{\ell},\ k = 1, \dots, \ell-1;$$

(3) $$\operatorname{ord}_{[\infty]_{\ell^2}} \bar{T}^*_\ell = d_\ell,\ \operatorname{ord}_{[0]_{\ell^2}} \bar{T}^*_\ell = -\frac{d_\ell}{\ell}, \text{ and}$$
$$\operatorname{ord}_{[k/\ell]_{\ell^2}} \bar{T}^*_\ell = -\frac{d_\ell}{\ell},\ k = 1, \dots, \ell-1.$$

Proof. The statements are implied by [26, Thms. 1.64 and 1.65]. □

The Fundamental Lemma owes its existence to the following theorem.

Theorem 5.2. *For any prime $\ell \geq 5$ and ν_ℓ as in (29) there exists a polynomial $F_\ell(X,Y) \in \mathbb{Q}[X,Y]$ which is irreducible over $\mathbb{C}$ and which is monic in x of degree $d_\ell = \nu_\ell(\ell-1)\ell/24$ such that*

$$F_\ell(\bar{t}_\ell, \bar{T}_\ell) = F\Big(\Big(\frac{\eta(\ell\tau)}{\eta(\tau)}\Big)^{\nu_\ell}, \Big(\frac{\eta(\ell^2\tau)}{\eta(\ell\tau)}\Big)^{\nu_\ell}\Big) = 0.$$

Remark 5.3. Being *monic* and having *polynomial* coefficients the modular relation $F_\ell(X,Y)$ is of particular algorithmic importance. Therefore we give a detailed proof of its existence in Section 7. Nevertheless, for a fixed prime $\ell \geq 5$ the polynomial $F_\ell(X,Y)$ can be computed explicitly as in the proof of Theorem 5.4. In other words, for our proof of the Ramanujan congruences (Cor. 4.16) one does not need Theorem 5.2, respectively the proofs presented in Section 7.

Theorem 5.4. *For $\ell \in \{5,7,11\}$ the uniquely determined polynomials as in Theorem 5.2 are of the form*

$$F_\ell(X,Y) = X^{d_l} + \sum_{i=0}^{d_\ell-1} a_i^{(\ell)}(Y) X^i$$

where

$$\begin{aligned} F_5(X,Y) = X^5 + a_4^{(5)}(Y)X^4 + \cdots + a_1^{(5)}(Y)X + a_0^{(5)}(Y) = X^5 \\ - (5^{12}Y^5 + 6\cdot 5^{10}Y^4 + 63\cdot 5^7 Y^3 + 52\cdot 5^5 Y^2 + 63\cdot 5^2 Y)X^4 \\ - (5^9 Y^4 + 6\cdot 5^7 Y^3 + 63\cdot 5^4 Y^2 + 52\cdot 5^2 Y)X^3 \\ - (5^6 Y^3 + 6\cdot 5^4 Y^2 + 63\cdot 5Y)X^2 - (5^3 Y^2 + 6\cdot 5Y)X - Y; \end{aligned} \tag{30}$$

(31)

$$\begin{aligned}
F_7(X,Y) = X^7 + \sum_{i=0}^{6} a_i^{(7)}(Y)X^i = X^7 \\
&- (82 \cdot 7^2Y + 176 \cdot 7^4Y^2 + 845 \cdot 7^5Y^3 + 272 \cdot 7^7Y^4 \\
&\qquad + 46 \cdot 7^9Y^5 + 47^{11}Y^6 + 7^{12}Y^7)X^6 \\
&- (176 \cdot 7^2Y + 845 \cdot 7^3Y^2 + 272 \cdot 7^5Y^3 + 46 \cdot 7^7Y^4 + 4 \cdot 7^9Y^5 + 7^{10}Y^6)X^5 \\
&- (845 \cdot 7Y + 272 \cdot 7^3Y^2 + 46 \cdot 7^5Y^3 + 4 \cdot 7^7Y^4 + 7^8Y^5)X^4 \\
&- (272 \cdot 7Y + 46 \cdot 7^3Y^2 + 4 \cdot 7^5Y^3 + 7^6Y^4)X^3 - (46 \cdot 7Y + 4 \cdot 7^3Y^2 + 7^4Y^3)X^2 \\
&- (4 \cdot 7Y + 7^2Y^2)X - Y;
\end{aligned}$$

and

$$F_{11}(X,Y) = X^{55} + \sum_{i=0}^{54} a_i^{(11)}(Y)X^i, \tag{32}$$

where the polynomials $a_i^{(11)}(Y)$ *are listed explicitly at*

`https://www.risc.jku.at/people/sradu/powers11`

Remark 5.5. We prove our claim by invoking a *reduction algorithm*. Its steps have been described in the proof of Lemma 9.6; its strategy is used at various other places in this article. We stress the following aspect: By running the reduction algorithm for $\ell = 5, 7, 11$ one detects that the principal parts indeed reduce to zero, which is sufficient to prove Theorem 5.4. But the *guaranty* that this always happens (not only for $\ell = 5, 7, 11$ but also for all primes $\ell \geq 5$) is provided by Theorem 5.2. Before entering the algorithmic reduction argument, we begin the proof of Theorem 5.4 by making this existence aspect explicit.

Proof of Theorem 5.4. By Theorem 5.2 there exist polynomials $a_i^{(\ell)}(Y) \in \mathbb{Q}[Y]$ such that $F_\ell(\bar{t}_\ell, \bar{T}_\ell) = 0$ for

$$F_\ell(X,Y) := X^{d_\ell} + a_{d_\ell-1}^{(\ell)}(Y)X^{d_\ell-1} + \cdots + a_1^{(\ell)}(Y)X + a_0^{(\ell)}(Y); \tag{33}$$

in addition, $F_\ell(X,Y) \in \mathbb{Q}[X,Y]$ is irreducible in $\mathbb{C}[X,Y]$. To compute the $a_i^{(\ell)}(Y)$ recall Lemma 5.1 which implies that $t_\ell T_\ell = q^{-(d_\ell+d_\ell/\ell)} + O(q^{-(d_\ell+d_\ell/\ell)+1})$ and $T_\ell = q^{-d_\ell} + O(q^{-d_\ell+1})$ have their only pole at ∞ with multiplicity $d_\ell + d_\ell/\ell$, respectively d_ℓ. Hence, analogously to the proof of Theorem 5.2, Theorem 7.1 implies that there exist polynomials $p_j^{(\ell)}(Y) \in \mathbb{Q}[Y]$ such that

$$(t_\ell T_\ell)^{d_\ell} + p_1^{(\ell)}(T_\ell)(t_\ell T_\ell)^{d_\ell-1} + \cdots + p_{d_\ell}^{(\ell)}(T_\ell) = 0. \tag{34}$$

Define

$$G_\ell(X,Y) := X^{d_\ell} + p_1^{(\ell)}(Y)X^{d_\ell-1} + \cdots + p_{d_\ell}^{(\ell)}(Y) \in \mathbb{Q}[Y][X]. \tag{35}$$

Rewrite

$$G_\ell(t_\ell T_\ell, T_\ell) = (t_\ell T_\ell)^{d_\ell}\Big(1 + \frac{p_1^{(\ell)}(T_\ell)}{T_\ell}\bar{t}_\ell + \frac{p_2^{(\ell)}(T_\ell)}{T_\ell^2}\bar{t}_\ell^2 + \cdots + \frac{p_{d_\ell}^{(\ell)}(T_\ell)}{T_\ell^{d_\ell}}\bar{t}_\ell^{d_\ell}\Big)$$
$$= \frac{(t_\ell T_\ell)^{d_\ell}}{\bar{T}_\ell^m}\Big(\frac{1}{T_\ell^m} + \frac{p_1^{(\ell)}(T_\ell)}{T_\ell^{m+1}}\bar{t}_\ell + \frac{p_2^{(\ell)}(T_\ell)}{T_\ell^{m+2}}\bar{t}_\ell^2 + \cdots + \frac{p_{d_\ell}^{(\ell)}(T_\ell)}{T_\ell^{m+d_\ell}}\bar{t}_\ell^{d_\ell}\Big),$$

where $m \geq 0$ is such that y^m is the maximal non-negative power of y occuring in the summands of the Laurent polynomials $p_i^{(\ell)}(y)/y^i$, $i = 1, \ldots, d_\ell$; if all occuring powers of y are negative, set $m := 0$. Owing to the choice of m, the $p_i^{(\ell)}(y)/y^{m+i}$ are polynomials in $1/y$ over $\mathbb{Q}$. Hence by the irreducibility of $F_\ell(X,Y)$ there exists a $c \in \mathbb{Q}$ such that

$$a_i(Y) = c \cdot Y^{m+i} p_i^{(\ell)}(1/Y),\ \ i = 1, \ldots, d_\ell - 1, \ \text{ and } \ a_0(Y) = c \cdot Y^m. \tag{36}$$

As a consequence, to compute $F_\ell(X,Y)$ it is sufficient to determine the polynomial $G_\ell(X,Y)$ which can be computed by iterated reduction until the principal part vanishes.

The reduction procedure. For instance, for $\ell = 5$ the first reduction step is

$$(t_5T_5)^5 - T_5^6 = -30q^{-29} + 405q^{-28} - 3190q^{-27} + O(q^{-26}).$$

For the next step one determines non-negative integers $a = 4, b = 1$ such that

$$6\ a + 5\ b = 29.$$

Then

$$(t_5T_5)^5 - T_5^6 + 30(t_5T_5)^4T_5^1 = -315q^{-28} + 4370q^{-27} - 28500q^{-26} + O(q^{-26}).$$

One finds $a = 3, b = 2$ such that

$$6\ a + 5\ b = 28;$$

resulting in the reduction,

$$(t_5T_5)^5 - T_5^6 + 30(t_5T_5)^4T_5^1 + 315(t_5T_5)^3T_5^2 = -1300q^{-27} + O(q^{-26}).$$

By iterating this reduction and setting $S := t_5T_5$ and $T := T_5$, one arrives at

$$\begin{aligned} G_5(X,Y) &= X^5 + p_1^{(5)}(Y)X^4 + \cdots + p_4^{(5)}(Y)X + p_5^{(5)}(Y) \\ &= X^5 + (5^3 + 5 \cdot 6 \cdot Y)X^4 + (5^6 + 5^4 \cdot 6 \cdot Y + 5 \cdot 63 \cdot Y^2)X^3 \\ &\quad + (5^9 + 5^7 \cdot 6 \cdot Y + 5^4 \cdot 63 \cdot Y^2 + 5^2 \cdot 52 \cdot Y^3)X^2 \\ &\quad + (5^{12} + 5^{10} \cdot 6 \cdot Y + 5^7 \cdot 63 \cdot Y^2 + 5^5 \cdot 52 \cdot Y^3 + 5^2 \cdot 63 \cdot Y^4)X - Y^6; \end{aligned}$$

i.e., $d_5 = 5$, $m = 1$ and $c = -1$. By (36),

$$a_0^{(5)}(Y) = -Y, \text{ and } a_i^{(5)}(Y) = -Y^{1+i}p_i^{(5)}(1/Y) \text{ for } i = 1,2,3,4, \tag{37}$$

which gives $F_5(X,Y)$ as in (30). The cases $\ell = 7$ and $\ell = 11$ work analogously. $\square$

By inspection we have

Corollary 5.6. *Let $\ell \in \{5,7,11\}$. With A_ℓ as in (26) and ν_ℓ as in (29), the polynomials $a_i^{(\ell)}(Y)$ in Theorem 5.4 are of the form*

$$a_i^{(\ell)}(Y) = \sum_{k=\lceil \frac{d_\ell - i}{\ell} \rceil}^{\frac{d_\ell}{\ell}+i} s_\ell(i,k)\ell^{\lceil \frac{A_\ell}{\ell}(k\ell+i-d_\ell)\rceil} Y^k \tag{38}$$

where the $s_\ell(i,k)$ are integers.

Remark 5.7. Writing $a_j^{(\ell)}(Y)$ as in (38) to reveal divisibility by powers of ℓ of its coefficients is of help in the proof of Lemma 6.3 and is inspired by [5].

Lemma 5.8 ("Fundamental Lemma")**.** *Let $\ell \in \{5,7,11\}$, d_ℓ as in (29), and $a_i^{(\ell)}(Y)$ as in (38). Then for any $w : \mathbb{H} \to \mathbb{C}$ and arbitrary $j \in \mathbb{Z}$:*

$$U_\ell(w\,\bar{t}_\ell^j) = -\sum_{i=0}^{d_\ell - 1} a_i^{(\ell)}(\bar{t}_\ell)\,U_\ell(w\,\bar{t}_\ell^{j+i-d_\ell}).$$

Proof. Let $\lambda \in \{0,\dots,\ell-1\}$. Theorem 5.4, after substituting $\tau \mapsto \frac{\tau+\lambda}{\ell}$ and using $\bar{t}_\ell(\tau+\lambda) = \bar{t}_\ell(\tau)$, implies

$$\bar{t}_\ell\Big(\frac{\tau+\lambda}{\ell}\Big)^{d_\ell} + \sum_{i=0}^{d_\ell-1} a_i^{(\ell)}(\bar{t}(\tau))\,\bar{t}_\ell\Big(\frac{\tau+\lambda}{\ell}\Big)^i = 0.$$

Now, after multiplying both sides with $w\left(\frac{\tau+\lambda}{\ell}\right)\bar{t}_\ell\left(\frac{\tau+\lambda}{\ell}\right)^{j-d_\ell}$, summation over all λ from $\{0,\dots,\ell-1\}$ completes the proof of the lemma. $\square$

6. Proving the Main Theorem

Throughout this section we assume that ℓ is a fixed prime chosen from $\{5,7,11\}$. We need to prepare with some lemmas.

Lemma 6.1. *Let d_ℓ be as in (29) and $w : \mathbb{H} \to \mathbb{C}$. Suppose for some $l \in \mathbb{Z}$ and all k with $l \le k \le l + d_\ell - 1$ there exist Laurent polynomials $p_k^{(0)}(X), \dots, p_k^{(r)}(X) \in \mathbb{Z}[X, X^{-1}]$, functions $v_0, \dots, v_r : \mathbb{H} \to \mathbb{C}$, and integers $\sigma_0, \dots, \sigma_r$ such that*

$$U_\ell(w\bar{t}_\ell^k) = \sum_{i=0}^r v_i p_k^{(i)}(\bar{t}_\ell) \tag{39}$$

where

$$\mathrm{ord}_{\bar{t}_\ell}\left(p_k^{(i)}(\bar{t}_\ell)\right) \ge \left\lceil \frac{k+\sigma_i}{\ell} \right\rceil, \quad i \in \{0,\dots,r\}. \tag{40}$$

For each $i=0,\dots,r$ extend $\{p_k^{(i)}(X)\}_{l\le k\le l+d_\ell-1}$ to the infinite set $\{p_k^{(i)}(X)\}_{k\ge l}$ by defining for successive N and starting with $N=l+d_\ell$:

$$p_N^{(i)}(X) := -\sum_{j=0}^{d_\ell-1} a_i^{(\ell)}(X)\, p_{N+j-d_\ell}^{(i)}(X) \in \mathbb{Z}[X,X^{-1}]. \tag{41}$$

Then

$$\text{(39) and (40) hold for all } k\ge l \text{ and } i=0,\dots,r.$$

Remark 6.2. The definition (41) is a natural consequence of extending the set $\{p_k^{(i)}(X)\}_{l\le k\le l+d_\ell-1}$ by inductive use of the action of the U-operator. For example, if $N=\ell+d_\ell$ then by (39) and Lemma 5.8:

$$\begin{aligned} U_\ell(w\bar{t}^N) &= -\sum_{j=0}^{d_\ell-1} a_j^{(\ell)}(\bar{t}_\ell)\, U_\ell(w\,\bar{t}_\ell^{N+j-d_\ell}) = -\sum_{j=0}^{d_\ell-1} a_j^{(\ell)}(\bar{t}_\ell) \sum_{i=0}^{r} v_i p_{N+j-d_\ell}^{(i)}(\bar{t}_\ell) \\ &= -\sum_{i=0}^{r} v_i \sum_{j=0}^{d_\ell-1} a_j^{(\ell)}(\bar{t}_\ell)\, p_{N+j-d_\ell}^{(i)}(\bar{t}_\ell) = \sum_{i=0}^{r} v_i p_N^{(i)}(\bar{t}_\ell), \end{aligned} \tag{42}$$

where the last equality is by (41).

Proof of Lemma 6.1. In view of (42) the definition of the $p_k^{(i)}$ is such that (39) is satisfied also for all $k\ge \ell+d_\ell$. To show (40), we proceed by mathematical induction assuming that N is an integer with $N>l+d_p-1$. Moreover, suppose that (40) holds for $l\le k\le N-1$ and

$$p_k^{(i)}(X) = \sum_{n\ge \lceil \frac{k+\sigma_i}{\ell}\rceil} c_i(k,n)X^n, \quad l\le k\le N-1,$$

with integers $c_i(k,n)$. Then,

$$\begin{aligned} &-p_N^{(i)}(X) = \\ &\sum_{j=0}^{d_\ell-1} a_i^{(\ell)}(X)\, p_{N+j-d_\ell}^{(i)}(X) = \sum_{j=0}^{d_\ell-1} a_i^{(\ell)}(X) \sum_{n\ge \lceil \frac{N+j-d_\ell+\sigma_i}{\ell}\rceil} c_i(N+j-d_\ell,n)X^n \\ &= \sum_{j=0}^{d_\ell-1} a_j^{(\ell)}(X) \sum_{n-\lceil \frac{d_\ell-j}{\ell}\rceil \ge \lceil \frac{N+j-d_\ell+\sigma_i}{\ell}\rceil} c_i\Big(N+j-d_\ell, n-\Big\lceil \frac{d_\ell-j}{\ell}\Big\rceil\Big) X^{n-\lceil \frac{d_\ell-j}{\ell}\rceil} \\ &= \sum_{j=0}^{d_\ell-1} a_j^{(\ell)}(X)X^{-\lceil \frac{d_\ell-j}{\ell}\rceil} \sum_{n\ge \lceil \frac{d_\ell-j}{\ell}\rceil + \lceil \frac{N+j-d_\ell+\sigma_i}{\ell}\rceil} c_i\Big(N+j-d_\ell, n-\Big\lceil \frac{d_\ell-j}{\ell}\Big\rceil\Big) X^n \end{aligned}$$

Finally, recall that $a_j^{(\ell)}(X)X^{-\left\lceil\frac{d_\ell-j}{\ell}\right\rceil}$ for $0\leq j\leq d_\ell-1$ is a polynomial in X (Corollary 5.6). Hence, owing to $\lceil x\rceil+\lceil y-x\rceil\geq\lceil y\rceil$, the Laurent polynomials $p_N^{(i)}(X)$ satisfy (40) for all $N\geq l$. □

Lemma 6.3. *Let d_p be as in* (29) *and $w:\mathbb{H}\to\mathbb{C}$. Suppose for some $l\in\mathbb{Z}$ and all k with $l\leq k\leq l+d_\ell-1$ there exist Laurent polynomials $p_k^{(0)}(X),\dots,p_k^{(r)}(X)\in\mathbb{Z}[X,X^{-1}]$ and functions $v_0,\dots,v_r:\mathbb{H}\to\mathbb{C}$ such that*

$$U_\ell(w\bar{t}_\ell^k)=\sum_{i=0}^r v_i p_k^{(i)}(\bar{t}_\ell) \tag{43}$$

and

$$p_k^{(i)}(X)=\sum_n c_i(k,n)\ell^{\left\lceil\frac{A_\ell}{\ell}(\ell n+\gamma_i-k)\right\rceil}X^n \text{ for integers } c_i(k,n) \text{ and } \gamma_i, \tag{44}$$

where A_ℓ is as in (26). *For each $i=0,\dots,r$ extend $\{p_k^{(i)}(X)\}_{l\leq k\leq l+d_\ell-1}$ to the infinite set $\{p_k^{(i)}(X)\}_{k\geq l}$ by defining for successive N and starting with $N=l+d_\ell$:*

$$p_N^{(i)}(X):=-\sum_{j=0}^{d_\ell-1}a_i^{(\ell)}(X)\,p_{N+j-d_\ell}^{(i)}(X)\in\mathbb{Z}[X,X^{-1}]. \tag{45}$$

Then

(43) and (44) hold for all $k\geq l$ and $i=0,\dots,r$.

Proof. In view of (42) the definition of the $p_k^{(i)}$ is such that (43) is satisfied also for all $k\geq\ell+d_\ell$. To show (44), we proceed by mathematical induction assuming that N is an integer with $N>l+d_p-1$. Then using (45), (38), and (44) as the induction hypothesis,

$$\begin{aligned}
-p_N^{(i)}(X)&=\sum_{j=0}^{d_\ell-1}a_j^{(\ell)}(X)\,p_{N+j-d_\ell}^{(i)}(X)\\
&=\sum_{j=0}^{d_\ell-1}\sum_{k=1}^{d_\ell+j}s_\ell(j,k)\ell^{\left\lceil\frac{A_\ell}{\ell}(k\ell+j-d_\ell)\right\rceil}X^k\\
&\qquad\times\sum_n c_i(N+j-d_\ell,n)\ell^{\left\lceil\frac{A_\ell}{\ell}(\ell n+\gamma_i-N-j+d_\ell)\right\rceil}X^n\\
&=\sum_{j=0}^{d_\ell-1}\sum_{k=1}^{d_\ell+j}\sum_n s_\ell(j,k)c_i(N+j-d_\ell,n-k)\\
&\qquad\times\ell^{\left\lceil\frac{A_\ell}{\ell}(\ell(n-k)+\gamma_i-N-j+d_\ell)\right\rceil+\left\lceil\frac{A_\ell}{\ell}(\ell k+j-d_\ell)\right\rceil}X^n.
\end{aligned}$$

The induction proof of (44) is completed by bounding the exponent of ℓ from below with $y:=\left\lceil\frac{A_\ell}{\ell}(\ell n+\gamma_i-N)\right\rceil$, again using $\lceil y-x\rceil+\lceil x\rceil\geq\lceil y\rceil$. □

Before proving the Main Theorem, Theorem 4.15, we need three more lemmas and the integers,

$$n_\ell := \frac{\ell-1}{\gcd(\ell-1,12)} = \frac{d_\ell}{\ell}, \quad \ell = 5,7,11. \tag{46}$$

In addition, for the next lemma we need to define integer maps.

Definition 6.4. *For $\ell \in \{5,7,11\}$ and $s \in \{1,2\}$ define integer maps $\mu_{s,\ell}, \nu_{s,\ell} : \{0,\dots,n_\ell-1\}^2 \to \mathbb{Z}$ as follows: For $s=1$,*

$$\mu_{1,5}(0,0) := -3, \mu_{1,7}(0,0) := -6, \text{ and } \nu_{1,5}(0,0) := 1, \nu_{1,7}(0,0) := 2, \text{ and}$$

$\mu_{1,11}$	0	1	2	3	4
0	-3	1	3	5	9
1	-3	1	3	4	8
2	-3	1	2	4	8
3	-3	0	2	4	8
4	-3	0	2	4	7

, *and*

$\nu_{1,11}$	0	1	2	3	4
0	1	-1	-3	-5	-7
1	2	-1	-3	-5	-7
2	2	0	-3	-5	-7
3	2	0	-2	-5	-7
4	2	0	-2	-4	-7

.

For example, $\mu_{1,11}(1,2) = 3$. For $s=2$,

$$\mu_{2,\ell}(i,j) := \mu_{1,\ell}(i,j) + \frac{\ell}{2A_\ell}, \text{ and } \nu_{2,\ell}(i,j) := \nu_{1,\ell}(i,j) - \frac{\ell}{2A_\ell},$$

where A_ℓ is as in (26).

For our convention for $J_{i,\ell}$ recall Definition 4.13. Also recall that $\bar{t}_\ell = z_\ell$ with z_ℓ as in (11). For the rest of this section we prefer to use z_ℓ.

Lemma 6.5. *Let $\ell \in \{5,7,11\}$ and n_ℓ as in* (46)*. Then for $(s,m,k) \in \{1,2\} \times \{0,\dots,n_\ell-1\} \times \mathbb{Z}_{\geq 0}$ there exist discrete arrays $a^{(s,\ell)}_{m,k}$ such that*

$$U^{(s)}_\ell(J_{m,\ell} z^k_\ell) = \sum_{i=0}^{n_\ell-1} J_{i,\ell} \sum_{j \geq N^{(s,\ell)}_{m,k}(i)} a^{(s,\ell)}_{m,k}(i,j)\ell^{M^{(s,\ell)}_{m,k}(i,j)} z^j_\ell, \tag{47}$$

where

$$N^{(s,\ell)}_{m,k}(i) = \left\lceil \frac{k+\nu_{s,\ell}(m,i)}{\ell} \right\rceil \text{ and } M^{(s,\ell)}_{m,k}(i,j) = \left\lceil \frac{A_\ell}{\ell}(\ell j - k + \mu_{s,\ell}(m,i)) \right\rceil \tag{48}$$

with $\nu_{s,\ell}$ and $\mu_{s,\ell}$ as in Definition 6.4.

Remark 6.6. In particular, if $s=1$,

$$N^{(1,5)}_{m,k}(i) = N^{(1,5)}_{0,k}(0) = \left\lceil \frac{k+1}{5} \right\rceil, N^{(1,7)}_{m,k}(i) = N^{(1,7)}_{0,k}(0) = \left\lceil \frac{k+2}{7} \right\rceil,$$

and,

$$N^{(1,11)}_{m,k}(i) = \left\lceil \frac{k+\nu_{1,11}(m,i)}{11} \right\rceil \geq \left\lceil \frac{k-7}{11} \right\rceil, m,i \in \{0,\dots,4\}.$$

If $s=2$,

$$N_{m,k}^{(2,5)}(i)=N_{0,k}^{(2,5)}(0)=\left\lceil\frac{k}{5}\right\rceil,\ N_{m,k}^{(2,7)}(i)=N_{0,k}^{(2,7)}(0)=\left\lceil\frac{k}{7}\right\rceil,$$

and,

$$N_{m,k}^{(2,11)}(i)=\left\lceil\frac{k-1+\nu_{1,11}(m,i)}{11}\right\rceil\geq\left\lceil\frac{k-8}{11}\right\rceil,\ m,i\in\{0,\dots,4\}.$$

Sketch of Proof of Lemma 6.5. For a fixed $s\in\{1,2\}$ let

$$p_k^{(i)}(X):=\sum_{j\geq N_{m,k}^{(s,\ell)}(i)} a_{m,k}^{(s,\ell)}(i,j)\ell^{M_{m,k}^{(s,\ell)}(i,j)}z_\ell^j,\quad i=0,\dots,n_\ell-1. \tag{49}$$

Recall that $U_\ell^{(1)}(J_{m,\ell}z_\ell^k)=U_\ell(\bar{u}_\ell J_{m,\ell}z_\ell^k)$ and $U_\ell^{(2)}(J_{m,\ell}z_\ell^k)=U_\ell(J_{m,\ell}z_\ell^k)$. Consequently, for either choice of $s\in\{1,2\}$ the conditions (47) and (48) fit the pattern of (39) and (40), and (43) and (44) with $v_i:=J_{i,\ell}$. Thus to prove the properties claimed by Lemma 6.5 we can invoke Lemma 6.1 and Lemma 6.3. Concretely, to complete the proof of Lemma 6.5 one only has to verify that in the given setting there exist Laurent polynomials $p_k^{(i)}(X)\in\mathbb{Z}[X,X^{-1}]$ of the form as in (49) which satisfy the conditions (47) and (48) for all $(s,m,k)\in\{1,2\}\times\{0,\dots,n_\ell-1\}\times\{-d_\ell+1,\dots,0\}$. To this end we compute explicity all such relations and check the conditions (47) and (48) by inspection. This procedure proves Lemma 6.5; details are given in Section 12.2 (Appendix). □

Definition 6.7. *For $m\in\mathbb{Z}\setminus\{0\}$ and any prime p let $v_p(m)\geq 0$ be the p-adic valuation of m; i.e., the maximal non-negative integer power of p arising as a factor of m.*

Lemma 6.8. *Let $\ell\in\{5,7,11\}$, n_ℓ as in (46), and $r\in\{0,\dots,n_\ell-1\}$. Let $f\in M_{\mathbb{Z}}^0(\ell)$ be of the form*

$$f=\sum_{m=0}^{n_\ell-1}J_{m,\ell}\sum_{n=0}^{\infty}b(m,n)\ell^{\left\lceil\frac{A_\ell}{\ell}(\ell n+\epsilon_{m,\ell})\right\rceil}z_\ell^n$$

with fixed $\epsilon_{m,\ell}\in\mathbb{Z}$ and where b is a discrete array with $b(m,0)=0$ for $0\leq m\leq r$. Then $U_\ell^{(s)}(f)$, $s\in\{1,2\}$, is of the form

$$U_\ell^{(s)}(f)=\sum_{i=0}^{n_\ell-1}J_{i,\ell}\sum_{j=0}^{\infty}c^{(s,\ell)}(i,j)z_\ell^j,$$

where $c^{(s,\ell)}$ is a discrete array with $c^{(s,\ell)}(0,0)=0$. Moreover, for the given r:

$$v_\ell(c^{(s,\ell)}(i,j))\geq\min_{\substack{m\in\{0,\dots,r\}\\ m'\in\{r+1,\dots,n_\ell-1\}}}\left(M_{m,1}^{(s,\ell)}(i,j)+\left\lceil\frac{A_\ell}{\ell}(\ell+\epsilon_{m,\ell})\right\rceil,M_{m',0}^{(s,\ell)}(i,j)+\left\lceil\frac{A_\ell}{\ell}\epsilon_{m',\ell}\right\rceil\right),$$

using the notation from (48).

Proof. Utilizing Lemma 6.5 we write $U^{(s)}(f)$ as

$$U_\ell^{(s)}(f) = \sum_{m=0}^{n_\ell-1}\sum_{n=0}^{\infty}\sum_{i=0}^{n_\ell-1}\sum_{j\geq N_{m,n}^{(s,\ell)}(i)} c_{m,n}^{(s,\ell)}(i,j)J_{i,\ell}z_\ell^j$$

with

$$c_{m,n}^{(s,\ell)}(i,j) = a_{m,n}^{(s,\ell)}(i,j)b(m,n)\ell^{\left\lceil \frac{A_\ell}{\ell}(\ell n+\epsilon_{m,\ell})\right\rceil + M_{m,n}^{(s,\ell)}(i,j)}.$$

Splitting the sum with respect to the given r, and noticing that $b(0,0)=0$, gives rise to summation bounds as follows:

$$\begin{aligned} U_\ell^{(s)}(f) = {} & J_{0,\ell}\sum_{j=1}^{\infty} z_\ell^j \sum_{m=0}^{r}\sum_{n=1}^{\infty} c_{m,n}^{(s,\ell)}(0,j) + \sum_{i=1}^{n_\ell-1} J_{i,\ell}\sum_{j=0}^{\infty} z_\ell^j\sum_{m=0}^{r}\sum_{n=1}^{\infty} c_{m,n}^{(s,\ell)}(i,j) \\ & + J_{0,\ell}\sum_{j=1}^{\infty} z_\ell^j\sum_{m=r+1}^{n_\ell-1}\sum_{n=0}^{\infty} c_{m,n}^{(s,\ell)}(0,j) + \sum_{i=1}^{n_\ell-1} J_{i,\ell}\sum_{j=0}^{\infty} z_\ell^j\sum_{m=r+1}^{n_\ell-1}\sum_{n=0}^{\infty} c_{m,n}^{(s,\ell)}(i,j). \end{aligned}$$

From this representation the lemma follows immediately by inspection. □

Lemma 6.9. *For $\ell \in \{5,7,11\}$ and $X^{(s,\ell)}$ as in Definition 4.12 we have*

$$f \in X^{(1,\ell)} \quad \textit{implies} \quad \ell^{-1}U^{(2)}(f) \in X^{(2,\ell)} \tag{50}$$

and

$$f \in X^{(2,\ell)} \quad \textit{implies} \quad \ell^{\chi(\ell)}\ell^{-1}U^{(1)}(f) \in X^{(1,\ell)}, \tag{51}$$

where $(\chi(5),\chi(7),\chi(11)) := (0,1,0)$.

Proof. Recall the map $\xi^{(s,\ell)}$ from Definition 4.12. Applying Lemma 6.8 with $r=0$, $s=2$ and $\epsilon_{i,\ell} := \xi_i^{(1,\ell)}$ we obtain

$$v_\ell(c^{(2,\ell)}(i,j)) \geq \left\lceil \frac{A_\ell}{\ell}(\ell j + \xi_i^{(2,\ell)})\right\rceil + 1.$$

Similarly, applying Lemma 6.8 with $r=0$, $s=1$ and $\epsilon_{i,\ell} = \xi_i^{(2,\ell)}$ we obtain

$$v_\ell(c^{(1,\ell)}(i,j)) \geq \left\lceil \frac{A_\ell}{\ell}(\ell j + \xi_i^{(1,\ell)})\right\rceil + 1 - \chi(\ell).$$

□

Now we are ready for the proof of the Main Theorem.

Proof of Theorem 4.15 ("Main Theorem"). We proceed by mathematical induction on β. For $\beta = 1$ the statement is settled by the fundamental relations providing representations of $U_{11}^1(1)$:

$$L_{1,5} = U_5^{(1)}(1) = 5\ z_5\ J_{0,5}, \quad L_{1,7} = U_7^{(1)}(1) = 7(z_7 + 7z_7^2)J_{0,7}$$

and

$$L_{1,11} = U_{11}^{(1)}(1) = 11(11^3 z_{11} J_{0,11} \ + J_{1,11} + 2{\cdot}11 J_{2,11} + 11^2 J_{3,11}).$$

These identites are entries of the tables in Section 12.2 (Appendix); there it is also explained how to obtain them algorithmically. The induction step will be carried out as follows: In the first step we show that the correctness of (27) for $N = 2\beta - 1$, $\beta \in \mathbb{Z}_{>0}$, implies (28) for $N+1 = 2\beta$, which in the second step is shown to imply the correctness of (27) for $N+2 = 2\beta+1$.

For the first step we recall (20). Assuming the induction hypothesis (27) and applying (50) from Lemma 6.9 we obtain

$$U_\ell^{(2)}(L_{2\beta-1,\ell}) = \ell^{p_\ell(2\beta-1)} U_\ell^{(2)}(f_{2\beta-1}) = \ell^{p_\ell(2\beta-1)} \cdot \ell f_{2\beta}$$

for some $f_{2\beta} \in X^{(2,\ell)}$. Next we assume (28) and apply (51) in Lemma 6.9 to obtain

$$U_\ell^{(1)}(L_{2\beta}, \ell) = \ell^{p_\ell(2\beta)} U_\ell^{(1)}(f_{2\beta}) = \ell^{p_\ell(2\beta)} \cdot \ell^{1-\chi(\ell)} f_{2\beta+1}$$

for some $f_{2\beta+1} \in X^{(1,\ell)}$. This completes the proof of the Main Theorem on the basis of having established the fundamental relations for Lemma 6.5 which is done in Section 12.2 (Appendix). □

7. The Fundamental Polynomials

In this section we prove Theorem 7.1 which, as shown below, implies the existence of the fundamental polynomials $F_\ell(X, Y)$ stated in Theorem 5.2. To this end we need to recall a couple of notions from Riemann surfaces; see for instance [12].

We let $M(S)$ denote the field of meromorphic functions $f : S \to \hat{\mathbb{C}}$ on a Riemann surface S.[1] Let $f \in M(S)$ be non-constant: then for every neighborhood U of $x \in S$ there exist neighborhoods $U_x \subseteq U$ of x and V of $f(x)$ such that the set $f^{-1}(v) \cap U_x$ contains exactly k elements for every $v \in V \setminus \{f(x)\}$. This number k is called the multiplicity of f at x; notation: $k = \mathrm{mult}_x(f)$. If S is compact, $f \in M(S)$ is surjective and each $v \in \hat{\mathbb{C}}$ has the same number of preimages, say n, counting multiplicities; i.e., $n = \sum_{x \in f^{-1}(v)} \mathrm{mult}_x(f)$; see, e.g., [12, Thm. 4.24]. This number n is called the degree of f; notation: $n = \mathrm{Deg}(f)$. $\mathrm{RamiPts}(f) := \{x \in S : \mathrm{mult}_x(f) \geq 2$ denotes the set of ramification points of f; $\mathrm{BranchPts}(f) := f(\mathrm{RamiPts}(f)) \subseteq \hat{\mathbb{C}}$ denotes the set of branch points of f.

Theorem 7.1. *Given a compact Riemann surface S, let $G \in M(S)$ be non-constant with $n := \mathrm{Deg}(G)$. Let $F \in M(S)$ be such that for $p \in S$:*

$$p \ \text{a pole of} \ F \Rightarrow p \ \text{a pole of} \ G. \tag{52}$$

[1]In this context $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is understood to be a compact Riemann surface isomorphic to the Riemann sphere.

Then there exist POLYNOMIALS $c_1, \dots, c_n \in \mathbb{C}[Y]$ *such that*

$$F^n + c_1(G)F^{n-1} + \cdots + c_n(G) = 0. \tag{53}$$

Remark 7.2. Without the pole condition (52) this theorem is folklore. In this case the c_j are *rational functions*; see for instance [12, §8.2 and §8.3] and also Lemma 7.3 below. The given version where condition (52) forces the c_j to be *polynomials* is algorithmically relevant, but it seems to be less known. A special instance of Theorem 7.1 where F is allowed to have poles only at one point is used in [30].

Proof of Theorem 7.1. The meromorphic function $G \in M(S)$ can be viewed as a holomorphic map between Riemann surfaces which, by the compactness of S, is also proper.[2] Hence one can apply the Theorems 8.2 and 8.3 from [12] which imply the existence of $c_1, \dots, c_n \in M(\hat{\mathbb{C}})$ such that (53). Namely, G being a non-constant proper holomorphic map implies that G is an n-sheeted covering map. This means, for each $x \in \hat{\mathbb{C}} \setminus \mathrm{BranchPts(G)}$ the set $G^{-1}(x)$ contains exactly n elements, say $G^{-1}(x) = \{a_1, \dots, a_n\}$ for a fixed x; moreover, there exist neighborhoods U_j of the a_j, containing no pole of G with the possible exception of a_j itself, and a neighborhood V of x containing no branch point of G, such that $G^{-1}(V) = \cup_{j=1}^{n} U_j$ as a disjoint union, and where the local restrictions $G|U_j \to V$ are bi-holomorphic maps. Using the elementary symmetric functions $e_j(X_1, \dots, X_n)$, $1 \leq j \leq n$, in n variables, for $v \in V$ the c_j are defined as

$$c_j(v) := (-1)^j e_j(F \circ (G|U_1)^{-1}(v), \dots, F \circ (G|U_n)^{-1}(v)).$$

These locally defined c_j are glued together to define global meromorphic functions $c_j : \hat{\mathbb{C}} \setminus \mathrm{BranchPts}(G) \to \hat{\mathbb{C}}$. By applying Riemann's Removable Singularity Theorem [12, Thm. 1.8] and the Identity Theorem [12, Thm. 1.11] one can show that these functions can be extended meromorphically also to the branch points of G; i.e., to meromorphic functions $c_j : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $j = 1, \dots, n$. Classical complex analysis tells that $M(\hat{\mathbb{C}}) = \mathbb{C}(z)$, the field of rational functions. To show that the c_j are indeed polynomials, consider $x \in \hat{\mathbb{C}}$ such that $c_\ell(x) = \infty$ for some $\ell \in \{1, \dots, n\}$ and $G^{-1}(x) = \{a_1, \dots, a_k\}$.[3]

Case A: $x \notin \mathrm{BranchPts}(G)$. In this case we have $n = k$ and

$$\infty = c_\ell(x) = (-1)^\ell e_\ell(F(a_1), \dots, F(a_n)).$$

Hence $F(a_j) = \infty$ for some $j \in \{1, \dots, n\}$, which by (52) implies $\infty = G(a_j) = x$. This means, the only pole of c_ℓ is at $x = \infty$.

[2]A continuous mapping $f : X \to Y$ between two locally compact spaces is called proper if the preimage of every compact set is compact.

[3]$n = \sum_{j=1}^{k} \mathrm{mult}_{a_j}(G)$; if x is no branch point of G then all $\mathrm{mult}_{a_j}(G) = 1$ and $n = k$.

Case B: $x \in \mathrm{BranchPts}(G)$. Suppose all values $F(a_1), \dots, F(a_k)$ are in $\mathbb{C}$. Then F is bounded in neighborhoods Y_j of a_j, $j = 1, \dots, k$, and again owing to G being non-constant, proper and holomorphic, there is a neighborhood W of x, containing no further branch point of G, such that $G^{-1}(W) \subseteq Y_1 \cup \cdots \cup Y_k$. Consequently, for $v \in W \setminus \{x\}$ the values

$$|c_\ell(v)| = |e_\ell(F \circ (G|U_1)^{-1}(v), \dots, F \circ (G|U_n)^{-1}(v))|$$

have a common bound; recall that as described above the c_j are defined locally as functions on suitable neighborhoods $V \subseteq W$ of v together with neighborhoods U_j of the n preimages of v. The bound for $|c_\ell(v)|$ on $W \setminus \{x\}$ is a contradiction to $c_\ell(x) = \infty$. Hence as in Case A the only pole of c_ℓ is at $x = \infty$.

Summarizing, we have proven that for $j = 1, \dots, n$ the only possible poles of the $c_j : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are at ∞. Hence the c_j are polynomials. □

Lemma 7.3. *Given a compact Riemann surface S, let $G \in M(S)$ and $F \in M(S)$ be non-constant meromorphic functions with $n := \mathrm{Deg}(G)$ and $m := \mathrm{Deg}(F)$. If $\gcd(m, n) = 1$ then there exists a polynomial*

$$p(X, Y) = X^n + c_1(Y)X^{n-1} + \cdots + c_n(Y) \in \mathbb{C}(Y)[X]$$

which is irreducible over $\mathbb{C}(Y)$, and where the $c_j(Y)$ are rational functions in $\mathbb{C}(Y)$ such that $p(F, G) = 0$.

Proof. See for instance [30, p. 485, Lem. 1]. □

Proof of Theorem 5.2. To determine $\mathrm{Deg}(f)$ of a meromorphic function f one can count the number of its poles, alternatively its zeros, with their multiplicities. By Lemma 5.1, $\bar{t} = \bar{t}_\ell$ and $\bar{T} = \bar{T}_\ell$ can be viewed as meromorphic functions $F := \bar{t}^*$ and $G := \bar{T}^*$ on the compact Riemann surface $X_0(\ell^2)$. They are both holomorphic and non-zero at all points of $[\tau] \in X_0(\ell^2)$ with $\tau \in \mathbb{H}$. At the $\ell + 1$ cusps Lemma 5.1 gives:[4]

$$\begin{aligned} \mathrm{ord}_{[\infty]} \bar{t}^* &= n_\ell, & \mathrm{ord}_{[\infty]} \bar{T}^* &= d_\ell, \\ \mathrm{ord}_{[0]} \bar{t}^* &= -d_\ell, & \mathrm{ord}_{[0]} \bar{T}^* &= -n_\ell, \\ \mathrm{ord}_{[k/\ell]} \bar{t}^* &= n_\ell, & \mathrm{ord}_{[k/\ell]} \bar{T}^* &= -n_\ell. \end{aligned}$$

Consequently, we see that the pole condition (52) of Theorem 7.1 is fulfilled. The resulting algebraic relation between $\bar{t}^*$ and $\bar{T}^*$ on $X_0(\ell^2)$ induces on $\mathbb{H}$ the relation:

$$\bar{t}^n + c_1(\bar{T})\bar{t}^{n-1} + \cdots + c_n(\bar{T}) = 0 \tag{54}$$

for some $c_j(Y) \in \mathbb{C}[Y]$ where $n := \mathrm{Deg}(\bar{T}^*) = d_\ell$, the number of zeros of $\bar{T}^*$ in $X_0(\ell^2)$. We still need to verify that the $c_j(Y)$ are polynomials in $\mathbb{Q}[Y]$ and not

[4] Recall $n_\ell := \frac{d_\ell}{\ell} = \frac{\ell-1}{\gcd(\ell-1,12)}$ from (46).

in $\mathbb{C}[Y]$. But this is straightforward by using the same induction argument as in the proof of Lemma 9.7.

Finally we prove that the polynomial

$$F_\ell(X,Y) := X^n + c_1(Y)X^{n-1} + \cdots + c_n(Y) \in \mathbb{Q}[Y][X]$$

is indeed irreducible in $\mathbb{C}[X,Y]$. To this end, we apply Theorem 7.1 again, now with $F := (t^*T^*)^\ell - (T^*)^{\ell+1}$ and $G := T^*$.[5] From the $\mathrm{ord}_{[*]}$-scheme above we see that both F and G have poles only at $[\infty]$. A simple calculation shows that $m := \mathrm{Deg}(F) = -\mathrm{ord}_{[\infty]} F = (\ell+1)d_\ell - 1$ and $n := \mathrm{Deg}(G) = -\mathrm{ord}_{[\infty]} G = d_\ell$. Hence Theorem 7.1 implies the existence of a polynomial

$$p_\ell(X,Y) = X^{d_\ell} + \gamma_1(Y)X^{d_\ell-1} + \cdots + \gamma_{d_\ell}(Y) \in \mathbb{C}[Y][X]$$

such that $p_\ell(F,G) = 0$. Owing to $\gcd(m,n) = 1$, Lemma 7.3 implies that $p_\ell(X,Y)$ is irreducible over $\mathbb{C}(Y)$. Hence $[\mathbb{C}(F,G) : \mathbb{C}(G)] = d_\ell$ or, equivalently, $[\mathbb{C}(\bar{t}^\ell, \bar{T}) : \mathbb{C}(\bar{T})] = d_\ell$. Since $\mathbb{C}(\bar{t}, \bar{T}) \supseteq \mathbb{C}(\bar{t}^\ell, \bar{T})$ it follows that

$$[\mathbb{C}(\bar{t}, \bar{T}) : \mathbb{C}(\bar{T})] \geq d_\ell. \tag{55}$$

But now, owing to (54), we must have equality in (55). This proves the irreducibility of $F_\ell(X,Y) \in \mathbb{Q}[Y][X]$ as a polynomial in X over $\mathbb{C}(Y)$. As such the polynomial $F_\ell(X,Y)$ is monic; this means, it has 1 as leading coefficient. Consequently, $F_\ell(X,Y)$ is irreducible also in $\mathbb{C}[X,Y]$; i.e., it has no proper non-trivial factor in $\mathbb{C}[x,y]$. □

8. Conclusion

As pointed out in the introduction, algorithmic aspects were a major driving force for the development of the framework presented in the main part of this paper. Despite new results like the derivation of new compact representations of Atkin's basis functions g_i in Section 9, for the sake of better readability we put various constructions and propositions of algorithmic relevance in a separate part, the Appendix, starting with Section 9. In fact, the main part up to Section 7 in its essence is independent from the material in the Appendix with the only exception of Section 12, where we describe the algorithmic derivation and proof of the fundamental relations needed to prove the crucial Lemma 6.5.

We conclude the main part of this paper with a conjecture on lower bounds for the ℓ-adic valuation of the coefficients of the modular equation for all primes $\ell \geq 5$.

[5]Note that $\bar{\bar{f}} = f$.

Conjecture 8.1. *Write*

$$F_\ell(X,Y) = X^{d_\ell} + \sum_{i=0}^{d_\ell-1} a_i^{(\ell)}(Y) X^i \tag{56}$$

for the unique polynomials $F_\ell(X,Y) \in \mathbb{Q}[Y][X]$ determined by Theorem 5.2. Then the statement of Corollary 5.6 is valid not only for $\ell \in \{5,7,11\}$ but for all primes $\ell \geq 5$.

9. Appendix 1: New representations of Atkin's generators

In this section we prove Theorem 3.9; see Theorem 9.8 together with (100). Moreover, we give a detailed description of the derivation of our new representations of Atkin's basis functions g_i using an approach which is close to an algorithm. To this end, we focus on properties of module generators, in particular, the concept of an Atkin basis in Definition 59 of Section 9.1.

In order to work with compact representations of the modular functions involved, starting with (65) from Section 9.2 we make use of special instances of a trace operator formula; see, e.g., [19, (1)]. In Section 9.3 we describe a classical reduction procedure to obtain relations between modular functions. This reduction strategy is applied at various places in this article, in particular, in Section 12 to obtain the fundamental relations. Finally, Section 9.4 completes the "reconstruction" of Atkin's g_i. Besides proving Theorem 3.9, we explain that there is some freedom in the choice of (16). For instance, one could omit the summand z_{11} there.

9.1. **General construction strategy.** First, we recall that the Fricke involution $W_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ which normalizes $\Gamma_0(N)$; i.e., $W_N \Gamma_0(N) W_N^{-1} = W_N^{-1}\Gamma_0(N) W_N = \Gamma_0(N)$. In particular, for all $f \in M(N)$,

$$(f \mid W_N) \mid \gamma = f \mid W_N \text{ for all } \gamma \in \Gamma_0(N), \text{ and} \tag{57}$$

$$\operatorname{ord}_{[0]_N}(f \mid W_N)^* = \operatorname{ord}_{[\infty]_N} f^* \text{ and } \operatorname{ord}_{[\infty]_N}(f \mid W_N)^* = \operatorname{ord}_{[0]_N} f^*. \tag{58}$$

For the case $N = 11$ the short hand notation $W := W_{11}$ will be convenient.

Example 9.1. Recall that $\operatorname{ord}_{[\infty]_{11}} z_{11}^* = 5$. By Lemma 3.5(3) we have that $(z_{11} \mid W)(\tau) = 11^{-6} z_{11}(\tau)^{-1}$. Hence $\operatorname{ord}_{[0]_{11}} z_{11}^* = \operatorname{ord}_{[\infty]_{11}} (z_{11} \mid W)^* = -5$.

Definition 9.2. *We say that the functions $\phi_2, \phi_3, \phi_4, \phi_6 \in M_{\mathbb{Z}}^0(11)$ form an Atkin basis iff the following three conditions hold:*

$$M_{\mathbb{Z}}^0(11) = \langle 1, \phi_2, \phi_3, \phi_4, \phi_6 \rangle_{\mathbb{Z}[z_{11}]}; \tag{59}$$

$$(\operatorname{ord}_{[\infty]_{11}} \phi_2^*, \operatorname{ord}_{[\infty]_{11}} g_\phi^*, \operatorname{ord}_{[\infty]_{11}} \phi_4^*, \operatorname{ord}_{[\infty]_{11}} \phi_6^*) = (1,2,3,4); \tag{60}$$

$$(\mathrm{ord}_{[0]_{11}}\,\phi_2^*, \mathrm{ord}_{[0]_{11}}\,\phi_3^*, \mathrm{ord}_{[0]_{11}}\,\phi_4^*, \mathrm{ord}_{[0]_{11}}\,\phi_6^*) = (-2,-3,-4,-6). \tag{61}$$

We note that $\mathrm{ord}_{[0]_{11}}\,\phi_i^* = \mathrm{ord}_{[\infty]_{11}}(\phi_i \mid W)^*$. Moreover, owing to (60), $M^0_{\mathbb{Z}}(11)$ is a $\mathbb{Z}[z_{11}]$-module which is freely generated by the basis elements ϕ_i.

Example 9.3. Atkin's functions g_i from Example 3.8 form an Atkin basis. In [6] Atkin uses the notation $g_i \mid W = 11^{-i}G_i$, $i = 2,3,4,6$; the first terms of the q-expansions of the g_i and G_i are given explicitly in Table 1 of [6, Appendix].[6]

Inspecting the orders in (61) one observes that the minimal pole order at 0 is 2. Indeed, as a consequence of Riemann surface theory one can prove that there is no $f \in M(11)$ having exactly one pole of order 1. An elementary proof is given in the Appendix, Section 11.

Atkin [6] defined the functions g_i in a skillful manner by following Newman [23]. In this section we present a method to find these functions which is more close to an algorithm. More precisely, in a first step and in view of (61) we will construct modular functions $J_i \in M^0_{\mathbb{Q}}(11)$ such that

$$M^0(11) = \langle 1, J_2, J_3, J_4, J_6\rangle_{\mathbb{C}[z_{11}]} \quad \text{with} \quad \mathrm{ord}_{[0]_{11}}\,J_i^* = -i, \quad i = 2,3,4,6. \tag{62}$$

Subsequently, using these J_i we construct an Atkin basis $(h_2 \mid W, h_3 \mid W, h_4 \mid W, h_6 \mid W)$ and relate it to Atkin's g_i in (100).

A first natural choice to choose such J_i is an ansatz in the form of eta quotients $\prod_{\delta|N} \eta_\delta(\tau)^{r_\delta(i)}$. Newman [25] gave a criterion for membership of such quotients in $M(N)$; see also [27, Thm. 5.1]. For $N = 11$ this criterion translates into the following linar system of Diophantine equations: $r_1(i) + r_{11}(i) = 0$, $r_1(i) + 11r_{11}(i) = 24a(i)$, $11r_1(i) + r_{11}(i) = 24b(i)$, and $r_{11}(i)$ even. In addition, Ligozat's order formula, e.g., [27, Lemma 5.2], translates $\mathrm{ord}_{[0]_{11}}\,J_i^* = -i$ into $11r_1(i) + r_{11}(i) = -24i$. For $i = 2,3,4,6$ the resulting linear Diophantine system has no integer solutions. For $i = 5$ there is exactly one solution z_{11}, because then $r_1(5) = -12$, $r_{11}(5) = 12$ with $a(5) = 5$, $b(5) = -5$, and $c(5) = 6$. As a consequence, to construct the desired J_i we try as the next nearest choice $N = 22$. The heuristical fundament for proceeding like this is

Conjecture 9.4 (Newmann's Conjecture, modified version)**.** *Let N be divisible by two distinct primes. Then each modular function from $M(N)$ can be written as a linear combination of eta quotients of the form $\prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ where $(r_\delta)_{\delta|N}$ is an integer sequence indexed by the positive divisors δ of N.*

[6] Atkin uses x instead of q.

Invoking the index formula $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N\prod_{p|N}\left(1+\frac{1}{p}\right)$ for $N = 11$ and $N = 22$ implies the existence of a decomposition into three disjoint cosets such that

$$\Gamma_0(11) = \Gamma_0(22)\gamma_0 \cup \Gamma_0(22)\gamma_1 \cup \Gamma_0(22)\gamma_2. \tag{63}$$

As a concrete choice we take $\gamma_0 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\gamma_1 := V$ and $\gamma_2 := V^2$ where $V := \left(\begin{smallmatrix}1&1\\11&12\end{smallmatrix}\right)$. This coset decomposition together with a coset decomposition of $\Gamma_0(11)$ in $\mathrm{SL}_2(\mathbb{Z})$, for example as in [18, Ch. 1, Cor. 7], implies that $X_0(22)$ has the four cusps $[\infty]_{22}$, $[0]_{22}$, $[1/2]_{22}$, and $[1/11]_{22}$ with widths 1, 22, 11, and 2, respectively.

In a first step, in view of (58), for $i = 2, 3, 4, 6$ we will construct

$$F_i \in M^{\infty}_{\mathbb{Q}}(11) \text{ such that } \operatorname{ord}_{[0]_{11}} F_i^* \geq 0 \text{ and } \operatorname{ord}_{[\infty]_{11}} F_i^* = -i. \tag{64}$$

Then the functions $J_i := F_i \mid W \in M^0_{\mathbb{Q}}(11)$ will have the desired properties (62).

9.2. **Constructing the F_i, resp. J_i.** As an ansatz for the construction we consider traces

$$F_i := f_i + f_i \mid V + f_i \mid V^2. \tag{65}$$

For any $f_i \in M(22)$ the decomposition (63) implies that $F_i \in M(11)$. The F_i are also in $M(22)$, and the cusp orders of the F_i in $\Gamma_0(22)$ connect to those in $\Gamma_0(11)$ by the following relations which are also straightforward to prove.

Lemma 9.5. *Let $F_i \in M^{\infty}(11)$ such that* (64)*. Then $F_i \in M^{\infty}(22)$ and*

$$\begin{aligned}
&\operatorname{ord}_{[\infty]_{22}} F_i^* = \operatorname{ord}_{[\infty]_{11}} F_i^* = -i;\\
&\operatorname{ord}_{[1/11]_{22}} F_i^* = 2\operatorname{ord}_{[\infty]_{11}} F_i^* = -2i;\\
&\operatorname{ord}_{[0]_{22}} F_i^* = 2\operatorname{ord}_{[0]_{11}} F_i^* \geq 0;\\
&\operatorname{ord}_{[1/2]_{22}} F_i^* = \operatorname{ord}_{[0]_{11}} F_i^* \geq 0.
\end{aligned}$$

When choosing the f_i as eta quotients $\prod_{\delta|22}\eta_\delta^{r_\delta}$, for fixed integers r_δ Ligozat's lemma, e.g. [27, Lemma 5.2], tells the orders of the f_i at the cusps. We set

$$a(i) := \operatorname{ord}_{[\infty]_{22}} f_i^*, b(i) := \operatorname{ord}_{[0]_{22}} f_i^*, c(i) := \operatorname{ord}_{[1/11]_{22}} f_i^*, d(i) := \operatorname{ord}_{[1/2]_{22}} f_i^*.$$

From (65) one obtains

$$\operatorname{ord}_{[\infty]_{22}} F_i^* \geq \min\{a(i), c(i)/2\}, \text{ with equality if } a(i) < c(i)/2; \tag{66}$$

$$\operatorname{ord}_{[1/11]_{22}} F_i^* \geq \min\{2\,a(i), c(i)\}, \text{ with equality if } a(i) < c(i)/2; \tag{67}$$

$$\operatorname{ord}_{[0]_{22}} F_i^* \geq \min\{b(i), 2\,d(i)\}, \text{ with equality if } 2d(i) < b(i); \tag{68}$$

$$\operatorname{ord}_{[1/2]_{22}} F_i^* \geq \min\{b(i)/2, d(i)\}, \text{ with equality if } 2d(i) < b(i). \tag{69}$$

As consequence of (66) to (69) and Lemma 9.5, without loss of generality we can restrict to choosing eta quotients $f_i \in M_{\mathbb{Z}}^{\infty}(22)$ such that

$$a(i) = \mathrm{ord}_{[\infty]_{22}} f_i^* = -i \ \text{ and } \ c(i) = \mathrm{ord}_{[1/11]_{22}} f_i^* \geq -2i.$$

Constructing the f_i, respectively the F_i: To find such f_i computationally, similarly to above one solves the linear system of Diophantine equations and inequalities determined by Newman [27, Thm. 5.1] and Ligozat [27, Lemma 5.2]. For $i = 2$ one finds

$$f_2(\tau) := \frac{\eta(\tau)\eta_2(\tau)^3}{\eta_{11}(\tau)^3\eta_{22}(\tau)} = \frac{1}{q^2} - \frac{1}{q} - 4 + 3q + 3q^2 + \cdots \in M_{\mathbb{Z}}^{\infty}(22) \tag{70}$$

with

$$a(2) = -2,\ b(2) = 2,\ c(2) = -3, \ \text{ and } \ d(2) = 3.$$

Consequently, (66) to (69) and Lemma 9.5 give $\mathrm{ord}_{[\infty]_{11}} F_2^* = -2$ and $\mathrm{ord}_{[0]_{11}} F_2^* \geq \min\{1, 3/2\} \geq 0$, as desired. For $i = 3$ one obtains

$$f_3(\tau) := \frac{\eta(\tau)^3\eta_2(\tau)}{\eta_{11}(\tau)\eta_{22}(\tau)^3} = \frac{1}{q^3} - \frac{3}{q^2} - \frac{1}{q} + 8 - q - \cdots \in M_{\mathbb{Z}}^{\infty}(22) \tag{71}$$

with

$$a(3) = -3,\ b(3) = 3,\ c(3) = -2, \ \text{ and } \ d(3) = 2.$$

Here (66) to (69) and Lemma 9.5 give $\mathrm{ord}_{[\infty]_{11}} F_3^* = -3$ and $\mathrm{ord}_{[0]_{11}} F_3^* \geq \min\{3/2, 2\} \geq 0$, as desired. Finally,

$$f_4(\tau) := f_2(\tau)^2 \in M_{\mathbb{Z}}^{\infty}(22) \ \text{ and } \ f_6(\tau) := f_3(\tau)^2 \in M_{\mathbb{Z}}^{\infty}(22)$$

give $\mathrm{ord}_{[\infty]_{11}} F_4^* = -4$ and $\mathrm{ord}_{[0]_{11}} F_4^* \geq 2$, resp. $\mathrm{ord}_{[\infty]_{11}} F_6^* = -6$ and $\mathrm{ord}_{[0]_{11}} F_6^* \geq 2$, as desired. The fact that the coefficients of the F_i are integers for $i = 2, 3, 6$, resp. half-integers for $i = 4$, is immediate from the following representations being more explicit than (65):

$$F_2(\tau) = f_2(\tau) - (U_2 f_3)(\tau) = q^{-2} + 2q^{-1} - 12 + 5q + 8q^2 + \ldots; \tag{72}$$

$$F_3(\tau) = f_3(\tau) - 4(U_2 f_2)(\tau) = q^{-3} - 3q^{-2} - 5q^{-1} + 24 - 13q - \ldots; \tag{73}$$

$$F_4(\tau) = f_2(\tau)^2 + \frac{1}{2}(U_2 f_3^2)(\tau) = q^{-4} - \frac{3}{2}q^{-3} - \frac{7}{2}q^{-2} - \frac{21}{2}q^{-1} + 48 - \ldots; \tag{74}$$

$$F_6(\tau) = f_3(\tau)^2 + 8(U_2 f_2^2)(\tau) = q^{-6} - 6q^{-5} + 7q^{-4} + 22q^{-3} - 41q^{-2} + \ldots. \tag{75}$$

These representations, which invoke the U-operator (see Def. 4.2), can be derived in a straightforward manner by applying the modular transformation properties of the eta function. Nevertheless, from a more general point of view we remark that they are special instances of a trace operator formula; see, e.g., [19, (1)]. A proof of (72) along this line is given in Section 10.1 (Appendix); the proofs of the other formulas work analogously.

Another advantage of invoking the formula [19, (1)] is that it facilitates (we leave the details to the reader) the application of the W-operator to finally obtain the desired module generators $J_i := F_i \mid W \in M^0_{\mathbb{Q}}(11)$ satisfying (62):

$$J_2(\tau) = -11^2\Big(\frac{1}{f_3} - U_2\frac{1}{f_2}\Big)(\tau) = 11^2(q + 5q^2 + \dots); \tag{76}$$

$$J_3(\tau) = -11^2\Big(\frac{1}{f_2} - 4\,U_2\frac{1}{f_3}\Big)(\tau) = 11^2(11q^2 + 99q^3 + \dots); \tag{77}$$

$$J_4(\tau) = 11^4\Big(\frac{1}{f_3^2} + \frac{1}{2}\,U_2\frac{1}{f_2^2}\Big)(\tau) = 11^4\Big(\frac{1}{2}q^2 + \frac{11}{2}q^3 + \dots\Big); \tag{78}$$

$$J_6(\tau) = 11^4\Big(\frac{1}{f_2^2} + 8\,U_2\frac{1}{f_3^2}\Big)(\tau) = 11^4(8q^3 + 233q^4 + \dots). \tag{79}$$

All these equalities are obtained as direct consequences of the formula [19, (1)]. A proof of (76) along this line is given in Section 10.1 (Appendix); the proofs of the other formulas work analogously.

9.3. **Module property of the F_i, resp. J_i.** For later it is important to note that the first coefficients of the q-expansion in (77) suggest that

$$11\Big|\Big(\frac{1}{f_2} - 4\,U_2\frac{1}{f_3}\Big), \tag{80}$$

meaning that 11 divides each coefficient in the q-expansion. This divisibility is immediate from a relation already used by Atkin [6, (56)]; namely,

$$J_3 = 11^3 F_3\, z_{11}. \tag{81}$$

This relation cannot only be proved but also derived algorithmically. To this end we introduce an elementary but useful

Lemma 9.6. *For $\bar{z}_{11} = 1/z_{11}$,*

$$M^\infty(11) = \langle 1, F_2, F_3, F_4, F_6\rangle_{\mathbb{C}[\bar{z}_{11}]}. \tag{82}$$

Proof. The non-trivial direction is to show that every non-constant function $F \in M^\infty(11)$ can be represented in the form

$$F = P_0(\bar{z}_{11}) + P_2(\bar{z}_{11})F_2 + P_3(\bar{z}_{11})F_3 + P_4(\bar{z}_{11})F_4 + P_6(\bar{z}_{11})F_6 \tag{83}$$

for some polynomials $P_i(X) \in \mathbb{C}[X]$. Suppose that $F(\tau) = c\,q^{-n} + O(q^{-n+1})$ for some $c \in \mathbb{C}$. Recall that there exists no principal modular function ("Hauptmodul") in $M(11)$ (Section 11), hence $n = 5j + i \geq 2$ with $j \geq 0$ and $i \in \{0, 2, 3, 4, 6\}$. Owing to $\mathrm{ord}_{[0]_{11}}\, \bar{z}^*_{11} = -5$ one can reduce F with $\bar{z}^j_{11}F_i$; more concretely,

$$F(\tau) - c\,\bar{z}_{11}(\tau)^j F_i(\tau) = d\,q^{-n+1} + O(q^{-n+2}) \text{ for some } d \in \mathbb{C}.$$

Consequently, owing to Lemma 3.2, by iterated reduction F finds a representation of the form (83) for some polynomials $P_i(X) \in \mathbb{C}[X]$. $\square$

Proof of (81). The only pole of J_3 is at 0 of multiplicity $3 = -\operatorname{ord}_{[0]_{11}} J_3^* = -\operatorname{ord}_{[\infty]_{11}}(J_3 \mid W)^* = -\operatorname{ord}_{[\infty]_{11}} F_3^*$; in addition, J_3 has a zero at infinity of multiplicity $2 = \operatorname{ord}_{[\infty]_{11}} J_3^*$. The only pole of $\bar{z}_{11}$ is at infinity of multiplicity 5; in addition, $\bar{z}_{11}$ has a zero at 0 of multiplicity $5 = \operatorname{ord}_{[0]_{11}} \bar{z}_{11}^* = \operatorname{ord}_{[\infty]_{11}}(\bar{z}_{11} \mid W)^* = \operatorname{ord}_{[\infty]_{11}}(11^6 z_{11})^*$. Hence $11^{-3}J_3\bar{z}_{11}$ has its only pole at infinity of multiplicity 3 which is confirmed by $11^{-3}J_3\bar{z}_{11} = q^{-3} - 3q^{-2} - 5q^{-1} + 24 - \cdots \in M^\infty(11)$. Thus (82) implies that $11^{-3}J_3\bar{z}_{11} \in \langle 1, F_2, F_3, F_4, F_6\rangle_{\mathbb{C}[\bar{z}_{11}]}$. To derive the corresponding representation, we can apply the reduction strategy as described in the proof of Lemma 9.6. Already the first reduction step gives

$$11^{-3}J_3(\tau)\bar{z}_{11}(\tau) - F_3(\tau) = 0 + 0 \cdot q + 0 \cdot q^2 + \dots,$$

which by Lemma 3.2 proves (81). □

The J_i we constructed satisfy the requirements of (62)*:* By construction the J_i are in $M^0_{\mathbb{Q}}(11)$ with $\operatorname{ord}_{[0]_{11}} J_i^* = \operatorname{ord}_{[\infty]_{11}}(J_i \mid W)^* = \operatorname{ord}_{[\infty]_{11}} F_i^* = -i$ for $i = 2, 3, 4, 6$. To show also the first part of (62); i.e., that every function $f \in M^0(11)$ can be represented in the form

$$f = p_0(z_{11}) + p_2(z_{11})J_2 + p_3(z_{11})J_3 + p_4(z_{11})J_4 + p_6(z_{11})J_6 \tag{84}$$

for some polynomials $p_i(X) \in \mathbb{C}[X]$, one proceeds as follows: For any non-constant $f \in M^0(11)$ the only pole sits at 0 with some multiplicity n; i.e., $(f \mid W)(\tau) = c\, q^{-n} + O(q^{-n+1}) \in M^\infty(11)$ for some $c \in \mathbb{C}$. Hence by Lemma 9.6 $f \mid W$ finds a representation of the form

$$f \mid W = P_0(\bar{z}_{11}) + P_2(\bar{z}_{11})F_2 + P_3(\bar{z}_{11})F_3 + P_4(\bar{z}_{11})F_4 + P_6(\bar{z}_{11})F_6.$$

for some polynomials $P_i(X) \in \mathbb{C}[X]$. Using $\bar{z}_{11} \mid W = 11^6 z_{11}$ (Lemma 3.5) and $F_i \mid W = J_i$ this relation turns into one as in (84).

From complex to rational numbers coefficients. If we want to show that every function $f \in M^0_{\mathbb{Q}}(11)$ can be represented in the form

$$f = p_0(z_{11}) + p_2(z_{11})J_2 + p_3(z_{11})J_3 + p_4(z_{11})J_4 + p_6(z_{11})J_6 \tag{85}$$

for some polynomials $p_i(X) \in \mathbb{Q}[X]$, it would be sufficient to have

$$\operatorname{ord}_{[\infty]_{11}} J_2^* = 1, \operatorname{ord}_{[\infty]_{11}} J_3^* = 2, \operatorname{ord}_{[\infty]_{11}} J_4^* = 3, \text{ and } \operatorname{ord}_{[\infty]_{11}} J_6^* = 4. \tag{86}$$

Because then mathematical induction would show that the $p_i(z_{11}) := P_i(\bar{z}_{11} \mid W)$ must have coefficients in $\mathbb{Q}$; i.e., $p_i(X) \in \mathbb{Q}[X]$.

The same induction argument would apply to showing that every function $f \in M^0_{\mathbb{Z}}(11)$ can be represented in the form

$$f = p_0(z_{11}) + p_2(z_{11})J_2 + p_3(z_{11})J_3 + p_4(z_{11})J_4 + p_6(z_{11})J_6 \tag{87}$$

for some polynomials $p_i(X) \in \mathbb{Z}[X]$, provided we would have

$$J_i(\tau) = q^{i-1} + O(q^i) \in M^0_{\mathbb{Z}}(11) \text{ for } i = 2, 3, 4, \text{ and } J_6(\tau) = q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11).$$

We summarize these observations:

Lemma 9.7. *Let* $h_2, h_3, h_4, h_6 \in M^{\infty}(11)$ *be such that*

$$\operatorname{ord}_{[\infty]_{11}} h_i^* = -i \text{ for } i = 2, 3, 4, 6, \tag{88}$$

and

$$\begin{aligned} (h_i \mid W)(\tau) &= q^{i-1} + O(q^i) \in M^0_{\mathbb{Z}}(11) \text{ for } i = 2, 3, 4, \quad \text{and} \\ (h_6 \mid W)(\tau) &= q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11). \end{aligned} \tag{89}$$

Then the functions $\phi_i := h_i \mid W$, $i = 2, 3, 4, 6$, *form an Atkin basis.*

Proof. The order properties of the ϕ_i are clear by their definition and (58). The remaining non-trival part of the proof is to show that any function $f \in M^0_{\mathbb{Z}}(11)$ can be represented in the form

$$f = p_0(z_{11}) + p_2(z_{11})\phi_2 + p_3(z_{11})\phi_3 + p_4(z_{11})\phi_4 + p_6(z_{11})\phi_6 \tag{90}$$

for some polynomials $p_i(X) \in \mathbb{Z}[X]$. Since $f \mid W \in M^{\infty}(11)$, using the same argument as in the proof of Lemma 9.6, one has that

$$f \mid W = P_0(\bar{z}_{11}) + P_2(\bar{z}_{11})h_2 + P_3(\bar{z}_{11})h_3 + P_4(\bar{z}_{11})h_4 + P_6(\bar{z}_{11})h_6 \tag{91}$$

for some polynomials $P_i(X) \in \mathbb{C}[X]$. Using $\bar{z}_{11} \mid W = 11^6 z_{11}$ this implies that

$$f = p_0(z_{11}) + p_2(z_{11})\phi_2 + p_3(z_{11})\phi_3 + p_4(z_{11})\phi_4 + p_6(z_{11})\phi_6$$

for polynomials $p_i(X) := P_i(11^6 X) \in \mathbb{C}[X]$. We show that all $p_i(X) \in \mathbb{Z}[X]$:

For $i = 0, 2, 3, 4, 6$ suppose $p_i(X) = a_i + b_i X + c_i X^2 + \dots$ with $a_i, b_i, c_i \in \mathbb{C}$. Then

$$\begin{aligned} f(\tau) = (a_0 &+ b_0 z_{11}(\tau) + \dots) \\ &+ (a_2 + b_2 z_{11}(\tau) + \dots)(q + O(q^2)) + (a_3 + b_3 z_{11}(\tau) + \dots)(q^2 + O(q^3)) \\ &+ (a_4 + b_4 z_{11}(\tau) + \dots)(q^3 + O(q^4)) + (a_6 + b_6 z_{11}(\tau) + \dots)(q^4 + O(q^5)). \end{aligned}$$

Since $f \in M^0_{\mathbb{Z}}(11)$, one has $f(\tau) = \sum_{n=0}^{\infty} f_n q^n$ with all $f_n \in \mathbb{Z}$. Because of $z_{11}(\tau) = q^5 + O(q^5)$ coefficient comparison implies $a_0, a_2, a_3, a_4, a_6 \in \mathbb{Z}$. Hence

$$\begin{aligned} (f - a_0 - a_2\phi_2 - a_3\phi_3 - a_4\phi_4 - a_6\phi_6)\bar{z}_{11} &= (b_0 + c_0 z_{11}(\tau) + \dots) \\ + (b_2 + c_2 z_{11}(\tau) + \dots)(q + O(q^2)) &+ (b_3 + c_3 z_{11}(\tau) + \dots)(q^2 + O(q^3)) \\ + (b_4 + c_4 z_{11}(\tau) + \dots)(q^3 + O(q^4)) &+ (b_6 + c_6 z_{11}(\tau) + \dots)(q^4 + O(q^5)). \end{aligned}$$

The left hand side is again in $M^0_{\mathbb{Z}}(11)$, hence $b_0, b_2, b_3, b_4, b_6 \in \mathbb{Z}$. Iterating this argument proves $p_i(X) \in \mathbb{Z}[X]$. $\square$

9.4. **Atkin's functions reconstructed.** The functions $h_2 := F_2/11^2$ and $h_3 := F_3/11^3$ are in $M^\infty(11)$ and satisfy all properties requested in Lemma 9.7; in particular,

$$(h_2 \mid W)(\tau) = J_2(\tau)/11^2 = q + q^5 + \cdots \in M^0_{\mathbb{Z}}(11) \tag{92}$$

and

$$(h_3 \mid W)(\tau) = J_3(\tau)/11^3 = q^2 + 9q^3 + \cdots \in M^0_{\mathbb{Z}}(11). \tag{93}$$

The fact that these functions are in $M^0_{\mathbb{Z}}$ is owing to the representations (76) and (77) in terms of the U_2-operator and to (80). In the next step we try to determine $a, b, c \in \mathbb{Q}$ such that

$$\begin{aligned} a\,\frac{J_4(\tau)}{11^4} + b\,\frac{J_3(\tau)}{11^3} + c\,\frac{J_2(\tau)}{11^2} &= c\,q + \Big(\frac{a}{2} + b + 5c\Big)q^2 + \Big(\frac{11a}{2} + 9b + 19c\Big)q^3 + \dots \\ &= q^3 + O(q^4) \in M^0_{\mathbb{Z}}(11). \end{aligned}$$

The linear system $c = 0$, $a/2 + b + 5c = 0$, and $11a/2 + 9b + 19c = 1$ has a unique solution $a = 1$, $b = -(1/2)$, and $c = 0$. Looking at further terms in the q-expansion

$$\frac{J_4}{11^4} - \frac{1}{2}\,\frac{J_3}{11^3} = q^3 + 14q^4 + 102q^5 + 561q^6 + 2563q^7 + 10285q^8 + 37349q^9 + \dots$$

supports to conjecture that

$$\frac{J_4}{11^4} - \frac{1}{2}\,\frac{J_3}{11^3} \in M^0_{\mathbb{Z}}(11). \tag{94}$$

In Section 10.2 (Appendix) we prove that this is indeed the case. Thus we can define

$$\begin{aligned} h_4(\tau) &:= \frac{F_4(\tau)}{11^4} - \frac{1}{2}\,\frac{F_3(\tau)}{11^3} \in M^\infty_{\mathbb{Q}}(11) \\ &= \frac{1}{11^4}\Big(q^{-4} + 4q^{-3} - 20q^{-2} - 38q^{-1} + 180 - 86q - \dots\Big), \end{aligned} \tag{95}$$

which then has the properties requested in Lemma 9.7. In particular,

$$h_4 \mid W = \frac{F_4 \mid W}{11^4} - \frac{1}{2}\,\frac{F_3 \mid W}{11^3} = \frac{J_4}{11^4} - \frac{1}{2}\,\frac{J_3}{11^3} = q^3 + O(q^4) \in M^0_{\mathbb{Z}}(11). \tag{96}$$

Finally, proceeding analogously to above, we try to find $a, b, c, d \in \mathbb{Q}$ such that

$$\begin{aligned} a\,\frac{J_6(\tau)}{11^4} + b\,\frac{J_4(\tau)}{11^4} + c\,\frac{J_3(\tau)}{11^3} + d\,\frac{J_2(\tau)}{11^2} &= d\,q + \Big(\frac{b}{2} + c + 5d\Big)q^2 \\ &+ \Big(8a + \frac{11b}{2} + 9c + 19d\Big)q^3 + \Big(233a + \frac{77b}{2} + 49c + 63d\Big)q^4 + \dots \\ &= q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11). \end{aligned}$$

The linear system $d = 0$, $b/2 + c + 5d = 0$, $8a + (11b)/2 + 9c + 19d = 0$ and $233a + (77b)/2 + 49c + 63d = 1$ has a unique solution $a = 11^{-2}$, $b = -8/11^{-2}$, $c = 4/11^{-2}$, and $d = 0$. Looking at further terms in the q-expansion

$$\frac{J_6}{11^6} - 8\,\frac{J_4}{11^6} + 4\,\frac{J_3}{11^5} = q^4 + 18q^5 + 179q^6 + 1310q^7 + 7853q^8 + \dots$$

supports to conjecture that

$$\frac{J_6}{11^6} - 8\,\frac{J_4}{11^6} + 4\,\frac{J_3}{11^5} \in M^0_{\mathbb{Z}}(11). \tag{97}$$

In Section 10.2 (Appendix) we prove that this is indeed the case. Thus we can define

$$\begin{aligned} h_6(\tau) &:= \frac{F_6}{11^6} - 8\,\frac{F_4}{11^6} + 4\,\frac{F_3}{11^5} \in M^\infty_{\mathbb{Q}}(11) \\ &= \frac{1}{11^6}\Big(q^{-6} - 6q^{-5} - q^{-4} + 78q^{-3} - 145q^{-2} - 206q^{-1} + 864 - \dots\Big), \end{aligned} \tag{98}$$

which then has the properties requested in Lemma 9.7. In particular,

$$\begin{aligned} h_6 \mid W &= \frac{F_6 \mid W}{11^6} - 8\,\frac{F_4 \mid W}{11^6} + 4\,\frac{F_3 \mid W}{11^5} \\ &= \frac{J_6}{11^6} - 8\,\frac{J_4}{11^6} + 4\,\frac{J_3}{11^5} = q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11). \end{aligned}$$

New representations of the Atkin functions. We summarize in the form of a theorem.

Theorem 9.8. *The functions h_i, $i = 2, 3, 4, 6$, constructed above satisfy*

(i) $11^i h_i \in M^\infty_{\mathbb{Z}}(11)$;

(ii) $h_i \mid W \in M^0_{\mathbb{Z}}(11)$;

(iii) *the $h_i \mid W$ form an Atkin basis.*

We have seen that following the framework set up by Lemma 9.7 one is forced to define h_2, h_3, and h_4 as we did. With h_6 the situation is slightly different; its definition is up to adding constant multiples of powers of $\bar{z}_{11} = q^{-5} + O(q^{-4}) \in M^\infty_{\mathbb{Z}}$. For example, for $m \in \mathbb{Z}$:

$$H_6(\tau) := h_6(\tau) + \frac{m}{11^6}\,\bar{z}_{11}(\tau) = \frac{1}{11^6}\Big(q^{-6} - (6-m)q^{-5} + \dots\Big)$$

with $11^6 H_6 \in M^\infty_{\mathbb{Z}}(11)$, and by using again (106) and $z_{11} = q^5 + O(q^6)$,

$$\begin{aligned} (H_6 \mid W)(\tau) &= (h_6 \mid W)(\tau) + \frac{m}{11^6}\,(\bar{z}_{11} \mid W)(\tau) = (h_6 \mid W)(\tau) + m z_{11} \\ &= (h_6 \mid W)(\tau) + mq^5 + O(q^6) = q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11). \end{aligned}$$

In fact, Atkin's setting [6] corresponds to the choice $m = 1$; concretely, instead of using (h_2, h_3, h_4, h_6) he worked with

$$\Big(h_2, h_3, h_4, h_6 + \frac{\bar{z}_{11}}{11^6}\Big) =: \Big(\frac{G_2}{11^2}, \frac{G_3}{11^3}, \frac{G_4}{11^4}, \frac{G_6}{11^6}\Big), \tag{99}$$

with the G_i being the notation used by Atkin. Consequently, instead of the Atkin basis $h_i \mid W$, $i = 2, 3, 4, 6$, Atkin worked with the slightly different Atkin basis

$$(h_2 \mid W, h_3 \mid W, h_4 \mid W, h_6 \mid W + z_{11}) =: (g_2, g_3, g_4, g_6). \tag{100}$$

Subsequently we will work with this Atkin basis; moreover, we will use Atkin's notation g_i as in (100).

10. Appendix 2: Proofs of Formulas for F_i, resp. J_i

In Subsection 10.1 we prove the formulas (72) and (76) for F_2 and J_2, respectively. The proofs of the other formulas in these families work analogously. In Subsection 10.2 we prove that the modular functions in (94) and (97) involving J_i have integer coefficients in their q-expansions.

10.1. **Proofs of** (72) **and** (76)**.** The following lemma states a special instance of formula [19, (1)] adapted to our situation.

Lemma 10.1. *Let* $\Gamma_0(11) = \Gamma_0(22)\gamma_0 \cup \Gamma_0(22)\gamma_1 \cup \Gamma_0(22)\gamma_2$ *with* $\gamma_i \in \Gamma_0(11)$ *be a decomposition of* $\Gamma_0(11)$ *into disjoint cosets. Then the trace map*

$$\mathrm{tr} : M(22) \to M(11), f \mapsto \mathrm{tr}(f) := f \mid \gamma_0 + f \mid \gamma_1 + f \mid \gamma_2$$

can be written as

$$\mathrm{tr}(f)(\tau) = f(\tau) + 2\ U_2\big((f \mid V)(2\tau)\big) \ \text{ with } \ V = \left(\begin{smallmatrix} 1 & 1 \\ 11 & 12 \end{smallmatrix}\right). \tag{101}$$

Relevant for our applications are the following actions of V and $W = \left(\begin{smallmatrix} 0 & -1 \\ 11 & 0 \end{smallmatrix}\right)$ which are straightforward consequences of the transformation formula for the η function.

Lemma 10.2. *For* $f_2, f_3 \in M^{\infty}_{\mathbb{Z}}(22)$ *as in* (70) *and* (71)*:*

$$(f_2 \mid V)(2\tau) = -\frac{1}{2} f_3(\tau) \ \text{ and } \ (f_3 \mid V)(2\tau) = -2 f_2(\tau), \tag{102}$$

and

$$(f_2 \mid W)(2\tau) = \frac{11^2}{2}\frac{1}{f_2(\tau)} \ \text{ and } \ (f_3 \mid W)(2\tau) = 2 \cdot 11^2 \frac{1}{f_3(\tau)}. \tag{103}$$

Proof of (72). In (63) we used $\gamma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\gamma_1 = V$, and $\gamma_2 = V^2$. Hence by (102),

$$F_2 = \mathrm{tr}(f_2) = f_2 + 2\ U_2\big((f_2 \mid V)(2\tau)\big) = f_2 - \ U_2 f_3;$$

this is (72). $\square$

The representations (73), (74), and (75) are derived analogously.

Proof of (76). Consider
$$J_2(\tau) = (F_2 \mid W)(\tau) = (f_2 \mid W)(\tau) + (f_2 \mid VW)(\tau) + (f_2 \mid V^2W)(\tau).$$
From the functions on the right side, using the rules of Lemma 10.2, the first and the third evaluate to functions in the argument $\tau/2$. For the remaining function these rules give
$$(f_2 \mid VW)(\tau) = -\frac{1}{2} f_3\Big(\frac{W\tau}{2}\Big) = -\frac{1}{2}\cdot 2\cdot 11^2 \frac{1}{(f_3 \mid W)(W\tau)} = -11^2\frac{1}{f_3(\tau)}.$$
This motivates to express the $f_2 \mid W$ trace as a $\varphi := -11^2/f_3$ trace. To this end, observing that $VW = WX$ for $X = \left(\begin{smallmatrix} 12 & -1 \\ -11 & 1 \end{smallmatrix}\right)$ one rewrites
$$f_2 \mid V^2W = f_2 \mid VW \mid X = \varphi \mid X \text{ and } f_2 \mid W = f_2 \mid VW \mid X^{-1} = \varphi \mid X^{-1}.$$
Consequently,
$$J_2 = \varphi + \varphi \mid X + \varphi \mid X^{-1}.$$
Since $\Gamma_0(11) = \Gamma_0(22) \cup \Gamma_0(22)X \cup \Gamma_0(22)X^{-1}$ is a decomposition into disjoint cosets, formula (101) gives
$$\begin{aligned} J_2(\tau) = \varphi(\tau) + 2\, U_2\big((\varphi \mid V)(2\tau)\big) &= -\frac{11^2}{f_3(\tau)} - 2\cdot 11^2 U_2 \frac{1}{(f_3 \mid V)(2\tau)} \\ &= -\frac{11^2}{f_3(\tau)} + 11^2 U_2 \frac{1}{f_2(\tau)}; \end{aligned}$$
the last equality is by (102). This completes the proof of (76). □

The representations (77), (78), and (79) are derived analogously.

10.2. **Proofs of** (94) **and** (97)**.** The proofs are straightforward; they are included for the sake of completeness.

Proof of (94). The functions J_3 and J_4 from (77) and (78), respectively, are in $M^0_{\mathbb{Q}}(11)$ and have their only pole at 0 with multiplicity $i = -\operatorname{ord}_{[0]_{11}} J_i^* = -\operatorname{ord}_{[\infty]_{11}}(J_i \mid W)^* = -\operatorname{ord}_{[\infty]_{11}} F_i^*$, $i = 3, 4$. The only pole of $\bar{z}_{11}$ is at infinity of multiplicity 5; in addition, $\bar{z}_{11}$ has a zero at 0 of multiplicity $5 = \operatorname{ord}_{[0]_{11}} \bar{z}_{11}^* = \operatorname{ord}_{[\infty]_{11}}(\bar{z}_{11} \mid W)^* = \operatorname{ord}_{[\infty]_{11}}(11^6 z_{11})^*$. Hence $(J_4/11^4 - \frac{1}{2}J_3/11^3)\bar{z}_{11} \in M^\infty(11)$ has its only pole at infinity of multiplicity 3 as given by the q-expansion
$$\Big(\frac{J_4(\tau)}{11^4} - \frac{1}{2}\,\frac{J_3(\tau)}{11^3}\Big)\bar{z}_{11} = q^{-2} + 2q^{-1} - 12 + 5q + 8q^2 + \dots$$
Now (82) implies that $(J_4/11^4 - \frac{1}{2}J_3/11^3)\bar{z}_{11} \in \langle 1, F_2, F_3, F_4, F_6\rangle_{\mathbb{C}[\bar{z}_{11}]}$. To derive the corresponding representation, we can apply the reduction strategy as

described in the proof of Lemma 9.6. As in the proof of (81), already the first reduction step gives

$$\Big(\frac{J_4(\tau)}{11^4}-\frac{1}{2}\,\frac{J_3(\tau)}{11^3}\Big)\bar{z}_{11}-F_2(\tau)=0+0\cdot q+0\cdot q^2+\dots, \tag{104}$$

which by Lemma 3.2 proves (94) because $F_2\in M_{\mathbb{Z}}^{\infty}(11)$. $\square$

Proof of (97). The functions $J_3, J_4, J_6 \in M_{\mathbb{Q}}^{0}(11)$ from (77), (78), and (79), respectively, have their only pole at 0 with multiplicities 3, 4, and 6, respectively. E.g., $6=-\operatorname{ord}_{[0]_{11}} J_6^*=-\operatorname{ord}_{[\infty]_{11}}(J_6\mid W)^*=-\operatorname{ord}_{[\infty]_{11}} F_6^*$. We follow the same strategy as in the proof of (94) where using $\bar{z}_{11}$ we modified the expression in question such to express it as a linear combination of F_i. Recall that the only pole of $\bar{z}_{11}$ is at infinity of multiplicity 5 and its only zero at 0 also with multiplicity 5. Hence $(J_6/11^6-8J_4/11^6+4J_3/11^5)\bar{z}_{11}^2\in M^{\infty}(11)$ has its only pole at infinity with multiplicity 6 as given by the q-expansion

$$\Big(\frac{J_6}{11^6}-8\,\frac{J_4}{11^6}+4\,\frac{J_3}{11^5}\Big)\bar{z}_{11}^2=q^{-6}-6q^{-5}-q^{-4}+78q^{-3}-145q^{-2}-206q^{-1}+\dots.$$

Now (82) implies that $(J_6/11^6-8J_4/11^6+4J_3/11^5)\bar{z}_{11}^2\in\langle 1,F_2,F_3,F_4,F_6\rangle_{\mathbb{C}[\bar{z}_{11}]}$. To derive the corresponding representation, we can apply the reduction strategy as described in the proof of Lemma 9.6. The first reduction step gives

$$\Big(\frac{J_6}{11^6}-8\,\frac{J_4}{11^6}+4\,\frac{J_3}{11^5}\Big)\bar{z}_{11}^2-F_6=-8q^{-4}+56q^{-3}-104q^{-2}-136q^{-1}+\dots$$

The second reduction step results in

$$\Big(\frac{J_6}{11^6}-8\,\frac{J_4}{11^6}+4\,\frac{J_3}{11^5}\Big)\bar{z}_{11}^2-F_6+8F_4=44q^{-3}-132q^{-2}-220q^{-1}+1056-\dots$$

Finally,

$$\Big(\frac{J_6}{11^6}-8\,\frac{J_4}{11^6}+4\,\frac{J_3}{11^5}\Big)\bar{z}_{11}^2-F_6+8F_4-44F_3=0+0\cdot q+0\cdot q^2+\dots,$$

which, by Lemma 3.2, proves

$$\Big(\frac{J_6}{11^6}-8\,\frac{J_4}{11^6}+4\,\frac{J_3}{11^5}\Big)\bar{z}_{11}^2=F_6-8F_4+44F_3.$$

Finally observe that F_6 is in $M_{\mathbb{Z}}^{\infty}(11)$, owing to its definition, respectively to the property of the U_2-operator. Finally, also $-8F_4+44F_3$ is in $M_{\mathbb{Z}}^{\infty}(11)$ since applying the W-operator to both sides of (104) gives

$$F_4(\tau)-\frac{11}{2}F_3(\tau)=\frac{1}{11^2}\bar{z}_{11}J_2(\tau)\in M_{\mathbb{Z}}^{\infty}(11). \tag{105}$$

Here one uses again the fact

$$\bar{z}_{11}\mid W=11^6 z_{11} \tag{106}$$

from Lemma 3.5(3). This completes the proof of (97). $\square$

11. Appendix 3: There is no prinicpal modular function on $\Gamma_0(11)$

As announced in Section 9 we give an elementary proof of the non-existence of a principal modular function ("Hauptmodul") on $\Gamma_0(11)$. By (57) and (58) it is sufficient to prove the following version.

Lemma 11.1. *There exists no modular function on $\Gamma_0(11)$ which has only one single pole at the cusp $[\infty]_{11}$ and no pole at $[0]_{11}$.*

Proof. Suppose there exists a modular function $g \in M(11)$ having a pole only at infinity with pole order 1; i.e., $\operatorname{ord}_q g = \operatorname{ord}_{[\infty]_{11}} g^* = -1$ and $\operatorname{ord}_{[0]_{11}} g^* = \operatorname{ord}_{[\infty]_{11}} (g \mid W)^* \geq 0$ for $W = \left(\begin{smallmatrix} 0 & -1 \\ 11 & 0 \end{smallmatrix}\right)$, where we recall (58). Using g we will construct a non-zero modular form h of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$ which cannot exist; see e.g. [3, Thm. 6.4]. To this end, consider the following group action of $\mathrm{GL}_n(\mathbb{Z})$ on meromorphic functions defined on $\mathbb{H}$: for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_n(\mathbb{Z})$, $k \in \mathbb{Z}$,

$$(f \mid_k \gamma)(\tau) := \det(\gamma)^{k/2} (c\tau + d)^{-k} f\Big(\frac{a\tau + b}{c\tau + d}\Big).$$

For $f(\tau) := (\eta(\tau)\eta(11\tau))^2$, using the standard η transformation formula, one can verify that $f \mid_2 W = -f$ and $f \mid_2 \gamma = f$ for all $\gamma \in \Gamma_0(11)$. The latter invariance holds also for fg. The Atkin-Lehner operator $\left(\begin{smallmatrix} 11 & \alpha \\ 11 & 11\beta \end{smallmatrix}\right)$ with $11\beta - \alpha = 1$ can be written as the product of $\left(\begin{smallmatrix} -\alpha & 1 \\ -11\beta & 1 \end{smallmatrix}\right) \in \Gamma_0(11)$ with W. Hence the trace operator (1) from [19] applied to fg gives

$$h := fg + U_{11}(fg \mid_2 W) \text{ such that } h \mid_2 \gamma = h \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Owing to $\operatorname{ord}_q(fg) = -1 + \operatorname{ord}_q f = 0$, and $\operatorname{ord}_q(fg \mid W) \geq 1$ which implies $\operatorname{ord}_q U_{11}(fg \mid W) \geq 1$, we obtain the desired contradiction; i.e., the modular form h of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$ is non-zero. $\square$

12. Appendix 4: The Fundamental Relations for Lemma 6.5

In order to compute the fundamental relations to prove Lemma 6.5, we need two q-expansions at 0 derived in Section 12.1. The usage of these expansions is described in Section 12.2.

12.1. **Expansions at zero.**

The first q-expansion at 0 is proven in [21, (8.81)]:

Lemma 12.1. *For any prime $p \in \{5, 7, 11\}$ and $f \in M(p)$:*

$$(U_p f)\Big(\frac{-1}{p\tau}\Big) = (U_p f)(p\tau) - \frac{1}{p} f(\tau) + \frac{1}{p} f\Big(\frac{-1}{p^2\tau}\Big). \tag{107}$$

Because of $U_p^{(1)}(f) := U_p(\bar{u}_p f)$ we need a second q-expansion formula at 0. For the rest of this section we use the abbreviations $u := u_p$ and $\bar{u} := \bar{u}_p$; $\left(\frac{n}{p}\right)$ denotes the Jacobi symbol.

Lemma 12.2. *For any prime $p \geq 5$ and $f \in M(p)$ with $f(\tau) = \sum_{n=m}^{\infty} b(n)q^n$:*

$$\begin{aligned} U_p(\overline{u}f)\Big(\frac{-1}{p\tau}\Big) &= \frac{1}{p^2}u(\tau)f\Big(\frac{-1}{p^2\tau}\Big) \\ &\quad + \frac{1}{p}\left(\frac{-3}{p}\right)\prod_{k=1}^{\infty}(1-q^k)\sum_{n=m}^{\infty} a(n)\Big(\frac{24n-1}{p}\Big)q^n \end{aligned} \tag{108}$$

where the $a(n)$ are defined from the q-expansion of f and the partition generating function:

$$\sum_{n=m}^{\infty} a(n)q^n := \sum_{n=m}^{\infty} b(n)q^n \times \sum_{n=0}^{\infty} p(n)q^n.$$

Proof.

$$\begin{aligned} pU_p(\overline{u}f)(\tau) &= \sum_{\lambda=0}^{p-1} \frac{\eta(p(\tau+\lambda))}{\eta\Big(\frac{\tau+\lambda}{p}\Big)} f\Big(\frac{\tau+\lambda}{p}\Big) \\ &= \sum_{\lambda=0}^{p-1} \frac{\eta(p\tau)}{\eta\Big(\frac{\tau+24\lambda}{p}\Big)} f\Big(\frac{\tau+24\lambda}{p}\Big) \end{aligned} \tag{109}$$

Next we note that for any integers x and y such that $24^2\lambda y - px = 1$,

$$\frac{24\lambda\tau - 1}{p\tau} = A_\lambda \frac{\tau - 24y}{p} \quad \text{where } A_\lambda = \begin{pmatrix} 24\lambda & x \\ p & 24y \end{pmatrix} \in \Gamma_0(p).$$

Using this together with the standard η-transformation formula we find that

$$\begin{aligned}
&pU_p(\overline{u}f)\Big(\frac{-1}{\tau}\Big)\\
=&\frac{\eta(-1/(\tau/p))}{\eta(-1/p\tau)}f\Big(\frac{-1}{p\tau}\Big)+\sum_{\lambda=1}^{p-1}\frac{\eta(-1/(\tau/p))}{\eta\Big(\frac{24\lambda\tau-1}{p\tau}\Big)}f\Big(\frac{24\lambda\tau-1}{p\tau}\Big)\\
=&\frac{1}{p}\frac{\eta(\tau/p)}{\eta(p\tau)}f\Big(\frac{-1}{p\tau}\Big)+\sum_{\lambda=1}^{p-1}\frac{(-i(\tau/p))^{1/2}\eta(\tau/p)}{\epsilon(24\lambda,x,p,24y)(-i\tau)^{1/2}\eta\Big(\frac{\tau-24y}{p}\Big)}f\Big(\frac{\tau-24y}{p}\Big) \qquad (110)\\
=&\frac{1}{p}\frac{\eta(\tau/p)}{\eta(p\tau)}f\Big(\frac{-1}{p\tau}\Big)+\frac{e^{(p-1)i\pi/4}}{\sqrt{p}}\sum_{\lambda=1}^{p-1}\frac{\eta(\tau/p)}{\eta\Big(\frac{\tau-24y}{p}\Big)}\Big(\frac{24y}{p}\Big)f\Big(\frac{\tau-24y}{p}\Big)\\
=&\frac{1}{p}\frac{\eta(\tau/p)}{\eta(p\tau)}f(\Big(\frac{-1}{p\tau}\Big)+\frac{e^{(p-1)i\pi/4}}{\sqrt{p}}\sum_{\lambda=1}^{p-1}\frac{\eta(\tau/p)}{\eta\Big(\frac{\tau-24\lambda}{p}\Big)}\Big(\frac{24\lambda}{p}\Big)f\Big(\frac{\tau-24\lambda}{p}\Big).
\end{aligned}$$

The second last equality follows from a classical formula for $\epsilon(a,b,c,d)$; e.g., [18, Ch. 4, Thm. 2]. The last equality follows from the following observation: For $1\le\lambda\le p-1$ let $(x,y)=(x(\lambda),y(\lambda))$ be such that $24^2\lambda y-px=1$, then $\{y(1),\dots,y(p-1)\}\equiv\{1,\dots,p-1\}$ (mod p). Finally, by Lemma 12.3, using that $e^{\pi i(p-1)/2}=(-1)^{(p-1)/2}=\big(\frac{-1}{p}\big)$, we obtain

$$pU_p(\overline{u}f)\Big(\frac{-1}{\tau}\Big)=\frac{1}{p}\frac{\eta(\tau/p)}{\eta(p\tau)}f\Big(\frac{-1}{p\tau}\Big)+\left(\frac{-3}{p}\right)\frac{\eta(\tau/p)}{e^{\pi i\tau/(12p)}}\sum_{n=m}^{\infty}a(n)\left(\frac{24n-1}{p}\right)e^{\frac{2\pi in\tau}{p}}.$$

Substituting τ with $p\tau$ gives the desired result. ∎

Lemma 12.3. *Let $g(\tau)=q^{-1/24}\sum_{n=m}^{\infty}a(n)q^n$ where $q=e^{2\pi i\tau}$. Then for any prime $p\ge5$,*

$$\sum_{\lambda=0}^{p-1}g\left(\frac{\tau-24\lambda}{p}\right)\left(\frac{24\lambda}{p}\right)=\sqrt{p}\left(\frac{3}{p}\right)e^{\frac{\pi i(p-1)}{4}}e^{-\frac{\pi i\tau}{12p}}\sum_{n=m}^{\infty}a(n)\left(\frac{24n-1}{p}\right)e^{\frac{2\pi in\tau}{p}}.$$

Proof.

$$\begin{aligned}
\sum_{\lambda=0}^{p-1}g\left(\frac{\tau-24\lambda}{p}\right)\left(\frac{24\lambda}{p}\right)&=\sum_{\lambda=0}^{p-1}g\left(\frac{\tau+24\lambda}{p}\right)\left(\frac{-24\lambda}{p}\right)\\
&=\sum_{\lambda=0}^{p-1}e^{-\frac{\pi i}{12}\frac{\tau+24\lambda}{p}}\sum_{n=m}^{\infty}a(n)e^{2\pi in\frac{\tau+24\lambda}{p}}\left(\frac{-24\lambda}{p}\right)\\
&=e^{-\frac{\pi i\tau}{12p}}\sum_{n=m}^{\infty}a(n)e^{\frac{2\pi in\tau}{p}}s(n,p),
\end{aligned}$$

where

$$s(n,p) := \sum_{\lambda=0}^{p-1} \left(\frac{-24\lambda}{p}\right) e^{\frac{2\pi i\lambda}{p}(24n-1)}. \tag{111}$$

For $p \nmid (24n-1)$, under transformation of equivalent residue systems, the sum $s(n,p)$ rewrites to a classical Gauss sum:

$$\left(\frac{-24}{p}\right)\left(\frac{24n-1}{p}\right) s(n,p) = \sum_{\lambda=0}^{p-1} \left(\frac{\lambda(24n-1)}{p}\right) e^{\frac{2\pi i\lambda(24n-1)}{p}} = \sum_{\lambda=0}^{p-1}\left(\frac{\lambda}{p}\right) e^{\frac{2\pi i\lambda}{p}}.$$

It is convenient to represent the evaluation of the Gauss sum in the form

$$\sum_{\lambda=0}^{p-1}\left(\frac{\lambda}{p}\right) e^{\frac{2\pi i\lambda}{p}} = \sqrt{p}\left(\frac{-2}{p}\right) e^{\frac{\pi i(p-1)}{4}},$$

which immediately implies

$$s(n,p) = e^{\frac{\pi i(p-1)}{4}}\sqrt{p}\left(\frac{3}{p}\right)\left(\frac{24n-1}{p}\right). \tag{112}$$

Because of $\sum_{\lambda=0}^{p-1}\left(\frac{\lambda}{p}\right) = 0$, the $s(n,p)$ sum as defined in (111) evaluates to 0 if $p \mid (24n-1)$. Hence (112) is valid also in this case. Substitution of this formula for $s(n,p)$ completes the proof. $\square$

12.2. **How to derive the fundamental relations.** In this section we explain how one computes the fundamental relations needed for the proof of Lemma 6.5. As pointed out in the "Sketch of Proof of Lemma 6.5", the task to prove the existence of the infinitely many relations of type (47) can be restricted to computing only finitely many of them. More precisely, if $\ell = 5$ then $n_5 = 1$ and one needs to compute two times $d_5 = 5$ relations: for each $k \in \{-4,\dots,0\}$ and with $s = 1$, and another 5 relations for the same k but with $s = 2$. If $\ell = 7$ then $n_7 = 1$ and one needs to compute two times $d_7 = 7$ relations: for each $k \in \{-6,\dots,0\}$ and with $s = 1$, and another 7 relations for the same k but with $s = 2$.

Fundamental relations, $\ell = 5$ and $\ell = 7$ in Lemma 6.5: Since $\ell = 7$ works completely analogously, we restrict to discuss $\ell = 5$. For $-4 \le k \le 0$ we have to derive relations of the form,

$$\begin{aligned} U_5^{(1)}(z_5^k) = U_5(\bar{u}_5 z_5^k) &= J_{0,5} \sum_{j \ge N_{0,k}^{(1,5)}(0)} a_{0,k}^{(1,5)}(0,j) 5^{M_{0,k}^{(1,5)}(0,j)} z_5^j \\ &= \sum_{j \ge \lceil \frac{k+1}{5} \rceil} a_{0,k}^{(1,5)}(0,j) 5^{\lceil \frac{1}{2}(5j-k-3)\rceil} z_5^j, \end{aligned}$$

and

$$U_5^{(2)}(z_5^k) = U_5(z_5^k) = J_{0,5} \sum_{j \geq N_{0,k}^{(2,5)}(0)} a_{0,k}^{(2,5)}(0,j) 5^{M_{0,k}^{(2,5)}(0,j)} z_5^j$$
$$= \sum_{j \geq \lceil \frac{k}{5} \rceil} a_{0,k}^{(2,5)}(0,j) 5^{\lceil \frac{1}{2}(5j-k-2) \rceil} z_5^j.$$

Because of $\bar{u}_5, z_5^k \in M_{\mathbb{Z}}^0(5^2)$, Lemma 4.4(3) and (18) give $U_5^{(s)}(z_5^k) \in M_{\mathbb{Z}}^0(5)$. By inspection one sees that for $-4 \leq k \leq 0$ and $j = 0, 1$:

$$\operatorname{ord}_{[\infty]_5} U_5(\bar{u}_5^j z_5^k)^* = \begin{cases} 1, & \text{if } (j,k) = (1,0), \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, since $U_5^{(s)}(z_5^k)$ is analytic on $\mathbb{H}$, the only possible pole of $U_5^{(s)}(z_5^k)^*$ must sit at 0 with some multiplicity. Consequently, to derive the desired relation we first compute the q-expansion at 0, using (108) if $s = 1$ and (107) if $s = 2$. Next, in view of $\operatorname{ord}_{[\infty]_5} \bar{z}_5^* = -1$ and of Lemma 3.2, we reduce the obtained q-expansion (i.e., the sum of sufficiently many terms) with respect to powers of $\bar{z}_5$ until the principal part is 0. Finally, using Lemma 3.5(3) we translate the computed relation which, as we note, is presented at $-\frac{1}{5\tau}$ instead at τ, into the desired relation.

Example 12.4. With (108) we compute

$$U_5^{(1)}(z_5^{-2})\Big(\frac{-1}{5\tau}\Big) = U_5(\bar{u}_5 z_5^{-2})\Big(\frac{-1}{5\tau}\Big) = -\frac{1}{5}q^{-2} + \frac{1}{5}q^{-1} - \frac{43}{5} - \frac{11}{5}q - \frac{11}{5}q^2 + \dots$$

First reduction step:

$$U_5^{(1)}(z_5^{-2})\Big(\frac{-1}{5\tau}\Big) + \frac{1}{5}\bar{z}_5(\tau)^2 = -\frac{11}{5}q^{-1} + \frac{11}{5} - \frac{99}{5}q - 22q^2 + \dots$$

Next reduction step:

$$U_5^{(1)}(z_5^{-2})\Big(\frac{-1}{5\tau}\Big) + \frac{1}{5}\bar{z}_5(\tau)^2 + \frac{11}{5}\bar{z}_5(\tau) = -11 + 0\cdot q + 0\cdot q^2 + \dots$$

This proves that

$$U_5^{(1)}(z_5^{-2})\Big(\frac{-1}{5\tau}\Big) = -\frac{1}{5}\bar{z}_5(\tau)^2 - \frac{11}{5}\bar{z}_5(\tau) - 11.$$

To obtain the desired relation one applies the W-operator to both sides which, using Lemma 3.5(3), gives

$$U_5^{(1)}(z_5^{-2})(\tau) = -11 - 11\cdot 5^2 z_5(\tau) - 5^5 z_5(\tau)^2.$$

This relation matches the pattern

$$U_5^{(1)}(z_5^k) = \sum_{j \geq \lceil \frac{k+1}{5} \rceil} a_{0,k}^{(1,5)}(0,j) 5^{\lceil \frac{1}{2}(5j-k-3) \rceil} z_5^j \tag{113}$$

predicted for $k = -2$ by Lemma 6.5.

Below the 5 times 2 relations for $\ell = 5$, and the 7 times 2 relations for $\ell = 5$ are listed explicitly. All these relations have been computed in the same manner as in Example 12.4.

Fundamental relations, $\ell = 11$ in Lemma 6.5: If $\ell = 11$ then $n_{11} = 5$ and more work has to be done. In particular, one has to be careful with the domain for k. For $\ell = 5, 7$ we could use $k \in \{-d_\ell + 1, \dots, 0\}$ owing to $\operatorname{ord}_{[0]_\ell} U_\ell(\bar{u}_\ell^j z_\ell^k)^* \geq 0$ for k from this domain and $j = 0, 1$. But, in contrast, if $j \in \{0, 1\}$:

$$\operatorname{ord}_{[0]_{11}} U_{11}(\bar{u}_{11}^j z_{11}^k)^* \geq 0 \text{ only for } k \geq -2.$$

For example, for $j = 0$, using (107) one can compute

$$\operatorname{ord}_{[0]_{11}} U_{11}(z_{11}^{-3})^* = -15.$$

In addition,

$$\operatorname{ord}_{[\infty]_{11}} U_{11}(z_{11}^{-3})^* = -1.$$

As a consequence, one runs into functions having poles both at 0 and ∞. To avoid special treatment of these cases we choose $\{0, \dots, 54\}$ as the domain for k. More precisely, to settle the $s = 1$ case of (47), we compute $d_{11} = 55$ times 5 relations: for each $k \in \{0, \dots, 54\}$ and with varying $m \in \{0, \dots, 4\}$. The same number of relations result from the $s = 2$ case of (47). Despite the larger number of fundamental relations for $\ell = 11$, the algorithmic derivation works as straightforward and along the same lines as for $\ell = 5, 7$. These 550 relations, partitioned into five groups according to $m \in \{0, \dots, 4\}$, are presented at the web page

`https://www.risc.jku.at/people/sradu/powers11`

Notation used there: $u := \bar{u}_{11}, t := z_{11} =: J[0], J[1] := g_2 = J_{2,11}, J[2] := g_3 = J_{3,11}, J[3] := g_4 = J_{4,11}$, and $J[4] := g_6 = J_{6,11}$.

To illustrate the computation, we restrict to one example. Because of the size of the relations for $k \geq 0$, we present an example with $k = -2$; in this case the function under consideration still has a pole only at 0. The computations for $k \geq 0$ work entirely the same.

Example 12.5. We derive the relation

$$U_{11}^{(2)}(J_{3.11} z_{11}^{-2}) = -9204 J_{0,11} - 11^7 z_{11} J_{1,11}. \tag{114}$$

We note that this matches the requirements of Lemma 6.5 with

$$N_{3,-2}^{(2,11)}(i) = \left\lceil \frac{-2 + \nu_{2,11}(3,i)}{11} \right\rceil = \left\lceil \frac{-3 + \nu_{1,11}(3,i)}{11} \right\rceil = 0 \text{ for } i = 0, 1, 2, 3, 4,$$

and

$$M_{3,-2}^{(2,11)}(i,j) = \left\lceil \frac{1}{2}(11j + 2 + \mu_{2,11}(3,i)) \right\rceil = \left\lceil \frac{1}{2}(11j + 3 + \mu_{1,11}(3,i)) \right\rceil,$$

which for the 11-power in question gives $M^{(2,11)}_{3,-2}(1,1) = 7$.

To derive the right side of (114) we apply (107) with $f = J_{3,11}\bar{z}_{11}^2$,

$$U^{(2)}_{11}(f)\Big(\frac{-1}{11\tau}\Big) = (U_{11}f)\Big(\frac{-1}{11\tau}\Big) = (U_{11}f)(11\tau) - \frac{1}{11}f(\tau) + \frac{1}{11}f\Big(\frac{-1}{11^2\tau}\Big)$$
$$= -\frac{1}{11}q^{-7} + \frac{10}{11}q^{-6} - \frac{18}{11}q^{-5} - \frac{169}{11}q^{-4} + O(q^{-3}).$$

To expand $f(-1/(11^2\tau))$ we use the relation

$$f \mid W = (g_4 \mid W)(\bar{z}_{11} \mid W)^2 = 11^{12}\ h_4\ z_{11}^2,$$

which is by (106) and (100). The first reduction step already gives

$$U^{(2)}_{11}(f)\Big(\frac{-1}{11\tau}\Big) + 11\ \bar{z}_{11}h_2 = -9204 + 0\cdot q + 0\cdot q^2 + \cdots = 0.$$

Applying the W-operator to both sides, and using again (106) and (100), results in

$$U^{(2)}_{11}(f)(\tau) = -9204 - 11^7 z_{11}J_{1,11}.$$

which is (114).

12.3. **The Fundamental Relations for $\ell = 5$.** $U^{(1)}_5(z^k) = U_5(\bar{u}_5 z^k)$ representations, $z := z_5$ and $k \in \{-4,\dots,0\}$:

$$\begin{aligned} U^{(1)}_5(1) &= 5z,\\ U^{(1)}_5(z^{-1}) &= 1,\\ U^{(1)}_5(z^{-2}) &= -11 - 11{\cdot}5^2 z - 5^5 z^2,\\ U^{(1)}_5(z^{-3}) &= 119 + 51{\cdot}5^3 z + 34{\cdot}5^5 z^2 + 5^8 z^3,\\ U^{(1)}_5(z^{-4}) &= -253{\cdot}5 - 759{\cdot}5^3 z - 92{\cdot}5^6 z^2 + 5^{11} z^4. \end{aligned}$$

$U^{(2)}_\ell(z^k) = U_5(z^k)$ representations, $z := z_5$ and $k \in \{-4,\dots,0\}$:

$$\begin{aligned} U^{(2)}_5(1) &= 1,\\ U^{(2)}_5(z^{-1}) &= -5^2 t - 6,\\ U^{(2)}_5(z^{-2}) &= -5^5 t^2 + 54,\\ U^{(2)}_5(z^{-3}) &= -5^8 t^3 - 102\cdot 5,\\ U^{(2)}_5(z^{-4}) &= -5^{11} t^4 + 966\cdot 5. \end{aligned}$$

12.4. **The Fundamental Relations for $\ell = 7$.** $U_7^{(1)}(z^k) = U_7(\bar{u}_7 z^k)$ representations, $z := z_7$ and $k \in \{-6, \dots, 0\}$:

$$\begin{aligned}
U_7^{(1)}(1) &= 7z + 7^2 z^2,\\
U_7^{(1)}(z^{-1}) &= -7z,\\
U_7^{(1)}(z^{-2}) &= 1 + 7^2 z,\\
U_7^{(1)}(z^{-3}) &= -11 - 7{\cdot}11z + 11{\cdot}7^3 z^2 + 7^5 z^3,\\
U_7^{(1)}(z^{-4}) &= 90 - 20{\cdot}7^2 z - 90{\cdot}7^3 z^2 + 7^7 z^4,\\
U_7^{(1)}(z^{-5}) &= -627 + 209{\cdot}7^2 z - 7^4{\cdot}19 z^2 - 38{\cdot}7^6 z^3 - 38{\cdot}7^7 z^4 - 7^9 z^5,\\
U_7^{(1)}(z^{-6}) &= 3795 - 667{\cdot}7^2 z + 1955{\cdot}7^4 z^2 + 874{\cdot}7^6 z^3 + 874{\cdot}7^7 z^4 + 46{\cdot}7^9 z^5 + 7^{11} z^6.
\end{aligned}$$

$U_7^{(2)}(z^k) = U_7(z^k)$ representations, $z := z_7$ and $k \in \{-6, \dots, 0\}$:

$$\begin{aligned}
U_7^{(2)}(1) &= 1\\
U_7^{(2)}(z^{-1}) &= -4 - 7z,\\
U_7^{(2)}(z^{-2}) &= 20 - 7^3 z^2,\\
U_7^{(2)}(z^{-3}) &= -88 - 7^5 z^3,\\
U_7^{(2)}(z^{-4}) &= 260 - 7^7 z^4,\\
U_7^{(2)}(z^{-5}) &= 68{\cdot}7 - 7^9 z^5,\\
U_7^{(2)}(z^{-6}) &= -2392{\cdot}7 - 7^{11} z^6.
\end{aligned}$$

12.5. **The Fundamental Relations for $\ell = 11$.** The representations for $U_{11}^{(1)}(J_{m,11} t^k) = U_{11}(\bar{u}_{11} J_{m,11} t^k)$ and $U_{11}^{(2)}(J_{m,11} t^k) = U_{11}(J_{m,11} t^k)$, $m \in \{0, \dots, 4\}$, and $k \in \{0, \dots, 54\}$ are displayed at

`https://www.risc.jku.at/people/sradu/powers11`

At this web page these 550 relations are partitioned into five groups according to $m \in \{0, \dots, 4\}$. Notation used there: $u := \bar{u}_{11}, t := z_{11} =: J[0], J[1] := g_2 = J_{2,11}, J[2] := g_3 = J_{3,11}, J[3] := g_4 = J_{4,11}$, and $J[4] := g_6 = J_{6,11}$.

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