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MS06: Symbolic computation and special functions

An algorithmic summation theory for indefinite nested sums and products

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A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$f(j) = g(j+1) - g(j)$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!(S_1(j) - S_1(j+k) - S_1(j+n) + S_1(j+k+n))}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0) \\ = \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a) - S_1(a+k) - S_1(a+n) + S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \\ + \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a \rightarrow \infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

$$\text{In[2]:= mySum} = \sum_{j=0}^a \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!(-S_1[j] + S_1[j+k] + S_1[j+n] - S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!} \right);$$

In[1]:= << Sigma.m

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$$\text{In[2]:= mySum} = \sum_{j=0}^a \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!(-S_1[j] + S_1[j+k] + S_1[j+n] - S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!} \right);$$

In[3]:= res = SigmaReduce[mySum]

$$\text{Out[3]=} \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1[a] - S_1[a+k] - S_1[a+n] + S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k] + S_1[n] - S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

$$\text{In[2]:= mySum} = \sum_{j=0}^a \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!(-S_1[j] + S_1[j+k] + S_1[j+n] - S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!} \right);$$

In[3]:= res = SigmaReduce[mySum]

$$\text{Out[3]=} \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1[a] - S_1[a+k] - S_1[a+n] + S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k] + S_1[n] - S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= SigmaLimit[res, {n}, a]

$$\text{Out[4]=} \frac{1}{n!} \frac{S_1[k] + S_1[n] - S_1[k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Telescoping

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$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$:

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for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution 😞

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.**no solution** 

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma computes: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a [c_0(n)f(n, k) + c_1(n)f(n+1, k)]}$$

Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a c_0(n) f(n, k) + \sum_{k=1}^a c_1(n) f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

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for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n) \sum_{k=1}^a f(n, k) + c_1(n) \sum_{k=1}^a f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n)A(n) + c_1(n)A(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\begin{aligned} \boxed{g(n, a+1) - g(n, 1)} &= \boxed{c_0(n)A(n) + c_1(n)A(n+1)} \\ \parallel & \qquad \qquad \qquad \parallel \\ \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} & \qquad \qquad \qquad - nA(n) + (2+n)A(n+1) \\ + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} & \qquad \qquad \qquad \end{aligned}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

\in

$$\left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)} \mid c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory

see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a-c/2007/2008/2010a-c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$= 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

$$\text{ln[5]:= mySum} = \sum_{k=1}^a \frac{S[1, k] + S[1, n] - S[1, k + n]}{kn(k + n + 1)};$$

$$\text{In[5]:= mySum} = \sum_{k=1}^a \frac{S[1, k] + S[1, n] - S[1, k + n]}{kn(k + n + 1)};$$

Compute a recurrence

In[6]:= rec = GenerateRecurrence[mySum, n][[1]]

$$\text{Out[6]= } n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S[1, a] + S[1, n] - S[1, a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

$$\text{In[5]:= mySum} = \sum_{k=1}^a \frac{S[1, k] + S[1, n] - S[1, k + n]}{kn(k + n + 1)};$$

Compute a recurrence

$$\text{In[6]:= rec} = \text{GenerateRecurrence}[\text{mySum}, n][[1]]$$

$$\text{Out[6]= } n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

$$\text{In[7]:= rec} = \text{LimitRec}[\text{rec}, \text{SUM}[n], \{n\}, a]$$

$$\text{Out[7]= } -n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S[1,n] + 1}{(n+1)^3}$$

$$\text{In}[5]:= \text{mySum} = \sum_{k=1}^a \frac{S[1, k] + S[1, n] - S[1, k + n]}{kn(k + n + 1)};$$

Compute a recurrence

$$\text{In}[6]:= \text{rec} = \text{GenerateRecurrence}[\text{mySum}, n][[1]]$$

$$\text{Out}[6]= n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S[1, a] + S[1, n] - S[1, a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

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Solve a recurrence

$$\text{In}[8]:= \text{recSol} = \text{SolveRecurrence}[\text{rec}, \text{SUM}[n]]$$

$$\text{Out}[8]= \left\{ \left\{ 0, \frac{1}{n(n+1)} \right\}, \left\{ 1, \frac{S[1, n]^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)} \right\} \right\}$$

$$\text{In[5]:= mySum} = \sum_{k=1}^a \frac{S[1, k] + S[1, n] - S[1, k + n]}{kn(k + n + 1)};$$

Compute a recurrence

$$\text{In[6]:= rec} = \text{GenerateRecurrence}[\text{mySum}, n][[1]]$$

$$\text{Out[6]= } n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

$$\text{In[7]:= rec} = \text{LimitRec}[\text{rec}, \text{SUM}[n], \{n\}, a]$$

$$\text{Out[7]= } -n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S[1,n] + 1}{(n+1)^3}$$

Solve a recurrence

$$\text{In[8]:= recSol} = \text{SolveRecurrence}[\text{rec}, \text{SUM}[n]]$$

$$\text{Out[8]= } \left\{ \left\{ 0, \frac{1}{n(n+1)} \right\}, \left\{ 1, \frac{S[1,n]^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)} \right\} \right\}$$

Combine the solutions

$$\text{In[9]:= FindLinearCombination}[\text{recSol}, \{1, \{1/2\}, n, 2\}$$

$$\text{Out[9]= } \frac{S[1,n]^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)} \\ = \frac{1}{n!} \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(n, k, j)}$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$A(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a **recurrence** for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
 indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND **all solutions** expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a “closed form”

$A(n)$ =combined solutions in terms of **indefinite nested** sums.

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \left[\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!} \right]$$

||

$$\left(\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\parallel$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\parallel$$

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1) (2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1) (2-n)_j} + \frac{(-1)^{j+n} (-j-2) (-j+n-2)!}{(j-n+1) (n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

||

$$\frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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$$\text{In[4]:= mySum} = \sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum**[mySum, {}, {n}, {1}]

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In[5]:= **EvaluateMultiSum**[mySum, {}, {n}, {1}]

$$\text{Out[5]=} \frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S[-2, n]}{n+1} + \frac{S[1, n]}{(n+1)^2} + \frac{S[2, n]}{-n-1}$$

Sigma.m is based on difference ring/field theory

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The underlying framework

- difference ring theory

[a gentle introduction]

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}} [s]}_{\text{polynomial ring}}$

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{aligned} \text{ev}' : \mathbb{Q}(x) \times \mathbb{N} &\rightarrow \mathbb{Q} \\ \left(\frac{p(x)}{q(x)}, k\right) &\mapsto \begin{cases} \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\begin{aligned} \text{ev} : \mathbb{A} \times \mathbb{N} &\rightarrow \mathbb{Q} \\ \left(\sum_{i=0}^d f_i s^i, k\right) &\mapsto \sum_{i=0}^d \text{ev}'(f_i, k) S_1(k)^i \quad \text{ev}(s, k) = S_1(k) \end{aligned}$$

Definition: (\mathbb{A}, ev) is called an eval-ring

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \rightarrow \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{lll} \sigma' : \mathbb{Q}(x) & \rightarrow & \mathbb{Q}(x) \\ r(x) & \mapsto & r(x+1) \end{array}$$

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \rightarrow \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{lcl} \sigma' : \mathbb{Q}(x) & \rightarrow & \mathbb{Q}(x) \\ & & r(x) \mapsto r(x+1) \end{array}$$

$$\sigma : \mathbb{Q}(x)[s] \rightarrow \mathbb{Q}(x)[s]$$

$$s \mapsto s + \frac{1}{x+1}$$

$$S_1(k+1) = S_1(k) + \frac{1}{k+1}$$

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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$$\begin{aligned} \sigma : \mathbb{Q}(x)[s] &\rightarrow \mathbb{Q}(x)[s] & s &\mapsto s + \frac{1}{x+1} \\ \sum_{i=0}^d f_i s^i &\mapsto \sum_{i=0}^d \sigma'(f_i) \left(s + \frac{1}{x+1} \right)^i & S_1(k+1) &= S_1(k) + \frac{1}{k+1} \end{aligned}$$

Definition: (\mathbb{A}, σ) is a difference ring

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

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Definition: (\mathbb{A}, σ) is a difference ring with

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$ -ring

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \rightarrow \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \rightarrow \mathbb{A}$

ev and σ interact:

$$\text{ev}(\sigma(s), k) = \text{ev}\left(s + \frac{1}{x+1}, k\right) = S_1(k) + \frac{1}{k+1} = \text{ev}(s, k+1)$$

Simplify

$$\sum_{k=0}^a S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
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$$\text{ev}(\sigma(s), k) = \text{ev}\left(s + \frac{1}{x+1}, k\right) = S_1(k) + \frac{1}{k+1} = \text{ev}(s, k+1)$$

$$\Downarrow$$

$$\boxed{\underbrace{\mathbb{K}(x)[s]}_{=\mathbb{A}} \simeq \underbrace{\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}]}_{\text{rat. seq.}}_{=\mathbb{B}}}$$

Simplify

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$$\text{ev}(\sigma(s), k) = \text{ev}\left(s + \frac{1}{x+1}, k\right) = S_1(k) + \frac{1}{k+1} = \text{ev}(s, k+1)$$

$$\Downarrow$$

$$\boxed{\underbrace{(\mathbb{K}(x)[s], \sigma)}_{=\mathbb{A}} \simeq \underbrace{(\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}], S)}_{\substack{=\mathbb{B} \\ \text{rat. seq.}}}}$$

$$\sum_{k=0}^a S_1(k) = ?$$

$$\underbrace{(\mathbb{K}(x)[s], \sigma)}_{=A} \simeq \underbrace{(\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}], S)}_{\substack{=B \\ \text{rat. seq.}}}$$

$$\sum_{k=0}^a S_1(k) = ?$$

Given: $F(k) = S_1(k)$

Find: $G = \langle G(k) \rangle_{k \geq 0} \in \mathbb{B}$ s.t.

$$G(k+1) - G(k) = S_1(k)$$

$$\underbrace{(\mathbb{K}(x)[s], \sigma)}_{=A} \simeq \underbrace{(\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}], S)}_{\substack{=B \\ \text{rat. seq.}}}$$

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Find: $g \in \mathbb{A}$:

$$\sigma(g) - g = s$$

$$\sum_{k=0}^a S_1(k) = ?$$

Given: $F(k) = S_1(k)$

Find: $G = \langle G(k) \rangle_{k \geq 0} \in \mathbb{B}$ s.t.

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$$\sigma(g) - g = s$$

Output: $g = x s - x$

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Given: $F(k) = S_1(k)$

Find: $G = \langle G(k) \rangle_{k \geq 0} \in \mathbb{B}$ s.t.

$$G(k+1) - G(k) = S_1(k)$$

Output: $G(k) = k S_1(k) - k$



Find: $g \in \mathbb{A}$:

$$\sigma(g) - g = s$$

Output: $g = x s - x$

$$\sum_{k=0}^a S_1(k) = G(a+1) - G(0)$$

Given: $F(k) = S_1(k)$

Find: $G = \langle G(k) \rangle_{k \geq 0} \in \mathbb{B}$ s.t.

$$G(k+1) - G(k) = S_1(k)$$

Output: $G(k) = k S_1(k) - k$

\Updownarrow

Find: $g \in \mathbb{A}$:

$$\sigma(g) - g = s$$

Output: $g = x s - x$

$$\sum_{k=0}^a S_1(k) = G(a+1) - G(0) = (a+1)S_1(a+1) - (a+1)$$

Given: $F(k) = S_1(k)$

Find: $G = \langle G(k) \rangle_{k \geq 0} \in \mathbb{B}$ s.t.

$$G(k+1) - G(k) = S_1(k)$$

Output: $G(k) = k S_1(k) - k$

\Updownarrow

Find: $g \in \mathbb{A}$:

$$\sigma(g) - g = s$$

Output: $g = x s - x$

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

↓ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

► such that

$$A(\lambda) = B(\lambda)$$

for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
(δ can be computed explicitly)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

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$B(k)$: nested product-sum expression (sums/products not in the denominator)

- ▶ such that

$$A(\lambda) = B(\lambda) \quad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

- ▶ and such that

the arising sums and products in $B(k)$ (except the alternating sign) are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

↓ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

↓ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Application 2: We solve the zero-recognition problem.

$A(k)$ evaluates to 0 from a certain point on $\Leftrightarrow B(k) = 0$

The underlying framework

- special function algorithms

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remiddi, Blümlein, . . .)

$$\sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remm, Blümlein, . . .)

$$\sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx,$$

$$\zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

Computer algebra and special functions:

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n) - \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta_3 + O\left(\frac{\ln(n)}{n^6}\right)$$

limit computations

numerical evaluation

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0) \\ = \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a) - S_1(a+k) - S_1(a+n) + S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \\ + \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a \rightarrow \infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

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Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0) \\ \stackrel{a \rightarrow \infty}{=} \frac{1}{n!} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

► Generalized algorithms for generalized harmonic sums

$$\begin{aligned}
 \sum_{k=1}^n \frac{2^k \sum_{i=1}^k \frac{2^{-i} \sum_{j=1}^i S_1(j)}{j}}{k} &= -\frac{21\zeta_2^2}{20} \frac{1}{n} + \frac{1}{8n^2} + \frac{295}{216n^3} - \frac{1115}{96n^4} + O(n^{-5}) \\
 &+ \left(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5}) \right) \zeta_2 \\
 &+ 2^n \left(\frac{3}{2n} + \frac{3}{2n^2} + \frac{9}{2n^3} + \frac{39}{2n^4} + O(n^{-5}) \right) \zeta_3 \\
 &+ \left(\frac{1}{n} + \frac{3}{4n^2} - \frac{157}{36n^3} + \frac{19}{n^4} + O(n^{-5}) \right) (\log(n) + \gamma) \\
 &+ \left(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5}) \right) (\log(n) + \gamma)^2
 \end{aligned}$$

► Generalized algorithms for cyclotomic harmonic sums

$$\begin{aligned}
 \sum_{k=1}^n \frac{\sum_{j=1}^k \frac{1}{1+2i}}{j^2} &= \left(-3 + \frac{35\zeta_3}{16}\right)\zeta_2 - \frac{31\zeta_5}{8} \\
 &+ \frac{1}{n} - \frac{33}{32n^2} + \frac{17}{16n^3} - \frac{4795}{4608n^4} + O(n^{-5}) \\
 &+ \log(2)\left(6\zeta_2 - \frac{1}{n} + \frac{9}{8n^2} - \frac{7}{6n^3} + \frac{209}{192n^4} + O(n^{-5})\right) \\
 &+ \left(-\frac{7}{4} - \frac{7}{16n} + \frac{7}{16n^2} - \frac{77}{192n^3} + \frac{21}{64n^4} + O(n^{-5})\right)\zeta_3 \\
 &+ \left(\frac{1}{16n^2} - \frac{1}{8n^3} + \frac{65}{384n^4} + O(n^{-5})\right)(\log(n) + \gamma)
 \end{aligned}$$

[Ablinger, Blümlin, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for nested binomial sums

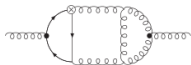
$$\sum_{j=1}^n \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{n} \left\{ \left[-\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} \right. \right. \\ \left. \left. + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} \right. \right. \\ \left. \left. + O(n^{-10}) \right] \ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} \right. \\ \left. - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} \right. \\ \left. + O(n^{-10}) \right\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

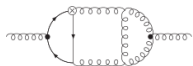
Application: The simplification of
Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles

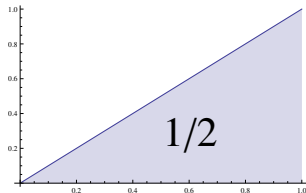


$$\int \Phi(N, \epsilon, x) dx$$

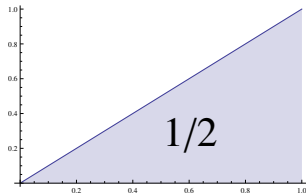
Feynman integrals

$$\int_0^1 x dx = ?$$

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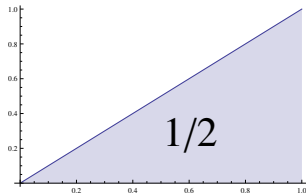


$$\int_0^1 x^1 dx =$$

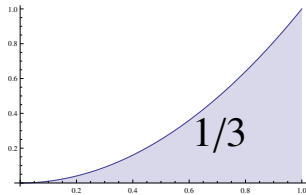


$$\int_0^1 x^2 dx = ?$$

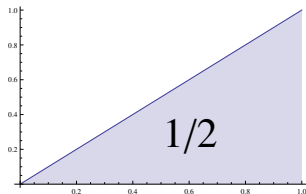
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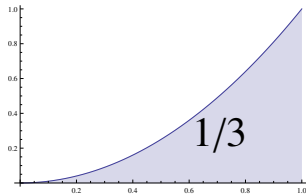
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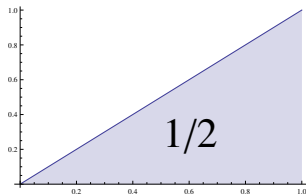


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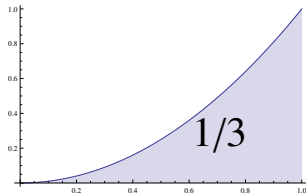


$$\int_0^1 x^3 dx = ?$$

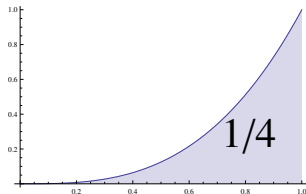
$$\int_0^1 x^1 dx =$$



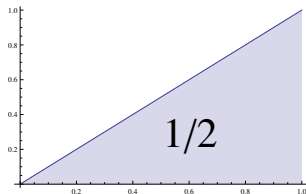
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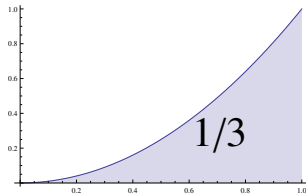
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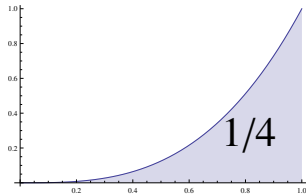
$$\int_0^1 x^1 dx =$$



$$\int_0^1 x^2 dx =$$



$$\int_0^1 x^3 dx =$$



$$\int_0^1 x^N dx = \frac{1}{N+1}$$

für $N = 1, 2, 3, 4, \dots$

Feynman integrals

$$\int_0^1 x^N dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N dx$$

Feynman integrals

$$\int_0^1 \frac{x^N (1+x)^N}{(1-x)^{1+\varepsilon}} dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1 dx_2$$

Feynman integrals

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Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1 dx_2 dx_3 dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1 dx_2 dx_3 dx_4 dx_5$$

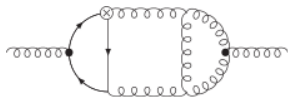
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1 dx_2 dx_3 dx_4 dx_5 dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3} \sum_{k=0}^j \binom{N-1}{j+2} \binom{j+1}{k+1} \\ \times \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}} \dots dx_1 dx_2 dx_3 dx_4 dx_5 dx_6$$

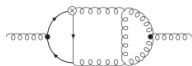
Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\begin{aligned}
 & \sum_{j=0}^{N-3} \sum_{k=0}^j \binom{N-1}{j+2} \binom{j+1}{k+1} \quad || \\
 & \times \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon} \\
 & (1-x_2)^{-\varepsilon} x_4^{\varepsilon/2-1} (1-x_4)^{\varepsilon/2-1} x_5^{\varepsilon-1} x_6^{-\varepsilon p/2} \\
 & \left[\begin{aligned}
 & [-x_3(1-x_4) - x_4(1-x_5-x_6 + x_5x_1 + x_6x_3)]^k \\
 & + [x_3(1-x_4) - (1-x_4)(1-x_5-x_6 + x_5x_1 + x_6x_3)]^k
 \end{aligned} \right] \\
 & \times (1-x_5-x_6 + x_5x_1 + x_6x_3)^{j-k} (1-x_2)^{N-3-j} \\
 & \times [x_1 - (1-x_5-x_6) - x_5x_1 - x_6x_3]^{N-3-j} dx_1 dx_2 dx_3 dx_4 dx_5 dx_6
 \end{aligned}$$

Evaluation of Feynman Integrals



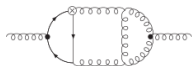
Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

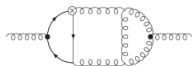
Feynman integrals

DESY
(J. Blümlein)

$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

DESY
(J. Blümlein)



$$\sum f(N, \epsilon, k)$$

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expression in
special functions

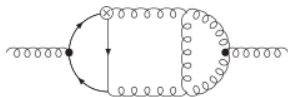


RISC
(Sigma-package)

Example 1:

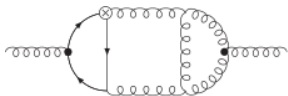
massive 3-loop ladder integrals

Feynman integrals

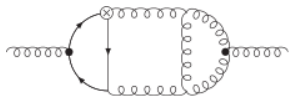


a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\begin{aligned}
 & \sum_{j=0}^{N-3} \sum_{k=0}^j \binom{N-1}{j+2} \binom{j+1}{k+1} \quad || \\
 & \times \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon} \\
 & (1-x_2)^{-\varepsilon} x_4^{\varepsilon/2-1} (1-x_4)^{\varepsilon/2-1} x_5^{\varepsilon-1} x_6^{-\varepsilon p/2} \\
 & \left[\begin{aligned}
 & [-x_3(1-x_4) - x_4(1-x_5-x_6 + x_5x_1 + x_6x_3)]^k \\
 & + [x_3(1-x_4) - (1-x_4)(1-x_5-x_6 + x_5x_1 + x_6x_3)]^k
 \end{aligned} \right] \\
 & \times (1-x_5-x_6 + x_5x_1 + x_6x_3)^{j-k} (1-x_2)^{N-3-j} \\
 & \times [x_1 - (1-x_5-x_6) - x_5x_1 - x_6x_3]^{N-3-j} dx_1 dx_2 dx_3 dx_4 dx_5 dx_6
 \end{aligned}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

||

Simplify

$$\sum_{j=0}^{N-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+N-3} \sum_{s=1}^{-l+N-q-3} \sum_{r=0}^{-l+N-q-s-3} (-1)^{-j+k-l+N-q-3} \times$$

$$\times \frac{(j+1)(k)(N-1)(-j+N-3)(-l+N-q-3)(-l+N-q-s-3)r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\left[4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k) \right.$$

$$\left. - (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)) \right.$$

$$\left. + 2S_1(s-1) - 2S_1(r+s) \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\boxed{F_0(N)} =$$

$$\begin{aligned} & \frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(-\frac{4(13N+5)}{N^2(N+1)^2} + \left(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \right. \\ & + \left(2 + 2(-1)^N \right) S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} S_1(N) + \left(\frac{3}{4} + (-1)^N \right) S_2(N)^2 \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N+1} \right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)} \right. \\ & + \left. \frac{4(3N-1)}{N(N+1)} \right) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left(\frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6 + 5(-1)^N) S_{-4}(N) \\ & + \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

$$\boxed{F_0(N)} =$$

$$\begin{aligned} & \frac{7}{12} S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(- \frac{1^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \\ & + \left(2 + \frac{28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}}{S_1(N)} + \left(\frac{3}{4} + (-1)^N \right) S_2(N)^2 \right. \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N+1} \right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)} \right. \\ & + \left. \frac{4(3N-1)}{N(N+1)} \right) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left(\frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6 + 5(-1)^N) S_{-4}(N) \\ & + \left(- \frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\ & + 32 S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

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$$S_1(N) = \sum_{i=1}^N \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\begin{aligned} & \frac{7}{12} S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(\frac{(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \\ & + \left(2 + \frac{20(-1)^N}{N^2(N+1)} \right) S_2(N)^2 \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26+4(-1)^N) S_2(N) \right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) \left(10S_1(N)^2 + \frac{8(-1)^N(2N+1)}{N(N+1)} \right) \\ & + \frac{4(3N-5)}{N(N+1)} S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N}{N(N+1)} - \frac{1}{N} \right) S_2(N) + (-6+5(-1)^N) S_{-4}(N) \\ & + \left(\frac{(-1)^N}{2(N+1)} - \frac{1}{2} \right) S_{-2,-2}(N) + (-17+13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N}{N(N+1)} S_{-2,1}(N) - (24+4(-1)^N) S_{-3,1}(N) + (3-5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

$$S_1(N) = \sum_{i=1}^N \frac{1}{i}$$

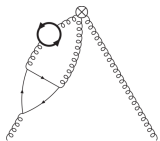
$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^N \frac{(-1)^i \sum_{k=1}^i \frac{1}{k}}{i^2}$$

Example 2:

2-mass 3-loop Feynman integrals

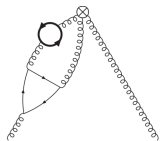
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

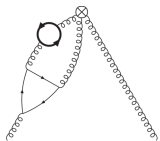


Mellin-Barnes-
and ${}_pF_q$ -technologies

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
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Mellin-Barnes-
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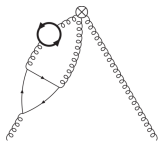
- 150 single sums
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- 12160 triple sums
- 1555 quadruple sums

Typical triple sum:

$$\sum_{j=0}^N \sum_{i=0}^j \sum_{k=0}^i \frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon} \times 2^{-2+\varepsilon} e^{-\frac{3\varepsilon\gamma}{2}} \eta^k \times$$

$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
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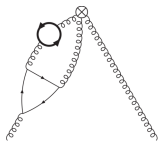
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

6 hours for this sum

\sim 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and ${}_pF_q$ -technologies \rightarrow

expression (95 MB) with

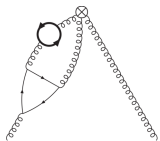
- 150 single sums
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\downarrow SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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Mellin-Barnes-
and pF_q -technologies

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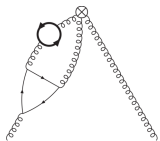
↓ EvaluateMultiSums.m

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
 (arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation	number of indef. sums
$\sum_{i_4=2}^{N-3} \sum_{i_3=0}^{i_4-2} \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s (2.1 days)	1188
$\sum_{i_3=3}^{N-4} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s (11.4 days)	747
$\sum_{i_2=3}^{N-4} \sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s (6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s (8.0 days)	44
$\sum_{i_4=2}^{N-3} \sum_{i_3=0}^{i_4-2} \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s (3.5 days)	1933
$\sum_{i_3=3}^{N-4} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s (8.2 days)	621
$\sum_{i_2=3}^{N-4} \sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s (5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s (2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and pF_q -technologies

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

↓ EvaluateMultiSums.m
(3 month)

expression (154 MB)
consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
 (arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

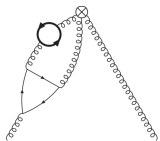
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^N 2^{-2h} (1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^h \frac{2^{2i} (1-\eta)^{-i}}{i \binom{2i}{i}} \right) \left(\sum_{i=1}^h \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^h \frac{2^{2i} (1-\eta)^{-i} \sum_{j=1}^i \frac{\sum_{k=1}^j (1-\eta)^k}{k}}{i \binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and pF_q -technologies

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

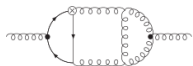
↓ EvaluateMultiSums.m
(3 month)

expression (8.3 MB)
consisting of
74 indefinite sums

← Sigma.m (32 days)

expression (154 MB)
consisting of 4110 indefinite sums

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

DESY
(J. Blümlein)



$$\sum f(N, \epsilon, k)$$

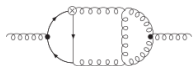
complicated
multi-sums

expression in
special functions



RISC
(Sigma-package)

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals



LHC at CERN

DESY
(J. Blümlein)



$$\sum f(N, \epsilon, k)$$

complicated multi-sums

applicable

expression in
special functions

RISC
(Sigma-package)

