OPSFA 2019, Hagenberg, Austria, July 22

MS11: Developments in q-series and the theory of partitions

Symbolic Summation, difference ring algorithms and q-applications

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Der Wissenschaftsfonds

Indefinite summation

Simplify

$$\sum_{k=0}^{a} \big(1 + (n-2k)\, S_1(k)\big) \binom{n}{k} = \text{?},$$

where $S_1(k) := \sum_{i=1}^{k} \frac{1}{i} \ (= H_k)$.

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

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FIND $g(k)$:

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My summation package Sigma computes

$$g(k) = (k\,S_1(k) - 1)\binom{n}{k}$$

GIVEN $f(k) = \big(1 + (n - 2k)\, S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Summing the telescoping equation over k from 0 to a gives

$$\boxed{\sum_{k=0}^{a} \big(1 + (n - 2k)\, S_1(k)\big)\binom{n}{k}} = \boxed{g(a+1) - g(0)}$$

$$= 1 + (n-a)S_1(a)\binom{n}{a}.$$

GIVEN $f(k)=\big(1+(n-2k)\,S_1(k)\big)\binom{n}{k}$

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$$=1+(n-a)S_1(a)\binom{n}{a}.$$

based difference ring algorithms $\longrightarrow$ see my talk on Wednesday (11:30)

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A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n} \left(1 + \alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

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$\alpha = 1$:

$$\sum_{k=0}^{a}\big(1+(n-2k)S_1(k)\big)\binom{n}{k} = 1+(n-a)S_1(a)\binom{n}{a}$$

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$$\boxed{\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?}$$

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$$\sum_{k=0}^{n}\big(1+(n-2k)S_1(k)\big)\binom{n}{k} = 1$$

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$\alpha = 1$:

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$\alpha = 2$:

$$\sum_{k=0}^{a}\big(1+2(n-2k)S_1(k)\big)\binom{n}{k}^2 = \frac{(a-n)^2(1+2nS_1(a))}{n^2}\binom{n}{a}$$

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$\alpha=3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3}=?$$

$\alpha=4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=?$$

$\alpha=5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=?$$

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Telescoping

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1 + 5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{\quad f(n,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

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FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

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$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Sigma *computes*: $c_0(n),\dots,c_4(n) \in \mathbb{Q}[n]$

$$g(n,k) := \binom{n}{k}^5 \frac{p_1(k,n,S_1(k))}{(k-n-4)^5(k-n-3)^5(k-n-2)^5(k-n-1)^5},$$

$$g(n,k+1) := \binom{n}{k}^5 \frac{p_2(k,n,S_1(k))}{(k-n-3)^5(k-n-2)^5(k-n-1)^5}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1 + 5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \cdots + c_4(n)\, f(n+4,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1) - g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n) + \\ c_1(n)\left[\mathsf{SUM}(n+1) - f(n+1,n+1)\right] \\ \vdots \\ c_4(n)\left[\mathsf{SUM}(n+4) - f(n+4,n+1) - f(n+4,n+2) - \cdots - f(n+4,n+4)\right]. \end{array}}$$

Zeilberger's creative telescoping paradigm

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

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$$\boxed{g(n,n+1)-g(n,0)} =$$

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$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

extended → Krattenthaler/Rivoal 07

$\alpha=1$:
$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k}=1$$

$\alpha=2$:
$$\sum_{k=0}^{n}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2}=0$$

$\alpha=3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3}=(-1)^n$$

$\alpha=4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=(-1)^n\binom{2n}{n}$$

$\alpha=5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=(-1)^n\sum_{j=0}^{n}\binom{n}{j}^{2}\binom{n+j}{j}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:
$$\sum_{k=0}^{a}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=\frac{(a+1)S_1(a)+1}{\binom{n}{a}}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

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$\alpha=-1$:
$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:
$$\sum_{k=0}^{a}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$
$$=\frac{(n+1)^2}{(n+2)^2}+\frac{(a+1)(-a+2n+2(a+1)(n+2)S_1(a)+3)}{(n+2)^2\binom{n}{a}^{-2}}$$

The other direction:

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$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

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$\alpha=-3$:

$$\sum_{k=0}^{n}(1-3(n-2k)S_1(k))\binom{n}{k}^{-3} = 1+6(1+n)S_1(n)$$

$$+5(1+n)^3(-1)^n\sum_{i=1}^{n}\frac{(-1)^i}{i^3}-6(1+n)^3(-1)^n\sum_{i=1}^{n}\frac{(-1)^iS_1(i)}{i^2}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha=-4$:

$$\sum_{k=0}^{n}\left(1-4(n-2k)S_1(k)\right)\binom{n}{k}^{-4} = \frac{(10(n+1)S_1(n)+3)(n+1)}{2n+3}$$
$$+\frac{(-1)^n\binom{2n}{n}^{-1}(n+1)^5}{(4n(n+2)+3)}\left(\frac{7}{2}\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}}{i^3}-5\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}S_1(i)}{i^2}\right)$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

Example 1: CA for partition theory conjectures

(joint with Ali Uncu and Jakob Ablinger)

Kanade–Russel Conjectures coming from partition theory

Conjecture (Kanade-Russel 2018) - one of 21

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

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Proof:

1. Guess

$$H_1(x) + (-1 - qx - q^2x + q^3x)H_1(q^2x)$$
$$-q^3x(1-q^2x+q^3x+q^4x)H_1(q^4x)+q^8x^2(-1+q^4x)H_1(q^6x) = 0$$

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2. Solve the recurrence and show that the triple sum equals the solution
$\to$ many steps of formal, but skillful/artistic manipulations

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2. Solve the recurrence and show that the triple sum equals the solution $\rightarrow$ many steps of formal, but skillful/artistic manipulations

3. Tracing back the manipulations with extra transformations shows (*)

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **qObjects.m**

qObjects by Ali Uncu and C. Schneider © RISC-Linz

In[3]:= << **qFunctions.m**

qObjects by Jakob Ablinger and Ali Uncu © RISC-Linz

In[4]:= $\mathbf{summand} = \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k};$

In[5]:= $\mathbf{GuessQShiftEquation[summand,}$

$$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\},H_1[x],6,\{3,12\},50]$$

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Out[5]= $H_1[x] - (1+qx+q^2x-q^3x)H_1[q^2x] - q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x] - q^8x^2(1-q^4x)H_1[q^6x]$

Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k \geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Guess ☹

$$H_1(x) + (-1 - qx - q^2x + q^3x) H_1(q^2x)$$
$$- q^3x(1 - q^2x + q^3x + q^4x) H_1(q^4x) + q^8x^2(-1 + q^4x) H_1(q^6x) = 0$$

2. Solve the recurrence and show that the triple sum equals the solution
   $\rightarrow$ many steps of formal, but skillful/artistic manipulations

3. Tracing back the manipulations with extra transformations shows (*)

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In[5]:= **GuessQShiftEquation[summand,**

$$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$$

Out[5]= $H_1[x] - (1+qx+q^2x-q^3x)H_1[q^2x] - q^3x(1-q^2x+q^3x+q^4x)H_1[q^4x] - q^8x^2(1-q^4x)H_1[q^6x]$

$$\Updownarrow \quad H_1[x] = \sum_{m=0}^{\infty} x^m h_1[m]$$

In[6]:= $\mathbf{QSEToQRE[H1qShift[[1]], H_1[x], h_1[m]]}$

Out[6]= $q^{(12+6m)}h_1[m] - q^{(}9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
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$$\Big\Updownarrow \quad H_1[x] = \sum_{m=0}^{\infty} x^m \underbrace{\sum_{k,j\geq 0} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{=h_1[m]}$$

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Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

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1. Recurrences for $F[k]$

In[7]:= $\mathbf{innerSum} = \sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k};$ $\quad (= \boldsymbol{F[k]})$

In[8]:= **recK = GenerateRecurrence[innerSum, k]/.SUM → F**

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In[9]:= **recKM = GenerateRecurrence[innerSum, k, OneShiftIn → m]/.SUM → F**

Out[9]= $F[m+1,k] == a_0[m,k]F[k] + a_1[m,k]F[k+1] + a_2[m,k]F[k+2] + a_3[m,k]F[k+3]$

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2. A recurrence for $h_1[m]$:

In[10]:= $\mathbf{doubleSum} = \sum_{k=0}^{\infty} F[k];$ $\quad (= h_1[m])$

In[11]:= **GenerateRecurrence[doubleSum, m, recK, F[k], recKM]/.SUM →** h_1

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

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In[11]:= **GenerateRecurrence[doubleSum, m, recK, F[k], recKM]/.SUM →** $\mathbf{h_1}$

Out[11]= $q^{(12+6m)}h_1[m] - q^{(}9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
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In[5]:= **GuessQShiftEquation[summand,**

$$\{\{i,0,30\},\{j,0,30\},\{k,0,30\}\}, H_1[x], 6, \{3,12\}, 50]$$

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☺

Out[6]= $q^{(12+6m)} h_1[m] - q^(9+4m)(-1+q+q^2+q^{(5+2m)})h_1[1+m]$

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Kanade–Russel Conjectures coming from partition theory

Theorem (Bringman–Jennings-Shaffer–Mahlburg 2019)

$$H_1(1) = \frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \qquad (*)$$

with

$$H_1(x) := \sum_{i,j,k \geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q;q)_k} x^{i+2j+3k}.$$

Proof:

1. Find and prove in one stroke

$$H_1(x) + (-1 - qx - q^2x + q^3x)H_1(q^2x)$$
$$- q^3x(1 - q^2x + q^3x + q^4x)H_1(q^4x) + q^8x^2(-1 + q^4x)H_1(q^6x) = 0$$

2. Solve the recurrence and show that the triple sum equals the solution
   → many steps of formal, but skillful/artistic manipulations

3. Tracing back the manipulations with extra transformations shows (*)

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In[4]:= **summand** $= \frac{(-1)^k q^{(i+2j}}{}$ … $6k$ … $i+2j+3k$;

In[5]:= **GuessQS**

Out[5]= $\mathtt{H_1[x}$ … $\mathtt{q^4x)H_1[q^6x]}$

m^2

$q^4)_j(q;q)_k$

$= h_1[m]$

In[6]:= **QSEToQRE[H1qShift[[1]],**

Out[6]= $\mathtt{q^{(12+6m)}h_1[m]-q^{(9+4m)}(-1+q+q^2+q^{(5+2m)})h_1[1+m]}$
$\mathtt{-q^{(5+2m)}(1+q-q^2+q^{(6+2m)})h_1[2+m]-(-1+q^{(3+m)})(1+q^{(3+m)})h_1[3+m]}$

Remark 1: The qFunctions package contains many other interesting features to support the discovery of identities in the context of partition theory

see Ali Uncu's talk (Tuesday, 11:30)

Finding the recurrence (together with proof certificates)

$$h_1[m] = \sum_{k=0}^{\infty} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k}}_{F[k]}$$

1. Recurrences for $F[k]$

In[7]:= $\textbf{innerSum} = \sum_{j=0}^{\infty} \frac{(-1)^k q^{4j+3k+3k^2+m^2}}{(q;q)_{m-2j-3k}(q^4;q^4)_j(q;q)_k};$ $\quad (= F[k])$

In[8]:= **recK = Gene**

Out[8]= $a_0[m,k]F[k$ … $k]F[k+4] == 0$

In[9]:= **recKM** … **SUM → F**

Out[9]= $F[m+$ … $3]$

2. A recur

In[10]:= **doubleSum =** $\sum_{k=0}$

Remark 2: These Σ-tools have been used/developed to obtain the first computer-assisted proof of Stembridge's TSPP Theorem

(joint with G.E. Andrews and P. Paule, 2005)

In[11]:= **GenerateRecurrence[doubleSum, m, recK, F[k], recKM]/.SUM → h_1**

Out[11]= $q^{(12+6m)}h_1[m] - q^{(9+4m)}(-1+q+q^2+q^{(5+2m)})h_1[1+m]$
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Example 2: Exploring the Calkin–identities

G.E. Andrews, P. Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums, From the Andrews Festschrift, Springer, Berlin (2001), pp. 189-208.
N.J. Calkin. A curious binomial identities Discrete Math., 131 (1994), pp. 335-337.
M. Hirschhorn Calkin's binomial identity Discrete Math., 159 (1996), pp. 273-278.
C. Schneider. C. Schneider Symbolic Summation Assists Combinatorics. Sem. Lothar. Combin. 56, pp. 1-36. 2007.
J. Wang, Z.Z. Zhang. On extensions of Calkin's binomial identities Discrete Math., 274 (2004), pp. 331-342.
Z.Z. Zhang. A kind of curious binomial identity Discrete Math., 306 (2006), pp. 2740-2754.

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1} = (2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

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- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2} = (2+n)2^{-1+2n} - \frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}=(2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}=(2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}=(2+n)2^{-1+3n}-3n2^{-2+n}\binom{2n}{n}$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k \left(\sum_{i=0}^{k} \binom{n}{i}\right)^1$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k \left(\sum_{i=0}^{k} \binom{2n}{i}\right)^2$$

$$\sum_{k=0}^{2n+1}(-1)^k \left(\sum_{i=0}^{k} \binom{2n+1}{i}\right)^2$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k \left(\sum_{i=0}^{k} \binom{2n+1}{i}\right)^3$$

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- Case 2:

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$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2=-2^{1+4n}+(-1)^{1+n}\binom{2n}{n}$$

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$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

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$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2=-2^{1+4n}+(-1)^{1+n}\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3=-2^{2+6n}-3(-1)^n 2^{2n}\binom{2n}{n}$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3=2^{-1+6n}+\frac{(-1)^{1+n}4^{-2+3n}\sum_{i=0}^{-1+n}64^{-i}(3+11i)\binom{2i}{i}^2\binom{3i}{i}}{n\binom{2n}{n}}$$

Case 1:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = ?$$

Case 1:

- $x \neq 1$
$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

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q-Case 1:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

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$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} (-1)^i q^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{(q;q)_{1+n}}{1-q^{1+n}} - \frac{(q;q)_{1+n}}{1-q^{1+n}} \sum_{i=1}^{n} \frac{1}{-1+q^i}$$

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$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i \binom{n}{i} \right)^2 = \text{?}$$

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- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}, a = n$:

$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} \binom{n}{i} \right)^2 = \frac{-2 + \bar{x}^{1+n} + \bar{x}^{2+n}}{-1 + \bar{x}} \bar{x}^{1-n} (1 + \bar{x})^{-2+2n} + \frac{\binom{2n}{n}}{(1 + \bar{x})^2} - \bar{x}^{-n} (1 + \bar{x})^{-2+2n} \sum_{i=1}^{n} \bar{x}^i (1 + \bar{x})^{-2i} \binom{2i}{i}$$

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- $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}, a = n$:

$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left(\sum_{i=0}^{k} \frac{(-1)^i}{\bar{x}^i} \binom{n}{i}\right)^2 = \frac{\bar{x}^2 (-1+\bar{x})^{-1+2n}}{1+\bar{x}} - \frac{2(-1)^n (-1+\bar{x})^{-2+2n} \bar{x}^{1-n}}{1+\bar{x}} + \frac{\binom{2n}{n}}{(-1+\bar{x})^2} + \frac{(-1)^{1+n} (-1+\bar{x})^{-2+2n}}{\bar{x}^n} \sum_{i=1}^{n} \frac{(-1)^i \bar{x}^i \binom{2i}{i}}{(-1+\bar{x})^{2i}}$$

- $x = 1$
  - $y \neq -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = -\frac{2(a-n)y^{1+a}\binom{n}{a}\sum_{i=0}^{a} y^i\binom{n}{i}}{1+y} - \frac{ny\sum_{i=0}^{a} y^{2i}\binom{n}{i}^2}{1+y} + \frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a} y^i\binom{n}{i}\Big)^2 + \frac{(-1+y)\sum_{i=0}^{a} i y^{2i}\binom{n}{i}^2}{1+y}$$

- $x=1$
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$$\downarrow \quad a=n,\ y=1$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2=(2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

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$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \frac{-2a^2-2a^3+4an+7a^2n-2n^2-8an^2+3n^3}{2n^2(-1+2n)}\binom{n}{a}^2 + \frac{n\sum_{i=0}^{a}\binom{n}{i}^2}{2(-1+2n)}$$

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$$+\frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a}y^i\binom{n}{i}\Big)^2+\frac{(-1+y)\sum_{i=0}^{a}iy^{2i}\binom{n}{i}^2}{1+y}$$

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$$\sum_{k=0}^{a}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2=\frac{(-a+n)^2(-1)^a\binom{n}{a}^2}{2n^2}+\frac{1}{2}\sum_{i=0}^{a}(-1)^i\binom{n}{i}^2-\frac{\sum_{i=0}^{a}(-1)^i i\binom{n}{i}^2}{n}$$

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$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

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$$\downarrow \quad a=n,\ y=1$$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2=\begin{cases}2^{-1+2n} & n \text{ even}\\ -2^{2n-1}+(-1)^{\frac{n+1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd}\end{cases}$$

- $x = -1$
  - $y = -1$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

  - $y = 1$

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \begin{cases} 2^{-1+2n} & n \text{ even} \\ -2^{2n-1} + (-1)^{\frac{n+1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 =?$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = ?$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\begin{aligned}
\sum_{k=0}^{n} (\bar{x}^2)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 &= \frac{q^2\bar{x}^3(1+q^2-2q\bar{x}^2)\left(-\frac{1}{q\bar{x}};q\right)_{1+n}\left(-\frac{\bar{x}}{q};q\right)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})\left(-q+\bar{x}^2\right)\left(-1+q\bar{x}^2\right)} \\
&+ \frac{\left(-\frac{1}{q};q\right)_{1+n}\left(\frac{1}{q};q^2\right)_{1+n}}{(-1+q)(1+q)\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)(q;q)_{1+n}}\left(-q^3-q^3\bar{x}-q^4\bar{x}-q^3\bar{x}^2\right) \\
&+ \frac{\left(-\frac{1}{q};q\right)_{1+n}\left(\frac{1}{q};q^2\right)_{1+n}}{(-1+q)(1+q)\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)(q;q)_{1+n}}\left(q^4+q^3\bar{x}+q^4\bar{x}+q^4\bar{x}^2\right)q^n \\
&+ \frac{q^2\left(\left(-\frac{1}{q\bar{x}};q\right)_{1+n}\right)^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2}\bar{x}^{4+2n} \\
&+ \frac{q^2(1+q^2)\bar{x}\left(-\frac{1}{q\bar{x}};q\right)_{1+n}\left(-\frac{\bar{x}}{q};q\right)_{1+n}}{\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)}\sum_{i=1}^{n}\frac{\left(-\frac{1}{q};q\right)_{1+i}\left(\frac{1}{q};q^2\right)_{1+i}}{(q;q)_{1+i}\left(-\frac{1}{q\bar{x}};q\right)_{1+i}\left(-\frac{\bar{x}}{q};q\right)_{1+i}} \\
&- \frac{q^3(1+q^2)\bar{x}\left(-\frac{1}{q\bar{x}};q\right)_{1+n}\left(-\frac{\bar{x}}{q};q\right)_{1+n}}{\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)}\sum_{i=1}^{n}\frac{\left(-\frac{1}{q};q\right)_{1+i}\left(\frac{1}{q};q^2\right)_{1+i}}{(q+q^i)(q;q)_{1+i}\left(-\frac{1}{q\bar{x}};q\right)_{1+i}\left(-\frac{\bar{x}}{q};q\right)_{1+i}}
\end{aligned}$$

q-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\begin{aligned}
\sum_{k=0}^{n} (\bar{x}^2)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = {} & \frac{q^2 \bar{x}^3 (1+q^2-2q\bar{x}^2)\big(-\frac{1}{q\bar{x}};q\big)_{1+n}\big(-\frac{\bar{x}}{q};q\big)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})\big(-q+\bar{x}^2\big)\big(-1+q\bar{x}^2\big)} \\
& + \frac{\big(-\frac{1}{q};q\big)_{1+n}\big(\frac{1}{q};q^2\big)_{1+n}}{(-1+q)(1+q)\big(q-\bar{x}^2\big)\big(-1+q\bar{x}^2\big)(q;q)_{1+n}}\big(-q^3-q^3\bar{x}-q^4\bar{x}-q^3\bar{x}^2\big) \\
& + \frac{\big(-\frac{1}{q};q\big)_{1+n}\big(\frac{1}{q};q^2\big)_{1+n}}{(-1+q)(1+q)\big(q-\bar{x}^2\big)\big(-1+q\bar{x}^2\big)(q;q)_{1+n}}\big(q^4+q^3\bar{x}+q^4\bar{x}+q^4\bar{x}^2\big)q^n \\
& + \frac{q^2\big(\big(-\frac{1}{q\bar{x}};q\big)_{1+n}\big)^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2}\bar{x}^{4+2n} \\
& + \frac{q^2(1+q^2)\bar{x}\big(-\frac{1}{q\bar{x}};q\big)_{1+n}\big(-\frac{\bar{x}}{q};q\big)_{1+n}}{\big(q-\bar{x}^2\big)\big(-1+q\bar{x}^2\big)} \sum_{i=1}^{n} \frac{\big(-\frac{1}{q};q\big)_{1+i}\big(\frac{1}{q};q^2\big)_{1+i}}{(q;q)_{1+i}\big(-\frac{1}{q\bar{x}};q\big)_{1+i}\big(-\frac{\bar{x}}{q};q\big)_{1+i}} \\
& - \frac{q^3(1+q^2)\bar{x}\big(-\frac{1}{q\bar{x}};q\big)_{1+n}\big(-\frac{\bar{x}}{q};q\big)_{1+n}}{\big(q-\bar{x}^2\big)\big(-1+q\bar{x}^2\big)} \sum_{i=1}^{n} \frac{\big(-\frac{1}{q};q\big)_{1+i}\big(\frac{1}{q};q^2\big)_{1+i}}{(q+q^i)(q;q)_{1+i}\big(-\frac{1}{q\bar{x}};q\big)_{1+i}\big(-\frac{\bar{x}}{q};q\big)_{1+i}}
\end{aligned}$$

- $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}$: similar

q-Case 2: $$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = ?$$

Observation: (Indefinite) summable for
($x = q^r$ [$r \neq 0$] and $y = q^s$) or ($x = q^r$ and $y = -1$)

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E.g., $x = q$, $y = 1$

$$\sum_{k=0}^{a} q^k \left(\sum_{i=0}^{k} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \frac{-1 + 2q^{1+a} - q^n}{2(-1+q)} \Big(\big(\sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \big)^2 - \big(-1 + q^n \big) \sum_{i=0}^{a} q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + 2\big(-q^a + q^n \big) q^{\frac{1}{2}(-1+a)a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix} \Big)$$

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$$- \big(-1+q^n\big) \sum_{i=0}^{a} q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$
$$+ 2\big(-q^a+q^n\big) q^{\frac{1}{2}(-1+a)a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix}\Big)$$

$$\downarrow \quad a = n$$

$$\left(q^2\big(-\frac{1}{q};q\big)_{1+n}\big((1+q)\big(-1+q^n\big)\big(-1+q^{1+n}\big)\big(\frac{1}{q};q^2\big)_{1+n}\right.$$
$$\left. +(-1+q)\big(-1+q^n(-1+2q)\big)\big(-\frac{1}{q};q\big)_{1+n}(q;q)_{1+n}\big)\right) / \big(2(-1+q)^2(1+q)^2(q;q)_{1+n}\big)$$

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$$+ \left(1 - q^{1+n}\right) \sum_{i=0}^{a} q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1+q) \sum_{i=0}^{a} q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$
$$+ 2\big(-q^a + q^n \big) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \Bigg) \Big/ (-1+q)(1+q)$$

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$$+ \left(1 - q^{1+n}\right) \sum_{i=0}^{a} q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1+q) \sum_{i=0}^{a} q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$

$$+ 2\left(-q^a + q^n\right) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \Bigg) \Big/ (-1+q)(1+q)$$

$$\downarrow \quad a = n$$

$$\Bigg((-1+q)^2 \left(-1 + q^{2+n}\right) \left((-1;q)_{1+n}\right)^2 - \frac{2(-1+q)^2 \left(1 + q^{2+2n}\right) (-1;q)_{1+n} (q;q^2)_{1+n}}{\left(-1 + q^{1+n}\right)(q;q)_{1+n}}$$

$$+ \frac{4(1+q^2) + \left((q;q)_{1+n}\right)^2}{-1 + q^{1+n}} + \frac{4(-1+q)^2 q \left((q;q)_{1+n}\right)^2}{-1 + q^{1+n}} \sum_{i=1}^{n} \frac{q^i (-1;q)_{1+i} (q;q^2)_{1+i}}{\left((q;q)_{1+i}\right)^3} \Bigg) \Big/ \left(4(-1+q)^3(1+q)\right)$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}=(2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}=(2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}=(2+n)2^{-1+3n}-3n2^{-2+n}\binom{2n}{n}$$

Sigma

Further generalization with a generic sequence X_n

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^1$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^2$$

Further generalization with a generic sequence X_n

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^1 = (1+n)\sum_{i=0}^{n}X_i - \sum_{i=0}^{n}iX_i$$

see M. Kauers/CS in Discrete Math. 306(17), 2006.

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^2$$

Further generalization with a generic sequence X_n

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^1=(1+n)\sum_{i=0}^{n}X_i-\sum_{i=0}^{n}iX_i$$

see M. Kauers/CS in Discrete Math. 306(17), 2006.

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^2=(-c+n)\Big(\sum_{i=0}^{n}X_i\Big)^2+(-1-c)\sum_{i=0}^{n}X_i^2+\sum_{i=0}^{n}iX_i^2 \\ -\sum_{i=0}^{n}X_{1+i}Z_i-X_0Z_{-1}+\Big(\sum_{i=0}^{n}X_i\Big)Z_n+X_{1+n}Z_n$$

for an arbitrary sequence Z_n satisfying

$$Z_{1+n}-Z_n=(c-1)2X_{1+n}$$

see P. Paule/CS, in Elliptic Integrals, Elliptic Functions and Modular Forms in QFT, 2019.

Specializations:

For $X_k = \binom{n}{k}$ we can compute $c = \frac{2-n}{2}$ and $Z_k = \binom{n}{k}(-k+n)$ s.t.

$$Z_{1+n} - Z_n = (c-1)2X_{1+n}$$

Specializations:

For $X_k = \binom{n}{k}$ we can compute $c = \frac{2-n}{2}$ and $Z_k = \binom{n}{k}(-k+n)$ s.t.

$$Z_{1+n} - Z_n = (c-1)2X_{1+n}$$

This gives

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \binom{n}{a}(-a+n)\sum_{i=0}^{a}\binom{n}{i} + \frac{1}{2}(2+2a-n)\Big(\sum_{i=0}^{a}\binom{n}{i}\Big)^2 - \frac{1}{2}n\sum_{i=0}^{a}\binom{n}{i}^2$$

Specializations:

Similarly one can discover, e.g.,

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\frac{x^j}{\binom{n}{j}}\Big)^2 = \frac{1+n+x}{x+1}\sum_{k=0}^{a}\frac{x^{2k}}{\binom{n}{k}^2}+\frac{x-1}{x+1}\sum_{k=0}^{a}\frac{kx^{2k}}{\binom{n}{k}^2} + \frac{a-n+2x+ax}{x+1}\Big(\sum_{k=0}^{a}\frac{x^k}{\binom{n}{k}}\Big)^2-\frac{2(a+1)x^{a+1}}{(x+1)\binom{n}{a}}\sum_{k=0}^{a}\frac{x^k}{\binom{n}{k}},$$

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\frac{(-1)^j}{\binom{n}{j}}\Big)^2 = \frac{(n+1)^2(4an^2+22an+30a+3n^2+23n+38)}{2(n+2)^2(n+3)(2n+5)}+\frac{2(-1)^a(a+1)(a+2)(n+1)}{(n+2)^2(n+3)}\frac{1}{\binom{n}{a}} + \frac{(a+1)^2(6+2a+n)}{2(n+2)^2(2n+5)}\frac{1}{\binom{n}{a}^2}+\frac{n+2}{2(2n+5)}\sum_{k=0}^{a}\frac{1}{\binom{n}{k}^2}$$

for all $x\in\mathbb{K}\setminus\{-1\}$ and $a,n\in\mathbb{N}$ with $a\leq n$.