University of Vienna, Austria, April 12

SFB Colloquium

# Symbolic Summation and (q-)Applications

Carsten Schneider

Research Institute for Symbolic Computation
Johannes Kepler University Linz

JOHANNES KEPLER UNIVERSITY LINZ

# Some of the available summation tools:


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⋮

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Koornwinder, T.H.: On Zeilberger's algorithm and its $q$-analogue. J. Comp. Appl. Math. **48**, 91–111 (1993)

Koutschan, C.: Creative telescoping for holonomic functions. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 171–194. Springer (2013). ArXiv:1307.4554 [cs.SC]

Paule, P.: Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)

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Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. **14**(2-3), 243–264 (1992)

Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$. A. K. Peters, Wellesley, MA (1996)

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Zeilberger, D., 1990. A holonomic systems approach to special functions identities. J. Comput. Appl. Math. 32, 321–368.

Zeilberger, D.: The method of creative telescoping. J. Symbolic Comput. **11**, 195–204 (1991)

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Here I will restrict to the setting of difference rings/fields.

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \quad (=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Bigg)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

↑ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1)-g(0)$$

$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$ :

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

**no solution** ☹

# Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

# Zeilberger's creative telescoping paradigm

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FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a}\Big[c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)\Big]}$$

# Zeilberger's creative telescoping paradigm

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Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\,f(n,k) + \sum_{k=1}^{a} c_1(n)\,f(n+1,k)}$$

# Zeilberger's creative telescoping paradigm

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Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

# Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad = \quad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} \in \boxed{\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

## Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0\times\frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\,\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Toolbox 1: Indefinite summation

Toolbox 2: Definite summation

Toolbox 3: Special function algorithms

# Toolbox 1: Indefinite summation

# Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

# Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Sigma compute

$$g(k) = (S_1(k) - 1)k.$$

# Telescoping

GIVEN $f(k) = S_1(k)$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over $k$ from $1$ to $n$ gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

# Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

Consider a ring

$$\mathbb{A}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

# Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

Consider a ring

$$\mathbb{A} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

# Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

# Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)[h]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

# Telescoping in the given difference ring

FIND $\gamma \in \mathbb{A}$:

$$\boxed{\sigma(\gamma) - \gamma = h.}$$

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Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

# Toolbox 1: Indefinite summation – the basic tactic

(inspired by Karr's algorithm, 1981)

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

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$$\sigma(k) = k + 1$$

**CONSTRUCT** a difference ring $(\mathbb{A},\sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(k) = k+1$$

Sk!=(k+1)k! $\quad\leftrightarrow\quad$ $\sigma(p_1) = (k+1)p_1$

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

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(q–)hyper-geometric $\quad \leftrightarrow \quad \sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(k)^*$

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(k) = k + 1$$

(q–)hyper-geometric $\leftrightarrow$

$$\sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(k)^*$$

$$\sigma(p_2) = a_2 p_2 \qquad a_2 \in \mathbb{K}(k)(p_1)^*$$

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)\dots(p_e)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(k) = k + 1$$

(q–)hyper- geometric $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(k)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(k)(p_1)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(k)(p_1, \dots, p_{e-1})^* \end{aligned}$$

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)\dots(p_e)[x]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(k) = k + 1$$

| | | | |
|---|---|---|---|
| (q–)hyper-geometric | $\leftrightarrow$ | $\sigma(p_1) = a_1\, p_1$ | $a_1 \in \mathbb{K}(k)^*$ |
| | | $\sigma(p_2) = a_2 p_2$ | $a_2 \in \mathbb{K}(k)(p_1)^*$ |
| | | $\vdots$ | |
| | | $\sigma(p_e) = a_e p_e$ | $a_e \in \mathbb{K}(k)(p_1, \dots, p_{e-1})^*$ |
| $(-1)^k$ | $\leftrightarrow$ | $\sigma(x) = -x$ | $x^2 = 1$ |

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)\dots(p_e)[x][s_1]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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| | | | |
|---|---|---|---|
| (q–)hyper-geometric | $\leftrightarrow$ | $\sigma(p_1) = a_1\, p_1$ | $a_1 \in \mathbb{K}(k)^*$ |
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| | | $\sigma(p_e) = a_e p_e$ | $a_e \in \mathbb{K}(k)(p_1, \dots, p_{e-1})^*$ |
| $(-\mathbf{1})^{\mathbf{k}}$ | $\leftrightarrow$ | $\sigma(\mathbf{x}) = -\mathbf{x}$ | $\mathbf{x}^{\mathbf{2}} = \mathbf{1}$ |
| $\mathcal{S} S_1(k) = S_1(k) + \frac{1}{k+1}$ | $\leftrightarrow$ | $\sigma(s_1) = s_1 + \frac{1}{k+1}$ | |

**CONSTRUCT** a difference ring $(\mathbb{A},\sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)\dots(p_e){\color{blue}[x]}[s_1]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\begin{array}{rcrcll}
 & & \sigma(k) & = & k+1 & \\
\text{(q–)hyper-} & \leftrightarrow & \sigma(p_1) & = & a_1\,p_1 & a_1 \in \mathbb{K}(k)^* \\
\text{geometric} & & \sigma(p_2) & = & a_2 p_2 & a_2 \in \mathbb{K}(k)(p_1)^* \\
 & & & \vdots & & \\
 & & \sigma(p_e) & = & a_e p_e & a_e \in \mathbb{K}(k)(p_1,\dots,p_{e-1})^* \\
{\color{blue}(-\mathbf{1})^{\mathbf{k}}} & \leftrightarrow & {\color{blue}\sigma(\mathbf{x})} & {\color{blue}=} & {\color{blue}-\mathbf{x}} & {\color{blue}\mathbf{x}^{\mathbf{2}} = \mathbf{1}} \\
\text{sum} & \leftrightarrow & \sigma(s_1) & = & s_1 + f_1 & f_1 \in \mathbb{K}(k)(p_1,\dots,p_e)[x]
\end{array}$$

**CONSTRUCT** a difference ring $(\mathbb{A}, \sigma)$:

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)\dots(p_e)[x][s_1][s_2]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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 & & \vdots & &
\end{array}$$

**CONSTRUCT** a $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$: (Karr81,...,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)\dots(p_e)[x][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(k) = k+1$$

(q–)hyper-geometric $\leftrightarrow$

$$\sigma(p_1) = a_1\,p_1 \qquad a_1 \in \mathbb{K}(k)^*$$
$$\sigma(p_2) = a_2 p_2 \qquad a_2 \in \mathbb{K}(k)(p_1)^*$$
$$\vdots$$
$$\sigma(p_e) = a_e p_e \qquad a_e \in \mathbb{K}(k)(p_1,\dots,p_{e-1})^*$$

$\mathbf{(-1)^k}$ $\leftrightarrow$ $\boldsymbol{\sigma(\mathbf{x}) = -\mathbf{x}} \qquad \mathbf{x^2 = 1}$

sum $\leftrightarrow$

$$\sigma(s_1) = s_1 + f_1 \qquad f_1 \in \mathbb{K}(k)(p_1,\dots,p_e)[x]$$
$$\sigma(s_2) = s_2 + f_2 \qquad f_2 \in \mathbb{K}(k)(p_1,\dots,p_e)[x][s_1]$$
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$$\vdots$$

such that $\mathrm{const}_\sigma\mathbb{A} = \{c\in\mathbb{A}\,|\,\sigma(c)=c\} = \mathbb{K}$.

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- a ring (containing $\mathbb{Q}$)

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$(-1)^{\mathbf{k}}$

**GIVEN** $f\in\mathbb{A}$;
**FIND**, in case of existence, a $g\in\mathbb{A}$ such that

$$\sigma(g)-g=f.$$

sum … $_e)[x]$

$$\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(k)(p_1,\dots,p_e)[x][s_1]$$

$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(k)(p_1,\dots,p_e)[x][s_1][s_2]$$

$$\vdots$$

such that $\mathrm{const}_\sigma\mathbb{A} = \{c\in\mathbb{A}\,|\,\sigma(c)=c\} = \mathbb{K}$.

# Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A $R\Pi\Sigma^*$-ring for the summand**

$\text{const}_\sigma \mathbb{A} = \mathbb{Q}$

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)[h]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

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FIND $\gamma \in \mathbb{A}$:

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Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

# Toolbox 1: Improved indefinite summation – symbolic simplification

## For algorithmic details see:


- CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
- CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
- CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
- CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
- S.A. Abramov, M. Petkovšek. Polynomial ring automorphisms, rational $(w, \sigma)$-canonical forms, and the assignment problem. *J. Symbolic Comput.*, 45(6): 684–708, 2010.
- CS, A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In: A. Carey, D. Ellwood, S. Paycha, S. Rosenberg (eds.) Motives, Quantum Field Theory, and Pseudodifferential Operators, Clay Mathematics Proceedings, vol. 12, pp. 285–308. Amer. Math. Soc (2010). ArXiv:0808.2543
- CS, Parameterized Telescoping Proves Algebraic Independence of Sums. Ann. Comb. **14**(4), 533–552 (2010). [arXiv:0808.2596]
- CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
- CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. To appear in *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].


## For special cases see:


- S.A. Abramov. On the summation of rational functions. Zh. vychisl. mat. Fiz., 11: 1071-1074, 1971.
- P. Paule. Greatest factorial factorization and symbolic summation, J. Symbolic Comput., 20(3): 235-268, 1995.

# The basic difference ring approach

GIVEN a $R\Pi\Sigma^*$-ring/$\Pi\Sigma^*$-field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

FIND, in case of existence, $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

# A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring/$\Pi\Sigma^*$-field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

2. FIND $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

# A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring/$\Pi\Sigma^*$-field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

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# A symbolic summation approach

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$$\sigma(g) - g = f.$$

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{a}\Big(\frac{-2+k}{10(1+k^2)} + \frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)} + \frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$=?$$

# A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring/$\Pi\Sigma^*$-field $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

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Example:

$$\sum_{k=1}^{a}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)}\Big)$$
$$=\frac{a^2+4a+5}{10(a^2+2a+2)}S_1(a)-\frac{(a-1)(a+1)}{5(a^2+2a+2)}S_3(a)-\frac{2}{5}\sum_{k=1}^{a}\frac{1}{k^2}$$

# A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring/$\Pi\Sigma^*$-field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = ?$$

# A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring/$\Pi\Sigma^*$-field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n} \frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i^3}$$

depth 3 depth 1

# Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad\qquad \text{for almost all } \lambda \in \mathbb{N}$$

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- such that

$$A(\lambda) = B(\lambda) \qquad \text{for almost all } \lambda \in \mathbb{N}$$

- such that all the sums in $B(k)$ are **simplified** as above
- and such that the arising sums in $B(k)$ are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

# Toolbox 2: Definite summation

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

# 1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
$n$: extra parameter

FIND a recurrence for $A(n)$

# Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k)+(-n-1)S_1(n)-kS_1(k+n)-2}{(k+n+1)(n+1)^2}$$

# Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

$$||\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n)+(2+n)\mathsf{A}(n+1)$$

# Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

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$$\lim_{a\to\infty} \| \qquad\qquad\qquad\qquad \|$$

$$\frac{(n+1)S_1(n)+1}{(n+1)^3} \qquad\qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
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FIND a recurrence for $A(n)$

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$a_0(n), \ldots, a_d(n), h(n)$:
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(d'Alembertian solutions)
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**Note:** the sum solutions are highly nested
(possibly with denominators of high degrees)

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3. Simplify the solutions (using difference ring/field theory) s.t.
   - the sums are algebraically independent;
   - the sums are flattened;
   - the sums can be given in terms of special functions.

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4. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums in $n$.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{A} \frac{S_1[k] + S_1[n] - S_1[k+n]}{kn(k+n+1)};$$

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## Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S_1[a]+S_1[n]-S_1[a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S_1[n]+1}{(n+1)^3}$

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In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to False]}$

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## Solve a recurrence

In[5]:= **recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → False]**

Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{\sum_{i=1}^{n} \frac{S_1[i]}{i}}{n(n+1)}\}\}$

In[1]:= $<<$ **Sigma.m**

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In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to True]}$

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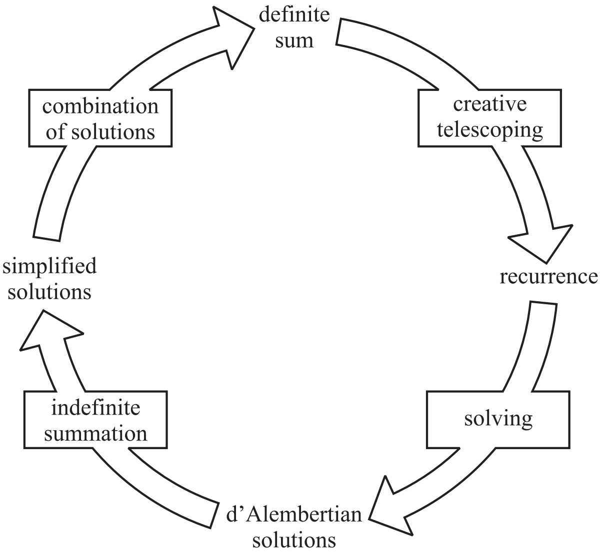
## Combine the solutions

In[6]:= **FindLinearCombination[recSol, {1, {1/2}, n, 2]**

Out[6]= $\frac{S_1[n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

# Sigma's summation spiral



# Toolbox 3: Special function algorithms

# Computer algebra and special functions:

**Harmonic sums** (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,...)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

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**Integral representation:**

$$= \int_0^1 \frac{x^n - 1}{1-x} \left( \int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

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**Asymptotic expansion:**

$$=\left(\frac{1}{30n^5}-\frac{1}{6n^3}+\frac{1}{2n^2}-\frac{1}{n}\right)\ln(n)$$
$$-\tfrac{1}{100n^5}-\tfrac{1}{6n^4}+\tfrac{13}{36n^3}-\tfrac{1}{4n^2}-\tfrac{1}{n}+2\zeta_3+O\left(\tfrac{\ln(n)}{n^6}\right).$$

**limit computations** **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$
\begin{aligned}
\sum_{k=1}^{N}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{S_1(j)}{j}}{i}}{k}
&= -\frac{21\zeta_2^2}{20}\frac{1}{N}+\frac{1}{8N^2}+\frac{295}{216N^3}-\frac{1115}{96N^4}+O(N^{-5})\\
&+\Big(\frac{1}{2N}-\frac{3}{4N^2}+\frac{19}{12N^3}-\frac{5}{N^4}+O(N^{-5})\Big)\zeta_2\\
&+2^N\Big(\frac{3}{2N}+\frac{3}{2N^2}+\frac{9}{2N^3}+\frac{39}{2N^4}+O(N^{-5})\Big)\zeta_3\\
&+\Big(\frac{1}{N}+\frac{3}{4N^2}-\frac{157}{36N^3}+\frac{19}{N^4}+O(N^{-5})\Big)(\log(N)+\gamma)\\
&+\Big(\frac{1}{2N}-\frac{3}{4N^2}+\frac{19}{12N^3}-\frac{5}{N^4}+O(N^{-5})\Big)(\log(N)+\gamma)^2) \quad \blacksquare
\end{aligned}
$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{N} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta_3}{16}\Big)\zeta_2 - \frac{31\zeta_5}{8}$$
$$+\frac{1}{N} - \frac{33}{32N^2} + \frac{17}{16N^3} - \frac{4795}{4608N^4} + O(N^{-5})$$
$$+\log(2)\Big(6\zeta_2 - \frac{1}{N} + \frac{9}{8N^2} - \frac{7}{6N^3} + \frac{209}{192N^4} + O(N^{-5})\Big)$$
$$+\Big(-\frac{7}{4} - \frac{7}{16N} + \frac{7}{16N^2} - \frac{77}{192N^3} + \frac{21}{64N^4} + O(N^{-5})\Big)\zeta_3$$
$$+\Big(\frac{1}{16N^2} - \frac{1}{8N^3} + \frac{65}{384N^4} + O(N^{-5})\Big)(\log(N)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{N}\Bigg\{\Bigg[-\frac{2}{N} + \frac{5}{12N^2} - \frac{21}{320N^3} - \frac{223}{10752N^4} + \frac{671}{49152N^5}$$
$$+ \frac{11635}{1441792N^6} - \frac{1196757}{136314880N^7} - \frac{376193}{50331648N^8} + \frac{201980317}{18253611008N^9}$$
$$+ O(N^{-10})\Bigg]\ln(\bar{N}) - \frac{4}{N} + \frac{5}{18N^2} - \frac{263}{2400N^3} + \frac{579}{12544N^4} + \frac{10123}{1105920N^5}$$
$$- \frac{1705445}{71368704N^6} - \frac{27135463}{11164188672N^7} + \frac{197432563}{7927234560N^8} + \frac{405757489}{775778467840N^9}$$
$$+ O(N^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

# The full machinery:

# Toolbox 1 + Toolbox 2 + Toolbox 3

# The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum**[

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}\Big),\{n\},\{1\}]$$

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Out[4]= $\dfrac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$

# Example 1: Super-congruences

(S. Ahlgren, E. Mortenson, R. Osburn, Sigma)

# Sigma's contribution to harmonic number congruences

- S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \mod p$$

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- E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \mod p$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \left(1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j\right) \equiv 0 \mod p$$

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- R. Osburn (2008):

$$p^2\, E_2(p) + p\, E_1(p) + p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+pE_1(p)$$

$$+p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

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$$+p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

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$$
\begin{aligned}
p^2\Big[\sum_{j=1}^{\frac{p-3}{2}}\Big(&\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}} \\
&+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big) \\
&\qquad + j^2\big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\big)\big)\Big] \\
+p\Big[\sum_{j=0}^{\frac{p-1}{2}}&(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big] \\
+p^0\sum_{j=0}^{\frac{p-1}{2}}&\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3
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$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\big)\Big)\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Big]$$

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$$\sum_{j=1}^{\frac{p-3}{2}} \Big( \frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \Big(1 + 4j\big(H_{j+\frac{p-1}{2}} - H_j\big)$$

$$+ j^2\Big(2\big(H_{j+\frac{p-1}{2}} - H_j\big)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)$$

$$\sum_{j=1}^{n-1} \Big(\frac{(-1)^j}{\binom{n}{j}\binom{j+n}{j}}$$

$$+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+4j\big(H_{j+n}-H_j\big)$$

$$+j^2\big(2\big(H_{j+n}-H_j\big)^2+H_j^{(2)}-H_{j+n}^{(2)}\big)\big)$$

$$\sum_{j=1}^{n-1} \Big(\frac{(-1)^j}{\binom{n}{j}\binom{j+n}{j}}$$

$$+\sum_{j=0}^{n}(-1)^j \binom{n}{j}\binom{j+n}{j}\big(1+4j\big(H_{j+n}-H_j\big)$$

$$+j^2\big(2\big(H_{j+n}-H_j\big)^2+H_j^{(2)}-H_{j+n}^{(2)}\big)\big)$$

$$||$$ summation spiral

$$(-1)^n((n+1)(2n+1)-\binom{2n}{n})$$

$$\sum_{j=1}^{\frac{p-3}{2}} \Big( \frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \Big(1 + 4j\big(H_{j+\frac{p-1}{2}} - H_j\big)$$

$$+ j^2\Big(2\big(H_{j+\frac{p-1}{2}} - H_j\big)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)$$

$$||$$

$$(-1)^{\frac{p-1}{2}} \Big( \big(\frac{p-1}{2} + 1\big)p - \binom{p-1}{\frac{p-1}{2}} \Big)$$

For a prime $p > 2$,

$$p^2\Big[\sum_{j=1}^{\frac{p-3}{2}}\Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\Big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[ \qquad (-1)^{\frac{p-1}{2}}\Big((\tfrac{p-1}{2}+1)p - \binom{p-1}{\frac{p-1}{2}}\Big) \qquad \Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[ - (-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}} \Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$-(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}$$

$$\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

$$\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(-2Hj+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}\big)\big)$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+j\big(-2Hj+H_{j+n}+H_{-j+n}\big)\big)$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+j\big(-2Hj+H_{j+n}+H_{-j+n}\big)\big)$$

$$||$$ summation spiral

$$-\frac{3}{2}(-1)^n n(n+1)\sum_{j=1}^{n}\frac{\binom{2j}{j}}{j}+(-1)^n(2n+1)\binom{2n}{n}$$

$$\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(-2Hj+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}\big)\big)$$

$$||$$

$$-\frac{3}{2}(-1)^{\frac{p-1}{2}}\Big(\frac{p^2}{4}-\frac{1}{4}\Big)\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}+(-1)^{\frac{p-1}{2}}p\binom{p-1}{\frac{p-1}{2}}$$

For a prime $p > 2$,

$$p^2\Big[\quad - (-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}} \quad\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$-(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}$$

$$\Big]$$

$$+p\Big[-\frac{3}{2}(-1)^{\frac{p-1}{2}}\Big(\frac{p^2}{4}-\frac{1}{4}\Big)\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}+(-1)^{\frac{p-1}{2}}p\binom{p-1}{\frac{p-1}{2}}\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[ \qquad 0 \qquad \Big]$$

$$+p\Big[-\frac{3}{2}(-1)^{\frac{p-1}{2}}\Big(\frac{p^2}{4}-\frac{1}{4}\Big)\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$0$$

$$\Big]$$

$$+p\Big[\frac{3}{8}(-1)^{\frac{p-1}{2}}\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[\sum_{j=1}^{\frac{p-3}{2}} \Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\Big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

# Sigma's contribution to harmonic number congruences

- S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} \left(H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}\right) \equiv 0 \mod p$$

- E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} \left(1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j\right) \equiv 0 \mod p$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \left(1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j\right) \equiv 0 \mod p$$

- R. Osburn/CS (2008):

$$p\,\frac{3}{8}(-1)^{\frac{p-1}{2}} \sum_{j=1}^{\frac{p-1}{2}} \frac{\binom{2j}{j}}{j} + \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

# Example 2: Exploring the Calkin identities

G.E. Andrews, P. Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums, From the Andrews Festschrift, Springer, Berlin (2001), pp. 189-208.

N.J. Calkin. A curious binomial identities Discrete Math., 131 (1994), pp. 335-337.

M. Hirschhorn Calkin's binomial identity Discrete Math., 159 (1996), pp. 273-278.

C. Schneider. C. Schneider Symbolic Summation Assists Combinatorics. Sem. Lothar. Combin. 56, pp. 1-36. 2007.

J. Wang, Z.Z. Zhang. On extensions of Calkin's binomial identities Discrete Math., 274 (2004), pp. 331-342.

Z.Z. Zhang. A kind of curious binomial identity Discrete Math., 306 (2006), pp. 2740-2754.

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1} = (2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}=(2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}=(2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1}=(2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2}=(2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3}=(2+n)2^{-1+3n}-3n2^{-2+n}\binom{2n}{n}$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1=(-1)^n2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1 = (-1)^n 2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2 = 2^{-1+4n}$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2 = -2^{1+4n}+(-1)^{1+n}\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3$$

- Case 1:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}\binom{n}{i}\right)^1 = (-1)^n 2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^2 = 2^{-1+4n}$$

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^2 = -2^{1+4n}+(-1)^{1+n}\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\binom{2n+1}{i}\right)^3 = -2^{2+6n}-3(-1)^n 2^{2n}\binom{2n}{n}$$

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\binom{2n}{i}\right)^3 = 2^{-1+6n} + \frac{(-1)^{1+n}4^{-2+3n}\displaystyle\sum_{i=0}^{-1+n}64^{-i}(3+11i)\binom{2i}{i}^2\binom{3i}{i}}{n\binom{2n}{n}}$$

# Case 1:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \text{?}$$

# Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

# Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

# Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$

$$\sum_{k=0}^{a} \sum_{i=0}^{k} y^i \binom{n}{i} = -\frac{(a-n)y^{1+a}\binom{n}{a}}{1+y} + \frac{1+a+y+ay-ny}{1+y} \sum_{i=0}^{a} y^i \binom{n}{i}$$

# Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$

$$\sum_{k=0}^{a} \sum_{i=0}^{k} y^i \binom{n}{i} = -\frac{(a-n)y^{1+a}\binom{n}{a}}{1+y} + \frac{1+a+y+ay-ny}{1+y} \sum_{i=0}^{a} y^i \binom{n}{i}$$

  - $y \neq -1$

$$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = (1+n+y)(1+y)^{-1+n}$$

# Case 1:

- $x \neq 1$

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^{a} y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^{a} x^i y^i \binom{n}{i}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$

$$\sum_{k=0}^{a} \sum_{i=0}^{k} y^i \binom{n}{i} = -\frac{(a-n)y^{1+a}\binom{n}{a}}{1+y} + \frac{1+a+y+ay-ny}{1+y} \sum_{i=0}^{a} y^i \binom{n}{i}$$

  - $y \neq -1$

$$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = (1+n+y)(1+y)^{-1+n}$$

  - $y = -1$

$$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = 0$$

# Case 1:

- $x \neq 1$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} y^i \binom{n}{i} = \frac{x^{1+n}(1+y)^n}{-1+x} - \frac{(1+xy)^n}{-1+x}$$

- $x = 1$
  - $y \neq -1$
    $$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = (1+n+y)(1+y)^{-1+n}$$
  - $y = -1$
    $$\sum_{k=0}^{n} \sum_{i=0}^{k} y^i \binom{n}{i} = 0$$

# $q$-Case 1:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

## $q$-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

# $q$-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

# $q$-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

- $x = 1$:

$$\sum_{k=0}^{a} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = (1+a) \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} i q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

# $q$-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

- $x = 1$:

$$\sum_{k=0}^{a} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = (1+a) \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} i q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

  - $y \neq -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{q+y} + \frac{q^2\left(-\frac{y}{q};q\right)_{1+n}}{q+y} \sum_{i=1}^{n} \frac{1}{q+q^i y}$$

# $q$-Case 1:

- $x \neq 1$:

$$\sum_{k=0}^{a} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

$$\downarrow \quad a = n$$

$$\sum_{k=0}^{n} x^k \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = -\frac{q\left(-\frac{xy}{q};q\right)_{1+n}}{(-1+x)(q+xy)} + \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{(-1+x)(q+y)} x^{1+n}$$

- $x = 1$:

$$\sum_{k=0}^{a} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = (1+a) \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^{a} i q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

  - $y \neq -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{q\left(-\frac{y}{q};q\right)_{1+n}}{q+y} + \frac{q^2\left(-\frac{y}{q};q\right)_{1+n}}{q+y} \sum_{i=1}^{n} \frac{1}{q+q^i y}$$

  - $y = -q$:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} q^{\frac{1}{2}(-1+i)i} (-1)^i q^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{(q;q)_{1+n}}{1-q^{1+n}} - \frac{(q;q)_{1+n}}{1-q^{1+n}} \sum_{i=1}^{n} \frac{1}{-1+q^i}$$

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- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}, a = n$:

$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left( \sum_{i=0}^{k} \frac{1}{\bar{x}^i} \binom{n}{i} \right)^2 = \frac{-2 + \bar{x}^{1+n} + \bar{x}^{2+n}}{-1 + \bar{x}} \bar{x}^{1-n} (1 + \bar{x})^{-2+2n} + \frac{\binom{2n}{n}}{(1+\bar{x})^2} - \bar{x}^{-n}(1+\bar{x})^{-2+2n} \sum_{i=1}^{n} \bar{x}^i (1+\bar{x})^{-2i} \binom{2i}{i}$$

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$$\sum_{k=0}^{n} \left(\bar{x}^2\right)^k \left(\sum_{i=0}^{k} \frac{(-1)^i}{\bar{x}^i} \binom{n}{i}\right)^2 = \frac{\bar{x}^2 (-1+\bar{x})^{-1+2n}}{1+\bar{x}} - \frac{2(-1)^n (-1+\bar{x})^{-2+2n} \bar{x}^{1-n}}{1+\bar{x}} + \frac{\binom{2n}{n}}{(-1 + \bar{x})^2} + \frac{(-1)^{1+n} (-1+\bar{x})^{-2+2n}}{\bar{x}^n} \sum_{i=1}^{n} \frac{(-1)^i \bar{x}^i \binom{2i}{i}}{(-1 + \bar{x})^{2i}}$$

- $x = 1$
  - $y \neq -1$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k} y^i \binom{n}{i}\right)^2 = -\frac{2(a-n)y^{1+a}\binom{n}{a}\sum_{i=0}^{a} y^i \binom{n}{i}}{1+y} - \frac{ny\sum_{i=0}^{a} y^{2i}\binom{n}{i}^2}{1+y} + \frac{1+a+y+ay-ny}{1+y}\Big(\sum_{i=0}^{a} y^i \binom{n}{i}\Big)^2 + \frac{(-1+y)\sum_{i=0}^{a} i y^{2i}\binom{n}{i}^2}{1+y}$$

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$$\sum_{k=0}^{n}(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{n}{i}\right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

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## $q$-Case 2:

$$\sum_{k=0}^{a} x^k \left( \sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

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- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\sum_{k=0}^{n} (\bar{x}^2)^k \left( \sum_{i=0}^{k} \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \frac{q^2 \bar{x}^3 (1 + q^2 - 2q\bar{x}^2) \left( -\frac{1}{q\bar{x}}; q \right)_{1+n} \left( -\frac{\bar{x}}{q}; q \right)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})\left(-q+\bar{x}^2\right)\left(-1+q\bar{x}^2\right)}$$
$$+ \frac{\left(-\frac{1}{q}; q\right)_{1+n} \left(\frac{1}{q}; q^2\right)_{1+n}}{(-1+q)(1+q)\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)(q;q)_{1+n}} \left(-q^3 - q^3\bar{x} - q^4\bar{x} - q^3\bar{x}^2\right)$$
$$+ \frac{\left(-\frac{1}{q}; q\right)_{1+n} \left(\frac{1}{q}; q^2\right)_{1+n}}{(-1+q)(1+q)\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)(q;q)_{1+n}} \left(q^4 + q^3\bar{x} + q^4\bar{x} + q^4\bar{x}^2\right) q^n$$
$$+ \frac{q^2 \left(\left(-\frac{1}{q\bar{x}}; q\right)_{1+n}\right)^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2} \bar{x}^{4+2n}$$
$$+ \frac{q^2(1+q^2)\bar{x}\left(-\frac{1}{q\bar{x}}; q\right)_{1+n}\left(-\frac{\bar{x}}{q}; q\right)_{1+n}}{\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)} \sum_{i=1}^{n} \frac{\left(-\frac{1}{q}; q\right)_{1+i}\left(\frac{1}{q}; q^2\right)_{1+i}}{(q;q)_{1+i}\left(-\frac{1}{q\bar{x}}; q\right)_{1+i}\left(-\frac{\bar{x}}{q}; q\right)_{1+i}}$$
$$- \frac{q^3(1+q^2)\bar{x}\left(-\frac{1}{q\bar{x}}; q\right)_{1+n}\left(-\frac{\bar{x}}{q}; q\right)_{1+n}}{\left(q-\bar{x}^2\right)\left(-1+q\bar{x}^2\right)} \sum_{i=1}^{n} \frac{\left(-\frac{1}{q}; q\right)_{1+i}\left(\frac{1}{q}; q^2\right)_{1+i}}{(q+q^i)(q;q)_{1+i}\left(-\frac{1}{q\bar{x}}; q\right)_{1+i}\left(-\frac{\bar{x}}{q}; q\right)_{1+i}}$$

# $q$-Case 2:

$$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = ?$$

- $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\sum_{k=0}^{n} (\bar{x}^2)^k \left(\sum_{i=0}^{k} \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = \frac{q^2\bar{x}^3(1+q^2-2q\bar{x}^2)(-\frac{1}{q\bar{x}};q)_{1+n}(-\frac{\bar{x}}{q};q)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})(-q+\bar{x}^2)(-1+q\bar{x}^2)}$$
$$+ \frac{(-\frac{1}{q};q)_{1+n}(\frac{1}{q};q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q;q)_{1+n}}(-q^3-q^3\bar{x}-q^4\bar{x}-q^3\bar{x}^2)$$
$$+ \frac{(-\frac{1}{q};q)_{1+n}(\frac{1}{q};q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q;q)_{1+n}}(q^4+q^3\bar{x}+q^4\bar{x}+q^4\bar{x}^2)q^n$$
$$+ \frac{q^2((-\frac{1}{q\bar{x}};q)_{1+n})^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2}\bar{x}^{4+2n}$$
$$+ \frac{q^2(1+q^2)\bar{x}(-\frac{1}{q\bar{x}};q)_{1+n}(-\frac{\bar{x}}{q};q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^{n} \frac{(-\frac{1}{q};q)_{1+i}(\frac{1}{q};q^2)_{1+i}}{(q;q)_{1+i}(-\frac{1}{q\bar{x}};q)_{1+i}(-\frac{\bar{x}}{q};q)_{1+i}}$$
$$- \frac{q^3(1+q^2)\bar{x}(-\frac{1}{q\bar{x}};q)_{1+n}(-\frac{\bar{x}}{q};q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^{n} \frac{(-\frac{1}{q};q)_{1+i}(\frac{1}{q};q^2)_{1+i}}{(q+q^i)(q;q)_{1+i}(-\frac{1}{q\bar{x}};q)_{1+i}(-\frac{\bar{x}}{q};q)_{1+i}}$$

- $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}$: similar

$q$-Case 2: $$\sum_{k=0}^{a} x^k \left(\sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r\ [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

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E.g., $x = q$, $y = 1$
$$\sum_{k=0}^{a} q^k \left(\sum_{i=0}^{k} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix}\right)^2 = \frac{-1+2q^{1+a}-q^n}{2(-1+q)} \Big(\Big(\sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}\Big)^2$$
$$-\big(-1+q^n\big)\sum_{i=0}^{a} q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$
$$+2\big(-q^a+q^n\big)q^{\frac{1}{2}(-1+a)a}\sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix}\Big)$$

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$$\sum_{k=0}^{a} q^k \left( \sum_{i=0}^{k} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \frac{-1 + 2q^{1+a} - q^n}{2(-1+q)} \Big( \big( \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \big)^2$$
$$- \big( -1 + q^n \big) \sum_{i=0}^{a} q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$
$$+ 2\big( -q^a + q^n \big) q^{\frac{1}{2}(-1+a)a} \sum_{i=0}^{a} q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix} \Big)$$

$$\downarrow \quad a = n$$

$$\Bigg( q^2 \big( -\frac{1}{q}; q \big)_{1+n} \big( (1+q) \big( -1 + q^n \big) \big( -1 + q^{1+n} \big) \big( \frac{1}{q}; q^2 \big)_{1+n}$$
$$+ (-1+q) \big( -1 + q^n(-1+2q) \big) \big( -\frac{1}{q}; q \big)_{1+n} (q;q)_{1+n} \big) \Bigg) / \big( 2(-1+q)^2 (1+q)^2 (q;q)_{1+n} \big)$$

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$(x = q^r \ [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

E.g., $x = q$, $y = q$

$$\sum_{k=0}^{a} q^k \left( \sum_{i=0}^{k} q^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \Bigg( \big( -1 + (1+q)q^{1+a} - q^{1+n} \big) \left( \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2$$

$$+ \left(1 - q^{1+n}\right) \sum_{i=0}^{a} q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1+q) \sum_{i=0}^{a} q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$

$$+ 2\big(-q^a + q^n\big) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \Bigg) \Big/ (-1+q)(1+q)$$

$q$-Case 2: $$\sum_{k=0}^{a} x^k \left( \sum_{i=0}^{k} y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

Observation: (Indefinite) summable for
$(x = q^r \; [r \neq 0]$ and $y = q^s)$ or $(x = q^r$ and $y = -1)$

E.g., $x = q$, $y = q$

$$\sum_{k=0}^{a} q^k \left( \sum_{i=0}^{k} q^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = \Bigg( \big( -1 + (1+q)q^{1+a} - q^{1+n} \big) \left( \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2$$
$$+ \left(1 - q^{1+n}\right) \sum_{i=0}^{a} q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1+q) \sum_{i=0}^{a} q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2$$
$$+ 2\big( -q^a + q^n \big) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^{a} q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \Bigg) \Big/ (-1+q)(1+q)$$

$$\downarrow \quad a = n$$

$$\Bigg( (-1+q)^2 \big( -1 + q^{2+n} \big) \big( (-1;q)_{1+n} \big)^2 - \frac{2(-1+q)^2 \big(1+q^{2+2n}\big) (-1;q)_{1+n} \big(q;q^2\big)_{1+n}}{\big( -1 + q^{1+n} \big) (q;q)_{1+n}}$$
$$+ \frac{4\big(1+q^2\big) + \big((q;q)_{1+n}\big)^2}{-1+q^{1+n}} + \frac{4(-1+q)^2 q \big((q;q)_{1+n}\big)^2}{-1+q^{1+n}} \sum_{i=1}^{n} \frac{q^i (-1;q)_{1+i} \big(q;q^2\big)_{1+i}}{\big((q;q)_{1+i}\big)^3} \Bigg) \Big/ \big(4(-1+q)^3(1+q)\big)$$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{1} = (2+n)2^{-1+n}$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{2} = (2+n)2^{-1+2n}-\frac{1}{2}n\binom{2n}{n}$$

- Case 3:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^{3} = (2+n)2^{-1+3n}-3n2^{-2+n}\binom{2n}{n}$$

Sigma

# Further generalization with a generic sequence $X_n$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^1$$

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^2$$

# Further generalization with a generic sequence $X_n$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^1 = (1+n)\sum_{i=0}^{n}X_i - \sum_{i=0}^{n}iX_i$$

see M. Kauers/CS in Discrete Math. 306(17), 2006.

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_i\right)^2$$

# Further generalization with a generic sequence $X_n$

- Case 1:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^1 = (1+n)\sum_{i=0}^{n} X_i - \sum_{i=0}^{n} iX_i$$

see M. Kauers/CS in Discrete Math. 306(17), 2006.

- Case 2:

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k} X_i\right)^2 = (-c+n)\Big(\sum_{i=0}^{n} X_i\Big)^2 + (-1-c)\sum_{i=0}^{n} X_i^2 + \sum_{i=0}^{n} iX_i^2 - \sum_{i=0}^{n} X_{1+i}Z_i - X_0 Z_{-1} + \Big(\sum_{i=0}^{n} X_i\Big)Z_n + X_{1+n}Z_n$$

for an arbitrary sequence $Z_n$ satisfying

$$Z_{1+n} - Z_n = (c-1)2X_{1+n}$$

see P. Paule/CS, in Elliptic Integrals, Elliptic Functions and Modular Forms in QFT, 2019.

# Specializations:

For $X_k = \binom{n}{k}$ we can compute $c = \frac{2-n}{2}$ and $Z_k = \binom{n}{k}(-k+n)$ s.t.

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# Specializations:

For $X_k = \binom{n}{k}$ we can compute $c = \frac{2-n}{2}$ and $Z_k = \binom{n}{k}(-k+n)$ s.t.

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This gives

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = \binom{n}{a}(-a+n)\sum_{i=0}^{a}\binom{n}{i} + \frac{1}{2}(2+2a-n)\Big(\sum_{i=0}^{a}\binom{n}{i}\Big)^2 - \frac{1}{2}n\sum_{i=0}^{a}\binom{n}{i}^2$$

# Specializations:

Similarly one can discover, e.g.,

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\frac{x^j}{\binom{n}{j}}\Big)^2 = \frac{1+n+x}{x+1}\sum_{k=0}^{a}\frac{x^{2k}}{\binom{n}{k}^2}+\frac{x-1}{x+1}\sum_{k=0}^{a}\frac{kx^{2k}}{\binom{n}{k}^2} + \frac{a-n+2x+ax}{x+1}\Big(\sum_{k=0}^{a}\frac{x^k}{\binom{n}{k}}\Big)^2-\frac{2(a+1)x^{a+1}}{(x+1)\binom{n}{a}}\sum_{k=0}^{a}\frac{x^k}{\binom{n}{k}},$$

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\frac{(-1)^j}{\binom{n}{j}}\Big)^2 = \frac{(n+1)^2(4an^2+22an+30a+3n^2+23n+38)}{2(n+2)^2(n+3)(2n+5)}+\frac{2(-1)^a(a+1)(a+2)(n+1)}{(n+2)^2(n+3)}\frac{1}{\binom{n}{a}} + \frac{(a+1)^2(6+2a+n)}{2(n+2)^2(2n+5)}\frac{1}{\binom{n}{a}^2}+\frac{n+2}{2(2n+5)}\sum_{k=0}^{a}\frac{1}{\binom{n}{k}^2}$$

for all $x \in \mathbb{K}\setminus\{-1\}$ and $a, n \in \mathbb{N}$ with $a \leq n$.