The Castle Game

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We focus on the Bernays-Schönfinkel-Ramsey fragment of first order logic, initially, the two quantifier prefix fragment of it, .i.e. $\exists\forall$. This game can be developed for a larger quantifier prefix, i.e. an $\exists$ block followed by a $\forall$ block, but this adds complexity which is not necessary for understanding the basic concepts of the game. One can of course introduce a similar game for other quantifier prefixes, i.e. $\forall\exists$, but then the interesting dependency is lost unless one introduces function symbol representatives.

What is special about BSR is that it captures the necessity of contraction within FOL, a rule which propositionally does not provide much logical meaning and is mostly structural. Another point to be made here is that BSR is a quite special fragment being that its satisfiability problem is decidable. One can even see satisfiability as a weird decoration of the box game. However, validity is more complex and has essential first order components not seen in propositional logic.

For example, one may have instances of clauses which are not logically equivalent, a concept which is completely missing in propositional logic. Of course this leads to an interesting aspect of a box game for first order validity, that is that one needs to find the right instances of the clauses comprising the given formula which for any assignment of **TRUE** and **FALSE** each clause evaluates to **TRUE**. While this seems to be equivalent to the concept of validity one captures with the box game, we need to add an additional level of abstraction to capture clausal instances. This is done by introducing the concept of "keys" locking the predicates. Because we are working in such a weak fragment of first order logic one can even imagine that we do not have free and bound variables nor quantifiers, but rather constants which must be interpreted with respect to a given list of constraints. Each of these concepts are captured by the game illustrated below.

1 The Castle Game

Imagine a castle with infinitely many floors, numbered $0, 1, 2, \cdots$ and each floor has the same number of rooms numbered $0, 1, 2, \cdots n$, each of which has the same number of chest labeled

$$C_0^0, \cdots, C_{m_0}^0, \cdots, C_0^i, \cdots, C_{m_i}^i, \cdots, C_0^n, \cdots, C_{m_n}^n.$$

labels need not be unique, i.e. $C_k^i = C_l^j$. Furthermore, each chest is made of metal or wood. Note that room i on floor j will always have the same number of chest each made of the same material and given the same labels as room i on floor k.

Each chest has a number of locks sealing it numbered $L_0, \cdots, L_r$. Some locks require two keys, i.e. double locks, to open and other locks require only one key, i.e. single locks. Luckily, you have an infinite bag of keys which has keys to all locks in the castle. Note that all chest with the same label are sealed by the same number of locks. Every wooden (metal) chest with a particular label has a duel chest made of metal (wood) with the same label. Note that the locks may differ but the number of locks must be the same. The keys open locks in the following way:

- All single locks on floor i are opened by a single unique key k_i not used on lower floors.
- The second key way of double locks on floor i is opened by k_i, the key which opened the single locks on floor i.
- The first key way of double locks on floor i cannot be opened by k_i nor any key which opens the single locks on a floor j such that $i < j$.

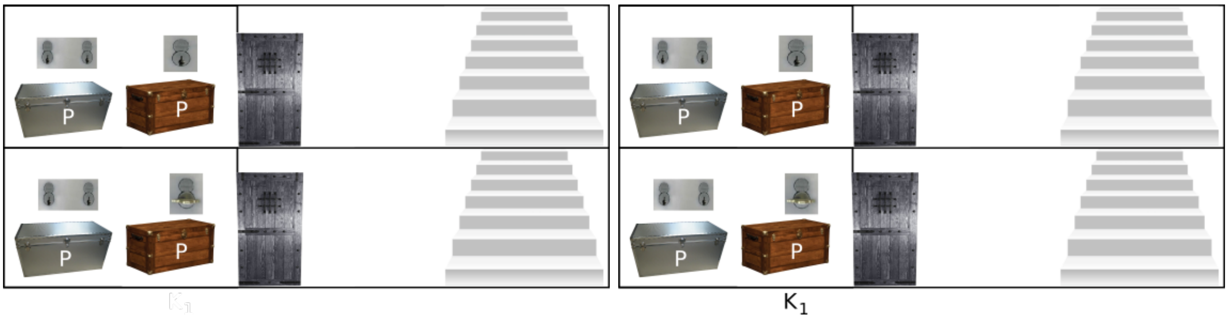
After opening the chest you find that they either contain Gold or Sliver. However, to obtain the gold and the silver the following condition must hold:

(a) If all wooden (metal) chests labeled L whose double locks are opened by keys $k_1, \cdots k_w$ and k and whose single locks are opened by k then all its duel chests labeled L whose double locks are opened by keys $k_1, \cdots k_w$ and k' and whose single locks are opened by k must contain silver.

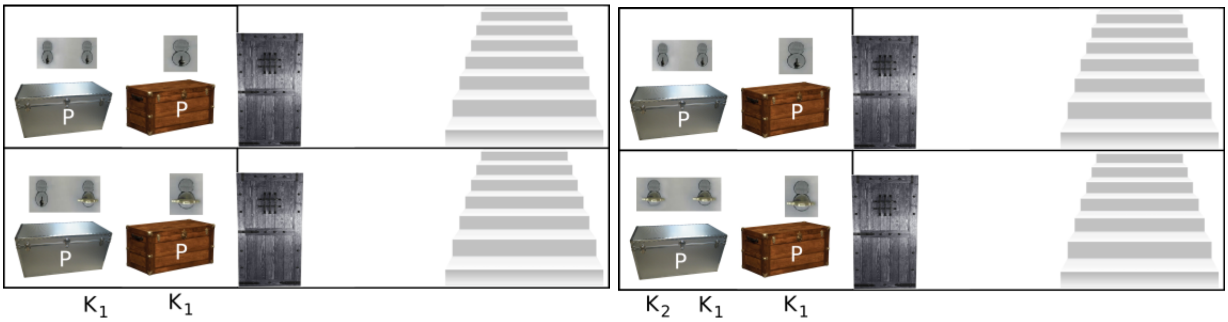
You win if no matter how gold and silver are contained within the chests (respecting condition (a)), There is a floor with open chest such that every room contains gold. A weaker winning condition is: There is a particular assignment of gold and silver to the chests (respecting condition (a)), s.t. every floor with open chest contains a room with gold.

2 example

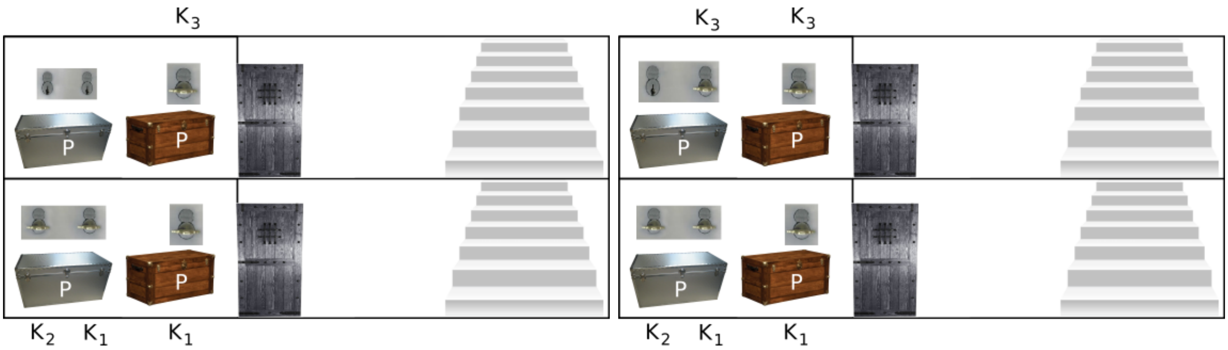
Steps 1 and 2



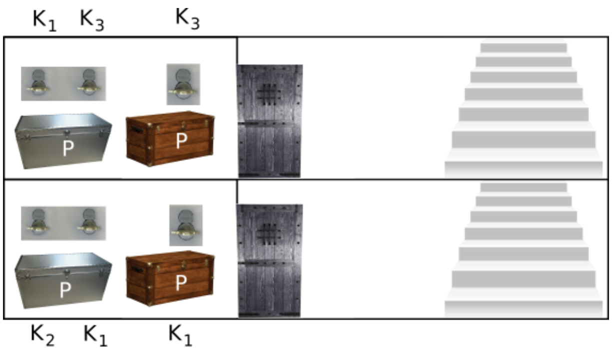
Steps 3 and 4



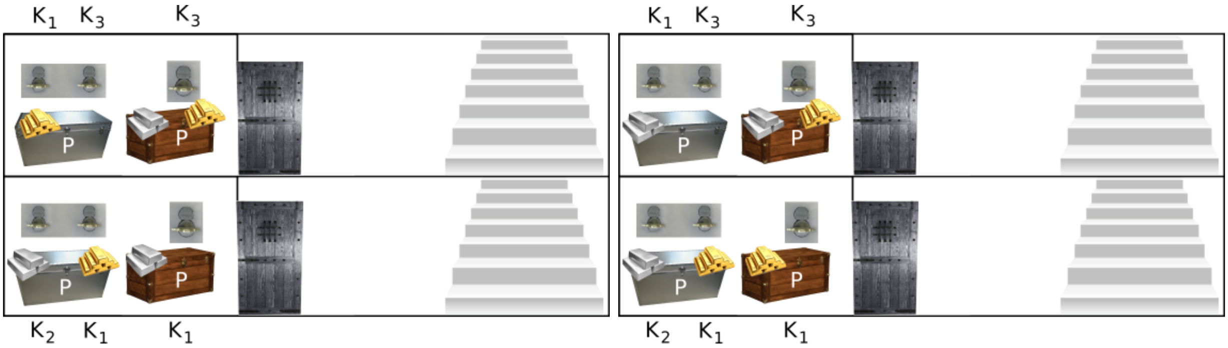
Steps 5 and 6



Step 7



Solutions



Notice that this example is none other then the drink's paradox:

$$\exists x \forall y(\neg P(x) \vee P(y))$$

which is proven by considering two instances of the clause $\neg P(x) \vee P(y)$, once with x instantiated by a and y by b and the second with x instantiated by b and y by c. Notice that this matches the keys opening the chest and the need for two floors. Also, in the solutions, it is only the assignment of gold or silver to the first chest on floor 1 and the second chest on floor 0 which matters. For the other two chest either gold or sliver may be contained.

Creating formulas which require an arbitrary high number floors is easy using the following formula schema

$$\exists x \forall y \left(\bigwedge_{i=1}^{n} \left(\left(\bigwedge_{j=i}^{n} P_j(x) \right) \Rightarrow P_i(y) \right) \right)$$

The number of floors need to win the game is precisely $n + 1$.