Differential Resultants

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Abstract. We review classical concepts of resultants of algebraic polynomials, and we adapt some of these concepts to objects in differential algebra, such as linear differential operators and differential polynomials.

1 Introduction

There are interesting notions of a differential polynomial resultant in the literature. Chardin [3] presented an elegant treatment of resultants and subresultants of (noncommutative) ordinary differential operators. Carra'-Ferro (see for example [1, 2]) published several works on differential resultants of various kinds, with firm algebraic foundations, but the relations to Zwillinger's [18] suggested notion of a differential resultant of a systems of two coupled algebraic ordinary differential equations (AODEs) and also to Chardin's theory might not be immediately clear from glancing through these works. Rueda and Sendra [14] define a linear complete differential resultant and investigate its application to the implicitization problem of systems of differential polynomial parametric equations. Zhang et al. [17] construct a matrix whose determinant contains the resultant of two generic ordinary differential polynomials as a factor.

Here we investigate the relation between the theories of Chardin and Carra'-Ferro. In fact we will see that these approaches are intimately related. It would appear that the common source for the essential basic notion of differential resultant can be traced to work of Ritt [13] in the 1930s. After reviewing relevant background material on elimination theory and differential algebra, in Sections 2 and 3, we will present in Section 4 the concepts of resultant for linear differential operators, for linear homogeneous differential polynomials, and finally for arbitrary differential polynomials. This could be viewed as a simpler and more streamlined account of Carra'-Ferro's theory. The problem posed by Zwillinger seems to be treatable by such methods; but this needs to be worked out in more detail.

Background material and more proof details may be found in [12].

2 Basics of elimination theory

In this section we briefly review basic facts from polynomial elimination theory and differential algebra.

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Let $\mathcal{R}$ be an integral domain (commutative ring with identity element 1, and no zero divisors), with quotient field $\mathcal{K}$. By $\overline{\mathcal{K}}$ we denote the algebraic closure of $\mathcal{K}$.

First we review the basic theory of the Sylvester resultant for algebraic polyomials, with an emphasis on the necessary requirements for the underlying coefficient domain. Let

$$f(x) = \sum_{i=0}^{m} a_i x^i, \quad g(x) = \sum_{j=0}^{n} b_j x^j$$

be polynomials of positive degrees m and n, respectively, in $\mathcal{R}[x]$. If f and g have a common factor $d(x)$ of positive degree, then they have a common root in the algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$; so the system of equations

$$f(x) = g(x) = 0 \tag{1}$$

has a solution in $\overline{\mathcal{K}}$.

On the other hand, if $\alpha \in \overline{\mathcal{K}}$ is a common root of f and g, then $\mathrm{norm}_{\mathcal{K}(\alpha):\mathcal{K}}(x-\alpha)$ is a common divisor of f and g in $\mathcal{K}[x]$. So, by Gauss' Lemma (for which we need $\mathcal{R}$ to be a unique factorization domain) on primitive polynomials there is a similar (only differing by a factor in $\mathcal{K}$) common factor of f and g in $\mathcal{R}[x]$. We summarize these observations as follows:

Proposition 1. *Let $\mathcal{R}$ be a unique factorization domain (UFD) with quotient field $\mathcal{K}$. For polynomials $f(x), g(x) \in \mathcal{R}[x]$ the following are equivalent:*

(i)f *and g have a common solution in $\overline{\mathcal{K}}$, the algebraic closure of $\mathcal{K}$,*

(ii)f *and g have a common factor of positive degree in $\mathcal{R}[x]$.*

So now we want to determine a necessary condition for f and g to have a common divisor of positive degree in $\mathcal{R}[x]$. Suppose that f and g indeed have a common divisor $d(x)$ of positive degree in $\mathcal{R}[x]$; i.e.,

$$f(x) = d(x)\overline{f}(x), \quad g(x) = d(x)\overline{g}(x). \tag{2}$$

Then for $p(x) := \overline{g}(x)$, $q(x) := -\overline{f}(x)$ we have

$$p(x)f(x) + q(x)g(x) = 0. \tag{3}$$

So there are non-zero polynomials p and q with $\deg p < \deg g, \deg q < \deg f$, satisfying equation (3). This leads to the linear system

$$\begin{pmatrix} p_{n-1} & \cdots & p_0 & q_{m-1} & \cdots & q_0 \end{pmatrix} \cdot \begin{pmatrix} A \\ \cdots \\ B \end{pmatrix} = 0\,, \tag{4}$$

where the $n \times (m+n)$ matrix A contains the coefficients of $f(x)$ $(a_m a_{m-1} \ldots a_0 0 \ldots 0)$ in the first row, which are shifted to the right by $(i-1)$ positions in the i-th row $(0 \ldots 0 a_m a_{m-1} \ldots a_0 0 \ldots 0)$; the $m \times (m+n)$ matrix B is constructed analogously with the coefficients of $g(x)$. Under the given conditions, the system (4) has a non-trivial solution. The matrix of this system (4) is called the *Sylvester matrix* of f and g. Thus, the determinant of the Sylvester matrix of f and g is 0. The *resultant* of f and g, $\mathrm{res}(f, g)$, is this determinant, and it is clear that the resultant is a polynomial expression of the coefficients of f and g, and therefore an element of the integral domain $\mathcal{R}$. This does not require $\mathcal{R}$ to be a UFD.

We summarize this in the following proposition.

Proposition 2. *Let $f, g \in \mathcal{R}[x]$, for $\mathcal{R}$ an integral domain.*
$\mathrm{res}(f, g) = 0$ *is a necessary condition for f and g to have a common factor of positive degree; and therefore a common solution in $\overline{\mathcal{K}}$.*

If we identify a polynomial of degree d with the vector of its coefficients of length $d + 1$, we may also express this in terms of the linear map

$$\begin{array}{rlcl} S: & \mathcal{K}^{m+n} & \longrightarrow & \mathcal{K}^{m+n} \\ & (p_{n-1}, \ldots, p_0, q_{m-1}, \ldots, q_0) & \mapsto & \text{coefficients of } pf + qg \end{array}$$

The existence of a non-trivial linear combination (3) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0.

A proof of the following may be found, for instance, in [4].

Proposition 3. *The resultant is a constant in the ideal generated by f and g in $\mathcal{R}[x]$; i.e. we can write*

$$\mathrm{res}(f, g) \;=\; u(x)f(x) + v(x)g(x), \tag{5}$$

with $u, v \in \mathcal{R}[x]$. Moreover, these cofactors satisfy the degree bounds $\deg(u) < \deg(g)$, $\deg(v) < \deg(f)$.

Another variation on defining the Sylvester resultant of two polynomials with coefficients in a UFD (unique factorization domain) $\mathcal{D}$ is to start instead with two *homogeneous* polynomials

$$F(x, y) = \sum_{i=0}^{m} a_i x^i y^{m-i} \quad , \quad G(x, y) = \sum_{j=0} b_j x^j y^{n-j} \; .$$

Let us similarly regard the coefficients a_i and b_j as indeterminates. Then the resultant of F and G is defined as $\mathrm{res}(F, G) = \mathrm{res}(f, g)$, where $f(x) = F(x, 1)$ and $g(x) = G(x, 1)$. Then the relation between the vanishing of the resultant and the existence of a common zero may be expressed as follows.

Proposition 4. *After assigning values to the coefficients from a UFD $\mathcal{D}$,* $\mathrm{res}(F, G) = 0$ *is a necessary and sufficient condition for $F(x, y)$ and $G(x, y)$ to have a common factor of positive degree over $\mathcal{D}$, hence for a common zero to exist over an extension of the quotient field of $\mathcal{D}$.*

The concept of resultant can be extended from 2 polynomials in 1 variable (or 2 homogeneous polynomials in 2 variables) to n homogeneous polynomials in n variables. This multipolynomial resultant is known as *Macaulay's resultant*. For n generic homogeneous polynomials $F_1, \ldots F_n$ in the n variables $x_1, \ldots, x_n$, of positive (total) degrees d_i, there exists a multipolynomial resultant R, which is a polynomial in the indeterminate coefficients of the F_i, with the following property. If the coefficients of the F_i are assigned values from a field K, then the vanishing of R is necessary and sufficient for a nontrivial common zero of the F_i to exist in some extension of K. Here we will only give a very brief survey, referring the reader to more comprehensive sources such as [5, 6, 10, 15, 16] for the full story. We will primarily follow the treatment of this topic in [16].

Given r homogeneous polynomials $F_1, \ldots, F_r$ in $x_1, \ldots, x_n$, with indeterminate coefficients comprising a set A, an integral polynomial T in these indeterminates (that is, $T \in \mathbb{Z}[A]$) is called an *inertia form* for $F_1, \ldots, F_r$ if $x_i^{\tau} T \in (F_1, \ldots, F_r)$, for suitable i and τ.

Van der Waerden observes that the inertia forms comprise an ideal $\mathcal{I}$ of $\mathbb{Z}[A]$, and he shows further that $\mathcal{I}$ is a *prime* ideal of this ring. It follows from these observations that we may take the ideal $\mathcal{I}$ of inertia forms to be a resultant system for the given $F_1, \ldots, F_r$ in the sense that for special values of the coefficients in $\mathcal{K}$, the vanishing of all elements of the resultant system is necessary and sufficient for there to exist a nontrivial solution to the system $F_1 = 0, \ldots, F_r = 0$ in some extension of $\mathcal{K}$.

Now consider the case in which we have n homogeneous polynomials in the same number n of variables. Let $F_1, \ldots, F_n$ be n generic homogeneous forms in $x_1, \ldots, x_n$ of positive total degrees $d_1, \ldots, d_n$. That is, every possible coefficient of each F_i is a distinct indeterminate, and the set of all such indeterminate coefficients is denoted by A. Let $\mathcal{I}$ denote the ideal of inertia forms for $F_1, \ldots, F_n$. Proofs of the following two propositions my be found in [12].

Proposition 5. *$\mathcal{I}$ is a nonzero principal ideal of $\mathbb{Z}[A]$: $\mathcal{I} = (R)$, for some $R \neq 0$.*

R is uniquely determined up to sign. We call R the *(generic multipolynomial) resultant* of $F_1, \ldots, F_n$.

Proposition 6. *The vanishing of $\mathcal{R}$ for particular $F_1, \ldots, F_n$ with coefficients in a field $\mathcal{K}$ is necessary and sufficient for the existence of a nontrivial zero of the system $F_1 = 0, \ldots, F_n = 0$ in some extension of $\mathcal{K}$.*

The above considerations also lead to the notion of a resultant of n non-nonhomogeneous polynomials in $n-1$ variables. For a given non-homogeneous $f(x_1, \ldots, x_{n-1})$ over $\mathcal{K}$ of total degree d, we may write $f = H_d + H_{d-1} + \cdots + H_0$, where the H_j are homogeneous of degree j. Then H_d is known as the *leading form* of f. Recall that the *homogenization* $F(x_1, \ldots, x_n)$ of f is defined by $F = H_d + H_{d-1}x_n + \cdots + H_0 x_n^d$.

Let $f_1, \ldots, f_n$ be particular non-homogeneous polynomials in $x_1, \ldots, x_{n-1}$ over $\mathcal{K}$ of positive total degrees d_i, and with leading forms H_{i,d_i}. We put

$$\operatorname{res}(f_1, \ldots, f_n) = \operatorname{res}(F_1, \ldots, F_n)\,,$$

where F_i is the homogenization of f_i. Then we have (see proof in [11]):

Proposition 7. *The vanishing of $\operatorname{res}(f_1, \ldots, f_n)$ is necessary and sufficient for either the forms H_{i,d_i} to have a common nontrivial zero over an extension of $\mathcal{K}$, or the polynomials f_i to have a common zero over an extension of $\mathcal{K}$.*

Observe that the common zeros of the f_i correspond to the affine solutions of the system, whereas the nontrivial common zeros of the leading forms correspond to the projective solutions on the hyperplane at infinity.

3 Basics of differential algebra

Now let us review some basic differential algebra. For this it is enough to assume $\mathcal{R}$ to be a commutative ring with 1. A *derivation* on $\mathcal{R}$ is a mapping $\partial : \mathcal{R} \to \mathcal{R}$ such that $\partial(a+b) = \partial(a) + \partial(b)$ and $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in \mathcal{R}$. That $\partial(0) = 0$ and $\partial(1) = 0$ follow readily from these axioms. We sometimes denote the derivative of a $\partial(a)$ by a'. Such a ring (or integral domain or field) $\mathcal{R}$ together with a derivation on $\mathcal{R}$ is called a *differential ring (or integral domain or field, respectively)*. In such a ring $\mathcal{R}$ elements r such that $r' = 0$ are known as *constants* and the set C of constants comprises a subring of $\mathcal{R}$. If $\mathcal{R}$ is a field, C is a subfield of $\mathcal{R}$. An ideal I of such a ring $\mathcal{R}$ is known as a *differential ideal* if $r \in I$ implies $r' \in I$. If $r_1, \ldots, r_n \in R$ we denote by $[r_1, \ldots, r_n]$ the differential ideal generated by $r_1, \ldots, r_n$, that is, the ideal generated by the r_i and all their derivatives.

Example 1. **(a)** *The familiar rings such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are differential rings if we set $\partial(a) = 0$ for all elements a.*
(b) *Let $\mathcal{K}$ be a field and t an indeterminate over $\mathcal{K}$. Then $\mathcal{K}[t]$, equipped with the derivation $\partial = d/dt$, is a differential integral domain and its quotient field $\mathcal{K}(t)$ is a differential field,*

again with standard differentiation as its derivation. $\mathcal{K}$ is the ring (field) of constants of $\mathcal{K}[t]$ ($\mathcal{K}(t)$).

Let $(\mathcal{R}, \partial)$ be a differential ring. Let $x = x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ be distinct indeterminates over $\mathcal{R}$. Put $\partial(x^{(i)}) = x^{(i+1)}$ for all $i \geq 0$. Then ∂ can be extended to a derivation on the polynomial ring $\mathcal{R}\{x\} := \mathcal{R}[x^{(0)}, x^{(1)}, \ldots]$ in a natural way, and we denote this extension also by ∂. The ring $\mathcal{R}\{x\}$ together with this extended ∂ is a differential ring, called the ring of *differential polynomials* in the differential indeterminate x over $\mathcal{R}$. An element $f(x) = \sum_{i=0}^{m} a_i x^{(i)}$ of $\mathcal{R}\{x\}$ with $a_m \neq 0$ has *order* m and *leading coefficient* a_m.

It may be helpful to think of elements of $\mathcal{R}$ and of $x, x^{(1)}, \ldots$ as functions of an indeterminate t, and to regard ∂ as differentiation with respect to t.) If $(\mathcal{K}, \partial)$ is a differential field then $\mathcal{K}\{x\}$ is a differential integral domain, and its derivation extends uniquely to the quotient field. We write $\mathcal{K}\langle x\rangle$ for this quotient field; its elements are *differential rational functions* of x over $\mathcal{K}$. We similarly denote by $\mathcal{K}\langle\eta\rangle$ the differential extension of $\mathcal{K}$ by η, where η lies in some differential field containing $\mathcal{K}$.

We consider the ring of linear differential operators $\mathcal{R}[\partial]$, where the application of $A = \sum_{i=0}^{m} a_i \partial^i$ to $r \in \mathcal{R}$ is defined as

$$A(r) = \sum_{i=0}^{m} a_i r^{(i)} \ .$$

Here $r^{(i)}$ denotes the i-fold application of ∂ (that is, $'$) to r. If $a_m \neq 0$, the *order* of A is m and a_m is the *leading coefficient* of A. Now the application of A can naturally be extended to $\mathcal{K}$, and to any extension of $\mathcal{K}$. If $A(\eta) = 0$, with η in $\mathcal{R}$, $\mathcal{K}$ or any extension of $\mathcal{K}$, we call η a *root* of the linear differential operator A.

The ring $\mathcal{R}[\partial]$ is non-commutative, satisfying the relation

$$\partial r = r\partial + r' \ .$$

From a linear homogeneous ODE $p(x) = 0$, with $p(x) \in \mathcal{R}\{x\}$,

$$p(x) = p_0(t)x + p_1(t)x' + \cdots + p_n(t)x^{(n)} = 0 \ ,$$

we can extract a linear differential operator

$$O(p) = A = \sum_{i=0}^{n} p_i \partial^i \ ,$$

such that the given ODE can be written as $A(x) = 0$, in which x is regarded as an unknown element of $\mathcal{R}$, $\mathcal{K}$ or some extension of $\mathcal{K}$. Such a linear homogeneous ODE always has the trivial solution $x = 0$; so a linear differential operator always has the trivial root 0. The ring $\mathcal{K}[\partial]$ is left-Euclidean (see [3]), so every left-ideal ${}_{\mathcal{K}}I$ of the form ${}_{\mathcal{K}}I = (A, B)$ is principal, and is generated by the right-gcd of A and B. As remarked in [3] with reference to [7], under suitable assumptions on $\mathcal{K}$, any linear differential operator of positive order has a root in some extension of $\mathcal{K}$. We state this result precisely.

Theorem 1 (Ritt-Kolchin). *Assume that the differential field $\mathcal{K}$ has characteristic 0 and that its field C of constants is algebraically closed. Then, for any linear differential operator A over $\mathcal{K}$ of positive order n, there exist n roots $\eta_1, \ldots, \eta_n$ in a suitable extension of $\mathcal{K}$, such that the η_i are linearly independent over C. Moreover, the field $\mathcal{K}\langle\eta_1, \ldots, \eta_n\rangle$ ($= \mathcal{K}\langle\eta_1\rangle \ldots \langle\eta_n\rangle$) contains no constant not in C.*

This result is stated and proved in [9] using results from [8] and [13]. The field $\mathcal{K}\langle\eta_1,\ldots,\eta_n\rangle$ associated with A is known as a *Picard-Vessiot extension* of $\mathcal{K}$ (for A). Henceforth assume the hypotheses of the Ritt-Kolchin Theorem.

It follows from the Ritt-Kolchin Theorem that if the operators $A, B \in \mathcal{K}[\partial]$ have a common factor F of positive order on the right, i.e.,

$$A = \overline{A} \cdot F, \quad \text{and } B = \overline{B} \cdot F, \tag{6}$$

then they have a non-trivial common root in a suitable extension of $\mathcal{K}$. On the other hand, if A and B have a non-trivial common root η in a suitable extension of $\mathcal{K}$, we see from the properties of a left-Euclidean ring, that they have a common right factor of positive order in $\mathcal{K}[\partial]$. We summarize this in the following proposition.

Proposition 8. *Assume that $\mathcal{K}$ has characteristic 0 and that its field of constants is algebraically closed. Let A, B be differential operators of positive orders in $\mathcal{K}[\partial]$. Then the following are equivalent:*

(i) *A and B have a common non-trivial root in an extension of $\mathcal{K}$,*

(ii)*A and B have a common factor of positive order on the right in $\mathcal{K}[\partial]$.*

The existence of a non-trivial factor (6) is equivalent to the existence of a non-trivial order-bounded linear combination

$$CA + DB = 0\,, \tag{7}$$

with order(C) < order(B) and order(D) < order(A), and $(C, D) \neq (0, 0)$.
For given $A, B \in \mathcal{K}[\partial]$, with $m = \text{order}(A), n = \text{order}(B)$, consider the linear map

$$\begin{array}{llll} S: & \mathcal{K}^{m+n} & \longrightarrow & \mathcal{K}^{m+n} \\ & (c_{n-1},\ldots,c_0,d_{m-1},\ldots,d_0) & \mapsto & \text{coefficients of } CA + DB \end{array} \tag{8}$$

Obviously the existence of a non-trivial linear combination (7) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0. Indeed we have the following result.

Proposition 9. $\det(S) = 0$ *if and only if A and B have a common factor (on the right) in $\mathcal{K}[\partial]$ of positive order.*

4 The differential Sylvester resultant

4.1 Resultant of two linear differential operators

So let us see which linear conditions on the coefficients of A and B we get by requiring that (7) has a non-trivial solution of bounded order, i.e.,

$$\text{order}(C) < \text{order}(B) \quad \text{and} \quad \text{order}(D) < \text{order}(A).$$

Example 2. We consider a differential operator A of order 2 and another operator B of order 3. We try to find an operator C of order 2 and an operator D of order 1 s.t. $C \cdot A + D \cdot B = 0$. We make an ansatz and compare coefficients of order 4 down to order 0.

$$(c_2\partial^2 + c_1\partial + c_0)(a_2\partial^2 + a_1\partial + a_0) \; + \; (d_1\partial + d_0)(b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0)$$

order 4:

$$c_2\partial^2 a_2\partial^2 + d_1\partial b_3\partial^3 = 0$$

$$\underline{a_2c_2\partial^4} + 2a_2'c_2\partial^3 + a_2''c_2\partial^2 + \underline{b_3d_1\partial^4} + b_3'd_1\partial^3 = 0$$

order 3:

$$(2a_2'c_2\partial^3 + a_2''c_2\partial^2 \text{ from above}) + c_2\partial^2 a_1\partial + c_1\partial a_2\partial^2 + (b_3'd_1\partial^3 \text{ from above}) + d_1\partial b_2\partial^2 + d_0b_3\partial^3 = 0$$

$$\underline{2a_2'c_2\partial^3} + \underline{a_1c_2\partial^3} + \underline{a_2c_1\partial^3} + a_2''c_2\partial^2 + 2a_1'c_2\partial^2 + a_1''c_2\partial + \underline{b_3'd_1\partial^3} + \underline{b_2d_1\partial^3} + \underline{b_3d_0\partial^3} + b_2'd_1\partial^2 = 0$$

order 2:

$$\begin{aligned}(a_2''c_2\partial^2 + 2a_1'c_2\partial^2 + a_1''c_2\partial + a_2'c_1\partial^2 \text{ from above}) + c_2\partial^2 a_0 + c_1\partial a_1\partial + c_0a_2\partial^2 &\\ +(b_2'd_1\partial^2 \text{ from above}) + d_1\partial b_1\partial + d_0b_2\partial^2 &= 0\end{aligned}$$

$$\begin{aligned}\underline{a_2''c_2\partial^2} + \underline{2a_1'c_2\partial^2} + a_1''c_2\partial + \underline{a_2'c_1\partial^2} + \underline{a_0c_2\partial^2} + 2a_0'c_2\partial + a_0''c_2 + \underline{a_1c_1\partial^2} + a_1'c_1\partial + \underline{a_2c_0\partial^2} &\\ +\underline{b_2'd_1\partial^2} + \underline{b_1d_1\partial^2} + b_1'd_1\partial + \underline{b_2d_0\partial^2} &= 0\end{aligned}$$

order 1:

$$(a_1''c_2\partial + 2a_0'c_2\partial + a_0''c_2 + a_1'c_1\partial \text{ from above}) + c_1\partial a_0 + c_0a_1\partial + (b_1'd_1\partial \text{ from above}) + d_1\partial b_0 + d_0b_1\partial = 0$$

$$\underline{a_1''c_2\partial} + \underline{2a_0'c_2\partial} + \underline{a_1'c_1\partial} + \underline{a_0c_1\partial} + a_0'c_1 + \underline{a_1c_0\partial} + a_0''c_2 + \underline{b_1'd_1\partial} + \underline{b_0d_1\partial} + b_0'd_1 + \underline{b_1d_0\partial} = 0$$

order 0:

$$(\underline{a_0'c_1} + \underline{a_0''c_2} \text{ from above}) + \underline{a_0c_0} + (\underline{b_0'd_1} \text{ from above}) + \underline{b_0d_0} = 0.$$

So, finally

$$\begin{pmatrix} c_2 & c_1 & c_0 & d_1 & d_0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & a_1 + 2a_2' & a_0 + 2a_1' + a_2'' & 2a_0' + a_1'' & a_0'' \\ 0 & a_2 & a_1 + a_2' & a_0 + a_1' & a_0' \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + b_3' & b_1 + b_2' & b_0 + b_1' & b_0' \\ 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}.$$

Observe, that the rows of this matrix consist of the coefficients of

$$\partial^2 A,\ \partial A,\ A, \partial B,\ B\,.$$

Theorem 2. *The linear map S in* (8) *corresponding to* (7) *is given by the matrix whose rows are $\partial^{n-1}A, \ldots, \partial A,\ A, \partial^{m-1}B, \ldots, \partial B,\ B$.*

Definition 1. *Let A, B be linear differential operators in $\mathcal{R}[\partial]$ of* $\mathrm{order}(A) = m, \mathrm{order}(B) = n$, *with $m, n > 0$.*
By $\partial\mathrm{syl}(A, B)$ we denote the (differential) Sylvester matrix*; i.e., the $(m+n) \times (m+n)$-matrix whose rows contain the coefficients of*

$$\partial^{n-1}A, \ldots, \partial A,\ A, \partial^{m-1}B, \ldots, \partial B,\ B\,.$$

The (differential Sylvester) resultant *of A and B, $\partial\mathrm{res}(A, B)$, is the determinant of $\partial\mathrm{syl}(A, B)$.*

From Propositions 8 and 9 the following analogue of Proposition 2 is immediate.
Theorem 3. *Assume that $\mathcal{K}$ has characteristic 0 and that its field of constants is algebraically closed. Let A, B be linear differential operators over $\mathcal{R}$ of positive orders. Then the condition $\partial\mathrm{res}(A, B) = 0$ is both necessary and sufficient for there to exist a common non-trivial root of A and B in an extension of $\mathcal{K}$.*

We close this subsection by stating an analogue of Proposition 3.
Theorem 4. *Let $A, B \in \mathcal{R}[\partial]$. The resultant of A and B is contained in (A, B), the ideal generated by A and B in $\mathcal{R}[\partial]$. Moreover, $\partial\mathrm{res}(A, B)$ can be written as a linear combination $\partial\mathrm{res}(A, B) = CA + DB$, with* $\mathrm{order}(C) < \mathrm{order}(B)$, *and* $\mathrm{order}(D) < \mathrm{order}(A)$.

4.2 Resultant of two linear homogeneous differential polynomials

The results for differential resultants which we have derived for linear differential operators can also be stated in terms of linear homogeneous differential polynomials. Such a treatment facilitates the generalization to the non-linear algebraic differential case.

Let $(\mathcal{R}, \partial)$ be a differential domain with quotient field $\mathcal{K}$. Then elements of $\mathcal{R}\{x\}$ can be interpreted as algebraic ordinary differential equations (AODEs). For instance, the differential polynomial

$$3\,x\,x^{(1)} + 2t\,x^{(2)} \;\in\; \mathbb{C}(t)\{x\}$$

corresponds to the AODE

$$3x(t)x'(t) + 2tx''(t) = 0\;.$$

The next proposition says that linear differential operators correspond to linear homogeneous differential polynomials $\mathcal{R}_{LH}\{x\}$ in a natural way. $\mathcal{R}[\partial]$ and $\mathcal{R}_{LH}\{x\}$ are isomporphic as left $\mathcal{R}$-modules and $\mathcal{K}[\partial]$ and $\mathcal{K}_{LH}\{x\}$ are isomporphic as left vector spaces over $\mathcal{K}$.

Definition 2. *Let $f(x)$ and $g(x)$ be elements of $\mathcal{R}_{LH}\{x\}$ of positive orders m and n, respectively. Then the* (differential) Sylvester matrix *of $f(x)$ and $g(x)$, denoted by $\partial\mathrm{syl}(f,g)$, is $\partial\mathrm{syl}(A,B)$, where $A = O(f)$ and $B = O(g)$. The* (differential Sylvester) resultant *of $f(x)$ and $g(x)$, denoted by $\partial\mathrm{res}(f,g)$, is $\partial\mathrm{res}(A,B)$.*

We may observe that the $m+n$ rows of $\partial\mathrm{syl}(f,g)$ contain the coefficients of

$$f^{(n-1)}(x), \ldots, f^{(1)}(x), f(x), g^{(m-1)}(x), \ldots, g^{(1)}(x), g(x).$$

The following analogue and slight reformulation of Theorem 3 is immediate.

Theorem 5. *Assume that $\mathcal{K}$ has characteristic 0 and that its field of constants is algebraically closed. Let $f(x), g(x)$ be linear homogeneous differential polynomials of positive orders over $\mathcal{R}$. Then the condition $\partial\mathrm{res}(f,g) = 0$ is both necessary and sufficient for there to exist a common non-trivial solution of $f(x) = 0$ and $g(x) = 0$ in an extension of $\mathcal{K}$.*

We have also an analogue and slight reformulation of Theorem 4:

Theorem 6. *Let $f(x), g(x) \in \mathcal{R}_{LH}\{x\}$. Then $x\partial\mathrm{res}(f,g)$ is contained in the differential ideal $[f,g]$.*

4.3 Resultant of two arbitrary differential polynomials

Finally we consider two arbitrary differential polynomials and we review Carra-Ferro's adaptation of the multipolynomial resultant to a pair of algebraic ordinary differential equations (AODEs) [2]. Such AODEs can be described by differential polynomials. We will deal with both homogeneous and non-homogeneous AODEs.

Suppose first we are given 2 homogeneous AODEs in the form of 2 homogeneous differential polynomial equations over a differential field $\mathcal{K}$. Observe that a homogeneous AODE of positive order always has the solution 0. So we are interested in determining whether such a pair of homogeneous AODEs has a non-trivial common solution. Denote the given homogeneous AODEs by:

$$\begin{aligned} F(x) &= 0, && \text{of order } m\,, \\ G(x) &= 0, && \text{of order } n\,. \end{aligned} \tag{9}$$

So the differential polynomial $F(x) \in \mathcal{K}\{x\}$ is of the form $F(x, x^{(1)}, \ldots, x^{(m)})$; and $G(x) \in \mathcal{K}\{x\}$ is of analogous form.

The system (9) has the same solution set as the system

$$\begin{array}{ll} F^{(n-1)}(x) = \cdots = F^{(1)} = F(x) = 0, & n \text{ equations}, \\ G^{(m-1)}(x) = \cdots = G^{(1)} = G(x) = 0, & m \text{ equations}. \end{array} \tag{10}$$

This system (10) contains the variables $x, x^{(1)}, \ldots, x^{(m+n-1)}$. So it is a system of $m+n$ homogeneous equations in $m+n$ variables. Considered as a system of homogeneous algebraic equations (with the $x^{(i)}$ considered as unrelated indeterminates), it has a multipolynomial resultant $\mathrm{res}(F^{(n-1)}, \ldots, F, G^{(m-1)}, \ldots, G)$ (defined in Subsection 4.4) whose vanishing gives a necessary and sufficient condition for the existence of a non-trivial solution over an extension of $\mathcal{K}$.

Definition 3. *For such homogeneous differential polynomials $F(x), G(x)$, we define the* (differential) resultant $\partial\mathrm{res}(F, G)$ *to be the multipolynomial resultant* $\mathrm{res}(F^{(n-1)}, \ldots, F, G^{(m-1)}, \ldots, G)$.

But, whereas a solution to the differential problem is also a solution to the algebraic problem, the converse is not true. So we do not expect the vanishing of this resultant to be a sufficient condition for the existence of a nontrivial solution to (9).

Example 3. Consider Example 4 in [2], p. 554.

$$\begin{array}{rcll} F(x) & = & xx^{(1)} - x^2 & = 0\,, \\ G(x) & = & xx^{(1)} & = 0\,. \end{array}$$

The corresponding system (10) would be the same in this case. Whereas the differential problem only has the trivial solution $x = x^{(1)} = 0$, the corresponding algebraic problem has also the non-trivial solutions $(0, a)$, for a in $\mathcal{K}$.

Indeed, in this case the differential resultant coincides with the Sylvester resultant: $\partial\mathrm{res}(F, G) = \mathrm{res}(F, G) = 0$. This reflects the fact that there are non-trivial algebraic solutions. But $x = 0$ does not lead to a non-trivial differential solution.

The following theorem follows from Proposition 6.

Theorem 7. *For such homogeneous differential polynomials $F(x), G(x)$, the vanishing of $\partial\mathrm{res}(F, G)$ is a necessary condition for the existence of a non-trivial common solution of the system $F(x) = 0, G(x) = 0$ in an extension of $\mathcal{K}$.*

Next we consider the more general case of a pair of non-homogeneous AODEs $f(x) = 0, g(x) = 0$ over $\mathcal{K}$, of orders m and n, respectively. This system has the same solution set as the system

$$\begin{array}{ll} f^{(n)}(x) = \cdots = f^{(1)} = f(x) = 0, & n+1 \text{ equations}, \\ g^{(m)}(x) = \cdots = g^{(1)} = g(x) = 0, & m+1 \text{ equations}. \end{array} \tag{11}$$

This system contains the variables $x, x^{(1)}, \ldots, x^{(m+n)}$. So it is a system of $m+n+2$ non-homogeneous equations in $m+n+1$ variables. Considered as a system of non-homogeneous algebraic equations (with the $x^{(i)}$ considered as unrelated indeterminates), it has a multipolynomial resultant $\mathrm{res}(f^{(n)}, \ldots, f, g^{(m)}, \ldots, g)$ (defined in Section 2) whose vanishing gives a necessary condition for the existence of a common solution to the system in an extension of $\mathcal{K}$.

Definition 4. *For such differential polynomials $f(x), g(x)$, we define the* (differential) resultant $\partial\mathrm{res}(f, g)$ *to be the multipolynomial resultant* $\mathrm{res}(f^{(n)}, \ldots, f, g^{(m)}, \ldots, g)$.

The following theorem follows from Proposition 7.

Theorem 8. *For such differential polynomials $f(x), g(x)$, the vanishing of $\partial\text{res}(f, g)$ is a necessary condition for the existence of a common solution of the system $f(x) = 0, g(x) = 0$ in an extension of $\mathcal{K}$.*

In the comprehensive resource [18] by Zwillinger on differential equations there appears a short section entitled "Differential Resultants". The concept of differential resultants is briefly introduced and an example of two coupled differential equations in two differential variables x and z is given, for which a differential resultant in only z is determined. However, no precise definition of the concept is given. Carra'-Ferro in [1] treats the same problem and arrives at the same answer. The treatment in both these papers is consistent with our Theorem 8.

5 Conclusion

We have reviewed formulations of different notions of resultants, for differential operators and differential polynomials. We have extended the definition of a resultant for linear homogeneous differential polynomials to the case of arbitrary differential polynomials. With this generalization, the formulations of Chardin and Carra'-Ferro are seen to be intimately related. The application of these ideas to the problem of Zwillinger for a system of coupled differential equations remains to be further investigated.

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